OPTION PRICING
WITH DISCRETE REBALANCING

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Abstract

We consider option pricing when dynamic portfolios are discretely rebalanced. The portfolio adjustments only occur after fixed relative variations of the stock price. The stock price follows a marked point process and the market is incomplete. We first characterize the equivalent martingale measures. An explicit pricing formula based on the minimal martingale measure is then provided together with the hedging strategy underlying portfolio adjustments. It is particularized on two examples : a marked Poisson process and a jump process driven by a latent geometric Brownian motion. We further examine the convergence to the Black-Scholes model when the triggering price increment shrinks to zero. For the empirical application we use IBM intraday transaction data, and compare option prices given by the marked Poisson model and the Black-Scholes model.

Key words : weak convergence, incomplete market, option pricing, minimal martingale measure, discrete rebalancing, marked point process.

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1 Introduction

Due to practical constraints portfolios are not continuously rebalanced. Traders usually readjust their positions after significant moves in the underlying asset price. It means that hedging portfolios are discretely rebalanced according to relative price movements, i.e. after an increase (or decrease) by a given percentage a. Price movements leading to these portfolio modifications are best described by a process whose logarithmic variations have fixed sizes (a or $-a$) and occur at random times. Such a process is a marked point process on the real line with one dimensional mark space. The real line depicts time while the marks describe the random jumps taking place at random times of occurrence of events. In this paper we analyse the pricing of an option when stock prices follow this kind of processes also called space-time point processes. The dynamics of such processes can be estimated on high frequency data. As will become apparent in the empirical section of the paper, our pricing methodology allows an adequate fit of the empirical features of the IBM stock data in top of being in phase with market practice.

The paper is organized as follows. In Section 2 we outline our framework and briefly portray the basic structure of a marked point process using the martingale approach. In Section 3 we characterize the set of equivalent martingale measures (the market is incomplete due to the presence of jumps and option prices are no more uniquely determined). In Section 4 we give a pricing formula when a particular choice is made within this set, namely the minimal martingale measure introduced by Föllmer and Schweizer (1991). An explicit form for the trading strategy is derived when this choice is adopted to build hedging portfolios. Both sections are closely related to the work of Buhlman, Delbaen, Embrechts and Shiryaev (1996) and Colwell and Elliott (1993). Two examples are further analyzed. The first example consists in a marked Poisson process with independent binomial increments. The second example corresponds to a jump process built from the random-crossings of a latent (hidden or unobservable) geometric Brownian motion. In Section 5, we examine the convergence of the minimal pricing model to the standard Black-Scholes (1973) model when the price increment shrinks to zero. To exemplify such a convergence, we carry out numerical simulations. In Section 6, an empirical application is provided using IBM intraday transaction data on the NYSE (New York Stock Exchange). Parameters of the marked Poisson case are estimated and used as input to compute European call option prices. These prices are compared with a Black-Scholes pricing based on an historical volatility estimate from daily closing prices. Section 7 concludes.

All proofs are gathered in an appendix.

2 Framework

Before reviewing the concepts needed to proceed further, we would like to refer the reader to Brémaud (1981), Karr (1986) and Last and Brandt (LB) (1995) for more extensive treatments of (marked) point processes. Besides the exposition of Jacod and Shiryaev (JS) (1987) is of paramount interest for the background of this paper in terms of notations and concepts.

Let us consider a sequence of positive random variables T_j satisfying $T_j < T_{j+1}$, $j = 1, \ldots$. Random elements Z_j, called the marks, take values in the mark space E, and are associated with the random times T_j. The random elements are defined on the same probability space : $(\Omega, \mathcal{F}, P)$. Each (T_j, Z_j) is said to be a marked point and the sequence $((T_j, Z_j))_j$ of marked points is referred to as a marked point process (MPP).

The mark space E is here equal to $\{a, -a\}$ and the MPP describes the dynamics of a stock price :

$$S_t = S_0 e^{X_t}, \tag{1}$$

through the random variable :

$$X_t = \sum_{j:T_j \leq t} Z_j, \tag{2}$$

built with the marked points (T_j, Z_j). The process X is a purely discontinuous process with jumps $Z_j = \Delta X_{T_j}$ at random times T_j . The jumps take the values a or $-a$, and so do the logarithmic variations of the stock price : $\Delta \log S_{T_j} = \Delta X_{T_j}$.

The process X takes the form of an integral process $x * \mu$ defined as the sum of jumps :

$$X_t = \sum_{j:T_j \leq t} \Delta X_{T_j} = x * \mu_t,$$

where $x * \mu_t$ is the usual notation for $\int_0^t \int_E x\mu(dt, dx)$.

The integer-valued random measure $\mu(dt, dx)$ on $\mathbb{R}_+ \times E$ is the counting measure associated to the marked point process. It is given by (JS p. 69) :

$$\mu(dt, dx) = \sum_{s \geq 0} \mathbb{1}_{\{\Delta X_s \neq 0\}} \epsilon_{(s, \Delta X_s)}(dt, dx),$$

where ϵ_v denotes the Dirac measure at point v.

If $E[\sum_{j:T_j \leq t} \mid Z_j \mid]$ is finite for all t (JS p. 72), such a process can be decomposed as :

$$X = x * (\mu - \nu) + x * \nu.$$

The measure ν is a predictable measure, called the compensator, with the property that $\mu - \nu$ is a local martingale measure. This measure can be disintegrated as (JS p. 67) :

$$\nu(dt, dx) = d\Lambda_t K(t, dx),$$

where Λ is a predictable integrable increasing process and K is a transition kernel. Martingales of the sort $x * (\mu - \nu)_t$ are also called compensated sums of jumps.

In the following we work with position-dependent marking (LB p. 18 and 186) :

Assumption 1 (compensator specification)
The compensator $\nu(dt, dx)$ on $\mathbb{R}_+ \times E$ satisfies :

$$\nu(dt, dx) = d\Lambda_t K(t, dx),$$

where Λ is a predictable integrable increasing process absolutely continuous w.r.t. time, K is a probability kernel on E such that $\int_E (x^2 \wedge 1) K(t, dx) < +\infty$, and $Z_1, Z_2, \ldots$ are conditionally independent given $(T_j)_j$ such that $P(Z_j \in dx | (T_j)_j) = K(T_j, dx)$.

The marked point process $(T_j, Z_j)_j$ is called a position-dependent K-marking of the point process $(T_j)_j$. The marks have a probability kernel K and the arrivals of the increments are governed by the process Λ. The process Λ represents the intensity of the arrival times of jumps and is the compensator associated to the counting measure of the point process $(T_j)_j$. The absolute continuity hypothesis ensures that the set of fixed times of discontinuity of X is empty and that the Radon-Nikodym derivative $\lambda_t = d\Lambda_t/dt$ is well defined. The process λ is called the directing intensity. Recall that the standard Poisson process corresponds to the constant directing intensity case :

$$\nu(dt, dx) = \lambda dt \epsilon_1(dx),$$

and $x * (\mu - \nu)_t = x * \mu_t - \lambda t$.

Eventually for the sake of interpretation let us remark that the compensator may also be rewritten :

$$\nu(dt, dx) = \sum_j \mathbb{1}_{\{T_j < t \leq T_{j+1}\}} \frac{G_j(dt, dx)}{G_j([t, +\infty[\times E)},$$

where $G_j(dt, dx)$ is the regular version of the conditional distribution of (T_{j+1}, Z_{j+1}) w.r.t. $\mathcal{F}_{T_j}$ (JS p. 136 or LB ch. 4 and 6). In particular if $F_j(dt) = G_j(dt \times E)$ the point process $(T_j)_j$ has a compensator $\Lambda_t = \nu([0, t] \times E)$ which reads as :

$$\Lambda_t = \sum_j \int_{T_j}^{T_{j+1} \wedge t} \frac{F_j(ds)}{F_j([s, +\infty[)}.$$

The distribution $F_j(dt)$ represents the conditional distribution of the arrival time T_{j+1} w.r.t. $\mathcal{F}_{T_j}$. The current information set $\mathcal{F}_{T_j}$ is made of the current and past realisations of the MPP. Hence λ_t can be interpreted as a conditional hazard function (probability density function divided by survival function). Besides the kernel $K(t, dx)$ is given by $\sum_j \mathbb{1}_{\{T_j < t \leq T_{j+1}\}} P(Z_{j+1} \in dx | \mathcal{F}_{T_j}, T_{j+1})$ which means that it corresponds to the conditional distribution of the mark Z_{j+1} w.r.t. $\mathcal{F}_{T_j}$ and T_{j+1}.

Now that the setting is given we may turn to the next step : the characterization of the equivalent martingale measures.

3 Equivalent martingale measures

From Harrisson and Kreps (1979), Harrisson and Pliska (1981), Delbaen and Schachermayer (1994), we know that in order to price derivative assets, we need to exhibit an equivalent measure under which discounted prices are martingales. We take here as discount factor or numéraire a savings account whose growth rate r_{T_j} on the random time interval : $]]T_{j-1}, T_j]]$ satisfies :

$$r_{T_j} = e^{\rho(T_j - T_{j-1})} - 1, \tag{3}$$

with $\rho > 0$.[2]

The discounted stock price is then equal to :

$$\tilde{S}_t = \frac{S_t}{\prod\limits_{j:T_j \le t} (1 + r_{T_j})},$$

or in a shorter notation using the Doléans-Dade exponential :

$$\tilde{S}_t = S_t / \mathcal{E}(R_t),$$

with $R_t = r * \mu_t$.

Let us introduce the discounted excess return process $\delta(t,x)$ such that for $t \in]]T_j, T_{j+1}]]$:

$$\delta(t,x) = \delta(T_j, Z_j) = \frac{e^{Z_j} - \left(1 + r_{T_j}\right)}{1 + r_{T_j}}. \tag{4}$$

The equivalent martingale measures are then characterized by the following proposition (see also Buhlman, Delbaen, Embrechts and Shiryaev (1996)).

Proposition 1 (equivalent martingale measures)
Under Assumption 1, the equivalent martingale measures Q are characterized by their density process η relative to P :

$$\eta_t = \mathcal{E}\left(\int_0^t \int_E H(s,x) d(\mu - \nu) \right), \tag{5}$$

where H is a predictable process satisfying $H + 1 > 0$ a.s. and

$$0 = \int_0^t \int_E (H(s,x) + 1) \delta(s,x) d\nu. \tag{6}$$

The process η is a P-martingale such that η_t is the Radon-Nikodym derivative dQ_t/dP_t. The process H has the interpretation of a jump risk premium process. Equation (6) introduces restrictions on this process but does not lead to a unique equivalent martingale measure (market incompleteness). In fact for $t \in]]T_j, T_{j+1}]]$ the process should satisfy $H(t,x) = h_j(x)$ with $h_j(x) + 1 > 0$ a.s. and

$$0 = \int_E (h_j(x) + 1) \delta(T_j, x) K(T_{j+1}, dx).$$

Then the discounted price $\tilde{S}$ will be a martingale under the corresponding measure Q.

[2] For a marked point process where the directing intensity λ_t and the kernel $K(t,dx)$ depend on time, we cannot use the usual deterministic savings account $\beta_t = \exp(\rho t)$. Arbitrage is not precluded in such a case.

4 Option pricing with the minimal martingale measure

A very popular measure among equivalent martingale measures is the minimal martingale measure introduced by FÖLLMER and SCHWEIZER (1991). This measure is minimal among all martingale measures in the sense that, apart from turning discounted prices into martingales, it leaves unchanged the remaining structure of the model. In particular orthogonality relations are kept in a continuous framework (see SCHWEIZER (1991,1992a) for a full discussion, and HOFMANN, PLATEN and SCHWEIZER (1992) for an illuminating use in stochastic volatility models). This measure is computationally convenient and induces nice convergence properties (RUNGGALDIER and SCHWEIZER (1995), PRIGENT (1995), MERCURIO and VORST (1996), LESNE, PRIGENT and SCAILLET (2000)). SCHWEIZER (1993) shows that in some cases the expectation of the final payoff under the minimal measure is equal to the value of the variance optimal hedging strategy (for the variance optimal measure see e.g. FÖLLMER and SONDERMANN (1986), BOULEAU and LAMBERTON (1989), DUFFIE and RICHARDSON (1991), SCHWEIZER (1992b,1994), GOURIÉROUX, LAURENT and PHAM (1998), LAURENT and SCAILLET (1998)). This value or approximation price gives the initial amount required to implement a risk-minimizing strategy. Such a strategy is in general not self-financing and involves a non-vanishing hedging cost. However as soon as the minimal measure is a probability measure, this value is an actual no-arbitrage price (it is the case below since the jump size is bounded). Concerning existence and uniqueness of the minimal measure, we refer to ANSEL and STRICKER (1992, 1993). We can now proceed to answer what form this measure takes in our framework (see also COLWELL and ELLIOTT (1993)).

Proposition 2 (minimal probability measure)
Under Assumption 1, the minimal martingale measure $\hat{P}$ is a probability measure characterized by its density process $\hat{\eta}$ relative to P :

$$\hat{\eta}_t = \mathcal{E}\left((-\alpha\tilde{S}\delta) * (\mu - \nu)_t\right), \tag{7}$$

where for $t \in]]T_j, T_{j+1}]]$:

$$\alpha_t = \frac{\int_E \delta(T_j, x) K(T_{j+1}, dx)}{\tilde{S}_{T_j} \int_E \delta^2(T_j, x) K(T_{j+1}, dx)}. \tag{8}$$

If we relate this proposition to Proposition 1, we have for $t \in]]T_j, T_{j+1}]]$, $\hat{H}(t,x) = \hat{h}_j(x)$ with :

$$\hat{h}_j(x) = -\frac{\int_E \delta(T_j, v) K(T_{j+1}, dv)}{\int_E \delta^2(T_j, v) K(T_{j+1}, dv)} \delta(T_j, x).$$

Once the Radon-Nikodym derivative $\hat{\eta}_t$ is computed, it is straightforward to derive the price $C_t = C(t, S_t)$ of a contingent claim with final payoff $C(T, S_T)$. For a European call option with maturity T and strike price K, the final payoff is $(S_T - K)_+ = \max(0, S_T - K)$. By taking its expectation under $\hat{P}$ after an adequate discounting, we get :

$$C(t, S_t) = E^{\hat{P}}\left[(S_T - K)_+ \frac{\mathcal{E}(R_t)}{\mathcal{E}(R_T)} | \mathcal{F}_t\right], \tag{9}$$

which leads to :

Proposition 3 (minimal option price)
Under Assumption 1, the call price given by the minimal martingale measure is :

$$C(t,S_t) = E^P\left[(S_T-K)_+\frac{\mathcal{E}(R_t)}{\mathcal{E}(R_T)}\frac{\hat{\eta}_T}{\hat{\eta}_t}|\mathcal{F}_t\right], \tag{10}$$

with : $\mathcal{E}(R_t)/\mathcal{E}(R_T) = \prod_{j:t<T_j\leq T}(1+r_{T_j})^{-1}$ *and :*

$$\frac{\hat{\eta}_T}{\hat{\eta}_t} = \prod_{j:t<T_j\leq T}(1+\hat{h}_j(Z_j))\exp\left(-\int_t^T\int_E \hat{H}(s,x)K(s,dx)d\Lambda_s\right).$$

Let us specialize this option pricing formula on two examples.

Example 1 : Marked Poisson process

The first example consists in a marked Poisson process with independent binomial marks. The corresponding compensator specification is :

Assumption 2 (marked Poisson process)
The compensator $\nu(dt,dx)$ *on* $I\!R_+\times\{a,-a\}$ *satisfies :*

$$\nu(dt,dx) = \lambda dt K(dx),$$

with

$$\begin{aligned} K(dx) &= p \qquad if \quad dx = a, \\ &= 1-p \qquad if \quad dx = -a. \end{aligned}$$

This example is particularly suited for empirical purposes since the constant parameters λ and p can easily be estimated from intraday price data (see the empirical illustration below). We get immediately from Proposition 3 :

Corollary 1 (option price : marked Poisson process)
Under Assumption 2, the minimal call price is given by (10) with :

$$\begin{aligned} \frac{\hat{\eta}_T}{\hat{\eta}_t} &= \prod_{j:t<T_j\leq T}\left(1-\frac{\delta(T_j,a)p+\delta(T_j,-a)(1-p)}{\delta(T_j,a)^2p+\delta(T_j,-a)^2(1-p)}\delta(T_j,Z_j)\right) \\ &\quad \exp\left(\lambda\left((T_{j+1}\wedge T)-T_j\right)\frac{(\delta(T_j,a)p+\delta(T_j,-a)(1-p))^2}{\delta(T_j,a)^2p+\delta(T_j,-a)^2(1-p)}\right). \end{aligned} \tag{11}$$

When the interest rate is set equal to zero ($\rho = 0$), we have an analytic formula for the call price. From (10) and (11), the call price is given by :

$$C(t, S_T) = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} e^{(-(1-\gamma)\lambda T)} \left(\sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!} (p\eta_u)^i \left((1-p)\,\eta_d\right)^{n-i} \left(S_0 e^{((2i-n)a)} - X\right)_+ \right),$$

where :

$$\begin{aligned}
\eta_u &= 1 - k\,(\exp(a) - 1)\,, \\
\eta_d &= 1 - k\,(\exp(-a) - 1)\,, \\
k &= \frac{(p\,(\exp(a)-1) + (1-p)\,(\exp(-a)-1))}{p\,(\exp(a)-1)^2 + (1-p)\,(\exp(-a)-1)^2}, \\
\gamma &= \frac{(p\,(\exp(a)-1) + (1-p)\,(\exp(-a)-1))^2}{p\,(\exp(a)-1)^2 + (1-p)\,(\exp(-a)-1)^2}.
\end{aligned}$$

The value of the call can thus be obtained by taking a weighted sum of recombining binomial trees. Some rewriting enables us to obtain the simpler form :

$$C(t, S_T) = \sum_{n=0}^{\infty} \frac{\left(\widehat{\lambda} T\right)^n}{n!} e^{-\widehat{\lambda} T} \left(\sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!} \widehat{p}^i\,(1-\widehat{p})^{n-i} \left(S_0 e^{((2i-n)a)} - X\right)_+ \right), \tag{12}$$

with :

$$\widehat{p} = \frac{p\eta_u}{(1-\gamma)} = \frac{1}{e^a + 1}, \tag{13}$$

$$\widehat{\lambda} = (1-\gamma)\,\lambda. \tag{14}$$

Formula (12) in fact corresponds to calculating the expectation of the payoff directly under the Minimal Martingale Measure $\widehat{P}$ as in equation (9). Indeed under $\widehat{P}$ we have that S_t is a marked Poisson process with intensity $\widehat{\lambda}$ and transition kernel $\widehat{K}(t, a) = \widehat{p}$. The mappings (13) and (14) can be obtained by applying Girsanov theorem for jumps (JS p.157). Interestingly, we also recover the property of the standard binomial model (Cox, Ross and Rubinstein (1979)) that the risk-neutral up-move ($\widehat{p}$) and down-move ($1 - \widehat{p}$) probabilities do not depend on historical probabilities. Formula (12) is a sum of standard binomial pricing formulae weighted by the probabilities of the exponential distribution. The pricing formula can thus be derived analytically by integrating w.r.t. to the exponential distribution and by weighting the different possible option payoffs by their binomial probabilities.

When $\rho \neq 0$ one cannot build recombining trees. However, the price of our option can be computed via Monte Carlo integration by simulating the stock price process directly under $\widehat{P}$ (cf equation (9)). This ensures better numerical stability than simulating under P and using equation (10). The intensity and transition kernel under $\widehat{P}$ are characterised by :

$$
\begin{aligned}
d\widehat{\Lambda}_t &= \widehat{\lambda} dt = (1-\gamma)\,\lambda dt, \\
\widehat{K}(t,a) &= \widehat{p} = \frac{p\left(\widehat{h}(T_j,a)+1\right)}{I(T_j,a)},
\end{aligned}
$$

with :

$$
\begin{aligned}
\widehat{h}(T_j,x) &= -\delta(T_{j+1},x)\,\frac{p\delta(T_{j+1},a)+(1-p)\,\delta(T_{j+1},-a)}{p\delta^2(T_{j+1},a)+(1-p)\,\delta^2(T_{j+1},-a)}, \\
I(T_j,a) &= 1-\frac{(p\delta(T_{j+1},a)+(1-p)\,\delta(T_{j+1},-a))^2}{p\delta^2(T_{j+1},a)+(1-p)\,\delta^2(T_{j+1},-a)}.
\end{aligned}
$$

Example 2 : Latent geometric Brownian motion

Let us constitute the set of barriers $B = \{S_0 \exp ja, j \in \mathbb{Z}\}$. The random crossings of such barriers by the continuous process $\bar{S}$ satisfying $d\bar{S}_t/\bar{S}_t = mdt + sdW_t$ define a price process S of the form (1)-(2). It is enough to take : $\exp X_t = \bar{S}_t \cap B$. The entire path of the geometric Brownian motion is assumed to be not observable and the continuous time process driving the jumps is thus latent (hidden). The distributional features of the (geometric) Brownian motion can nevertheless be used to identify the distribution of the jump process X.

The conditional distribution of arrival times is characterized by the probability that a Brownian motion with drift escapes the corridor $\{a, -a\}$. This probability can be deduced from the trivariate distribution of the running minimum, running maximum and end value of a Brownian motion (Revuz and Yor (1994) p. 104) after a suitable change of measure to incorporate the presence of a drift (see also Kunitomo and Ikeda (1992), Géman and Yor (1994), He, Keirstead and Rebholz (1998)). The conditional distribution of marks is given by the probability that a Brownian motion with drift hits one barrier before the other (Karlin and Taylor (1975) p. 361).

Consequently we replace in this example Assumption 1 by the more precise assumption:

Assumption 3 (latent geometric Brownian motion)
The compensator $\nu(dt,dx)$ on $\mathbb{R}_+ \times \{a,-a\}$ satisfies :

$$
\nu(dt,dx) = \lambda_t dt K(dx),
$$

where for $t \in]]T_j, T_{j+1}]]$:

$$
\lambda_t = \left[-\frac{d}{dt}S(t-T_j)\right]/S(t-T_j),
$$

and

$$
\begin{aligned}
K(dx) &= \frac{e^{m'a/s^2}}{e^{-m'a/s^2}+e^{m'a/s^2}} \qquad if \quad dx = a, \\
&= \frac{e^{-m'a/s^2}}{e^{-m'a/s^2}+e^{m'a/s^2}} \qquad if \quad dx = -a,
\end{aligned}
$$

with $m' = m - s^2/2$ *and*

$$S(u) = \sum_{k=-\infty}^{+\infty} e^{4km'a/s^2} \left\{ \left[\Phi\left(\frac{a - m'u - 4ka}{s\sqrt{u}}\right) - \Phi\left(\frac{-a - m'u - 4ka}{s\sqrt{u}}\right) \right] \right.$$
$$\left. - e^{-2am'/s^2} \left[\Phi\left(\frac{3a - m'u - 4ka}{s\sqrt{u}}\right) - \Phi\left(\frac{a - m'u - 4ka}{s\sqrt{u}}\right) \right] \right\}.$$

In this setting the survival function $S(u)$ is equal to the probability that a Brownian motion lives during a time period u between $-a$ and a, or equivalently to the probability that the running minimum and maximum stay above $-a$ and below a, respectively (see Duffie and Lando (1997) for similar computations of hazard rate processes in default event modelling). We are now able to give the option pricing formula.

Corollary 2 (option price : latent geometric Brownian motion)
Under Assumption 3, the minimal call price is given by (10) with :

$$\frac{\hat{\eta}_T}{\hat{\eta}_t} = \prod_{j:t<T_j\leq T} \left(1 - \frac{\delta(T_j,a)K(a) + \delta(T_j,-a)K(-a)}{\delta(T_j,a)^2K(a) + \delta(T_j,-a)^2K(-a)}\delta(T_j,Z_j)\right)$$
$$\exp\left(-\frac{(\delta(T_j,a)K(a) + \delta(T_j,-a)K(-a))^2}{\delta(T_j,a)^2K(a) + \delta(T_j,-a)^2K(-a)}\log S((T_{j+1}\wedge T) - T_j)\right).$$

To close this section it remains to exhibit the composition (ϕ_t, ψ_t) of the hedging strategy which generates $C(T, S_T)$. The processes ϕ and ψ represent the quantities held in the risky and riskless asset, respectively. The discounted price of the contingent claim is denoted : $\tilde{C}(t, S_t) = C(t, S_t)/\mathcal{E}(R_t)$.

Proposition 4 (minimal trading strategy)
Under Assumption 1, the minimal trading strategy (ϕ_t, ψ_t) *is given by :*

$$\phi_t = \frac{\int_E \tilde{\phi}_{T_j}(x)\delta(T_j,x)K(T_{j+1},dx)}{\tilde{S}_{T_j}\int_E \delta^2(T_j,x)K(T_{j+1},dx)}, \tag{15}$$

and :

$$\psi_t = \tilde{C}(t, S_t) - \phi_t\tilde{S}_t, \tag{16}$$

for $t \in]]T_j, T_{j+1}]]$ *with :* $\tilde{\phi}_{T_j}(x) = \tilde{C}(T_j, S_{T_j}e^x) - \tilde{C}(T_j, S_{T_j})$.

5 Convergence to the Black-Scholes model

Hitherto we did not put much structure on the distributions of arrival times and marks. Let us specify them in order to get convergence to the Black-Scholes model when the increment a shrinks to zero.

In the BLACK-SCHOLES model, the stock price evolves according to a geometric Brownian motion :

$$\bar{S}_t = S_0 \exp\left((m - \frac{s^2}{2})t + sW_t\right), \tag{17}$$

and the savings account value according to : $\exp(\rho t)$. We use $\stackrel{P}{\longrightarrow}$ for convergence in probability, and $\stackrel{\mathcal{L}(\mathbb{D})}{\longrightarrow}$ for weak convergence on $\mathbb{D}$ the space of cadlag functions endowed with the customary Skorokhod topology. We index all relevant quantities by a, and convergence should be understood when letting a go to zero.

Proposition 5 (convergence)
Under Assumption 1, if :

$$\delta * \nu_t^a \stackrel{P}{\longrightarrow} (m-\rho)t, \tag{18}$$

$$\delta^2 * \nu_t^a \stackrel{P}{\longrightarrow} s^2 t, \tag{19}$$

then :

$$\left(\tilde{S}_t^a, \hat{\eta}_t^a\right) \stackrel{\mathcal{L}(\mathbb{D}^2)}{\longrightarrow} \left(\bar{S}_t e^{-\rho t}, \mathcal{E}\left(\frac{m-\rho}{s} W_t\right)\right). \tag{20}$$

Proposition 5 embodies the convergence of the incomplete model based on the marked point process to the BLACK-SCHOLES model. Indeed the proposition states the joint convergence of the sequence of discounted stock prices and Radon-Nikodym derivatives of the minimal measure. We recognize in (20) the well-known change of measure from the historical probability to the risk neutral measure of the limiting complete model. As usual convergence of contingent claim prices will be ensured under appropriate continuity and equiintegrability conditions on the payoffs.

Now we analyze in some details the marked Poisson model. For computational convenience we set the interest rate equal to zero.

Proposition 6 (convergence : marked Poisson process)
Under Assumption 2 and $\rho = 0$, if $p^a - 1/2 \sim ma/(2s^2)$ and $\lambda^a \sim s^2/a^2$, then :

$$(S_t^a, \hat{\eta}_t^a) \stackrel{\mathcal{L}(\mathbb{D}^2)}{\longrightarrow} \left(S_0 \mathcal{E}\left(mt + sW_t\right), \mathcal{E}\left(\frac{m}{s} W_t\right)\right).$$

Proposition 6 gives the condition on the probability p^a and the directing intensity λ^a so that the marked Poisson model coincides with the BLACK-SCHOLES model in the limit.

Figure 1 shows the smooth form of the convergence obtained for European call option prices when using equation (12). The numerical example is designed with : $S_0 = 100$, $m = 5\%$, $s = 25\%$, $K = 100$, and $T = 0.25$. The straight line is the BLACK-SCHOLES price equal to 4.9835, while the dashed line gives the price computed by the marked Poisson model with $p^a = \left(ma^2/s^2 + (1-e^{-a})\right) / \left(e^a - e^{-a}\right)$ and $\lambda^a = s^2/a^2$.

Finally, we are able to deliver the analogous result to proposition (6) for the latent geometric Brownian motion case. Some numerical results show that the convergence is very similar but less smooth because of the use of simulations.

Proposition 7 (convergence : latent geometric Brownian motion)
Under Assumption 3 and $\rho = 0$:

$$\left(\tilde{S}_t^a, \hat{\eta}_t^a\right) \overset{\mathcal{L}(\mathbb{D}^2)}{\longrightarrow} \left(S_0 \mathcal{E}\left(mt + sW_t\right), \mathcal{E}\left(\frac{m}{s} W_t\right)\right).$$

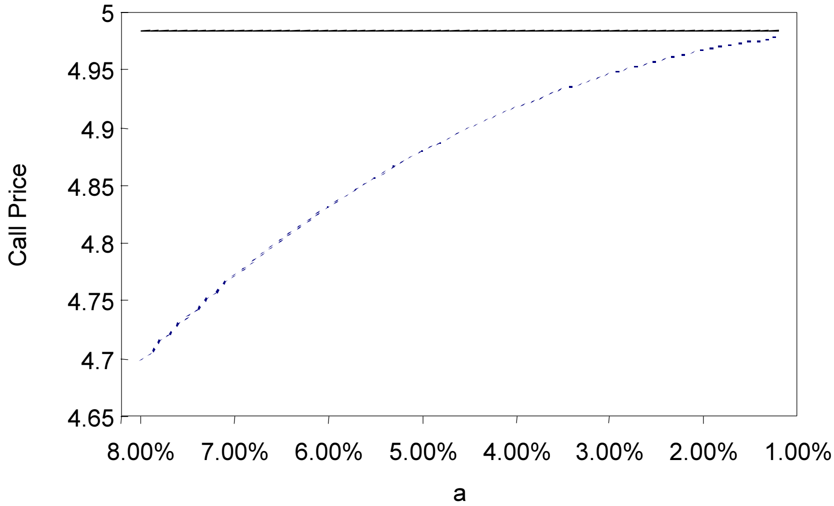

Figure 1: Convergence of option prices : marked Poisson process

6 An empirical illustration : IBM transaction data

This section illustrates the empirical application of the marked Poisson model to the pricing of European call options.[3] The model parameters λ and p are estimated from intraday transaction data. The data were extracted from the Trades and Quotes Database (TAQ Database) released by the NYSE. The observations are the trades of IBM stock recorded every second from market opening (9:30:00) to market closure (16:00:00). They consist of 486,513 transactions beginning on Thursday January 2th 1997 and ending on Wednesday September 30th 1997 (9 months).

The estimators are obtained by maximum likelihood. The estimator of λ is the inverse of the empirical average of durations between two successive relative price changes (recall that if events follow a Poisson distribution, time intervals are exponentially distributed). The estimator of p is the observed proportion of positive variations. To test the null hypothesis that the durations are exponentially distributed, we use a test statistic based on the equality between the mean and the standard deviation (see e.g. Chesher and Spady (1991) or Chesher, Dhaene, Gouriéroux, and Scaillet (1999)), and a Bootstrap procedure to account for finite sample level distortions. We consider models with absolute variation sizes a ranging from 0.25% to 5%. Common practice on the market is to rebalance after relative variations by 3% or 4%. An observed path of log S_t for the IBM stock is plotted on Figure 2. It corresponds to the period from January to September 97 with the jump size $a = 3\%$. Table 1 reports the estimators of λ and p, with their respective standard

[3]Gauss programs developed for this section are available on request.

deviations within brackets, and the number (Nb.) of observed relative price changes for each absolute variation size. The intensity estimate is in 1/sec. and should be multiplied by 60 (sec.) $\times$ 60 (min.) $\times$ 6.5 (hours) $\times$ 250 (days) = 5,850,000 in order to get an annualized intensity. The hypothetized exponential behavior of the durations is rejected at the 5% significance level for models with too narrow rebalancing bounds ($a \leq 2\%$). For the significant models, a jump is expected every 2, 4 and 6 days for $a = 3\%$, 4%, and 5%, respectively. The upmove probabilities are not far from one half.

Table 1 : Parameter estimates for IBM stock (in percent)

a	λ	p	Nb.
0.25	0.2849 (0.0054)	50.52 (0.45)	12501
0.5	0.0846 (0.0027)	50.87 (0.82)	3715
0.75	0.0407 (0.0017)	51.20 (1.18)	1789
1	0.0242 (0.0013)	51.55 (1.53)	1065
2	0.0050 (0.0004)	53.60 (3.35)	222
3	0.0021 (0.0002)	55.91 (5.15)	93
4	0.0010 (0.0001)	59.09 (7.41)	44
5	0.0007 (0.0001)	60.00 (8.94)	30

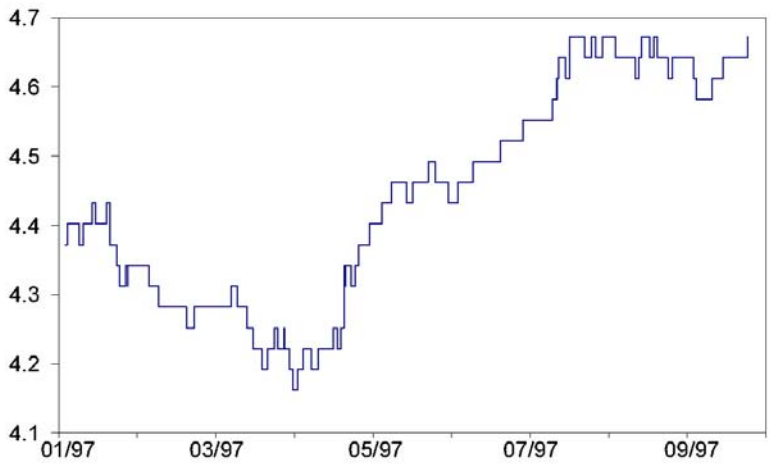

Figure 2 : MPP based on IBM stock price with a=3%

With these estimates we may compare option prices given by the marked Poisson model and the Black-Scholes model. We use an historical approach to estimate the volatility in the Black-Scholes model (unfortunately, we do not have option data to compute implied volatilities). The volatility estimator is the standard deviation of daily closing prices from Thursday January 2nd 1997 to Thursday March 27th 1997 (3 months). This choice corresponds to market standards. We get a volatility estimate of 2.19%, which gives an annualized volatility ($\times\sqrt{250}$) of 34.67%. This is also close to the annualized volatility computed on the whole dataset : 31.98%. The interest rate is taken equal to 5%. The maturity of the European call option is three months. The initial stock price S_0 is normalized to 100 and strike prices K expressed in percentage of spot price (spot moneyness degree) are between 90% and 110%. Table 2 only gives the results for the statistically significant models of Table 1. Monte-Carlo integration with 200,000 replications is used to calculate expectation (9). To reduce the variance of the simulations, we use $\tilde{S}_T$ as control variate device. Note that the empirical martingality of $\tilde{S}_T$ is satisfied up to a precision of at least 10^{-3}. All call prices are below their BLACK-SCHOLES equivalents. From put-call parity, it also implies that all put prices will be below their BLACK-SCHOLES counterparts. The prices for $a = 5\%$ are higher than for $a = 4\%$. This is due to the compensation of the lower intensity by the higher upmove probability.

Table 2 : Option prices (in percent)

a	K/S_0	Poisson	BS
3	90	0.13314	0.13529
	95	0.09976	0.10231
	100	0.07215	0.07503
	105	0.05097	0.05340
	110	0.03482	0.03693
4	90	0.12858	0.13529
	95	0.09415	0.10231
	100	0.06583	0.07503
	105	0.04491	0.05340
	110	0.02973	0.03693
5	90	0.13052	0.13529
	95	0.09660	0.10231
	100	0.06907	0.07503
	105	0.04805	0.05340
	110	0.03284	0.03693

7 Concluding remarks

This paper proposes explicit formulae for pricing contingent claims when the underlying price follows a marked point process. The formula relies on a minimal martingale approach. It is particularly suited for discrete hedging in which portfolios are only rebalanced after fixed relative price variations. The practical use of the pricing model is illustrated with a marked Poisson model estimated on high frequency transaction data. The marked Poisson model is shown to be able to provide a good fit for large variation sizes but not for small sizes. However, our theoretical framework encompasses very general dynamics for the stock price. A wide range of empirical applications based on more sophisticated econometric models, e.g. developed for microstructure theory, can thus be proposed. The relevance of such extensions is left to future research.

APPENDIX :

Proof of Proposition 1

The proof consists in finding the adequate restriction on the change of measure dQ/dP so that the discounted stock price $\tilde{S}$ is a local martingale under the new measure Q. It parallels the developments in BUHLMAN, DELBAEN, EMBRECHTS and SHIRYAEV (1996).

We denote by $\mathcal{M}_{loc}(Q)$ the set of all local martingales under Q. The density process η of Q relative to P can be rewritten thanks to the theorem of representation of martingales :

$$\eta_t = \mathcal{E}\left(\int_0^t \int_E H(s,x)d(\mu - \nu)\right),$$

where the predictable process H satifies some integrability conditions (JS p. 172) and $H + 1 > 0$ a.s. since η is strictly positive.

Furthermore $\tilde{S}_t \in \mathcal{M}_{loc}(Q)$ if and only if $\tilde{S}_t\eta_t \in \mathcal{M}_{loc}(P)$. From Ito's and Yor's formulas (PROTTER (1990) p. 78, REVUZ and YOR (1994) p. 354) we have :

$$\tilde{S}_t\eta_t = \mathcal{E}\left(\int_0^t \int_E \delta(s,x)d\mu + H(s,x)d(\mu-\nu) + H(s,x)\delta(s,x)d\nu\right).$$

This implies that $\int_0^t \int_E (H(s,x)+1)\delta(s,x)d\nu$ should be equal to zero in order to get $\tilde{S}_t\eta_t \in \mathcal{M}_{loc}(P)$, which ends the proof.

Proof of Proposition 2

We know from ANSEL and STRICKER (1992,1993) that if $1 - \alpha\Delta\tilde{M} > 0$ a.s. and $E^P\left[\sup_{0\le s\le T} \tilde{S}_s^2\right] < +\infty$, the minimal martingale measure is a probability measure characterised by its density process $\hat{\eta}$ relative to P :

$$\hat{\eta} = \mathcal{E}(-\alpha.\tilde{M}),$$

with $\tilde{M}$ a locally bounded integrable martingale and α a predictable process satisfying :

$$\tilde{S} = \tilde{M} + \alpha.\langle\tilde{M},\tilde{M}\rangle,$$

and where the dot denotes the stochastic integration of a predictable process w.r.t. a semimartingale. The predictable process α is given by the relation :

$$\alpha_t d\langle\tilde{M},\tilde{M}\rangle_t = d\tilde{A}_t, \tag{21}$$

where $\tilde{A}$ is the predictable bounded part of $\tilde{S}$.

In our case the martingale part of $\tilde{S}$ is equal to : $\tilde{M} = (\tilde{S}\delta) * (\mu - \nu)$ and its predictable quadratic variation (angle bracket) equal to : $\langle\tilde{M},\tilde{M}\rangle = (\tilde{S}\delta)^2 * \nu$ (JS 1.33 p. 73). Furthermore $\tilde{A} = (\tilde{S}\delta) * \nu$ which gives equation (8) using relation (21). Besides since the jump size is bounded, $\hat{\eta}$ is a strictly positive local martingale and the minimal martingale measure is a probability measure.

Proof of Proposition 4

We follow COLWELL and ELLIOTT (1993).

On the one hand, since $\tilde{C}$ is a $\hat{P}$-martingale, it comes from the martingale representation theorem and Ito's lemma that :

$$\tilde{C}(t, S_t) = \tilde{C}(0, S_0) + \tilde{\phi} * (\mu - \hat{\nu})_t, \tag{22}$$

with $\tilde{\phi}_t(x) = \tilde{C}(T_j, S_{T_j} e^x) - \tilde{C}(T_j, S_{T_j})$ for $t \in]]T_j, T_{j+1}]]$.

On the other hand we have (see e.g. SCHWEIZER (1991)) :

$$\tilde{C}(t, S_t) = \tilde{C}(0, S_0) + (\phi \tilde{S} \delta) * \mu_t + \Gamma_t, \tag{23}$$

where Γ_t is a martingale under P.

This process represents the cost associated with the trading strategy, and can be rewritten from (22) and (23) :

$$\Gamma = \tilde{\phi} * (\mu - \hat{\nu}) - (\phi \tilde{S} \delta) * \mu.$$

Since it is a martingale under P, we deduce from :

$$\tilde{\phi} * (\mu - \hat{\nu}) = \tilde{\phi} * (\mu - \nu) + \tilde{\phi} * (\nu - \hat{\nu}),$$

and :

$$(\phi \tilde{S} \delta) * \mu = (\phi \tilde{S} \delta) * (\mu - \nu) + (\phi \tilde{S} \delta) * \nu,$$

that the following equality should hold :

$$\tilde{\phi} * (\nu - \hat{\nu}) = (\phi \tilde{S} \delta) * \nu.$$

Using $\hat{\nu} = (\hat{H} + 1)\nu$ and the expression of $\hat{H}$, the result follows.

Proof of Proposition 5

Conditions (18)-(19) and the jump boundedness ensure the convergence of the first Doléans-Dade exponential $\mathcal{E}\left(\delta * \mu_t^a\right)$ to $\mathcal{E}\left((m - \rho)t + sW_t\right)$ (JS 3.11 p. 432). Since the martingale part of the discounted price is uniformly tight (the jumps are bounded and the predictable part is increasing) we conclude the convergence of the second Doléans-Dade exponential $\hat{\eta}_t^a = \mathcal{E}\left((-\alpha \tilde{S} \delta) * (\mu^a - \nu^a)_t\right)$ as in PRIGENT (1995) Proposition 3.3 or LESNE, PRIGENT and SCAILLET (2000) Proposition 1.

Proof of Proposition 6

Conditions (18) and (19) of Proposition 5 can be rewritten as :

$$\begin{array}{rcl}\lambda^a \left[(e^a-1)\,p^a + (e^{-a}-1)\,(1-p^a)\right] & \longrightarrow & m, \\ \lambda^a \left[(e^a-1)^2\,p^a + (e^{-a}-1)^2\,(1-p^a)\right] & \longrightarrow & s^2.\end{array}$$

From Taylor expansions, it can be verified that both conditions are satisfied if : $p^a - 1/2 \sim ma/(2s^2)$ and $\lambda^a \sim s^2/a^2$. The stated result is then a direct consequence of Proposition 5.

Proof of Proposition 7

i) The weak convergence of the stock price is immediately deduced from the construction of X^a since :

$$\sup_{t\in[0,T]} |X_t^a - ((m - \frac{s^2}{2})t + sW_t)| \leq a$$

(see JS 3.30 and 3.31 p. 316 with $t^a = X_t^a - (m - \frac{s^2}{2})t + sW_t$ and $Y_t^a = (m - \frac{s^2}{2})t + sW_t$).

ii) To deduce the weak convergence of the density process we need first to establish that X^a has the property UT (uniform tightness see Mémin and Slominsky (1991) for a definition). In order to do so, we will need the following key lemma which gives the behavior of the Laplace transform of the expectation of the number of jumps for small a. Its proof is given below.

Lemma

If N_t^a denotes the number of jumps on the time interval $[0,t]$,we have for small a :

$$\int_0^\infty e^{-\xi t} \mathbb{E}[N_t^a]dt \sim \frac{1}{\xi^2}\frac{s^2}{a^2}.$$

Now, recall that $X_t^a = (x * \nu^a)_t + x * (\mu^a - \nu^a)_t$. From Condition (ii) 2) of Theorem 1-4 of Mémin and Slominsky (1991), we have to check that $Var(B^a)$ is $\mathbb{P}^a$-stochastically bounded where B^a is the bounded variation part of X^a (i.e. $x * \nu^a$). Here, since $x * \nu^a$ is an increasing process, we have :

$$\sup_{[0,T]} Var(x * \nu^a) = (\int_E xK^a(dx)) \times \int_0^T \lambda_t^a dt.$$

Moreover, for all $L > 0$, we get :

$$\mathbb{P}[(\int_E xK^a(dx)) \times \int_0^T \lambda_t^a dt > L] \leq \frac{1}{L}(\int_E xK^a(dx)) \times \mathbb{E}[\int_0^T \lambda_t^a dt].$$

Recall that $\mathbb{E}[\int_0^t \lambda_s^a ds] = \mathbb{E}[N_t^a]$. Hence from the above lemma, we can deduce by continuity of the inverse Laplace transform that, for small a and fixed t, $\mathbb{E}[\int_0^t \lambda_s^a ds] \sim \frac{s^2}{a^2}t$.

Note also that $\int_E xK^a(dx) \sim \frac{m'}{s^2}a^2$ since $K(a) \sim \frac{1}{2} + \frac{1}{2}\frac{m'a}{s^2}$. Therefore we get :

$$\forall \epsilon > 0, \exists L, I\!P[\sup_{[0,T]} Var(x * \nu^a) > L] < \epsilon.$$

So $Var(B^a)$ is $I\!P^a$-stochastically bounded, and we conclude to the uniform tightness of X^a.

Second, since $(X_t)_t = (m't + sW_t)_t$ is continuous, and since X^a converges to X and satisfies the property UT, we deduce from Proposition 2.2 of MÉMIN and SLOMINSKY (1991) that the martingale part $x * (\mu^a - \nu^a)$ of X^a converges to the martingale $(sW_t)_t$ and that the bounded variation part $x * \nu^a$ converges to the bounded variation part $m't$. Therefore we get from direct Taylor expansions :

$$\begin{aligned} \delta * \nu_t^a &\xrightarrow{P} m't, \\ \delta^2 * \nu_t^a &\xrightarrow{P} s^2 t. \end{aligned}$$

The last condition guarantees the convergence of the predictable compensator of the martingales $\delta * (\mu^a - \nu^a)$. Moreover, since the jumps of these martingales are uniformly bounded, the sequence $(\delta * (\mu^a - \nu^a))_a$ satisfies the UT (uniform tightness) condition. From Equation (8), a standard Taylor expansion of α_t^a proves that α^a weakly converges to α with $\alpha_t = \frac{1}{\tilde{S}_t}\frac{m}{s^2}$.

From all these results the convergence of the second Doléans-Dade exponential $\hat{\eta}_t^a = \mathcal{E}\left((-\alpha\tilde{S}\delta) * (\mu^a - \nu^a)_t\right)$ follows immediately as in PRIGENT (1995) Proposition 3.3 or LESNE , PRIGENT and SCAILLET (2000) Proposition 1.

Proof of Lemma

Let us first compute the Laplace transform of $I\!E[N_t^a]$.

We denote by $T_{-a,+a}$ the first time the Brownian motion with drift escapes from the interval $[-a, +a]$:

$$T_{-a,+a} = \inf\{u > 0, W_u + m'u \notin [-a, +a]\}.$$

Consider the sequence $(\epsilon_k)_k$ of i.i.d. random variables where all ϵ_k have the same distribution as $T_{-a,+a}$. Then for all $k \in N^*$, the following equality is verified :

$$I\!P[N_t^a \geq k] = I\!P[\epsilon_1 + ... + \epsilon_k \leq t].$$

Moreover, recall that if $I\!P$ is a probability on $I\!R_*^+$ with a density p then :

$$\int_0^\infty e^{-\xi t}p(t)dt = \int_0^\infty e^{-\xi t}dP(t) = [e^{-\xi t}P(t)]_0^\infty + \xi\int_0^\infty e^{-\xi t}P(t)dt = \xi\int_0^\infty e^{-\xi t}P(t)dt.$$

Let us denote :

$$L_k(\xi) = \xi\int_0^\infty e^{-\xi t} I\!P[\epsilon_1 + ... + \epsilon_k \leq t]dt,$$

which is equal to $\xi \int_0^\infty e^{-\xi t} \mathbb{P}[N_t^a \geq k]dt$, and also equal to $(L_1(\xi))^k$ by independence of the ϵ_k.

Since :

$$\mathbb{P}[N_t^a = k] = \mathbb{P}[N_t^a \geq k] - \mathbb{P}[N_t^a \geq k+1],$$

we get :

$$\begin{aligned}
\xi \int_0^\infty e^{-\xi t} \mathbb{E}[N_t^a]dt &= \xi \int_0^\infty e^{-\xi t} (\textstyle\sum_{k=1}^\infty k\mathbb{P}[N_t^a = k])dt \\
&= \textstyle\sum_{k=1}^\infty k(\xi \int_0^\infty e^{-\xi t} \mathbb{P}[N_t^a \geq k]dt - \xi \int_0^\infty e^{-\xi t} \mathbb{P}[N_t^a \geq k+1]dt) \\
&= \textstyle\sum_{k=1}^\infty k(L_k(\xi) - L_{k+1}(\xi)) \\
&= \textstyle\sum_{k=1}^\infty kL_k(\xi) - \sum_{k=1}^\infty (k+1)L_{k+1}(\xi) + \sum_{k=1}^\infty L_{k+1}(\xi),
\end{aligned}$$

which finally gives :

$$\xi \int_0^\infty e^{-\xi t} \mathbb{E}[N_t^a]dt = \sum_{k=1}^\infty L_1(\xi)^k = \frac{L_1(\xi)}{1 - L_1(\xi)}.$$

For $m' = 0$, we have (exponential martingale) :

$$\mathbb{E}[\exp(\xi W_{T_{-a,+a}} - \frac{\xi^2}{2} T_{-a,+a})] = 1,$$

from which we deduce :

$$\mathbb{E}[\mathbb{E}[\exp(\xi W_{T_{-a,+a}})|\mathcal{F}_{T_{-a,+a}}] \exp(-\frac{\xi^2}{2} T_{-a,+a})] = 1,$$

and :

$$Ch(\xi a)\mathbb{E}[\exp(-\frac{\xi^2}{2} T_{-a,+a})] = 1.$$

Therefore :

$$L_1(\xi) = \frac{1}{Ch(a\sqrt{2\xi})}.$$

For $m' \neq 0$ and for all $t > 0$,

$$\mathbb{E}[\exp(-\xi T_{-a,+a}) \mathbb{1}_{T_{-a,+a} < t}] = \mathbb{E}[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}|_{\mathcal{F}_t} \exp(-\xi T_{-a,+a}) \mathbb{1}_{T_{-a,+a} < t}]$$

where $\tilde{W}_u = m'u + W_u$ is a Brownian motion under $\tilde{\mathbb{P}}$ by applying Girsanov theorem. Now, since $\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}|_{\mathcal{F}_t} = \exp(m'\tilde{W}_t - \frac{m'^2}{2}t)$, we get :

$\mathbb{E}[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}|_{\mathcal{F}_t} \exp(-\xi T_{-a,+a}) \mathbb{1}_{T_{-a,+a} < t}]$

$= \mathbb{E}[\mathbb{E}[\exp(m'\tilde{W}_t - \frac{m'^2}{2}t)|\mathcal{F}_{T_{-a,+a}}] \exp(-\xi T_{-a,+a}) \mathbb{1}_{T_{-a,+a} < t}]$

$= \mathbb{E}[\exp(m'\tilde{W}_{T_{-a,+a}} - (\xi + \frac{m'^2}{2})T_{-a,+a}) \mathbb{1}_{T_{-a,+a} < t}]$

$$= \mathbb{E}[\mathbb{E}[\exp(m'\tilde{W}_{T_{-a,+a}})|\mathcal{F}_{T_{-a,+a}}]\exp[-(\xi + \tfrac{m'^2}{2})T_{-a,+a}]\mathbb{1}_{T_{-a,+a}<t}]$$

$$= Ch(m'a)\mathbb{E}[\exp[-(\xi + \tfrac{m'^2}{2})T_{-a,+a})]\mathbb{1}_{T_{-a,+a}<t}].$$

Then, by convergence with respect to t, we get :

$$\mathbb{E}[\exp(-\xi T_{-a,+a})] = Ch(m'a)\mathbb{E}[\exp((-\xi - \frac{m'^2}{2})T_{-a,+a})] = \frac{Ch(m'a)}{Ch(a\sqrt{2\xi + m'^2})}.$$

Finally :

$$\int_0^\infty e^{-\xi t}\mathbb{E}[N_t^a]dt = \frac{1}{\xi}\frac{Ch(m'a)}{Ch(a\sqrt{2\xi + m'^2}) - Ch(m'a)},$$

and, for the process $(m't + sW_t)_t$ with the volatility s :

$$\int_0^\infty e^{-\xi t}\mathbb{E}[N_t^a]dt = \frac{1}{\xi}\frac{Ch(m'a/s^2)}{Ch(a/s\sqrt{2\xi + (m'^2/s^2)}) - Ch(m'a/s^2)}.$$

Then a Taylor expansion leads to the stated result :

$$\int_0^\infty e^{-\xi t}\mathbb{E}[N_t^a]dt \sim \frac{1}{\xi^2}\frac{s^2}{a^2}.$$

REFERENCES :

- Ansel J.P. and C. Stricker (1992) "Lois de martingale, densités et décomposition de Föllmer-Schweizer" *Annales de l'Institut Henri Poincaré*, **28**, 375 − 392.
- Ansel J.P. and C. Stricker (1993) "Unicité et existence de la loi minimale", *Lecture notes in math. Sem. Prob. XXVII.*, **1557**, 22-29, Springer-Verlag, Berlin.
- Black F. and M. Scholes (1973) "The pricing of options and corporate liabilities", *Journal of Political Economy*, **81**, 637-654.
- Bouleau N. and D. Lamberton (1989) "Residual risks and hedging strategies in Markovian markets", *Stochastic Processes and their Applications*, **33**, 131-150.
- Brémaud P. (1981), *Point processes and queues, martingale dynamics*, Springer-Verlag, Berlin.
- Buhlman H., F. Delbaen, P. Embrechts and A. Shiryaev (1996) "No-arbitrage, change of measure and conditional Esscher transforms in a semimartingale model of stock processes", *CWI Quaterly*, **9**, 291-317.
- Chesher A., Dhaene G., Gouriéroux C. and O. Scaillet (1999), "Bartlett Identities Tests", DP CREST 9932.
- Chesher A. and R. Spady (1991) "Asymptotic Expansions of the Information Matrix Test Statistic", *Econometrica*, **59**, 787-815.
- Colwell D. and R. Elliott (1993) "Discontinuous asset prices and non-attainable contingent claims", *Mathematical Finance*, **3**, 295-308.
- Delbaen F. and W. Schachermayer (1994) "A general version of the fundamental theorem of asset pricing", *Mathematische Annalen*, **300**, 463-520.
- Duffie D. and D. Lando (1997) "Term structures of credit spreads with incomplete accounting information", DP Stanford University.
- Duffie D. and H. Richardson (1991) "Mean-variance hedging in continuous time", *Annals of Applied Probability*, **1**, 1-15.
- Föllmer H. and D. Sondermann (1986) "Hedging of non-redundant contingent claims", in *Contributions to Mathematical Economics in Honor of Gérard Debreu*, eds. W. Hildenbrand and A. Mas-Colell, North-Holland, 205-223.
- Föllmer H. and M. Schweizer (1991) "Hedging of contingent claims under incomplete information", in *Applied Stochastic Analysis, Stochastics Monographs,* eds. M. H. A. Davis and R. J. Elliott, Gordon and Breach, **5**, London/New York, 389-414.

- Géman H. and M. Yor (1994) "Pricing and hedging double-barrier options : a probabilistic approach", *Mathematical Finance*, **6**, 365-378.

- Gouriéroux C., J.P. Laurent and H. Pham (1998) "Mean-Variance Hedging and Numéraire", *Mathematical Finance*, **8**, 179-200.

- Harrisson J. and D. Kreps (1979) "Martingale and arbitrage in multiperiod securities markets", *Journal of Economic Theory*, **20**, 348-408.

- Harrisson J. and S. Pliska (1981) "Martingales and stochastic integrals in the theory of continuous trading", *Stochastic Processes and their Applications*, **11**, 215-260.

- He H., W. Keirstead and J. Rebholz (1998) "Double lookbacks", *Mathematical Finance*, **8**, 201-228.

- Hofmann N., E. Platen and M. Schweizer (1992) "Option pricing under incompleteness and stochastic volatility", *Mathematical Finance*, **2**, 153-187.

- Jacod J. and A. Shiryaev (1987) *Limit theorems for stochastic processes*, Springer-Verlag, Berlin.

- Karlin S. and H. Taylor (1975) *A first course in stochastic processes*, 2nd ed., Academic Press, London.

- Karr A. (1986) *Point processes and their statistical inference*, Marcel Dekker Inc., New York.

- Kunitomo N. and M. Ikeda (1992) "Pricing options with curved boundaries", *Mathematical Finance*, **2**, 275-298.

- Last G. and A. Brandt (1995) *Marked point processes on the real line*, Springer-Verlag, Berlin.

- Laurent J.P. and O. Scaillet (1998) "Variance optimal caplet pricing models", DP CREST.

- Lesne J.P., J.L. Prigent and O. Scaillet (2000) "Convergence of discrete time option pricing models under stochastic interest rates", *Finance and Stochastics*, **4**, 81-93.

- Mémin J. and L. Slominsky (1991) "Condition UT et stabilité en loi des solutions d'équations différentielles stochastiques", *Lecture notes in math. Sem. Prob. XXV.*, **1485**, Berlin : Springer-Verlag.

- Mercurio F. and T. Vorst (1996) "Option pricing with hedging at fixed trading dates", *Applied Mathematical Finance*, **3**, 135-158.

- Prigent J.L. (1995) "Incomplete markets : Convergence of options values under the minimal martingale measure", DP THEMA 9526 forthcoming in *Advances in Applied Probability*, **31**.

- Protter P. (1990) *Stochastic integration and differential equations*, Springer Verlag, Berlin.
- Revuz D. and M. Yor (1994) *Continuous martingales and Brownian motion*, 2nd ed., Springer Verlag, Berlin.
- Runggaldier W. J. and M. Schweizer (1995) "Convergence of option values under incompleteness", in *Seminar on stochastic analysis, random fields and applications*, eds. E. Bolthausen, M. Dozzi and F. Russo, Birkhäuser, 365-384.
- Schweizer M. (1991) "Option hedging for semimartingales", *Stochastic Processes and their Applications*, **37**, 339-363.
- Schweizer M. (1992a) "Martingale densities for general asset prices", *Journal of Mathematical Economics*, **21**, 363-378.
- Schweizer M. (1992b) "Mean-variance hedging for general claims", *Annals of Applied Probability*, **2**, 171-179.
- Schweizer M. (1993) "Variance-optimal hedging in discrete time", *Mathematics of Operations Research*, **20**, 1-32.
- Schweizer M. (1994) "Approximation pricing and the variance-optimal martingale measure", *Annals of Probability*, **24**, 206-236.