Endogenous vs Exogenously Driven Fluctuations in Vintage Capital Models*

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Abstract

In this paper, we present a simple vintage capital growth model in which both exogenous and endogenous fluctuations sources are present. Indeed, it can be seen as a particular case of Caballero and Hammour (1996)'s creative destruction model, with the advantage that analytical characterization of the short run and asymptotic dynamics is partially allowed. In particular, we show that job creation follows a delayed-differential equation with periodic coefficients. The delay is equal to the optimal age of capital goods, and can be taken as a measure of the periodicity of the endogenous replacement echoes inherent to vintage models. The period of the coefficients is equal to the period of an exogenous profitability cycle. We mathematically show that job creation is asymptotically periodic, with the same period as the profitability cycle. Furthermore using an explicit numerical method, we find that replacement echoes generally dominate the short run dynamics. Finally, we find that the combination of the two fluctuations sources favors the appearance of asymmetries in job creation and job destruction patterns.

Keywords: Endogenous fluctuations, Exogenously driven fluctuations, Vintage capital models, Replacement echoes, Differential-difference equations, Floquet representations.

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1 Introduction

Explaining the economic fluctuations associated with the business cycle has always been one of the main concerns of economic theorists and practitioners. A distinction can be made between the so-called endogenous cycles literature (since Grandmont (1985)) and the exogenously driven fluctuations associated to the theory of Real Business Cycles (since Kydland and Prescott (1982)). RBC studies have become the dominating research program in the field and they have provided a highly successful methodological framework for the study of economic fluctuations. To analyze the predictions of their models RBC theorists are mainly concerned with volatilities and comovements of some detrended economic aggregates.[1] However, practitioners have also to be concerned with the *recurrence* of business fluctuations: a boom is necessarily followed by a recession and vice versa. Nevertheless, the neoclassical growth model, which is on the basis of RBC models, does not provide a good framework for the analysis of recurrent fluctuations, since convergence to the stationary solution is in general monotonic.[2] To overcome this problem, models generating periodic investment cycles should be incorporated to this literature.[3]

In this paper, we use a simple model to illustrate the potential scope of this approach. For the significance of our study, we choose an economically fundamental source of endogenous fluctuations, the replacement of the obsolete equipment. Actually, the endogenous fluctuations displayed by our model do not rely on the *amount* of non-linearities since both preferences and the production technology are taken linear. That is to say that the considered endogenous fluctuations are not the result of an artificial mathematical assumption but are rather naturally derived from a key economic decision. The most recent literature on embodied technical change has put forward the relevance of the so-called *replacement problem* in economic fluctuations (see for example Cooley *et al.* (1997)): Clearly, if technological progress is embodied in the new equipment, replacing the oldest capital goods by the most recent vintages becomes a key economic decision. This decision is at the basis of the so-called *replacement echoes* recently outlined in the vintage capital literature (see Benhabib and Rustichini (1991) and (1993), and Boucekkine *et al.* (1997-a) and (1998)). An interesting theoretical and empirical issue turns out to be how these replacement echoes interact with exogenously driven fluctuations.

In this paper, vintage capital and an exogenous profitability cycle are put together to provide an insight into this question. The considered model can be seen as a particular case of the creative destruction model built up by Caballero and Hammour (1996). Actually, we remove all the ingredients of the latter model except the vintage capital structure, the

[1]In general, trends are removed using the Holdrik-Prescott filter, which implicitly relates business cycles with fluctuations of a frequency lower than eighth years.

[2]Following the Cogley and Nason (1995) critique on the lack of propagation mechanisms in the standard RBC model, the RBC literature has put more emphasis on the search for endogenous propagation mechanisms.

[3]Dealing with the same problem, Wen (1998) has recently proposed a time-to-build model, slightly different from Kydland and Prescott (1982), which generates periodic investment cycles.

exogenous profitability cycle and a surplus appropiability problem yielding an inefficient decentralized equilibrium.[4] However, our objectives are completely different from those pursued by Caballero and Hammour. While these authors are interested in the cyclical properties of job creation and job destruction (in line with the empirical literature in this field, see Davis and Haltiwanger (1992) and Davis *et al.* (1996)), we aim at studying to which extent job creation and job destruction dynamics are driven by the exogenous cycle versus replacement echoes. Indeed, Caballero and Hammour have calibrated their model in order for the exogenous profitability variable to drive the cycle. By doing so, they have minimized the effects of replacement echoes, reducing them to some innocuous irregularities observable on the solution paths as mentioned by the authors themselves.[5] Obviously, this view is consistent with the objectives of the authors and as such it is not challenged here. We just argue that the interaction between replacement echoes and the exogenous cycles is much more complex than what it is reported in Caballero and Hammour's paper. Some analytical arguments will be presented to support our claim in sharp contrast to the purely computational setting adopted by the latter authors.

Indeed solving for the equilibrium of the economy under review, we find that the optimal replacement decision corresponds to a constant scrapping rule, and that job creation is driven by a delay differential equation (DDE) with periodic coefficients. The obtained DDE allows for a perfect visualization of the interaction between exogenous and endogenous fluctuations. While the delay of the (DDE) is exactly the value of the optimal scrapping, thus representing the frequency of replacement echoes, the period of the coefficients of the DDE is equal to the period of an exogenous profitability cycle. Then, using the available mathematical theory in the field (see Hale and Verduyn Lunel (1993), chapter 8), we analytically show that the exogenous cycle does not generally drive the short run fluctuations of job creation and job destruction. Instead short run fluctuations seem to be governed by replacement echoes. In the long run, two cases are examined. In a first case, the optimal scrapping time is taken to be a rational multiple of the period of the exogenous cycle. In such a case, there exists a sufficient mathematical theory to conclude: For any real-valued initial conditions, job creation converges to a periodic regime with periodicity equal to the period of the exogenous cycle. Therefore, in such a case, replacement echoes generally dominate in the short run, and the exogenous cycle is predominant in the long run. This analysis suggests that omitting replacement echoes in the analysis of the short run fluctuations of job creation and job destruction may be misleading.

When the optimal scrapping time is not a rational multiple of the period of the exogenous cycle, we do not have the sufficient theory to conclude analytically about the asymptotic behavior of the solutions.[6] Then, we have resorted to numerical simulations

[4] Indeed, Caballero and Hammour's model is much richer since it contains a job search problem and a somewhat complete story on the costs of job creation.

[5] See Caballero and Hammour (1996), footnote 19, page 820.

[6] This case is extremely difficult to handle analytically as it will be clear in the Section 5 of this paper. Huang and Mallet-Paret's (still unpublished) contribution (1992) is among the very rare attempts at

using the explicit method of steps (as recently applied to vintage models by Boucekkine *et al.* (1997-b)). Using very large resolution time horizons (about some thousands of periods), we find that replacement echoes again drive the short run dynamics while the exogenous cycle still dominates in the long run, although the computed long run regime is shown to present some important differences with respect to the first case. The adjustment from the short run "replacement echoes" regime to the long run "exogenous fluctuations" regime is also studied in both cases. Among our findings, we point out the occurrence of asymmetric job creation and destruction patterns in the short run, a feature which is shown to be caused by the simultaneous presence of the two fluctuations engines.

The paper is organized as follows. Next section describes briefly the example model we consider. Section 3 derives the (interior) optimal scrapping rule and the conditions on the initial state of the economy under which this rule is implementable from the initial period. Section 4 analyzes the dynamic outcomes of the model in the absence of the exogenous profitability cycle, and Section 5 shows how these outcomes are altered when the latter cycle is added. Section 6 concludes.

2 The model

Our model can be seen as a special case of Caballero and Hammour (1996)'s model. As mentioned in the introduction section, the latter model is representative to a large extent of the recent theoretical literature devoted to creative destruction. It is especially interesting because it includes an exogenous fluctuation source and this allows to tackle our principal issue: How may replacement echoes interact with exogenously driven fluctuations? Indeed, except for the specification of the exogenous fluctuation source, borrowed from Caballero and Hammour, our example is rather a canonical model of creative destruction with Leontieff technology and generalized Nash bargaining in the labor market. This will be pretty clear hereafter.

We model a two-sector decentralized economy with one produced good and one non-produced good. The former is used in consumption and investment, and the latter is consumed by households and employed as an intermediate input in production. At any time t, the economy is endowed with an amount $\overline{m}$ of the non-produced good, which price is exogenous and fluctuates deterministically. This is the Caballero and Hammour's story on exogenous fluctuations. The rest of the model is simple and standard. The production technology is Leontieff and combines capital, labor, and the intermediate input in fixed proportions. Labor augmenting technical progress is continuously embodied in new capital goods, which yields an endogenous process of creation and destruction through the replacement of the obsolete machines by the new (more productive) ones. Each capital unit of vintage t produces one unit of output and requires to be operated $\exp\{-\gamma t\}$ units

addressing the issues related to this case. Unfortunately, it does not seem to be of decisive help applied to our model.

of both labor and the non-produced input. Consequently, at time t each unit of labor associated with an operating vintage τ produces $\mathrm{e}^{\gamma\tau}$ units of good from one unit of the non-produced good.

If we denote by $T(t)$ the age of the oldest operating machines at time t (or scrapping time), aggregate output and employment are respectively given by

$$y(t) = \int_{t-T(t)}^{t} \mathrm{e}^{\gamma\tau} h(\tau)\,\mathrm{d}\tau, \tag{1}$$

and

$$l(t) = \int_{t-T(t)}^{t} h(\tau)\ \mathrm{d}\tau, \tag{2}$$

where $h(\tau)$ represents employment associated with vintage τ. By differentiating (2) with respect to time, we can see that at any time t changes in employment depend on **job creation,** $h(t)$, and **job destruction**, $h(t - T(t))(1 - T'(t))$. As we will show later, function $T(t)$ is constant at equilibrium, say equal to T, implying that job destruction at t corresponds to job creation at $t - T$.

The economy comprises a continuum of agents of measure one, with the same linear preferences over lifetime consumption:

$$\int_{0}^{\infty} (c(\tau) + p(\tau)e^{\gamma\tau} m(\tau))\ \mathrm{e}^{-\rho\tau}\ \mathrm{d}\tau,$$

where $\rho > 0$ is the subjective rate of time preference (equal to the interest rate due to the linearity of preferences), $c(\tau)$ is consumption of the produced good at time τ. $m(\tau)$ is consumption of the non-produced good, and $p(\tau)e^{\gamma\tau}$ represents its marginal utility. If we take the produced good as numeraire, the price of non-produced goods should be $p(\tau)e^{\gamma\tau}$ at equilibrium. There is no disutility of labor, so the labor supply is exogenous and we normalize it to 1. Aggregate unemployment at time t is given by

$$u(t) = 1 - l(t). \tag{3}$$

>From the Leontieff technology, the equilibrium conditions for the produced and non-produced goods markets are respectively:

$$y(t) = c(t) + \underbrace{\mathrm{e}^{\gamma t} h(t)}_{\text{investment}}$$

and

$$\overline{m} = m(t) + \underbrace{l(t)}_{\text{non-produced inputs}}.$$

In order to create a job at time t, a firm needs to invest $\mathrm{e}^{\gamma t}$ units of the produced good in the latest technology.[7] The corresponding appropiable surplus $\pi(t)$ is

$$\pi(t) = \int_{t}^{t+J(t)} \left(\mathrm{e}^{\gamma t} - \left(\widetilde{\omega}(\tau) + p(\tau) \right) \mathrm{e}^{\gamma \tau} \right) \mathrm{e}^{-\rho(\tau - t)} \, \mathrm{d}\tau. \tag{4}$$

It is equal to the present value, over the planned lifetime of the job, of instantaneous profits, which are defined as value added minus worker's shadow wages. Note that under perfect foresight, the planned lifetime $J(t)$ is related to the scrapping time $T(t)$ by $J(t) = T(t + J(t))$, or equivalently by $T(t) = J(t - T(t))$.

The optimal lifetime is obtained by maximizing the value of a job with respect to $J(t)$, which gives the condition:

$$p(t + J(t)) + \widetilde{\omega}(t + J(t)) = \mathrm{e}^{-\gamma J(t)}.$$

It can be rewritten in terms of $T(t)$:

$$p(t) + \widetilde{\omega}(t) = \mathrm{e}^{-\gamma T(t)}. \tag{5}$$

A job is destroyed when it becomes profitable to reallocate labor and non-produced resources to the latest technology.

In the labor market, the representative firm and the workers bargain over the appropiable surplus. A generalized Nash bargaining solution, with a share $\beta \in (0,1)$ of the surplus going to the worker and $(1-\beta)$ going to the firm, yields the following standard equilibrium conditions:

$$\widetilde{\omega}(t)\, \mathrm{e}^{\gamma t} = \frac{h(t)}{u(t)}\, \beta\, \pi(t). \tag{6}$$

and

$$\mathrm{e}^{\gamma t} = (1-\beta)\, \pi(t). \tag{7}$$

$>$From equation (6), the equilibrium shadow wage $\widetilde{\omega}(t) \exp\{\gamma t\}$ should be equal to the expected utility flow received by an unemployed worker, which is in turn equal to the flow probability $\frac{h(t)}{u(t)}$ of finding a job times the worker's share of the surplus. Equation (7) stipulates that the firm will create a job as long as the cost of creation is equal to its share of the surplus.

We are now able to define an equilibrium for our economy:

[7] Job creation costs are much richer in Caballero and Hammour (1996), since they include search and training costs in addition to hiring cost. In this respect, our production technology is much closer to the standard vintage model, as developed by Solow *et al.* (1966), than to Caballero and Hammour's specifications.

Definition 1 *For a given path of the exogenous price* $p(t)$ *and given initial conditions* $h(t) \geq 0, \forall t < 0$*, an equilibrium for this economy is a path for* $T(t)$*,* $J(t)$*,* $h(t)$*, and* $u(t)$*,* $t \geq 0$*, that satisfies the system of equations*

$$u(t) = 1 - \int_{t-T(t)}^{t} h(\tau) \; d\tau, \tag{8}$$

$$e^{-\gamma T(t)} - p(t) = \frac{h(t)}{u(t)} \frac{\beta}{1-\beta}, \tag{9}$$

$$\int_{t}^{t+J(t)} \left(1 - e^{-\gamma(t-\tau+T(\tau))}\right) \; e^{-\rho(\tau-t)} \; d\tau = \frac{1}{1-\beta}, \tag{10}$$

$$J(t) = T(t + J(t)), \tag{11}$$

and the inequalities $0 \leq u(t) \leq 1$ *and* $p(t) < e^{-\gamma T(t)}$.

Equation (8) is obtained from equations (2) and (3) and it states the resource constraint in the labor market. Equation (9) is obtained by combining (5), (6) and (7) so as to eliminate variables $\pi(t)$ and $\widetilde{\omega}(t)$. Equation (10) is an optimal condition that restates (7) using (4) and (5). The inequality $p(t) < \mathrm{e}^{-\gamma T(t)}$ is required for equation (9) to hold with positive job creation and unemployment. Indeed, this inequality also ensures the positivity of wages at equilibrium by (5).

The next sections are devoted to the dynamic analysis of the decentralized equilibrium described above when the exogenous price path is periodic of the form:

$$p(t) = p_0 + p_1 \sin(p_2 \, t), \tag{12}$$

where p_i are positive real numbers such that $p_0 \geq p_1$ to ensure the positivity of prices over time. We denote by $\Omega = \frac{2\pi}{p_2}$ the period of $p(t)$ and we will refer to it as the *exogenous period* hereafter.

3 Existence, uniqueness and implementability of the optimal scrapping rule

In the rest of the paper, we will characterize explicitly the dynamics of the economy described by the equilibrium conditions stated in Definition 1. First, note that as in Boucekkine *et al.* (1997-a), our equilibrium conditions show a clear recursive forward-looking sub-block, namely the sub-block formed by equations (10) and (11). This sub-block allows to solve for $T(t)$ and $J(t)$ independently of the other endogenous variables. The solution scheme for the former variables is very similar to the one adopted by Boucekkine *et al.* (1997-a). We apply it here, assuming that $T(t)$ and $J(t)$ are differentiable for all $t \geq 0$.

3.1 The optimal scrapping rule

The first step of the resolution scheme is given by the following proposition:

Proposition 1 *Equations (10) and (11) imply the existence of a function $F(.)$ such that for any $t \geq 0$*

$$T(t) = F(J(t))$$

with

$$F(x) = -\frac{1}{\gamma} \ln \left[1 - \frac{r-\gamma}{1-\beta} - \frac{\gamma}{r}(1 - \exp\{-rx\}) \right]$$

provided that $F(x)$ is defined $\forall x \geq 0$.

Given that $J(t)$ is differentiable at t, we can differentiate (10) and easily show Proposition 1 after some elementary manipulations. To make function $F(.)$ well-defined for any positive value, we restrict the parameters values as follows:

Assumption 1 *Parameters γ , β, and ρ check the following conditions:*

$$\begin{array}{lrcl} i) & \gamma & < & \rho \\ ii) & \rho & < & 1-\beta \end{array}$$

Condition i) is pretty standard in the growth literature: it is needed to guarantee that the individual's objective is bounded. Condition ii) states that the firm's surplus share is greater than the interest rate, which is necessary to get positive hiring at equilibrium. It is easy to check that Assumption 1 is sufficient for function $F(.)$ being strictly increasing and admitting a unique strictly positive fixed-point. It will allow us to use a fixed-point argument *à la* van Hilten (1991), exactly as in Boucekkine *et al.* (1997-a), in order to show that the optimal scrapping rule $T(t)$, and consequently $J(t)$, are constant and equal to the fixed-point of function $F(.)$ for any $t \geq 0$.

Proposition 2 *Under Assumption 1, the unique differentiable solutions $T(t)$ and $J(t)$, $t \geq 0$, are defined by*

$$T(t) = J(t) = T^{\circ}$$

with T° the unique positive fixed-point of function $F(.)$.

The proof of this proposition is identical to the proof of Proposition 2 in Boucekkine *et al.* (1997-a). Before studying the behavior of job creation, some comments on the central planner counterpart of the model can be useful. Since firms do not face any search cost, unemployment plays no positive role in our model: A central planner who cares about lifetime consumption should set unemployment to zero. By differentiating equation (2) under full-employment, job creation should be periodic with a period equal to the constant optimal scrapping time derived in Proposition 2. This is the occurrence of **replacement echoes** previously pointed out by Boucekkine *et al.* (1997-a). Since replacement echoes are endogenous and occur at a period equal to T°, we may call T° the *endogenous period* in contrast to the exogenous period Ω. The central planner counterpart of the model trivially yields the following result: Since unemployment should be zero, fluctuations on job creation are entirely driven by replacement echoes, despite the existence of a periodic exogenous environment. In this sense, the endogenous period T° dominates the exogenous period Ω. Indeed, the periodic exogenous price $p(t)$ only affects wages through equation (5), which still holds in the central planner model.[8] In the decentralized economy, things are much more complicated (since equilibrium unemployment needs not be zero) but we can still analyze job creation dynamics as the result of the interaction between the endogenous period (replacement echoes) and the exogenous period (price fluctuations). This will be clear in the next subsection.

3.2 Job creation and the implementability of the optimal scrapping rule

Once found out the solution of the forward-looking sub-block of the equilibrium conditions, we can derive the solution of the remaining sub-block, and compute the dynamics of job creation and unemployment.

Since the condition $p(t) < \mathrm{e}^{-\gamma T(t)}$ must be checked for all t (see Definition 1), we restrict the parameters of the forcing variable $p(t)$, specified in (12), and the parameters determining T^{o} to check:

Assumption 2 $p_0 + p_1 < \exp(-\gamma T^{\circ})$.

On the other hand, we can solve for job creation using equations (8) and (9):

$$h\left(t\right) = k_2(t) \left(1 - \int_{t-T^{\circ}}^{t} h\left(\tau\right)\ \mathrm{d}\tau\right), \tag{13}$$

for all $t \geq 0$, given $h\left(t\right) \geq 0$ for all $t < 0$, and $k_2(t)$ a periodic function of period Ω given by:

$$k_2(t) = \beta_0 \left(\mathrm{e}^{-\gamma T^{\circ}} - p(t)\right), \tag{14}$$

[8]The latter property, the periodicity of wages, is the unique relevant difference with respect to the purely endogenous fluctuations model of Boucekkine *et al.* (1997-a), in which (detrended) wages are constant.

where $\beta_0 = \frac{1-\beta}{\beta}$. Note that $k_2(t) > 0$ by Assumption 2, an important property that will be used later in the assessment of the stability outcomes of the model.

Differentiating (13), we get the following differential-difference equation (DDE) in $h(t)$

$$h'(t) = k_1(t)\, h(t) + k_2(t)\, h(t - T^\circ), \tag{15}$$

where $k_1(t)$ is another periodic function of period Ω given by:

$$k_1(t) = -k_2(t) + \frac{k_2'(t)}{k_2(t)}. \tag{16}$$

Equation (15) is a DDE with periodic coefficients of the same period Ω, the exogenous period. The deviating time argument appearing in the DDE is exactly the optimal scrapping value. Although the corresponding job creation dynamics are much more complicated than in the (trivial) case of the central planner counterpart of the model mentioned above, we can still analyze these dynamics in terms of the exogenous period Ω, the period of the coefficients of the DDE, and of the endogenous period T°, the delay of the DDE. Intuitively, this sounds straightforward: Beside the exogenous fluctuations source, the economy should move according to replacement investment activities, which optimally take place every T° units of time. Mathematically, this can be deduced from the DDE structure itself as shown in the following proposition:

Proposition 3 *Given an initial function $h(t) = h_0(t)$ for $t < 0$, the DDE (15) can be solved on the time interval $[0, +\infty)$ according to the following forward continuation process:*

For any $n \in \aleph$, $n \geq 1$, given $h_{n-1}(t)$, $h_n(t) = h(t)$ on $[(n-1)T^\circ, nT^\circ)$ is the solution of the ordinary differential equation:

$$h'(t) = k_1(t)\, h(t) + k_2(t)\, h_{n-1}(t - T^\circ)$$

with

$$h((n-1)T^\circ) = h_n((n-1)T^\circ) = k_2((n-1)T^\circ) \left(1 - \int_{(n-1)T^\circ}^{nT^\circ} h_{n-1}(\tau - T^\circ)\, d\tau\right).$$

Hence, for any $n \geq 1$, $h_n(t)$ may be written as:

$$h_n(t) = \mu_n(t) \left(h_n((n-1)T^\circ) + \int_{(n-1)T^\circ}^{t} \frac{k_2(\tau)}{\mu_n(\tau)} h_{n-1}(\tau - T^\circ)\, d\tau\right), \tag{17}$$

where $\mu_n(t) = \exp\left(\int_{(n-1)T^\circ}^{t} k_1(\tau)\, d\tau\right)$.

Proposition 3 is a formalization of the *method of steps* designed for explicitly solving DDE with constant delays (see Bellman and Cooke (1963), for a description of this device, and Boucekkine *et al.* (1997-b), for some extensions of the method). Note that the resolution scheme is explicit to the extent that no iterative technique is required. The latter property holds because the DDE is solved *via* successive resolutions of ordinary differential equations on the successive intervals $I_n = [(n-1)T^\circ, nT^\circ)$, a task that can be typically achieved without resorting to iterative techniques. Equation (17) is the solution in the integral form of the DDE on any interval I_n, and it illustrates perfectly the *echo principle* at work in the dynamics of job creation: the solution path on any interval I_n, of length T°, strongly depends upon the solution path on the anterior interval I_{n-1}. Hence, as argued just above, the (endogenous) replacement period T° is still at work in the case of the decentralized economy, in addition to the exogenous period Ω. However, unlike in the central planner counterpart of the model, it is not clear which kind of periodicity will dominate. The rest of the paper is mainly devoted to address this issue.

Before undertaking this task, some technical problems have to be tackled. The first problem concerns the continuity of the solution paths. By equation (13) and by Proposition 3, it is clear that provided the initial function $h_0(t)$ is continuous, the solution paths are also continuous except eventually at the so-called *meshpoints*, nT°, $n \geq 0$. One can show straightforwardly that the limits of the solution paths given by the integral expressions (17) at the meshpoints are indeed equal to the values of job creation $h_n((n-1)T^\circ)$ reported in Proposition 3. However, it is clear that in general the solution paths should exhibit a jump at $t = 0$ with respect to the initial function $h_0(t)$, since the limit of $h_0(t)$ when t goes to zero needs not be equal to the value of $h(0)$ given by (13) evaluated at $t = 0$. The occurrence of jumps at the initial period is indeed one of the most peculiar properties of the model as it will be shown later.

The second problem is more fundamental and has to do with the implementability of the optimal constant scrapping rule T°. In Definition 1, we have implicitly assumed that we can implement the interior solution $T(t) = T^\circ$, beginning at $t = 0$. Actually, given the structure of our model, the interior solution may not be implementable starting at $t = 0$ for certain initial profiles, i.e., for certain initial functions $h_0(t)$. For example, if function $h_0(t)$ takes huge values on the interval $[-T^\circ, 0)$, the condition "$u(t) \geq 0$" will be violated and we cannot implement the interior solution starting at $t = 0$. It is not difficult to show that in such cases the interior solution can be implemented after a finite time adjustment period (for some insight into this issue, see Boucekkine *et al.* (1997-a)). Given the objectives of this paper, we focus on the situations where the interior solution $T(t) = T^\circ$ is implementable from the beginning and to this end, we set the following condition on the initial distribution of jobs:

Assumption 3 *The initial function $h_0(t)$ is positive everywhere and checks*

$$0 < \int_0^{T^\circ} h_0\left(\tau - T^\circ\right)\ d\tau < 1.$$

By (8), Assumption 3 implies that $0 < u(0) < 1$. We show now that Assumption 3 implies that $0 < u(t) < 1$, $\forall t \geq 0$, or equivalently that $0 < h(t) < k_2(t)$, $\forall t \geq 0$, by (8)-(9).

Proposition 4 *Under Assumption 3, the solution paths of the DDE (15) check for any $t \geq 0$:*

$$0 < h(t) < k_2(t).$$

Proof: The strict positivity is obvious given the analytical form of the solutions produced by the method of steps. We prove the other inequality by induction on the successive intervals I_n. We will show that the proposition is true for I_1, the extension to the posterior intervals being trivial by construction. First note that $h(0) < k_2(0)$ by Assumption 3. The solution on $[0, T^\circ]$ (note that the continuity of the solution paths allows us to use closed intervals and we will do so hereafter) is given by equation (17) at $n = 1$:

$$h(t) = h_1(t) = \mu_1(t) \left(h(0) + \int_0^t \frac{k_2(\tau)}{\mu_1(\tau)} h_0(\tau - T^\circ) \, \mathrm{d}\tau \right)$$

with

$$h(0) = k_2(0) \left(1 - \int_0^{T^\circ} h_0(\tau - T^\circ) \, \mathrm{d}\tau \right),$$

and $\mu_1(t) = \exp\left(\int_0^t k_1(\tau) \, \mathrm{d}\tau \right)$.

Using (16), we can rewrite $\mu_1(t)$ and $h(t)$ in terms of $k_2(t)$ according to

$$\mu_1(t) = \frac{k_2(t)}{k_2(0)} \exp\left(- \int_0^t k_2(\tau) \, \mathrm{d}\tau \right),$$

and

$$\frac{h(t)}{k_2(t)} = \exp\left(- \int_0^t k_2(s) \, \mathrm{d}s \right)(1 - A) + \int_0^t \exp\left(- \int_\tau^t k_2(s) ds \right) h_0(\tau - T^\circ) \, \mathrm{d}\tau,$$

with $A = \int_0^{T^\circ} h_0(\tau - T^\circ) \, \mathrm{d}\tau$. By Assumption 3, we have $0 < A < 1$. Moreover, by definition of function $k_2(t)$- equation (14)- and under Assumption 2, $k_2(t) > 0$, $\forall t$. Thus, $\forall t$ in $[0, T^\circ]$,

$$\int_0^t \exp\left(- \int_\tau^t k_2(s) ds \right) h_0(\tau - T^\circ) \, \mathrm{d}\tau < \int_0^{T^\circ} h_0(\tau - T^\circ) \, \mathrm{d}\tau = A,$$

and

$$\frac{h(t)}{k_2(t)} < \exp\left(- \int_0^t k_2(s) \, \mathrm{d}s \right)(1 - A) + A < 1,$$

which proves the desired inequality on the interval $[0, T^{\circ}]$. By construction of the solution path (Proposition 3), the same arguments can be used to prove the inequality for the subsequent intervals.□

An obvious corollary of the previous proposition is:

Corollary 1 *Under Assumption 3, any solution of the DDE (15) is bounded.*

Since $k_2(t)$ is bounded, Proposition 4 implies that job creation solution paths should be bounded if Assumption 3 is satisfied.

4 Dynamics and asymptotic properties of job creation under constant p(t)

It remains to analyze the dynamics of the model, which are shown to depend on the values of the endogenous and exogenous periods, T° and Ω respectively. Since we also aim at discriminating somewhat between the resulting endogenous and exogenous fluctuations, we first study the decentralized economy in the absence of exogenous fluctuations, i.e. with $p(t) = p_0$, $\forall t$. The DDE (15) becomes:

$$h'(t) = \beta_1 \left(h(t - T^{\circ}) - h(t)\right), \tag{18}$$

with β_1 a positive constant equal to $\beta_0 \left(\mathrm{e}^{-\gamma T^{\circ}} - p_0\right)$. Obviously, we can use as well the method of steps to deal with this particular case. However, this approach is only useful for deriving the short run dynamics of our model and lacks interest for the asymptotic stability analysis of the solution paths. This task can be easily undertaken using Laplace transform techniques as detailed in Bellman and Cooke (1963), chapters 3 and 12 (see also Benhabib and Rustichini (1991), for earlier applications of these techniques to vintage models). This approach is useful for asymptotic stability assessment, because it consists in writing the solutions as sums of exponential terms. Applying these techniques to the particular DDE (18), it yields:

Proposition 5 *For any initial function $h_0(t)$, the solution path of equation (18) can be written as the sum of an exponential polynomial expansion of the form*

$$\sum p_r(t)\ e^{s_r t},$$

where $\{s_r\}$ is any sequence of roots of the transcendental function:

$$Q(s) = s + \beta_1 - \beta_1 e^{-sT^{\circ}},$$

and $p_r(t)$ is a polynomial of degree less than the multiplicity of s_r. Indeed:

i) *The roots of function $Q(s)$ are all simple, so that the polynomials $p_r(t)$ are constants determined by the initial function $h_0(t)$.*

ii) *Except the trivial root $s = 0$, all the roots of $Q(s)$ have strictly negative real part and nonzero imaginary part.*

Proof: The first part of the theorem, the exponential polynomial form of the solutions is a direct application of Theorem 3.4 of Bellman and Cooke (1963). Properties i) and ii) can be demonstrated using the following direct proof.[9] First, note that $Q(s)$ cannot admit multiple roots. A multiple root exists if and only if it exists a complex number s such that $Q(s) = Q'(s) = 0$, and this is impossible as long as $1 + \beta_1 T^\circ \exp(1 + \beta_1 T^\circ)$ is strictly positive. Since $\beta_1 > 0$, all the roots of $Q(s)$ are simple.

To prove property ii), we set as usual $s = x + \, i\, y$ with (x, y) a couple of real numbers and $i^2 = -1$. $Q(s) = 0$ implies

$$
\begin{aligned}
x &= \beta_1 \, e^{-x\, T^\circ} \cos(y\, T^\circ) - \beta_1 \\
y &= \beta_1 \, e^{-x\, T^\circ} \sin(y\, T^\circ).
\end{aligned}
$$

It is then trivial to see that $x > 0$ is impossible since it implies $\mathrm{e}^{x\, T^\circ} < \cos(y\, T^\circ)$. So, $x \leq 0$. We now turn to show that the unique real root is the trivial root $s = 0$. Indeed, if $y = 0$, we get $x = \beta_1(\mathrm{e}^{-xT^\circ} - 1)$, which is inconsistent with the strict negativity of x. □

The previous proposition allows us to conclude for the following asymptotic stability result:

Corollary 2 *For any initial positive function $h_0(t)$,*

$$\lim_{t \to \infty} h(t) = \overline{h},$$

with

$$\overline{h} = \frac{\beta_1}{1 + \beta_1 T^\circ}.$$

Moreover, convergence to $\overline{h}$ is oscillatory.

The corollary is a direct consequence of Proposition 5. $\overline{h}$ is the steady state value of job creation and $\overline{u} = \frac{1}{1+\beta_1 T^\circ}$ is the steady state value of the unemployment rate, being both computed simultaneously form the system (8)-(9) evaluated at $T(t) = T^\circ$, $u(t) = \overline{u}$ and $h(t) = \overline{h}$.

[9] A direct proof of property ii) is allowed here because of the special form of function $Q(s)$, which is in turn due to the special form of the DDE (18). For general scalar DDE with a single constant delay, one could use Hayes theorem (see Bellman and Cooke (1963), pages 143-144).

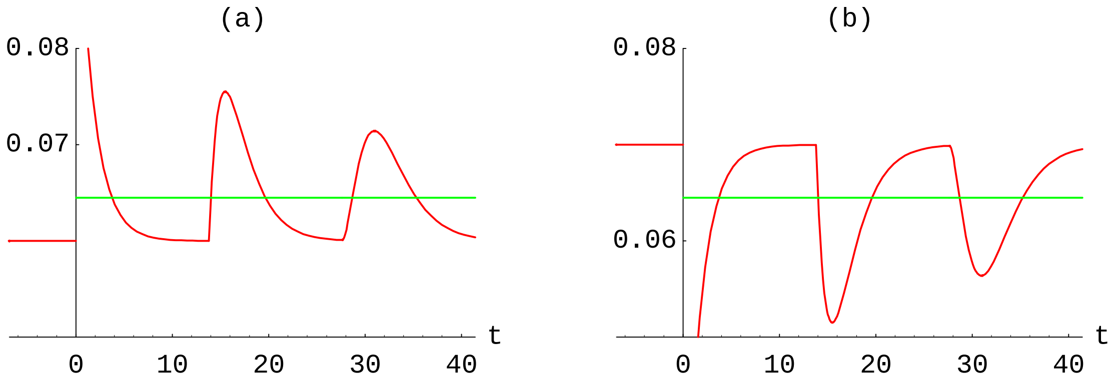

Figure 1: Transitional dynamics under constant initial conditions and constant $p(t)$
The grey line is the exogenous price and the dark line is job creation

Proposition 5 and Corollary 2 establish the existence of non-monotonic solution paths. Note that the obtained fluctuations are purely endogenous since they result exclusively from investment replacement activities. Figure 1 shows the short run adjustment to the steady state value $\overline{h}$ for two constant job creation initial profiles.[10] In panel (a), job creation was relatively low in the recent past history, which implies that at time zero unemployment is relatively high. An initially high unemployment rate stimulates job creation at the beginning. This initial boom is reproduced later due to replacement echoes. As time passes, echoes tend to vanish and job creation tends to its steady state solution. Panel (b) presents the opposite case, when job creation was relatively high in the recent past history.

It is worth pointing out that, in the case of the decentralized economy, these fluctuations vanish in the long run. In the central planner counterpart of the model, unemployment is zero and oscillations on job creation are everlasting, exactly as in Boucekkine *et al.* (1997-a). This is a **puzzling result**: Inefficiency implies that long run fluctuations disappear.

What happens if the decentralized economy is permanently affected by exogenous fluctuations? If the exogenous price $p(t)$ follows the periodic motion (12), one would expect that the job creation long run dynamics would be periodic with period Ω, the period of $p(t)$, since the endogenous replacement echoes will not operate in the long run according to the results of this section. In the following we study the mathematical relevance of this prediction. Indeed, replacement echoes and the exogenously driven fluctuations may interact in such a way that the resulting dynamics are much more complicated than

[10] In all the figures, the short run dynamics is solved using the method of steps formalized in Proposition 3. In all our numerical experiments, the time period is one year and the parameters are supposed to be $\beta = 0.5$, $\gamma = 0.03$ and $\rho = 0.05$, which implies that T^o is around 13.8 years. In this section, the exogenous price is supposed to be constant and equal to 0.0645. The corresponding value for $\overline{h}$ is around 0.0645. For presentation purposes, the exogenous price was chosen in order to have $\overline{h}$ approximately equal to it. In Figures 1 (a) and (b) $h_0(t)$ was assumed to be constant and equal to 0.06 and 0.07, respectively.

predicted above. In this respect, the ratio of the endogenous to the exogenous period, namely $\frac{T^{\circ}}{\Omega}$, is crucial, as it is detailed in the next section.

5 Dynamics of job creation under exogenous fluctuations

If $p(t)$ follows the periodic motion (12), job creation is given by the linear DDE (15), with periodic coefficients. Such equations have been considered in the mathematical literature since the early sixties (see Hahn (1961), Stokes (1962), and Zverkin (1963)). Hale and Verduyn Lunel (1993), chapter 8, provide a general treatment for periodic functional differential equations, a class of equations which includes DDEs. The main idea at the basis of the resolution scheme consists in the following observation: Given $h(t)$ a solution of (15), function $h(t+\Omega)$ is also a solution. Exploiting this invariance property, we define the functional operator K mapping the set of solutions of (15) into itself, such that $K(h)(t) = h(t+\Omega)$ for any positive t. An eigenvalue of $K(.)$ is a complex number λ such that there exists a function $h(.)$ checking : $K(h)(t) = \lambda\ h(t)$, $\forall t \geq 0$. Such a function $h(.)$ is called an eigenfunction of $K(.)$ associated with the eigenvalue λ. We can prove that the set of eigenvalues of $K(.)$ is at most countable, and is a compact set of the complex plane with the only possible accumulation point being zero (see Hale and Verduyn Lunel (1993), chapter 8). It can also be shown that any subspace formed by the eigenfunctions associated with a given eigenvalue (or eigenspace) is finite dimensional and **is invariant under** $K(.)$ (see also Hale and Verduyn Lunel (1993), chapter 8). Following a simple linear algebra intuition, one could try to obtain the solutions of the DDE under review as expansions which terms are the projections of the solutions into the eigenspaces generated by the computed eigenfunctions. Unfortunately, this decomposability issue is very far from trivial as it is made definitely clear in Hahn's seminal contribution. It can be related to the classical problem of Floquet representation arising in periodic dynamic systems (see Farkas (1994), chapter 2). To understand this, the following proposition is useful:

Proposition 6 *Any nonzero eigenvalue λ can be written as $\lambda = e^{\mu\ \Omega}$ with μ a conveniently chosen complex number. λ is an eigenvalue of the operator $K(.)$ if and only if the DDE (15) has a nonzero solution $h(t) = q(t)\ e^{\mu\ t}$ where $q(t)$ is Ω-periodic.*

The following corollary of this proposition will be most helpful later:

Corollary 3 *The DDE (15) has a nonzero Ω-periodic solution if and only if it has* 1 *as an eigenvalue. The Ω-periodic solutions of (15) are the eigenfunctions associated with the unit eigenvalue.*

Proposition 6 and its corollary are the statements of some well known properties (see for example Lemma 1.2, page 237, in Hale and Verduyn Lunel (1993), and Corollary 7.5.6, page 486, in Farkas (1994)) applied to our particular DDE (15). λ is sometimes called a Floquet multiplier and the representation $h(t) = q(t)\ \mathrm{e}^{\mu t}$ is called a Floquet representation. By the property of invariance of the eigenspaces under the operator K stated just above, one can easily show that if the DDE is initialized in an eigenspace, the solution will admit the Floquet representation given in Proposition 6. We say that the restriction of the DDE to each (finite dimensional) eigenspace has a Floquet representation. When the initialization is taken in the whole infinite dimensional space where the DDE operates, one may conjecture that the solution can be developable into a series which terms are of type $q(t)\ \mathrm{e}^{\mu t}$. We shall also refer to this as a Floquet representation. Unfortunately, the conjecture just above is false, a counter-example is provided by Hale and Verduyn Lunel (1993), page 250. This makes extremely hard, in general, the analytical study of asymptotic stability of periodic DDEs. Except in the case where the ratio T° is rational multiple of Ω, analytical assessment seems a daunting task. We provide below a theoretical analysis of the first case, and numerical experiments with the explicit method of steps will be used to study alternative cases.

5.1 The Case where $\frac{T^{\circ}}{\Omega}$ is integer

5.1.1 Long run dynamics

Set $\frac{T^{\circ}}{\Omega} = k$, k being an integer to ease our exposition. Indeed, k could be any rational number **greater than one** since some straightforward variable changes allow to transform the "rational multiple" case into an equivalent problem with an integer ratio. Such a property is provided for example in Hahn (1961). In our case, the eigenfunctions and eigenvalues of the operator can be computed straightforwardly using the fact that, by definition of the operator K, $h(t+T^{\circ}) = K^k(h)(t)$. Indeed, writing (15) at $t+T^{\circ}$ yields:

$$h'\left(t+T^{\circ}\right) = k_1(t+T^{\circ})\ h\left(t+T^{\circ}\right) + k_2(t+T^{\circ})\ h\left(t\right),$$

which implies the following characterization of an eigenfunction $h(t)$ associated with the eigenvalue λ

$$\lambda^k\ h'\left(t\right) = \lambda^k k_1(t)\ h\left(t\right) + k_2(t)\ h\left(t\right),$$

or

$$h'(t) = P_{\lambda}(t)\ h(t),$$

with $P_{\lambda}(t) = k_1(t) + \lambda^{-k}\ k_2(t)$. Hence, the eigenfunctions $h(t)$ have the form:

$$h(t) = \ h(0)\ \exp\left(\int_0^t P_{\lambda}(\tau)\ \mathrm{d}\tau\right), \tag{19}$$

and the associated eigenvalues, checking $h(t+\Omega) = \lambda\, h(t)$, are the roots of the transcendental function $f(s)$ given by

$$f(s) = s - \exp\left(\int_0^\Omega P_s(\tau)\ \mathrm{d}\tau\right).$$

Hahn (1961) shows that under certain conditions, the solution of (15) has a Floquet representation. We state this result as follows:

Proposition 7 *Assume that the roots of function $f(s)$ are all simple. Given the initial function $h_0(t)$, since $k_2(t)$ is permanently strictly positive under Assumption 2, the solution of the DDE (15) is developable into the following absolutely convergent series*

$$h(t) = \sum_i c_i \exp\left(\int_0^t P_{\lambda_i}(\tau)\ d\tau\right),$$

where c_i are constants determined by the initial function $h_0(t)$, and λ_i are the roots of $f(s)$.

Remark 1: Proposition 7 is indeed a kind of Floquet theorem for DDEs. This will be even clearer later once the eigenfunctions explicitly written using the exact expressions of $k_1(t)$ and $k_2(t)$.

Remark 2: The strict positivity of the coefficient affecting the lagged term of the DDE, here $k_2(t)$, is a sufficient condition for the absolute convergence of the series. See Hahn (1961) for a general treatment of this problem. See also Theorem 3.3 in Hale and Verduyn Lunel (1993), page 250.

Remark 3: Note that the proposition is stated under the assumption that all the eigenvalues are simple, which will be shown to hold in our specific case. In the presence of multiple roots, the constants c_i have to be replaced by polynomial terms in line with the Bellman-Cooke theorem for DDEs with constant coefficients used in the previous subsection.

We now turn to study the roots of the characteristic function $f(s)$. Given the analytical form of this function, we can conduct the following (traditional) transformation $f(s) = s\left(1 - s^{-1}\, e^{R(s^{-1})}\right)$ with $R(v) = \int_0^\Omega \left(k_1(\tau) + v^k k_2(\tau)\right) \mathrm{d}\tau$. Hence, the roots of $f(s)$ are the reciprocal zeros of $1 - v\ \mathrm{e}^{R(v)}$, or the reciprocal zeros of $\log(v) + R(v)$. Let us focus on the latter function. Putting $v = \mathrm{e}^{-w}$ or $s = \mathrm{e}^{w}$, we get a much simpler and standard equation

$$w - b_0 - b_1\ \mathrm{e}^{-kw} = 0,$$

with $b_0 = \int_0^\Omega\ k_1(\tau)\ \mathrm{d}\tau$ and $b_1 = \int_0^\Omega\ k_2(\tau)\ \mathrm{d}\tau$. Using the exact expressions of $k_1(t)$ and $k_2(t)$ and the periodic motion (12), we get

$$b_1 = -b_0 = \Omega\ \beta_0\ (\mathrm{e}^{-\gamma T^\circ} - p_0),$$

which ultimately allows to rewrite the characteristic equation as follows:

$$w + b_1 - b_1 \, \mathrm{e}^{-kw} = 0.$$

The latter equation has an identical structure as the characteristic equation of the DDE (18) studied in the previous subsection. In particular, properties i) and ii) of Proposition 5 also hold: The roots w are all simple, have negative real parts and nonzero imaginary parts, except the trivial root $w = 0$. Then, we can conclude for the following asymptotic behavior of the solutions:

Proposition 8 *For any initial function $h_0(t) \geq 0$, $\forall\, t < 0$, job creation solution paths are asymptotically Ω-periodic. More precisely, the solution paths converge to the limit cycle $\eta\, k_2(t)$, with $\eta = \frac{\overline{h}}{k_2(0)}$ and $\overline{h}$ defined in Corollary 2.*

Proof: By Proposition 7, the solution paths are developable into series which general term is $c_i \ \exp\left(\int_0^t P_{\lambda_i}(\tau)\ \mathrm{d}\tau\right)$, the product of a constant c_i determined by the initial conditions and the eigenfunction associated with an eigenvalue λ_i. Using the exact expressions of $k_1(t)$ and $k_2(t)$, we get

$$\int_0^t k_2(\tau)\ \ \mathrm{d}\tau = \beta_0\ (\mathrm{e}^{-\gamma T^\circ} - p_0)\ t + \frac{\beta_0 p_1}{p_2}\ (\cos(p_2 t) - 1),$$

and

$$\int_0^t k_1(\tau)\ \mathrm{d}\tau = -\int_0^t k_2(\tau)\ \mathrm{d}\tau + \log\left(\frac{k_2(t)}{k_2(0)}\right).$$

Thus, by definition of $P_{\lambda_i}(t)$, the eigenfunction $h_i(t)$ associated with the eigenvalue λ_i can be written as

$$h_i(t) = \exp(\ \beta_2\ (\lambda_i^{-k} - 1)\ t)\ \psi_i(t),$$

where $\beta_2 = \beta_0\ (\mathrm{e}^{-\gamma T^\circ} - p_0)$, which is strictly positive under Assumption 2, and $\psi_i(t)$ a periodic function of period Ω. From the analysis of the roots of the characteristic function $f(s)$, we know that $\mathrm{e}^w = \lambda_i$ should check

$$\lambda_i^{-k} = \mathrm{e}^{-kw} = 1 + \frac{w}{b_1}.$$

We can conclude that the term $\lambda_i^{-k} - 1$, appearing in the expression of the eigenfunctions, has always a negative real part since all the roots w have nonpositive real part and $b_1 > 0$. Hence, the eigenfunctions $h_i(t)$ vanish when t goes to infinity, except the eigenfunction associated with the trivial root $w = 0$ or $\lambda_0 = 1$. It is easy to check that the latter eigenfunction $h_0(t)$ is Ω-periodic since $\psi_0(t)$ is proportional to $k_2(t)$. Therefore, $h(t)$

should converge to a limit function of the form $\phi_0 + \phi_1\ k_2(t)$, $(\phi_0, \phi_1) \in R^2$. However, job creation should check the structural integral equation (13):

$$h(t) = k_2(t)\ \left(1 - \int_{t-T^\circ}^{t} h(\tau)\, d\tau\right),$$

$\forall t \geq 0$, and this equation is only consistent with $\phi_0 = 0$ and $\phi_1 = \eta = \frac{\overline{h}}{k_2(0)}$ when T° is a multiple of Ω. □

Proposition 8 gives the asymptotic behavior of the solutions of the DDE (15). Note that by Corollary 3, no solution is Ω-periodic from the initial period $t = 0$ except the eigenfunctions associated with the unitary eigenvalue. However, as it is clearly shown in the proof of Proposition 8, only the eigenfunction $\eta\ k_2(t)$ is consistent with the structural integral equation (13). By definition of the eigenfunctions of the functional operator $K(.)$, unless the initial function $h_0(t)$ is set equal to $\eta\ k_2(t)$, no Ω-periodic solution can be obtained from $t = 0$. That is to say that the set of initial conditions required to get Ω-periodic admissible solutions from $t = 0$ is of a null measure. Indeed, if the initial function does not correspond to function $\eta\ k_2(t)$, an adjustment process should occur. We study this adjustment process using the explicit method of steps as formalized by Proposition 3.

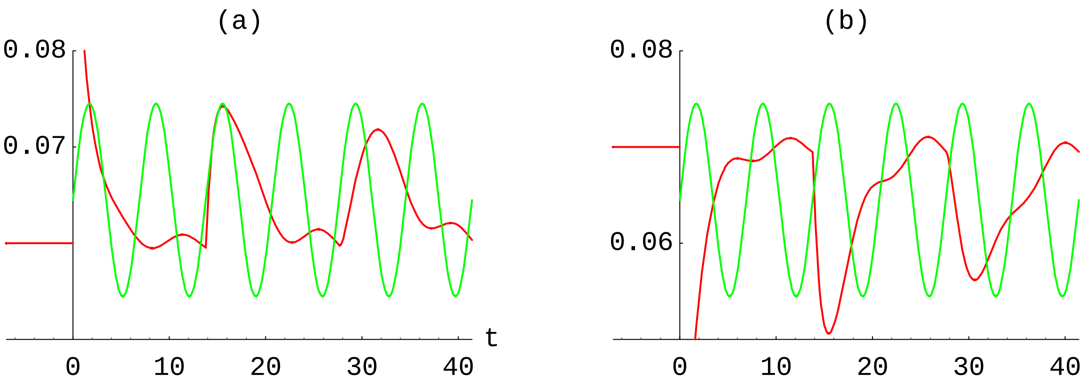

Figure 2: Transitional dynamics under constant initial conditions

5.1.2 Short run dynamics

In our case study, we fix $\Omega = \frac{T^\circ}{2}$. We set all parameters at the same values as in Section 4, so T° is around 13.8 and $\overline{h}$ is around 0.0645.[11] Assuming that no exogenous fluctuations have affected the economy in the past, Figures 2 (a) and (b) correspond to constant initial conditions lower and greater than the steady state value $\overline{h}$, respectively. As expected, the solution paths are not Ω-periodic from $t = 0$. Instead, we can observe a quite regular

[11] $p(t)$ is parameterized as follows: $p_0 = 0.0645$, $p_1 = 0.01$ and, obviously, $p_2 = \frac{4\pi}{T^\circ}$. In Figures 2 (a) and (b), we assume $h_0(t)$ constant and equal to 0.06 and 0.07, respectively.

behavior at a frequency equal to T, implying that, as in Figure 1, short run fluctuations are governed mainly by replacement echoes. Indeed, in our experiments for $T^\circ = 2\,\Omega$, the Ω-periodic long run regime is only reached after about two hundreds of periods. While the duration of the adjustment process obviously depends on the magnitude of certain parameters (in particular, on the ratio $\frac{T^\circ}{\Omega}$), the whole computational study clearly shows the predominance of replacement echoes in the short run.

There are two main differences with respect to the economy with a constant profitability cycle, study previously. First, by comparing panels (a) in Figures 1 and 2, we can observe in Figure 2, at the end of each T^o-period, a small cycle produced by the reaction of job creation to the exogenous fluctuations in profitability. As time passes, echoes tend to vanish and the fluctuations associated with the profitability cycle tend to dominate. Second, by comparing panels (a) and (b) of Figure 2, one can see that job creation dynamics are significantly asymmetric. Asymmetries are not observed in Figure 1 when $p(t)$ is constant. Thus, asymmetries are due to the combination of replacement echoes and the exogenous profitability cycle.

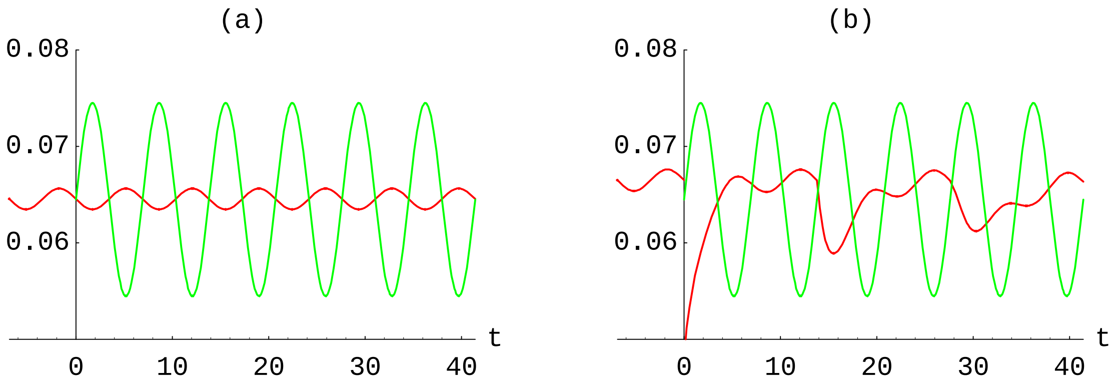

Figure 3: Transitional dynamics under periodic initial conditions

In Figure 3 we show the short run adjustment of the same parametrized economy with periodic initial conditions of the form $h_0(t) = ck_2(t)$, $c \in R$. Panel (a) corresponds to $c = \eta$, showing that if the economy was in the long run regime before zero, it stays there forever. In panel (b) we assume that initial conditions are just 3% larger than in the long run, i.e., $c = 1.03\,\eta$. This small perturbation makes a big difference: replacement echoes emerge from the beginning and they clearly dominate the short run dynamics. Since job creation was relatively high in the recent past history, unemployment is low at the very beginning, which generates a large reduction in job creation. A non-monotonic adjustment regime takes place according to the same principle as in Figure 2 (b). It is very important to notice that, to get Ω-periodic solution paths as in Caballero and Hammour (1996), we need very stringent initializations as in Figure 3 (a). Moreover, small perturbations of these initial conditions cause the economy to deviate clearly from an Ω-periodic regime in the short run (see Figure 3 (b)).

5.2 The case where $\frac{T^\circ}{\Omega} < 1$

It is easy to understand why the stability analysis is extremely difficult to undertake when T° is no longer a rational multiple of Ω. While the computation of eigenvalues and eigenfunctions is quite straightforward in the case $\frac{T^\circ}{\Omega} = k$, $k \in N$, one can easily see that even this task is considerably arduous when the latter assumption is dropped: The elementary but efficient algebraic techniques used to this end in the previous subsection are no longer valid. As argued in the introduction section, the mathematical literature does not provide so far sufficiently general analytical tools to perform the stability analysis of our DDE (15) for any T° and any Ω. To have an idea about the outcomes of our model if T° is no longer a rational multiple of Ω, we explicitly solve the model with $k \in \left\{\frac{1}{2}, \frac{1}{5}\right\}$.[12]

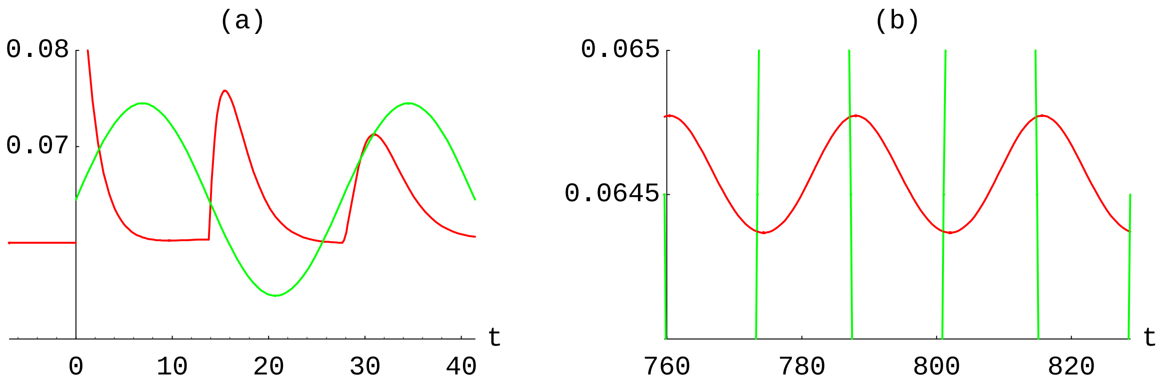

Figure 4: Transitional dynamics and long run regime with $\frac{T^o}{\Omega} = \frac{1}{2}$

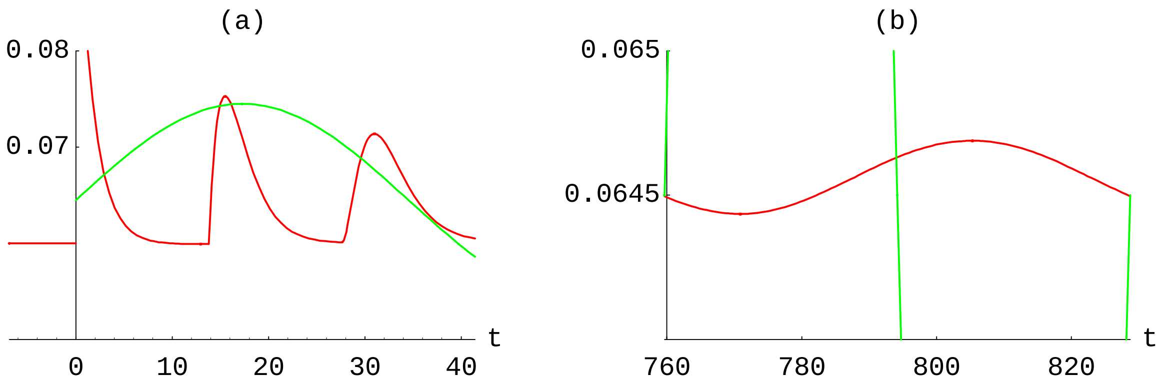

Figure 5: Transitional dynamics and long run regime with $\frac{T^o}{\Omega} = \frac{1}{5}$

[12]The other parameters are the same as in the previous subsections. To generate Figures 4, 5 and 6, we take $h_0(t) = 0.06$.

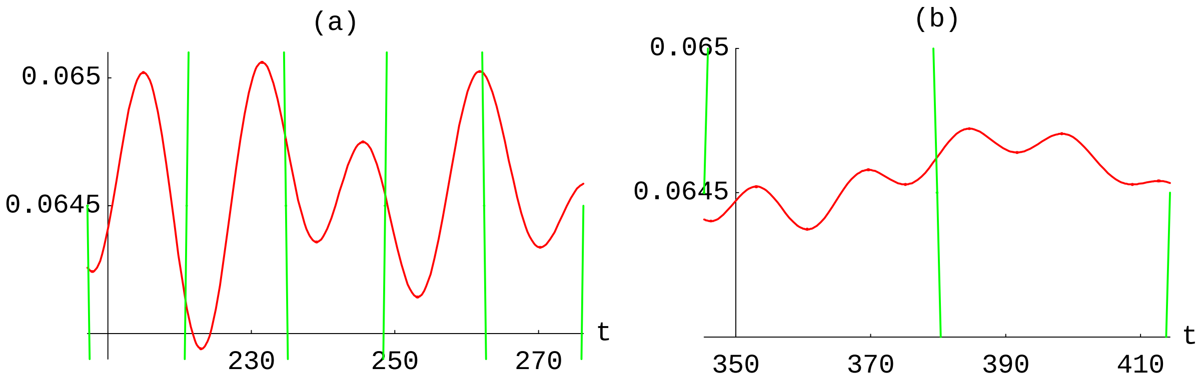

Figure 6: Adjustment to the long run regime: (a) $\frac{T^o}{\Omega} = \frac{1}{2}$ and (b) $\frac{T^o}{\Omega} = \frac{1}{5}$

>From Figures 4 and 5 we can deduce the following observations. First, from both panels (a), the short run fluctuations regime seems to be independent of the exogenous cycle. Actually, the solution paths in both panels are very similar to the one plotted in Figure 1 (a) when the profitability cycle is kept constant over time. Hence, **especially** in this case, replacement echoes again drive the short run dynamics. This is indeed the most robust finding of our numerical experiments. Second, while the long run fluctuations are again Ω-periodic, some clear differences appear with respect to the long run regime in Subsection 5.1. Indeed in this case, the smaller is k, the smaller is found to be the magnitude of long run fluctuations, see Figure 4(b) and Figure 5(b). This is a big difference with respect to the case $k \in N$, where this magnitude is independent of k. Another important difference resides in the fact that whereas the long run Ω-periodic solution is perfectly negatively correlated with the exogenous cycle (see Figure 3(a)), this property does not hold when $k < 1$ as it can be inferred from Figures 4(b) and 5(b). Finally, the adjustment process lasts much more time when $k < 1$. For example, the adjustment process is about three times longer in the case $k = \frac{1}{5}$ with respect to $k = 2$. Moreover, unlike in Subsection 5.1, extremely irregular patterns can emerge during the adjustment to the long run regime (see Figure 6).

6 Conclusion

Throughout this paper, we have tried to analyze how do interact endogenous and exogenous fluctuations in a canonical model of creative destruction yielding an inefficient decentralized equilibrium. We analytically show that job creation follows an economically very appealing delay differential equation (DDE) with periodic coefficients: The delay of the DDE is shown to be equal to the optimal scrapping time, the frequency of endogenous fluctuations, and the period of the coefficients is equal to the period of an exogenous profitability cycle. While available, we use the Floquet theory built up for DDEs to analyze the asymptotic behavior of the solutions. When that is not possible, we use the method of steps to numerically solve some complementary experiments. After all these mathematical and numerical exercises, we have brought out three main conclusions. First, replacement echoes seem to play a fundamental role in the short run dynamics of job creation and job destruction: they clearly dominate the exogenous profitability cycle. Secondly, in the long run, the solution paths are driven by the exogenous cycle. We mathematically prove this property when the optimal scrapping time is a rational multiple of the period of the exogenous cycle. We find numerically that this property also holds in some alternative cases. Finally, the adjustment from the *replacement echoes* short run fluctuations regime to the *exogenous* long run fluctuations regime is shown to display a number of interesting characteristics (asymmetries and highly irregular patterns).

Incidentally, the whole analysis conducted in this paper shows how merging exogenous and endogenous fluctuations may result in rich and economically interesting dynamics. Although the required mathematical treatment is not likely to be easy, the potential gain from this approach deserves consideration, as it offers a solid bridge between exogenously driven cycles, as in the RBC literature, and the most recent Schumpeterian set-ups.

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