Growth and Human Capital Accumulation under Uncertainty

Alexandra Rillaers[1]
IRES, Université catholique de Louvain.
3, Place Montesquieu.
B-1348 Louvain-la-Neuve. Belgium.
e-mail: rillaers@ires.ucl.ac.be

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Abstract

The aim of this paper is to understand the role of uncertainty in education choices and therefore in growth. We consider an overlapping generations model in which endogenous growth is introduced through human capital accumulation. We introduce uncertainty as to the individual returns to educational investment, and we assume markets to be incomplete. We study the effect of this uncertainty on the behavior of the risk averse individual regarding his effort in education. The analysis is carried out within a general equilibrium approach.

Keywords: Human capital, Growth, Education, Uncertainty, Overlapping Generations Model.

JEL classification: I29, O41.

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1 Introduction

In the literature on human capital it is nowadays commonly accepted that investment in education, and hence in human capital, is a risky activity, due to the uncertainty of its returns. Schultz (1961) already pointed out the risky nature of investment in education, due to the uncertainty individuals face in assessing their innate talents. Becker (1975) distinguished three sources of uncertainty in the returns to educational investment: uncertainty of people concerning their ability; uncertainty about the length of life which is an important element in the return; and uncertainty about the return to a person of given age and ability, due to other numerous unpredictable events.

However, initially the theory of investment in human capital has been developed under the assumption of perfect foresight; no attempt has been made to formally incorporate this uncertainty in the theory until Levhari and Weiss (1974) developed a first rigorous analysis of this problem. According to Levhari and Weiss (1974), the more risky nature of human capital compared to physical capital is due to the fact that the former cannot be bought or sold, nor can it be separated from its owner. They developed a two-period model of human capital formation, in which is shown that under the hypothesis of increasing risk, an increase in uncertainty in the return to human capital investment decreases the level of this investment. Snow and Warren (1990) extend the Levhari-Weiss model allowing for endogenous labor supply. Under certain assumptions their analysis obtains similar results, i.e. a negative effect of increasing uncertainty on educational investment.

From Levhari and Weiss (1974) the subsequent literature on human capital has further build on this idea of uncertain returns to human capital investment, allowing for several extensions. Williams (1978) examines the connection between investing in risky human capital and investing in marketable assets also characterized by risky returns. He too finds that an increase in uncertainty induces a decrease in educational investment. Another analysis of Williams (1979) includes uncertainty in a continuous time model of human capital formation, leading to the same findings. In this article Williams (1979) examines the properties of optimal time allocations to labor, leisure and education over the life cycle when individuals face several sources of uncertainty, among which one is linked to future wages. According to other studies however, like in Kodde (1986), the impact of uncertainty on the demand of education cannot be determined unambiguously, but depends on the way uncertainty is introduced in the earnings function. A common feature in above analyses is that they are all carried out in a partial equilibrium framework.

The uncertainty characterising the returns to education, as well as the externality which results from the aggregate level of human capital in human capital accumulation models, make educational efforts difficult to be correctly financed by free credit markets. The relevance of the issue is clear as soon as one accepts the role of human capital, and hence of education, in economic development —Schultz (1960), Schultz (1961), Becker (1962) and more recently Lucas (1990) stressed the importance of schooling in the formation of human capital as the engine of long run economic growth. The present paper considers a model in which human capital accumulation accounts for endogenous growth; education is assumed to be individually financed in an incomplete credit market (agents face some uninsured idiosyncratic shock). We aim to understand the role of uncertainty in education choices and hence on growth within a general equilibrium framework. The discussion will be carried out in an overlapping generations model building on previous work by Michel (1993) and de la Croix (1996).

In endogenous growth models the production of human capital results from the combination of the environment represented by the current average stock of human capital, as well as from some schooling effort (see Uzawa 1965). Human capital technology will be assumed to be of constant returns so that sustainable growth of human capital is feasible. Hence, any policy affecting individual incentives to invest in human capital will have an effect on growth by changing effort choices. However, as existing average human capital enters the production function of effective labour together with individual effort, the relative weight of these two inputs will determine the kind of growth we can expect. The less individual effort weights in the production function, the more this model will look as one of exogenous growth; one in which human capital accumulation would result from a simple externality, exogenously increasing the stock from one period to another.

In our model agents produce human capital in their first life period from the average current stock of human capital representing the environment, and from individual effort. This effort is measured in real terms and financed in the credit market. An individual-specific ability shock unknown ex ante will affect the final outcome next period. These individual shocks represent the ex ante non-observable ability of the agents to generate human capital from individual effort and from the environment; the shocks are assumed to independently affect a large number of agents so that there will be no aggregate uncertainty: ex ante individual probabilities will be ex post aggregate fractions of the population. Credit markets will only allow for borrowing-lending operations through a single asset, which implies that agents cannot insure themselves against individual uncertainty. We indeed assume markets to be incomplete, since this is a condition for uncertainty to have some effect on the individual's decision. As markets are supposed to be incomplete and as the Inada conditions imply positive consumption, the agents face a credit constraint. We find that uncertainty has a negative effect on individual educational effort and thus on growth, due to the individual's attitude towards risk. However, studying the consequences of uncertainty within a general equilibrium approach, as is done in this paper through numerical exercises, allows for other kinds of effects which in particular cases and for low levels of uncertainty, may reverse the usual negative effect of uncertainty on educational effort.

The paper is organized as follows: section 2 describes the benchmark model; the third section analyzes the behavior of this economy which includes the dynamics of the model and the effects of uncertainty on educational effort. Section 4 contains the concluding remarks.

2 The model

We essentially build on previous work by Michel (1993) and de la Croix (1996), an overlapping generations model in which human capital accumulation is responsible for endogenous growth through a constant returns to scale technology in the physical good sector. We introduce uncertainty at the individual level in the human capital accumulation function: the individual decides his level of educational effort without knowing its ex post return. It is assumed that each individual is endowed with a different ability to take advantage of his educational effort in terms of human capital formation. This ability is assumed to be not observable ex ante, and to be only revealed once the agent has decided upon his educational effort. As a consequence a same level of education will not necessarily provide each individual with a same level of human capital[2].

2.1 Human capital as an externality

In each period t a new generation consisting of N_t individuals is born. The number of individuals of each generation grows at a constant rate n; so we have that $N_t = (1+n)N_{t-1}$. Consequently the total population also grows at the rate n. Each generation lives three periods. Consider an agent born in $t-1$: in the first period he does not work, neither consumes, but decides how much to spend on educational effort e_t, in order to produce his human capital stock. This amount e_t is borrowed in the capital market. The young generation is affected by the environment created by the previous generations. This environment is formalized as a positive externality representing the existing average stock of human capital H_{t-1} in the production of next period's human capital. Constant returns to scale in the human capital production function will ensure sustainable accumulation of human capital and therefore long run growth.

Uncertainty is introduced at the individual level by means of an ability shock z_t affecting the human capital accumulation technology. z_t is a stochastic parameter which represents the individual's ability to generate human capital from educational effort and from the environment. This ability is ex ante not observable by the individual. By $f(z_t)$ we denote the density of z_t, which is assumed to be known. In short, the agent born in $t-1$ makes his effort decision e_t, the environment being represented by H_{t-1}. At the next period the ability shock z_t realizes resulting in some stock of individual

[2]This ability to accumulate human capital can be partially innate (See Schultz 1961 and Becker 1975), but it can also be assumed to be affected by external factors, as the quality of schooling (See Card and Krueger 1992 and Kodde 1986), or the family background (See Altonji and Dunn 1996).

human capital h_t given by

$$h_t = z_t H_{t-1}^{1-\beta} e_t{}^{\beta}, \tag{1}$$

where $\beta \in (0, 1)$ measures the relative weight of individual effort in the production function. Agents are faced with the same ex ante distribution of z_t. As a consequence decisions will be identical across individuals; this is the reason why we avoid the use of any agent indexation. Once the individual ability z_t is revealed however, agents become heterogenous as far as their human capital stock is concerned. Their resulting earnings will be different too, and so will be their respective savings and consumption level, and their utility.

Each individual ability shock realizes independently from other agent's shocks. It is assumed that the economy is populated by a large number of agents. As a consequence, the ex ante expectation of any random variable function of z_t can be reinterpreted as the ex post sample average of such function. In particular, the unconditional mean of h_t can be seen as the sample average human capital stock H_t in the economy. That is,

$$H_t = \int h_t(z_t) f(z_t)\, dz_t = \mu H_{t-1}^{1-\beta} e_t^{\beta}, \tag{2}$$

where μ is the unconditional mean of z_t. The absence of uncertainty at the aggregate level implies that prices, w_t and r_t, are deterministic.

2.2 The household's problem

In period t the agent with shock z_t supplies his units of efficiency labor inelastically; the labor market pays a real wage w_t. The agent also repays his debt $(1 + r_t)e_t$, consumes c_t, and saves s_t for retirement. In period $t + 1$ he will retire, spending his savings plus the interests, d_{t+1}. Individual preferences are assumed to be represented by a discounted sum of instantaneous utilities, each of them being CES, with $\theta > 0$ the inverse of elasticity of substitution. Recall that θ is also the coefficient of relative risk aversion. $\theta = 1$ is interpreted as the logarithmic case. Future utility is discounted at a constant exogenous rate $\rho \in (0, 1)$.

The problem is solved in two steps. In the first one the agent is young and chooses his investment in education. He will choose e_t so as to maximize the expectation of his indirect utility. In the second step, when adult, the agent chooses s_t for a given level of e_t and z_t. By that time individual ability z_t is revealed, so that h_t is known and the agents do not face any uncertainty anymore. We solve the problem backwards. For a given effort e_t and ability z_t, the agent chooses s_t in order to solve

$$\begin{aligned} \max_{s_t} \quad & \frac{{c_t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{{d_{t+1}}^{1-\theta}}{1-\theta} \\ \text{s.t.} \quad & c_t + s_t = w_t h_t - (1 + r_t)e_t \\ & d_{t+1} = (1 + r_{t+1})s_t \\ & h_t = z_t H_{t-1}^{1-\beta} e_t^{\beta} \end{aligned}$$

where r_t is the real interest rate and w_t denotes the real wage per unit of efficiency labor. H_{t-1}, r_t, w_t and z_t are known, and r_{t+1} is perfectly anticipated by the agent. As individual consumption and hence the resulting utility depend on the individual ability shock z_t, we are confronted with heterogenous agents. In regard of (1) it is clear that $z_t > 0$ is a necessary condition for this economy not to have a unique obvious equilibrium.

From above maximization problem it can easily be verified that an interior solution is characterized by[3]

[3]It should be clear to the reader that individual savings s_t is a function of z_t and hence differ among the agents.

$$(3) \qquad s_t = (h_t w_t - (1 + r_t)e_t)p_{t+1}$$

with

$$p_{t+1} = (1 + (1+\rho)^{\frac{1}{\theta}}(1 + r_{t+1})^{\frac{\theta-1}{\theta}})^{-1}$$

the propensity to save. Concavity will ensure that this is in fact the unique solution. After the uncertainty has been resolved, z_t and hence h_t are known, so for a given e_t we have that s_t is perfectly deterministic. Again, since we have a large number of agents, ex ante expectations are ex post interpreted as the sample mean over the total population. In particular

$$(4) \qquad S_t = \int s_t f(z_t)\ dz_t = (\mu H_{t-1}^{1-\beta} e_t^{\beta} w_t - (1 + r_t)e_t)p_{t+1}.$$

Average savings S_t become aggregate savings by multiplying by N_{t-1}. Substituting for s_t by (3) in the utility function, and using the household budget constraint, give us the indirect utility

$$u_t = (1-\theta)^{-1} s_t^{1-\theta} {p_{t+1}}^{\theta-1}(1 - p_{t+1})^{-\theta}.$$

When young, the agent will choose his educational effort e_t in order to maximize his expected indirect utility subject to his individual accumulation rule of human capital. Given that ρ and r_{t+1} are given to the individual, this problem is equivalent to

$$\max_{e_t} \frac{1}{1-\theta} \int [z_t H_{t-1}^{1-\beta} e_t^{\beta} w_t - (1 + r_t)e_t]^{1-\theta} f(z_t)\ dz_t.$$

The agent faces uncertain returns to his effort, since his ability z_t is unknown to him. As markets are incomplete, the agent cannot insure himself against the risk he faces by purchasing some portfolio. Since the Inada conditions imply positive consumption, effort should be such that $z_t H_{t-1}^{1-\beta} e_t^{\beta} w_t > (1 +$

$r_t)e_t$, for all z_t. In other words, the borrowing constraint should never bind, even in the worst of the situations tomorrow.

For a uniform distribution of the productivity shock z_t, defined on a support $[a, b]$, we have that

Proposition 1 *The optimal choice of effort is interior in the sense that* $z_t H_{t-1}^{1-\beta} e_t^{\beta} w_t > (1 + r_t)e_t$.

See Appendix A for a proof of this statement. An immediate implication of above proposition is that the borrowing constraint will never bind.

Given that the solution is interior, it is characterized by the following first order condition:

$$\begin{aligned} (5) \qquad \beta \left(\frac{e_t}{H_{t-1}} \right)^{\beta - 1} & w_t \int_a^b z_t (z_t H_{t-1}^{1-\beta} e_t^{\beta} w_t - (1 + r_t)e_t)^{-\theta} f(z_t) \; dz_t \\ &= (1 + r_t) \int_a^b (z_t H_{t-1}^{1-\beta} e_t^{\beta} w_t - (1 + r_t)e_t)^{-\theta} f(z_t) \; dz_t. \end{aligned}$$

The individual invests in education up to the point where the expected return of the last unit of educational effort is equal to its marginal cost. Also check that for the case of no uncertainty, this is when all probability is concentrated at $z_t = 1$, (5) tends to the same first order condition for e_t as in the deterministic case treated by Michel (1993) and de la Croix (1996)

$$e_t = h_{t-1} \left[\frac{w_t \beta}{1 + r_t} \right]^{\frac{1}{1-\beta}}$$

2.3 The Firms

The supply side of the economy is represented by an aggregative neoclassical production function: a single representative firm endowed with a constant returns to scale technology, behaving competitively and hiring the total supply of production factors. As there is no aggregate uncertainty the firm's problem is essentially deterministic.

In period $t-1$ the firm has to decide how much capital stock K_t it will use for production in period t; it will also hire effective labor L_t payed at a real wage w_t. Technology is represented by a Cobb-Douglas production function with $\alpha \in (0,1)$ being the share of capital in total income. The firm's problem is to maximize the sum of all revenues and costs discounted by the real interest rate r_t. Its problem is, therefore, to solve

$$\max K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - (1+r_t) K_t$$

over capital K_t and effective labor L_t choices. The first order conditions equalize the marginal product of each production factor to its marginal cost, that is, its price. In particular

$$\begin{aligned} w_t &= K_t^{\alpha}(1-\alpha) L_t^{-\alpha} \\ 1+r_t &= \alpha K_t^{\alpha-1} L_t^{1-\alpha}. \end{aligned}$$

2.4 Competitive equilibrium

At each period t aggregate effective labor supplied by the households is equal to the average H_t times the number of agents N_{t-1} of this generation. Hence the labor market clearing requires $L_t = N_{t-1} H_t$. The demand for credit consists of the firm's gross investment K_{t+1} and of the current young generation investment in human capital $N_t e_{t+1}$. The clearing of the capital market implies that aggregate savings, the supply of credit, should equal investment in physical and human capital. Thus

$$N_{t-1} S_t = K_{t+1} + N_t e_{t+1}. \tag{6}$$

The goods markets will clear as soon as the credit and labor markets do so and the individual budget constraint holds.

Let us state a complete system of equations describing competitive equilibria for this economy. Using these two equilibrium conditions we just derived, several individual optimality conditions can be rewritten in a more convenient way. For instance, the firm's first order conditions can be expressed as

$$w_t = (1-\alpha){k_t}^{\alpha} \tag{7}$$

$$1 + r_t = \alpha {k_t}^{\alpha-1} \tag{8}$$

where $k_t = K_t/(N_{t-1}H_t)$ is the quantity of capital per unit of efficiency labor. The law of motion of average human capital is given by (2). Substituting in (3), the first order condition for the savings contingent plan can be written as

$$s_t = (z_t H_{t-1}^{1-\beta} e_t^{\beta} w_t - (1+r_t)e_t)p_{t+1}. \tag{9}$$

The second optimality condition is given by (5). We must also consider the relationship between individual contingent plans and population average of savings (4). Finally, the credit market equilibrium condition (6) can be expressed in terms of these variables dividing by N_{t-1} in order to obtain

$$S_t = (1+n)[k_{t+1}H_{t+1} + e_{t+1}] = (1+n)[k_{t+1}\mu H_t^{1-\beta} e_{t+1}^{\beta} + e_{t+1}]. \tag{10}$$

Definition 1 *Given an initial capital stock K_0, an initial stock of human capital H_{-1}, and an initial population N_{-2}, an intertemporal equilibrium with perfect foresight is a sequence $\{e_t,\ K_t,\ L_t,\ h_t,\ H_t,\ s_t,\ S_t\}$ and a sequence of prices $\{(1+r_t),\ w_t\}$, which for all $t \geq 0$ satisfy (1), (2), (3), (4), (5), (7), (8), and the three market equilibrium conditions: (6), $L_t = N_{t-1}H_t$ and the clearing of the goods market.*

3 Dynamics

The optimality condition (5) will, in general, not allow to obtain an explicit relationship between contingent savings and effort. In this section we provide an analytical solution for a particular value of θ.

3.1 Balanced growth paths

In this section we shall prove existence, uniqueness and stability of a balanced growth path; a balanced growth path is defined as a competitive equilibrium path along which average human capital grows at a constant rate $g > 0$. Hence, our conjecture is that average variables will be growing at the same rate g while aggregate variables will be doing so at a rate $n + g$. In order to prove the existence of such a growth path, we shall transform our variables such that the system of equations is expressed in terms of variables that remain stationary along a balanced growth path. Let $\hat{e}_t = e_t / H_{t-1}$, individual effort per efficiency unit, while savings per efficiency unit are defined as $\hat{s}_t = s_t / H_t$ so that again $\hat{S}_t = \int \hat{s}_t f(z_t)\, dz_t$.

Capital stock per efficiency unit must already remain constant along a balanced growth path. Thus, the equations relative to the firm's first order conditions remain unchanged. The law of motion of average human capital gives us the growth g_t of human capital

$$\frac{H_t}{H_{t-1}} = 1 + g_t = \mu \hat{e}_t^{\beta}. \tag{11}$$

Dividing (10) by H_t we obtain

$$\hat{S}_t = (1+n)[k_{t+1} \mu \hat{e}_{t+1}^{\beta} + \hat{e}_{t+1}]. \tag{12}$$

The relationship between average and contingent savings constitutes the fifth equation of the system. Concerning the household, we have now

$$(1+g_t)\hat{s}_t = (z_t \hat{e}_t^{\beta} w_t - (1+r_t)\hat{e}_t)p_{t+1}, \tag{13}$$

and equation (5) is now given by

$$\begin{aligned} &\beta \hat{e}_t^{\beta-1} w_t \int_a^b z_t (z_t \hat{e}_t^{\beta} w_t - (1+r_t)\hat{e}_t)^{-\theta} f(z_t)\ dz_t \\ &= (1+r_t) \int_a^b (z_t \hat{e}_t^{\beta} w_t - (1+r_t)\hat{e}_t)^{-\theta} f(z_t)\ dz_t, \end{aligned} \tag{14}$$

which results from dividing the original equation by $H_{t-1}^{-\theta}$.

3.2 Solving for effort per efficiency unit

In order to obtain an analytical solution for $\hat{e}_t$, we consider the case in which $\theta = 3$, and we assume a uniform distribution for the productivity shock z_t. In particular we will assume that $z_t \sim U[a, b]$ where $a > 0$ so that the borrowing constraint does not annihilate the economy. The density function is therefore constant $f(z_t) = (b-a)^{-1}$ and we know that $a = \mu - \sqrt{3}\sigma$, and $b = \mu + \sqrt{3}\sigma$ where μ and σ are the mean and the standard deviation of the distribution respectively. After integrating (14) for $\theta = 3$, rearranging, and solving the resulting quadratic equation, we obtain the following two solutions for $\hat{e}_t$,

$$\hat{e}_t = \left[\frac{2w_t\beta(\mu^2 - 3\sigma^2)}{(1+r_t)[(1+\beta)\mu + \sqrt{\mu^2(1-\beta)^2 + 12\beta\sigma^2}]}\right]^{\frac{1}{1-\beta}} \tag{15}$$

$$\hat{e}_t = \left[\frac{2w_t\beta(\mu^2 - 3\sigma^2)}{(1+r_t)[(1+\beta)\mu - \sqrt{\mu^2(1-\beta)^2 + 12\beta\sigma^2}]}\right]^{\frac{1}{1-\beta}}. \tag{16}$$

For a proof see Appendix B.

In Appendix C we show that (15) satisfies the borrowing constraint, but (16) does not. Hence, (16) is not an optimal solution for $\hat{e}_t$, and we retain (15) as the only feasible first order condition for $\hat{e}_t$. From this solution we see that, for a given level of k_t, educational spending depends negatively on the variance: the individual will decrease his investment in education e_t when uncertainty, σ, increases. Risk aversion is the reason why when deciding upon his investment in education e_t, the agent takes into account not only the mathematical expectation of his return to education, but also the standard deviation σ.

Using the equilibrium equations we are able to express $\hat{e}_{t+1}$ implicitly as a function of $\hat{e}_t$. Indeed, compute the expectation of (13) with respect to z_t, use the definition of p_{t+1} and equations (7) and (8) to get

$$\hat{S}_t = \frac{1}{\mu \hat{e}_t^{\beta}} \left[\frac{\mu \hat{e}_t^{\beta}(1-\alpha)k_t^{\alpha} - \alpha k_t^{\alpha-1}\hat{e}_t}{1 + (1+\rho)^{1/3}(\alpha k_{t+1}^{\alpha-1})^{2/3}} \right], \tag{17}$$

where (11) has been used to substitute for g_t. We obtain the dynamic equation in terms of $\hat{e}$ of this economy by equating (17) and (12), and using (15) to substitute for k_t:

$$\xi(e_t, e_{t+1}) = 0. \tag{18}$$

See Appendix D for the explicit expression of (18).

Due to its non-linear nature, the dynamic equation (18) is difficult to solve analytically for $\hat{e}_{t+1}$.

Lemma 1 *The dynamic equation (18) ruling the behaviour of human capital investment is concave and satisfies*

$$\lim_{\hat{e}_t \to 0} \frac{d\hat{e}_{t+1}}{d\hat{e}_t} = \infty \ \textit{and} \ \lim_{\hat{e}_t \to \infty} \frac{d\hat{e}_{t+1}}{d\hat{e}_t} = 0$$

See Appendix E for a proof of this statement. The previous lemma implies the following proposition:

Proposition 2 *There exists a unique positive interior steady state which is globally stable from any interior initial condition.*

After having established the properties of existence, stability and uniqueness of the solution, we now study the impact of uncertainty on the balanced growth path.

3.3 The effects of uncertainty in the general equilibrium

Due to the complicated nature of the dynamic equation (18) we have to rely on a numerical analysis in order to study the effects of uncertainty on educational effort in the general equilibrium. In this numerical example we adopt the following settings for the parameters: population growth n is set to 0; the inverse of elasticity of substitution θ to 3; the share of capital in total income α is fixed to 0.3; the psychological discount rate ρ is fixed to 1.5. We further set $\mu = 3$, which gives us accumulated growth rates between 1.1 and 1.8; this implies annual growth rates between 0.3% and 2%, given the fact that each generation is assumed to live for 30 years. We let β vary between 0.1 and 0.2,[4] and σ between 0 and 1.6. The reason for including β in this graphical analysis is to study its influence on the effects of uncertainty. We obtain the following three-dimensional graphs representing the effects of uncertainty on the steady state level of educational effort per efficiency unit, figure (1), and on the growth rate, figure (2).

From figure (1) we can observe that the general trend consists in an

[4]We think that in OLG economies the role of inherited aggregate human capital is likely to be crucial for growth. One may think that investing in education in an economy without pre-existing knowledge would be totally inefficient. This implies that we will consider low β. Furthermore, numerical experiments indicate that higher β leads to numerical problems which seems to indicate that the range of values for β has to be restricted.

Figure 1: Steady state educational spending

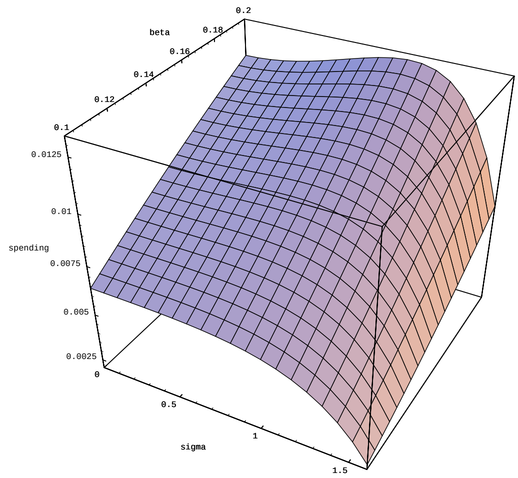
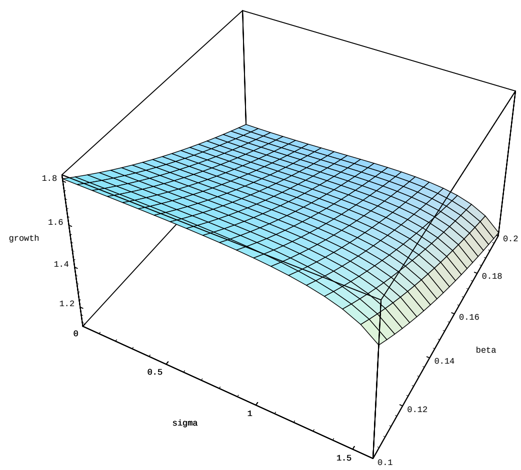


Figure 2: Growth



increase in uncertainty having a negative effect on individual educational effort per efficiency unit. This is due to the risk averse attitude of the agents, which induces them to invest less in risky assets.

Higher values of β however, initially allow for a positive response in terms of educational efforts to an increase in σ. But when uncertainty further increases, the effect becomes negative again. This argues for other effects besides that of risk aversion playing a role in a general equilibrium approach, when uncertainty increases. Indeed, a decrease in $\hat{e}$ resulting from an increase in uncertainty σ, implies on one hand a decrease in the supply of efficient labor, which will lead to a higher wage per efficiency unit w. On the other hand, a decrease in $\hat{e}$ implies a smaller demand for credit on the financial markets. This will induce a lower interest rate r, and hence stimulate the accumulation of physical capital: capital per efficiency unit k increases. Given (7) and (8), wages will further increase, while the interest rate decreases. These changes in prices will positively affect educational effort since a higher wage per efficiency unit implies higher returns in terms of wage to educational effort, while a decreased interest rate lowers its cost. While for small values of β these price effects are not important enough to reverse the initial decrease in $\hat{e}$, for higher values of β they may be. Recall that β stands for the relative weight of individual effort in the human capital accumulation rule. High values of β imply high returns to educational effort. A same increase in k will hence, through prices, induce an increase in effort which will be higher for large values of β than for small ones. Indeed, when computing $d\hat{e}/dk$ from (15), after replacing w_t and $(1+r_t)$ by their respective first order conditions (7) and (8), we get

$$\frac{d\hat{e}_t}{dk_t} = \frac{1}{1-\beta}k^{\frac{\beta}{1-\beta}}\left[\frac{2(1-\alpha)\beta(\mu^2-3\sigma^2)}{\alpha[(1+\beta)\mu+\sqrt{\mu^2(1-\beta)^2+12\beta\sigma^2}]}\right]^{\frac{1}{1-\beta}}$$

which positively depends on β. Hence for high values of β, the positive price effect on $\hat{e}$ may be large enough to more than compensate the initial decrease in effort.

However, when uncertainty raises too much, the negative effect due to risk aversion becomes again the dominating one. This points to a third factor playing a role, namely the level of k. Indeed, wages and interest rate exhibit negative second derivatives with respect to k. Stated differently, the respectively positive and negative effect of a marginal increase in k on wages and interest rate is decreasing in k, and so are thus the price effects. The latter will finally become too small to offset the initial negative effect of an increase of σ on $\hat{e}$, even for high values of β. Hence the ultimately negative effect of uncertainty on educational effort.

Figure (2) shows the effect of the same range of parameters on the growth rate. The effects of uncertainty on the growth rate go in the same direction as the ones on $\hat{e}$: for small values of β, the growth rate decreases monotonously with σ. For higher values of β, the growth rate increases with σ up to a certain point, after which it decreases again. These effects are directly related to the effects on $\hat{e}$. It is also interesting to note that the growth rate decreases with β, in spite of the fact that $\hat{e}$ increases with β. This is due to the fact that high values for β imply that the weight $1-\beta$ of the externality H_{t-1}, which ultimately accounts for any growth, is small[5].

Uncertainty has thus a non trivial impact on educational effort. When-

[5] The extreme case of $\beta = 1$ would result in a stationary economy in which no sustainable growth is feasible so that one should expect that g tends to 0.

ever the elasticity of individual human capital with regard to aggregate human capital is high, uncertainty exerts a clear negative effect on individual educational effort. Conversely, if this elasticity is low enough, uncertainty may be education enhancing, which should be thought of as a paradoxical situation. However, the first effect takes the upper hand for reasonable calibration. Indeed, as education is one of the main engines of growth in developed economies (see Jorgenson 1980 and Jorgenson and Fraumeni 1992), uncertainty has, in most cases, a negative effect on the rate of growth. Thus, mechanisms designed to reveal information on ability, and hence reducing uncertainty, should be growth enhancing.

4 Concluding remarks

We have presented an overlapping generations model in which human capital investment through educational effort accounts for endogenous growth. We introduced uncertainty as to the returns to individual educational effort, and studied its effect. We found that uncertainty has a negative effect on the individual effort, and hence on growth. The general equilibrium approach allows however, to take into account other, subsequent, effects which under certain circumstances may lead to a final increase of educational effort as a response to increasing uncertainty. In particular, for larger values of β, the relative weight of education in the human capital accumulation rule, price effects may be important enough to reverse the initial negative effect, resulting in a finally positive response in terms of educational effort. However, as uncertainty further increases, these price effects become less important due to the decreasing effect on prices of a marginal change in capital per efficiency unit. The price effects finally become insufficient to reverse the initial negative effect; hence educational effort decreases again as uncertainty rises.

A possible extension of the present paper is hence the study of policies which are likely to reduce uncertainty. Indeed, as human capital constitutes an important factor in sustainable long run growth, this negative impact of uncertainty on educational effort should be subject of concern on behalf of policy makers. Policies could be designed to help individuals in assessing their abilities — and therefore reducing uncertainty — in order to stimulate growth. One may also think of a policy consisting in reducing uncertainty linked to future earnings providing a minimum income like an unemployment benefit or a basic income. Reducing the individual cost of education by providing subsidies could also be a way to induce more educational effort, and hence stimulate growth.

Appendix

A. Proof of proposition 1

Define the objective function as

$$J(e_t) = \frac{1}{1-\theta} \int_a^b (z_t H_{t-1}^{1-\beta} e_t^\beta w_t - (1+r_t)e_t)^{1-\theta} f(z_t)\, dz_t \tag{19}$$

for all $e_t \in [0, \bar{e}]$ where $\bar{e}_t = \left(\frac{aw_t}{(1+r_t)}\right)^{\frac{1}{1-\beta}}$ is the maximum effort allowed by the borrowing constraint.

For all $e_t \in (0, \bar{e}_t)$ the derivative of $J(e_t)$ with respect to e_t is well defined and equal to

$$J'(e_t) = \int_a^b (z_t H_{t-1}^{1-\beta} e_t^\beta w_t - (1+r_t)e_t)^{-\theta} (z_t H_{t-1}^{1-\beta} e_t^{\beta-1} - (1+r_t)) f(z_t)\, dz_t.$$

Define $J'(0) = \lim_{e_t \to 0} J'(e_t)$ and $J'(\bar{e}_t) = \lim_{e_t \to \bar{e}_t} J'(e_t)$. Consider the case $0 < \theta < 1$, then $J(0) = 0$, while $J(e_t) > 0$ for $e_t \in (0, \bar{e}_t)$. In the case $\theta \geq 1$ we have $J(0) = -\infty$ while for $e_t \in (0, \bar{e}_t)$, $J(e_t) > -\infty$. We shall prove that $J'(\bar{e}_t) < 0$ for any $\theta > 0$ in which case there must be some interior $e*_t$ which verifies $J'(e*_t) = 0$; and this must be the optimal choice of effort.

For $z_t \sim U[a, b]$ we have that the density function $f(z_t) = \frac{1}{b-a}$. Hence we get

$$
\begin{aligned}
(20) \quad J'(e_t) &= \frac{1}{b-a} \int_a^b z_t H_{t-1}^{1-\beta} e_t^{\beta-1} w_t (z_t {H_{t-1}^{1-\beta}}^{1-\beta} e_t^{\beta} w_t - (1+r_t)e_t)^{-\theta} f(z_t)\, dz_t \\
&+ \frac{1}{b-a} \int_a^b (1+r_t)(z_t H_{t-1}^{1-\beta} e_t^{\beta} w_t - (1+r_t)e_t)^{-\theta}) f(z_t)\, dz_t.
\end{aligned}
$$

We distinguish two cases:

- $\theta \in (0, +\infty) \setminus \{1\}$

  After integrating (20) becomes

$$
\begin{aligned}
&\left[b(bH - t - 1^{1-\beta} e_t^{\beta} w_t - (1+r_t)e_t)^{1-\theta} - a(aH_{t-1}^{1-\beta} e_t^{\beta} w_t - (1+r_t)e_t)^{1-\theta} \right] \frac{\beta}{(b-a)(1-\theta)e_t} \\
- &\left[(bH_{t-1}^{1-\beta} e_t^{\beta} w_t - (1+r_t)e_t)^{2-\theta} - (aH_{t-1}^{1-\beta} e_t^{\beta} w_t - (1+r_t)e_t)^{2-\theta} \right] \frac{\beta}{(b-a)(1-\theta)(2-\theta)H_{t-1}^{1-\beta} e_t^{\beta-1} w_t} \\
- &\left[(bH_{t-1}^{1-\beta} e_t^{\beta} w_t - (1+r_t)e_t)^{1-\theta} - (aH_{t-1}^{1-\beta} e_t^{\beta} w_t - (1+r_t)e_t)^{1-\theta} \right] \frac{(1+r_t)}{(b-a)(1-\theta)H_{t-1}^{1-\beta} e_t^{\beta} w_t}
\end{aligned}
$$

  or, after rearranging terms:

$$
\begin{aligned}
&\frac{(bH_{t-1}^{1-\beta} e_t^{\beta} w_t - (1+r_t)e_t)^{1-\theta}}{(b-a)(1-\theta)} \left[\frac{\beta b}{e_t} - \frac{\beta(bH_{t-1}^{1-\beta} e_t^{\beta} w_t - (1+r_t)e_t)}{(2-\theta)H_{t-1}^{1-\beta} e_t^{\beta+1} w_t} - \frac{(1+r_t)}{H_{t-1}^{1-\beta} e_t^{\beta} w_t} \right] \\
- &\frac{(aH_{t-1}^{1-\beta} e_t^{\beta} w_t - (1+r_t)e_t)^{1-\theta}}{(b-a)(1-\theta)} \left[\frac{\beta a}{e_t} - \frac{\beta(aH_{t-1}^{1-\beta} e_t^{\beta} w_t - (1+r_t)e_t)}{(2-\theta)H_{t-1}^{1-\beta} e_t^{\beta+1} w_t} - \frac{(1+r_t)}{H_{t-1}^{1-\beta} e_t^{\beta} w_t} \right].
\end{aligned}
$$

We further have that

$$aH_{t-1}^{1-\beta}\bar{e_t}^{\beta}w_t - (1+r_t)\bar{e_t} = 0, \tag{21}$$

and hence that

$$a\beta H_{t-1}^{1-\beta}\bar{e_t}^{\beta-1}w_t - (1+r_t) < 0. \tag{22}$$

It is hence straightforward to see that $J'(\bar{e_t}) = -\infty$ for all $\theta > 1$, and that $J'(\bar{e_t}) < 0$ for all $\theta \in (0,1)$.

- $\theta = 1$

  Integrating (20) for $\theta = 1$ we get

$$\begin{aligned}
&\left[b\log(bH_{t-1}^{1-\beta}e_t^{\beta}w_t - (1+r_t)e_t) - a\log(aH_{t-1}^{1-\beta}e_t^{\beta}w_t - (1+r_t)e_t)\right]\frac{\beta}{(b-a)e_t}\\
- &\left[(bH_{t-1}^{1-\beta}e_t^{\beta}w_t - (1+r_t)e_t) - (aH_{t-1}^{1-\beta}e_t^{\beta}w_t - (1+r_t)e_t)\right]\frac{\beta}{(b-a)H_{t-1}^{1-\beta}e_t^{\beta-1}w_t}\\
- &\left[\log(bH_{t-1}^{1-\beta}e_t^{\beta}w_t - (1+r_t)e_t) - \log(aH_{t-1}^{1-\beta}e_t^{\beta}w_t - (1+r_t)e_t)\right]\frac{(1+r_t)}{(b-a)H_{t-1}^{1-\beta}e_t^{\beta}w_t}.
\end{aligned}$$

  Given (21) and (22) it is again straightforward to see that $J'(\bar{e_t}) = -\infty$.

  Hence we can conclude that the optimal choice of effort is interior.

B. Optimal individual effort

For $\theta = 3$, we have the following first order condition for e_t

$$\begin{aligned}
&\frac{\beta\hat{e}_t^{\beta-1}w_t}{1+r_t}\int_a^b z_t^i(z_t^i\hat{e}_t^{\beta}w_t - (1+r_t)\hat{e}_t)^{-3}f(z_t^i)\,dz_t^i\\
&\qquad = \int_a^b (z_t^i\hat{e}_t^{\beta}w_t - (1+r_t)\hat{e}_t)^{-3}f(z_t^i)\,dz_t^i
\end{aligned}$$

where $\hat{e}_t = \frac{e_t}{H_{t-1}}$, and since we consider a uniform distribution we have that the probability density function of the distribution $f(z) = \frac{1}{b-a}$, and thus independent of z; We can hence divide both sides by $f(z)$.

After integrating we get

$$\Big[b(b\hat{e}_t^{\beta}w_t - (1+r_t)\hat{e}_t)^{-2} - a(a\hat{e}_t^{\beta}w_t - (1+r_t)\hat{e}_t)^{-2}$$
$$+\frac{(b\hat{e}_t^{\beta}w_t - (1+r_t)\hat{e}_t)^{-1} - (a\hat{e}_t^{\beta}w_t - (1+r_t)\hat{e}_t)^{-1}}{\hat{e}_t^{\beta}w_t}\Big]\frac{\beta\hat{e}_t^{\beta-1}w_t}{(1+r_t)}$$
$$= (b\hat{e}_t^{\beta}w_t - (1+r_t)\hat{e}_t)^{-2} - (a\hat{e}_t^{\beta}w_t - (1+r_t)\hat{e}_t)^{-2}.$$

We divide both sides by $(1+r_t)^{-2}\hat{e}_t^{-2}$ and get

$$\left[b\left(b\frac{\hat{e}_t^{\beta-1}w_t}{(1+r_t)} - 1\right)^{-2} - a\left(a\frac{\hat{e}_t^{\beta-1}w_t}{(1+r_t)} - 1\right)^{-2}\right.$$
$$\left.+\frac{\left(b\frac{\hat{e}_t^{\beta-1}w_t}{(1+r_t)} - 1\right)^{-1} - \left(a\frac{\hat{e}_t^{\beta-1}w_t}{(1+r_t)} - 1\right)^{-1}}{\frac{\hat{e}_t^{\beta-1}w_t}{(1+r_t)}}\right]\frac{\beta\hat{e}_t^{\beta-1}w_t}{(1+r_t)}$$
$$= \left(b\frac{\hat{e}_t^{\beta-1}w_t}{(1+r_t)} - 1\right)^{-2} - \left(a\frac{\hat{e}_t^{\beta-1}w_t}{(1+r_t)} - 1\right)^{-2}.$$

We define $X \equiv \frac{\hat{e}_t^{\beta-1}w_t}{(1+r_t)}$, and replace in the above expression in order to get

$$\beta X[b(bX-1)^{-2} - a(aX-1)^{-2} + \frac{(bX-1)^{-1} - (aX-1)^{-1}}{X}]$$
$$= (bX-1)^{-2} - (aX-1)^{-2}.$$

After developing and rearranging terms we get the following quadratic equation in X:

$$2\beta ab(a-b)X^2 - (1+\beta)(a^2-b^2)X + 2(a-b) = 0.$$

We replace a and b by respectively $\mu - \sqrt{3}\sigma$ and $\mu + \sqrt{3}\sigma$. We solve for X and obtain two solutions:

$$X_{1,2} = \frac{(1+\beta)\mu \pm \sqrt{\mu^2(1-\beta)^2 + 12\beta\sigma^2}}{2\beta(\mu^2 - 3\sigma^2)}.$$

or

$$\left[\frac{\hat{e}_t^{\beta-1}w_t}{(1+r_t)}\right]_{1,2} = \frac{(1+\beta)\mu \pm \sqrt{\mu^2(1-\beta)^2+12\beta\sigma^2}}{2\beta(\mu^2-3\sigma^2)}.$$

The above solutions are real since the discriminant, $\triangle = \mu^2(1-\beta)^2+12\beta\sigma^2$ is positive. Indeed, both terms are positive, so $\triangle > 0$.

To be valid the solutions for $\frac{\hat{e}_t^{\beta-1}w_t}{(1+r_t)}$, should be positive. The denominator is always positive, since $\beta > 0$, and $\mu - \sqrt{3}\sigma > 0$ (because $a > 0$). As far as the nominator is concerned, we have two cases:

- $(1+\beta)\mu+\sqrt{\mu^2(1-\beta)^2+12\beta\sigma^2}$; we have the sum of two positive terms, which is hence positive
- $(1+\beta)\mu-\sqrt{\mu^2(1-\beta)^2+12\beta\sigma^2}$; using the fact that $\mu > \sqrt{3}\sigma$, and $\beta > 0$, it is easy to verify the positive sign of the above expression

Hence both solution are a priori valid.

We finally get

$$\hat{e}_{t(1,2)} = \left[\frac{2w_t\beta(\mu^2-3\sigma^2)}{(1+r_t)[(1+\beta)\mu \pm \sqrt{\mu^2(1-\beta)^2+12\beta\sigma^2}]}\right]^{\frac{1}{1-\beta}}.$$

C. Interiority of the optimal solution

In order to satisfy the borrowing constraint even in the worst case, -i.e. when $z_t = a$, we should have that

$$ah_{t-1}^{1-\beta}e_t^{\beta}w_t - (1+r_t)e_t > 0$$

or, after dividing by $e_t(1+r_t)$,

$$\frac{a\hat{e}_t^{\beta-1}w_t}{(1+r_t)} > 1.$$

From Appendix B we know that

$$\left[\frac{\hat{e}_t^{\beta-1} w_t}{(1+r_t)}\right]_{1,2} = \frac{(1+\beta)\mu \pm \sqrt{\mu^2(1-\beta)^2 + 12\beta\sigma^2}}{2\beta(\mu^2 - 3\sigma^2)}.$$

Replacing this in the above inequality, and substituting for $a = \mu - \sqrt{3}\sigma$ we obtain

$$\frac{(\mu - \sqrt{3}\sigma)[(1+\beta)\mu \pm \sqrt{\mu^2(1-\beta)^2 + 12\beta\sigma^2}]}{2\beta(\mu^2 - 3\sigma^2)} > 1$$

$$\iff (\mu - \sqrt{3}\sigma)[(1+\beta)\mu \pm \sqrt{\mu^2(1-\beta)^2 + 12\beta\sigma^2}]$$
$$> 2\beta(\mu + \sqrt{3}\sigma)(\mu - \sqrt{3}\sigma)$$

$$\iff \pm\sqrt{\mu^2(1-\beta)^2 + 12\beta\sigma^2} > 2\beta(\mu + \sqrt{3}\sigma) - (1+\beta)\mu.$$

Hence we have that

$$\sqrt{\mu^2(1-\beta)^2 + 12\beta\sigma^2} > 2\beta(m + \sqrt{3}\sigma) - (1+\beta)\mu \tag{23}$$

or

$$\sqrt{\mu^2(1-\beta)^2 + 12\beta\sigma^2} < (1+\beta)\mu - 2\beta(\mu + \sqrt{3}\sigma). \tag{24}$$

We verify (23). Raising both sides of (23) to a square gives us

$$\mu^2(1-\beta)^2 + 12\beta\sigma^2 > 4\beta^2(\mu + \sqrt{3})^2 + (1+\beta)^2\mu^2 - 4\beta(\mu + \sqrt{3})(1+\beta)\mu.$$

After developing and recollecting terms we find that

$$\beta > \beta^2, \tag{25}$$

which is always satisfied, since $0 < \beta < 1$. So we verified that (23) is always satisfied and that $2\beta(\mu + \sqrt{3}\sigma) - (1+\beta)\mu$ is smaller in absolute value than

$\sqrt{\mu^2(1-\beta)^2+12\beta\sigma^2}$; which proves that (24) is never true irrespective the sign of $2\beta(\mu+\sqrt{3}\sigma)-(1+\beta)\mu$. We can thus reject (16) since it does not satisfy the borrowing constraint, and we retain (15) as the only feasible solution for e_t.

D. Explicit expression of the dynamic equation

$$\xi(\hat{e}_t,\hat{e}_{t+1}) = X\hat{e}_{t+1}(1+Z\hat{e}_{t+1}^{\frac{2}{3}(1-\beta)(\alpha-1)}) - Y\hat{e}_t^{\alpha(1-\beta)} = 0, \tag{26}$$

with

$$\begin{cases} X &= (1+n)\left(\frac{\mu\alpha/\beta}{(1-\alpha)}C^{-1}+1\right) > 0 \\ Y &= (1-\alpha)C^{-\alpha}\left(\frac{\alpha/\beta}{(1-\alpha)}\right)^{\alpha}\left(1-\frac{C\beta}{\mu}\right) > 0 \\ Z &= (1+\rho)^{1/3}\left(\alpha\left(\frac{\alpha/\beta}{(1-\alpha)}\right)^{\alpha-1}\right)^{2/3} C^{(1-\alpha)2/3} > 0 \\ C &= \frac{2(\mu^2-3\sigma^2)}{(1+\beta)\mu+\sqrt{\mu^2(1-\beta)^2+12\beta\sigma^2}} > 0. \end{cases}$$

E. Proof of lemma 1

Due to its non-linear nature (26) is difficult to solve analytically for $\hat{e}_{t+1}$. It is however very easy to express $\hat{e}_t$ as a function of $\hat{e}_{t+1}$, which is the inverse function of (18). In order to proof Proposition 1, we use the inverse function of the dynamic equation (18), and show that this inverse function is monotonously increasing in $\hat{e}_{t+1}$, that it is convex and that it satisfies:

$$\lim_{\hat{e}_{t+1}\to 0}\frac{d\hat{e}_t}{d\hat{e}_{t+1}} = 0 \text{ and } \lim_{\hat{e}_{t+1}\to\infty}\frac{d\hat{e}_t}{d\hat{e}_{t+1}} = \infty.$$

Using the properties of the inverse function we know then that (18) is increasing in $\hat{e}_t$, concave and that it satisfies

$$\lim_{\hat{e}_t \to 0} \frac{d\hat{e}_{t+1}}{d\hat{e}_t} = \infty \text{ and } \lim_{\hat{e}_t \to \infty} \frac{d\hat{e}_{t+1}}{d\hat{e}_t} = 0.$$

The inverse function is

$$\hat{e}_t = \left[\frac{X}{Y} \hat{e}_{t+1} + \frac{ZX}{Y} \hat{e}_{t+1}^{\frac{2}{3}(1-\beta)(\alpha-1)+1} \right]^{\frac{1}{\alpha(1-\beta)}},$$

with

$$\begin{cases} X &= (1+n) \left(\frac{\mu\alpha/\beta}{(1-\alpha)} C^{-1} + 1 \right) > 0 \\ Y &= (1-\alpha) C^{-\alpha} \left(\frac{\alpha/\beta}{(1-\alpha)} \right)^{\alpha} \left(1 - \frac{C\beta}{\mu} \right) > 0 \\ Z &= (1+\rho)^{1/3} \left(\alpha \left(\frac{\alpha/\beta}{(1-\alpha)} \right)^{\alpha-1} \right)^{2/3} C^{(1-\alpha)2/3} > 0 \\ C &= \frac{2(\mu^2 - 3\sigma^2)}{(1+\beta)\mu + \sqrt{\mu^2(1-\beta)^2 + 12\beta\sigma^2}} > 0. \end{cases}$$

The first derivative is positive:

$$\begin{aligned} \frac{d\hat{e}_t}{d\hat{e}_{t+1}} &= \frac{1}{\alpha(1-\beta)} \left[\frac{X}{Y} \hat{e}_{t+1} + \frac{ZX}{Y} \hat{e}_{t+1}^{\frac{2}{3}(1-\beta)(\alpha-1)+1} \right]^{\left(\frac{1}{\alpha(1-\beta)} - 1\right)} \\ &\times \left[\frac{X}{Y} + \frac{ZX}{Y} [(1-\beta)(\alpha-1)\frac{2}{3} + 1] \hat{e}_{t+1}^{\frac{2}{3}(1-\beta)(\alpha-1)} \right] > 0. \end{aligned}$$

So $\hat{e}_t$ is monotonously increasing in $\hat{e}_{t+1}$.

The second derivative is

$$\begin{aligned} \frac{d^2\hat{e}_t}{d\hat{e}_{t+1}^2} &= \frac{1}{\alpha(1-\beta)} \left[\left(\frac{1}{\alpha(1-\beta)} - 1 \right) \left[\frac{X}{Y} \hat{e}_{t+1} + \frac{ZX}{Y} \hat{e}_{t+1}^{\frac{2}{3}(1-\beta)(\alpha-1)+1} \right]^{\left(\frac{1}{\alpha(1-\beta)} - 2\right)} \right. \\ &\times \left[\frac{X}{Y} + \frac{ZX}{Y} \left[(1-\beta)(\alpha-1)\frac{2}{3} + 1 \right] \hat{e}_{t+1}^{\frac{2}{3}(1-\beta)(\alpha-1)} \right]^2 \\ &+ \left[\frac{X}{Y} \hat{e}_{t+1} + \frac{ZX}{Y} \hat{e}_{t+1}^{\frac{2}{3}(1-\beta)(\alpha-1)+1} \right]^{\left(\frac{1}{\alpha(1-\beta)} - 1\right)} \\ &\times \frac{ZX}{Y} \left[(1-\beta)(\alpha-1)\frac{2}{3} + 1 \right] (1-\beta)(\alpha-1)\frac{2}{3} \hat{e}_{t+1}^{\frac{2}{3}(1-\beta)(\alpha-1)-1}. \end{aligned}$$

After developing and rearranging terms we get

$$
\begin{aligned}
\frac{d^2\hat{e}_t}{d\hat{e}_{t+1}^2} &= \frac{1}{\alpha(1-\beta)}\left[\frac{X}{Y}\hat{e}_{t+1} + \frac{XZ}{Y}\hat{e}_{t+1}^{\frac{2}{3}(1-\beta)(\alpha-1)+1}\right]^{\left(\frac{1}{\alpha(1-\beta)}-2\right)}\frac{X^2}{Y^2} \\
&\times \left[\left(\frac{1}{\alpha(1-\beta)}-1\right) + 2Z\left[(1-\beta)(\alpha-1)\frac{2}{3}+1\right]\hat{e}_{t+1}^{\frac{2}{3}(1-\beta)(\alpha-1)}\right. \\
&\times [\frac{1}{\alpha(1-\beta)} - 1 + \frac{(1-\beta)(\alpha-1)}{3}] \\
&+ Z^2\left[(1-\beta)(\alpha-1)\frac{2}{3}+1\right]\hat{e}_{t+1}^{\frac{4}{3}(1-\beta)(\alpha-1)} \\
&\times \left.\left[\left(\frac{1}{\alpha(1-\beta)}-1\right)\left[(1-\beta)(\alpha-1)\frac{2}{3}+1\right] + \frac{2(1-\beta)(\alpha-1)}{3}\right]\right].
\end{aligned}
$$

All terms on the first, second and fourth line of the above expression are clearly positive. At those on the third and the fifth line we will have to take a closer look. Let start with

$$\left[\frac{1}{\alpha(1-\beta)} - 1 + \frac{(1-\beta)(\alpha-1)}{3}\right];$$

when we set all terms on the smallest common denominator $3\alpha(1-\beta)$, which is positive, the nominator becomes

$$(1-\alpha)[3-\alpha(1-\beta)^2] + 3\alpha\beta$$

which is clearly positive, as α and $\beta < 1$. We do the same with the other term;

$$\left[\left(\frac{1}{\alpha(1-\beta)}-1\right)\left[(1-\beta)(\alpha-1)\frac{2}{3}+1\right] + \frac{2(1-\beta)(\alpha-1)}{3}\right]$$

and become as nominator

$$3-(1-\beta)(\alpha+2)$$

which is also positive. Hence we can conclude that the second derivative is positive too. The inverse function is thus increasing and convex, which, using the properties of inverse functions allow us to conclude that the original dynamic equation (18) is increasing and concave.

Finally we verify that

$$\lim_{\hat{e}_{t+1}\to 0} \frac{d\hat{e}_t}{d\hat{e}_{t+1}} = 0 \text{ and } \lim_{\hat{e}_{t+1}\to \infty} \frac{d\hat{e}_t}{d\hat{e}_{t+1}} = \infty.$$

We multiply the second and divide the third factor of $\frac{d\hat{e}_t}{d\hat{e}_{t+1}}$ by $\hat{e}_{t+1}^{\frac{2}{3}(1-\beta)(\alpha-1)}$, and we get

$$\begin{aligned}
\frac{d\hat{e}_t}{d\hat{e}_{t+1}} &= \frac{1}{\alpha(1-\beta)}\left[\frac{X}{Y}\hat{e}_{t+1}^{\,1-\frac{\frac{2}{3}(1-\beta)(1-\alpha)}{\frac{1}{\alpha(1-\beta)}-1}} + \frac{ZX}{Y}\hat{e}_{t+1}^{\,\frac{2}{3}(1-\beta)(\alpha-1)+1-\frac{\frac{2}{3}(1-\beta)(1-\alpha)}{\frac{1}{\alpha(1-\beta)}-1}}\right]^{\left(\frac{1}{\alpha(1-\beta)}-1\right)} \\
&\times \left[\frac{X}{Y}\hat{e}_{t+1}^{\frac{2}{3}(1-\beta)(1-\alpha)} + \frac{ZX}{Y}\left[(1-\beta)(\alpha-1)\frac{2}{3}+1\right]\right].
\end{aligned}$$

If we prove that all exponents of $\hat{e}_{t+1}$ in the above expression are positive, we have that

$$\lim_{\hat{e}_{t+1}\to 0} \frac{d\hat{e}_t}{d\hat{e}_{t+1}} = 0 \text{ and } \lim_{\hat{e}_{t+1}\to \infty} \frac{d\hat{e}_t}{d\hat{e}_{t+1}} = \infty.$$

It is trivial to see that $\frac{2}{3}(1-\beta)(1-\alpha)$ is positive. We next examine $\frac{2}{3}(1-\beta)(\alpha-1)+1-\frac{\frac{2}{3}(1-\beta)(1-\alpha)}{\frac{1}{\alpha(1-\beta)}-1}$; after multyplying by $(\frac{1}{\alpha(1-\beta)}-1)\alpha(1-\beta)$ and rearranging terms we get $\frac{1}{3}((1-\alpha)+\alpha\beta+2\beta)$, which is clearly positive. As far as the third exponent is concerned, since $1 > 1-\frac{2}{3}(1-\beta)(1-\alpha)$, it is straightforward to prove that this exponent is also positive.

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