MULTIREGIME
TERM STRUCTURE MODELS

C. GOURIEROUX[1] and O. SCAILLET[2]

March 1997

This version : December 1997

[1]CREST and CEPREMAP

[2]Université catholique de Louvain.

The authors have benefited of helpful comments from R. Anderson, E. Burgayran, J. Jasiak, J.P. Laurent and participants of CREST seminar, CORE workshop and AFFI meeting. Of course, the usual disclaimers apply.

The second author gratefully acknowledges the support of the grant "Actions de recherche concertées" 93/98-162 of the Ministry of Scientific Research (Belgian French Speaking Community).

MULTIREGIME TERM STRUCTURE MODELS

C. GOURIEROUX and O. SCAILLET

Résumé

Le modèle de Ho et Lee est pour l'étude de la structure par terme des taux d'intérêt l'analogue de l'arbre binomial introduit par Cox, Ross et Rubinstein pour le cas d'un seul actif risqué. Ce modèle ne permet cependant qu'un petit nombre de déformations de la structure par terme entre deux dates, ce qui le rend incompatible avec les données disponibles. Nous nous proposons de réconcilier les approches par arbre et l'inférence statistique. Pour cela nous considérons des modèles à régimes, où la déformation de la structure par terme peut dans chaque régime avoir une certaine variabilité. Les questions de contraintes provenant de l'absence d'opportunité d'arbitrage sont analysées dans un contexte d'information asymétrique entre les intervenants de marché et l'économètre chargé de l'étude.

Abstract

The Ho and Lee model is the analogue for the study of the term structure of interest rates of the binomial tree introduced by Cox, Ross and Rubinstein in the one risky asset case. This model allows only for a small number of deformations of the term structure between two successive dates, and is therefore incompatible with available data. We propose here to reconcile tree approaches and statistical inference. We consider regime models for which the deformation of the term structure may behave randomly in each regime. Questions about constraints induced by no arbitrage are also addressed in a context of asymmetric information between traders and the econometrician in charge with the estimation.

mots clés : modèle binomial, structure par terme, taux d'intérêt, information asymétrique.

key words : binomial model, term structure, interest rate, asymmetric information.

JEL : G13, C4.

MULTIREGIME TERM STRUCTURE MODELS

Abstract

The HO and LEE model is the analogue for the study of the term structure of interest rates of the binomial tree introduced by COX, ROSS and RUBINSTEIN in the one risky asset case. This model allows only for a small number of deformations of the term structure between two successive dates, and is therefore incompatible with available data. We propose here to reconcile tree approaches and statistical inference. We consider regime models for which the deformation of the term structure may behave randomly in each regime. Questions about constraints induced by no arbitrage are also addressed in a context of asymetric information between traders and the econometrician in charge with the estimation.

Introduction

In an influential paper, Ho and Lee (1986) introduced their binomial model for the dynamics of the term structure of interest rates. Such a binomial model is particularly useful for option pricing, especially for options with path dependent payoffs (see Turnbull and Milne (1991) for an extensive treatment of option pricing in the discrete time setting). The model is easy to understand and implement as it requires only basic computer programming skills. It is the counterpart, in interest rate theory, of the well-known binomial tree of Cox, Ross and Rubinstein (1979) developped for the pricing of stock options and serves as an introductory example in most finance text-books.

The model is based on the equality between the price of a discount bond one period ahead and its one period forward price, which holds in a deterministic environment. There are two states of the world that are envisaged and a disturbance function is introduced to accommodate the randomness. By standard no arbitrage arguments it is possible to exhibit restrictions on the values taken by the disturbance function.

However the Ho and Lee model is systematically incompatible with available data. It is due to the too small number of deformations of the term structure allowed between two successive dates. In this paper we propose to reconcile Ho and Lee type models and stastistical inference. Indeed as advocated by Bliss and Ronn (1989), the data should play a crucial role in the construction of a model in order to provide as accurate a representation of reality as possible. Bliss and Ronn (1989) adopt a method of classification of the states of their trinomial model based on a single maturity, and their empirical results are thus dependent on the chosen maturity. The methodology presented here avoids this feature, and allows for a classification of the type and number of regimes without any a priori by extracting the information provided by the data. Besides it leads to testing procedures of the restrictions induced by no arbitrage conditions on the selected model.

The paper is organized as follows. Section I briefly introduces the binomial model, discusses its statistical incompatibility with available data and the need for some extensions. A first extension is considered in Section II. We maintain the assumption of two regimes, but allow the term structure deformation to vary in each regime. We examine the constraints induced by the no arbitrage conditions and develop the corresponding inference procedures. In order to do so, we address the role of information available to market participants when they draw pricing implications from no arbitrage and assume that the econometrician is less informed. In Section III, we present an additional extension with more than two regimes. Section IV is devoted to an application to US term structure data where we reveal the number and features of existing regimes. The derivations of the no arbitrage conditions are gathered in the Appendix.

I Binomial models

The price at date t of the discount bond with time-to-maturity u is denoted by $B(t, t+u)$. This asset delivers one money unit at time $t+u$, $u = 1, \ldots, N$, $t = 1, \ldots, T$. The binomial term structure models are derived by multiplying the no arbitrage condition in a deterministic environment :

$$B(t, t+u) = \frac{B(t-1, t+u)}{B(t-1, t)}, \qquad \forall t, u,$$

by a random function $H_t(u)$ called the disturbance function. In the Ho and Lee model the disturbance function is :

$$H_t(u) = h(u)\epsilon_t + h^*(u)(1 - \epsilon_t),$$

where (ϵ_t) is a sequence of independent Bernoulli variables with $P(\epsilon_t = 1) = p$, $P(\epsilon_t = 0) = 1 - p$, and $h(u)$, $h^*(u)$, two time independent functions. Therefore the functional process $(H_t(u))$ is assumed stationary, time independent and admits only two values, functions of u : $h(u)$, $h^*(u)$:

$$H_t(u) = \begin{cases} h(u), & \text{with probability } p, \\ h^*(u), & \text{with probability } 1 - p. \end{cases}$$

A plot of the functions :

$$H_t(u) = \frac{B(t, t+u)B(t-1, t)}{B(t-1, t+u)}, \qquad u \text{ varying}, \tag{1}$$

for different dates, immediately reveals that such a strict time independent binomial assumption is not compatible with available data (see Figure 1 for US term structure data).

In fact, this basic model does not allow for enough randomness to accommodate the data behavior, and is misspecified.

It is thus essential to include some extra sources of randomness to improve the fit, and also to adapt the specification so that statistical inference becomes feasible. In this paper we modify the HO and LEE model first by allowing the $h(u)$ and $h^*(u)$ functions to take random values in the two regime framework, second by increasing the number of regimes.

II Inference for two regime models

A The two regime model

In this section we develop a method of inference for a data set such that, at each given date t, all values of the observed function $H_t(u)$, u varying, are either all bigger or all smaller than one. We may define without ambiguity the high and low regimes prevailing at different dates :

$$\epsilon_t = 1 \text{ (high regime)} \iff H_t(u) > 1, \qquad \forall u,$$
$$\epsilon_t = 0 \text{ (low regime)} \iff H_t(u) < 1, \qquad \forall u,$$

where ϵ_t is the regime indicator. In such a data set the regimes are easily observed ex post only by considering whether the observed disturbance function is smaller or larger than one. The disturbance function is now assumed to vary in each regime and obey a stochastic model for the function H_t. This function is specified in two steps. First we set the evolution of the latent processes : $(\epsilon_t,\, h_t,\, h_t^*)$ (or of a one-to-one function of these processes). Then we use the deterministic relation : $H_t(u) = \epsilon_t h_t(u) + (1-\epsilon_t)h_t^*(u)$, to represent the dynamics of the disturbance function.

In fact the latent functions h_t, h_t^* are not very convenient for modelling purposes, and we introduce the new parametrisation :

$$h_t(u) = 1 + \alpha_t(u), \qquad h_t^*(u) = 1 - \alpha_t^*(u), \tag{2}$$

which explicits the difference between the disturbance function and one.

The $N \times 1$ random vectors $\alpha_t = [\alpha_t(1), \ldots, \alpha_t(N)]', \alpha_t^* = [\alpha_t^*(1), \ldots, \alpha_t^*(N)]'$ take positive values and satisfy :

$$H_t - 1 = \epsilon_t \; \alpha_t - (1 - \epsilon_t) \; \alpha_t^*, \tag{3}$$

with $H_t = [H_t(1), \ldots, H_t(N)]'$.

It remains to specify the distributions of the latent processes : $(\epsilon_t, \alpha_t, \alpha_t^*)$. We assume :

A 1 *: $(\epsilon_t, \alpha_t, \alpha_t^*)$ are independent and identically distributed.*

A 2 *: ϵ_t and (α_t, α_t^*) are independent.*

A 3 *: (ϵ_t) is a sequence of independent Bernoulli variables with the same parameter p.*

A 4 *: $\log \alpha_t = [\log \alpha_t(1), \ldots, \log \alpha_t(N)]'$ and $\log \alpha_t^* = [\log \alpha_t^*(1), \ldots, \log \alpha_t^*(N)]'$, have continuous distributions on the set of admissible values with mean $\begin{pmatrix} m \\ m^* \end{pmatrix}$, and covariance matrix $\begin{pmatrix} \Sigma & C \\ C' & \Sigma^* \end{pmatrix}$.*

Let us remark that, for a given date t, only one regime prevails and a single type of disturbance function α or α^* is observed. Therefore we never have simultaneous observations of α_t and α_t^*, and the covariance matrix C is not identifiable.

B Arbitrage restrictions and the role of information

The previous modelling does not take into account the constraints implied by no arbitrage. In this section, we examine this issue when the traders (or the most informed market participants) and the econometrician may have different pieces of information (see Clément, Gouriéroux and Monfort (1997) for a similar approach). Indeed the no arbitrage implications are related to the traders' behavior and their information, whereas statistical inference depends on the econometrician's information.

Let us denote by ϵ_t the information on the regime, and by $a_t = (\alpha_t, \alpha_t^*)$ the information on the levels. Three cases may be distinguished depending on the degree of available information :

i) The market participants and the econometrician are informed on the future levels

In this framework the market participants and the econometrician know the future levels a_t, but do not know the future regime. This setup is equivalent to the HO and LEE model with deterministic time varying functions h_t and h_t^*. The no arbitrage constraints are :

$$\exists \pi_t \in [0,1] : \pi_t h_t(u) \ + \ (1-\pi_t)h_t^*(u) = 1, \qquad \forall u, \tag{4}$$

or equivalently :

$$\exists \pi_t \in [0,1] : \pi_t \alpha_t(u) \ - \ (1-\pi_t)\alpha_t^*(u) = 0, \qquad \forall u, \tag{5}$$

where π_t is a function of the information available at $t+1$ including a_t. Due to equation (5), π_t only depends on the admissible values of the disturbance function : $\pi_t = \pi(a_t)$. At each date the probability π_t must be the same for all u and is known with certainty.
The limiting case : $\Sigma = \Sigma^* = C = 0$ corresponds to the standard HO and LEE model, where the functions h and h^* are time invariant and constrained by :

$$\exists \pi \in [0,1] : \pi h(u) \ + \ (1-\pi)h^*(u) = 1, \qquad \forall u.$$

As already mentioned this model is statistically incompatible with the data and is rejected with probability one by the econometrician.

ii) The market participants and the econometrician are informed on neither the future regime, nor the future levels

It is proved in the Appendix that the no arbitrage assumption implies no additional restrictions on the model. This is due to the large number of additional random shocks introduced in the model at each date, i.e. one by maturity.

iii) The market participants are informed on the future levels and the econometrician is not

This is a situation of asymmetric information. Since the no arbitrage implications are relative to the traders and their lack of knowledge concerns only the regime, the constraints are :

$$\exists \pi(a_t) \in [0,1] : \pi(a_t)h(u;a_t) \; + \; (1-\pi(a_t))h^*(u;a_t) = 1, \qquad \forall u, \tag{6}$$

where $h(u;a_t)$ denotes component u of h_t which also depends on a_t.
Hence the no arbitrage constraints are similar to (4), and depend on the privileged information a_t.
Any statistical inference is now performed on the basis of the econometrician's knowledge only. If he does not know the future levels, the no arbitrage implications are : for any date t, there exists a *stochastic* risk neutral probability π_t such that : $\pi_t h_t(u) \; + \; (1-\pi_t)h_t^*(u) = 1, \qquad \forall u.$
Therefore he faces randomized constraints. Since $\pi_t = \pi(a_t)$ is a fixed function of the i.i.d. levels the stochastic risk neutral probabilities are also i.i.d., and the relation can be rewritten in terms of positive random variables λ_t such that :

$$\alpha_t^* = \lambda_t \alpha_t, \qquad \text{with : } \lambda_t = \frac{\pi_t}{1-\pi_t}, \tag{7}$$

or :

$$\forall u = 1, \ldots, N, \qquad \log \alpha_t^*(u) - \log \alpha_t(u) = \log \lambda_t. \tag{8}$$

This condition entails constraints on the joint distribution of $(\log \alpha_t, \log \alpha_t^*)$. First a condition on the first order moments is obtained by taking expectations of both sides of equation (8) :

$$\forall u = 1, \ldots, N, \qquad m^*(u) - m(u) = \mu, \;\; \text{independent of } u. \tag{9}$$

A constraint on the second order moment : $V[\log \alpha_t^*(u) - \log \alpha_t(u)] = V \log \lambda_t$, independent of u, can also be obtained. However it is not testable because of the nonidentifiability of C. Indeed, as we do not observe at the same date $\log \alpha_t$ and $\log \alpha_t^*$ simultaneously, it is not possible to identify the cross term C of the covariance matrix from the data.
To take into account second order features, we need additional identifiability conditions. They concern the properties of the stochastic risk neutral probability under the assumptions of no arbitrage and asymmetric information :

A 5 : *In the constrained model,* $\log \alpha_t$ *and* $\log \lambda_t$ *are independent.*

Under the constrained model, we have :

$$\exists \mu, \sigma^2 : m^* = m + \mu e, \quad \Sigma^* = \Sigma + \sigma^2 ee',$$

with e the $N \times 1$ vector with all components equal to one, μ and σ^2 the mean and variance of $\log \lambda_t$.

A symmetric assumption might also be introduced :

A 6 : *In the constrained model,* $\log \alpha_t^*$ *and* $\log \lambda_t$ *are independent.*

Such assumption leads to a change of sign in the relation between Σ^* and Σ :

$$\exists \sigma^2 : \Sigma^* = \Sigma - \sigma^2 ee'.$$

C Inference procedures

In this section, we present inference methods for the two regime model, which exploit the aforementioned moment conditions. We first cover the unconstrained estimation and next turn our attention to the constrained model.

i) Unconstrained estimation

The unconstrained moment estimators of the identifiable first and second order parameters are the sample moments :

$$\begin{aligned}
\widehat{p}_T &= \frac{1}{T}\sum_{t=1}^{T}\epsilon_t = \frac{1}{T}\sum_{t=1}^{T}\mathbf{1}_{(H_t>1)}, \\
\widehat{m}_T &= \frac{1}{T_1}\sum_{t:\epsilon_t=1}\log(H_t-1), \\
\widehat{\Sigma}_T &= \frac{1}{T_1}\sum_{t:\epsilon_t=1}\log(H_t-1)\log(H_t-1)' - \widehat{m}_T\,\widehat{m}_T', \\
\widehat{m}^*_T &= \frac{1}{T_0}\sum_{t:\epsilon_t=0}\log(1-H_t), \\
\widehat{\Sigma}^*_T &= \frac{1}{T_0}\sum_{t:\epsilon_t=0}\log(1-H_t)\log(1-H_t)' - \widehat{m}^*_T\,\widehat{m}^{*\prime}_T
\end{aligned}$$

with : $T_1 = \sum_{t=1}^{T} \epsilon_t$, $T_0 = T - T_1$. Note that $H_t > 1$ means $H_t(u) > 1$, $\forall u$, $\log(H_t - 1)$ is the vector with components $\log(H_t(u) - 1)$, and the regime indicators ϵ_t are observable.

In this setup the number of high regimes is equal to T_1, while they are observed with frequency $\widehat{p}_T$. The estimators $\widehat{m}_T, \widehat{\Sigma}_T$ are the sample mean and variance corresponding to this regime. Such estimators are easy to compute, since we can use the SURE routine (Seemingly Unrelated Regressions) included in most econometric packages and regress the values of the variables $\log(H_t(u)-1)$ corresponding to the high regime on the constant. The estimated intercept yields the estimated mean, and the covariance matrix of the residuals, the estimated covariance matrix. The asymptotic precision of the estimators computed by such routines are obtained under the normality assumption on $\log \alpha_t$ and $\log \alpha_t^*$. Under this assumption the estimators $\widehat{p}_T, \widehat{m}_T, \widehat{m}_T^*, \widehat{\Sigma}_T, \widehat{\Sigma}_T^*$ are asymptotically independent with the following asymptotic covariance matrices :

$$
\begin{aligned}
V_{as}\left[\sqrt{T}(\widehat{p}_T - p)\right] &= p(1-p), \\
V_{as}\left[\sqrt{T}(\widehat{m}_T - m)\right] &= p\Sigma, \\
V_{as}\left[\sqrt{T}(\widehat{m}_T^* - m^*)\right] &= (1-p)\Sigma^*, \\
V_{as}\left[\sqrt{T}(\text{vech } \widehat{\Sigma}_T - \text{vech } \Sigma)\right] &= p\Omega, \\
V_{as}\left[\sqrt{T}(\text{vech } \widehat{\Sigma}_T^* - \text{vech } \Sigma)\right] &= (1-p)\Omega^*,
\end{aligned}
$$

where the elements of the matrix $p\Omega$ are equal to Cov $(\sqrt{T}\hat{\sigma}_{ij,T}, \sqrt{T}\hat{\sigma}_{kl,T}) = p(\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk})$, for $i, j, k, l = 1, \ldots, N$, with $\Sigma = (\sigma_{ij})$ (see e.g. Gouriéroux, Scaillet and Szafarz (1997) chapter II). Note that vech (.) is the vector-half operator, which stacks the lower triangular elements of an $N \times N$ matrix in a $N(N+1)/2 \times 1$ vector. Since $\widehat{\Sigma}_T$ and $\widehat{\Sigma}_T^*$ are symmetric, vech $\widehat{\Sigma}_T$ and vech $\widehat{\Sigma}_T^*$ contain all the unique elements of $\widehat{\Sigma}_T$ and $\widehat{\Sigma}_T^*$, respectively. An estimate of the asymptotic covariance matrix is obtained by replacing Σ, Σ^* and p with their sample-based counterparts.

ii) Constrained estimation

Let us now turn to the model constrained by :

$$
H_0 = \{\exists \mu, \eta : m^* = m + \mu e, \quad \Sigma^* = \Sigma + \eta e e'\}, \tag{10}
$$

without imposing the positivity or negativity of η in order to be compatible with either identifiability condition A 5 or A 6.

Under the null hypothesis H_0, efficient moments estimators of μ and η are easily derived using weighted regressions. An estimator $\hat{\mu}_T$ of μ can be computed by applying quasi-generalized least squares to the linear model :

$$\widehat{m}_T^* - \widehat{m}_T = \mu e + v,$$

with :

$$\begin{aligned} Ev &= 0, \\ Vv &= (1-\widehat{p}_T)\widehat{\Sigma}_T^* + \widehat{p}_T\widehat{\Sigma}_T. \end{aligned}$$

An estimator $\hat{\eta}_T$ of η is obtained by a quasi-generalized least squares procedure applied to :

$$\text{vech } \widehat{\Sigma}_T^* - \text{vech } \widehat{\Sigma}_T = \eta e + \nu,$$

with :

$$\begin{aligned} E\nu &= 0, \\ V\nu &= (1-\widehat{p}_T)\widehat{\Omega}_T^* + \widehat{p}_T\widehat{\Omega}_T. \end{aligned}$$

The goodness of fit of each regression can be measured thanks to the residual sum of squares. More precisely the two statistics :

$$\begin{aligned} \xi_{1T} &= T(\widehat{m}_T^* - \widehat{m}_T - \hat{\mu}_T e)' \left[(1-\hat{p}_T)\hat{\Sigma}_T^* + \hat{p}_T\hat{\Sigma}_T\right]^{-1} (\hat{m}_T^* - \hat{m}_T - \hat{\mu}_T e), \quad (11) \\ \xi_{2T} &= T(\text{vech } \hat{\Sigma}_T^* - \text{vech } \hat{\Sigma}_T - \hat{\eta}_T e)' \\ &\qquad \left[(1-\widehat{p}_T)\widehat{\Omega}_T^* + \widehat{p}_T\widehat{\Omega}_T\right]^{-1} \left(\text{vech } \hat{\Sigma}_T^* - \text{vech } \hat{\Sigma}_T - \hat{\eta}_T e\right), \quad (12) \end{aligned}$$

are equal to T times the residual sum of squares provided in a standard output of linear regression packages. The general properties of least squares estimators lead to the following result (Gouriéroux and Monfort (1995) p.150).

Property 1 : *Under the no arbitrage restriction H_0, the two statistics ξ_{1T} and ξ_{2T} are asymptotically independent, and distributed as :*

$$\begin{aligned}
\xi_{1T} &\xrightarrow{d} \chi^2(N-1), \\
\xi_{2T} &\xrightarrow{d} \chi^2\left(\frac{N(N+1)}{2}-1\right) = \chi^2\left(\frac{N(N-1)}{2}\right).
\end{aligned}$$

The test procedure of the no arbitrage restriction H_0 consists in accepting the null hypothesis if $\xi_T = \xi_{1T}+\xi_{2T} < \chi^2_{95\%}\left(\frac{(N-1)(N+2)}{2}\right)$, where $\chi^2_{95\%}\left(\frac{(N-1)(N+2)}{2}\right)$ denotes the 95% percentile of the chi square distribution with $\frac{(N-1)(N+2)}{2}$ degrees of freedom, and in rejecting it otherwise. When the null hypothesis H_0 is rejected, we can further examine if it is due to the mean or the variance by comparing the values of the statistics ξ_{1T} and ξ_{2T} normalized by their critical values $\chi^2_{95\%}(N-1)$ and $\chi^2_{95\%}\left(\frac{N(N-1)}{2}\right)$, respectively.

III Multiregime models

A The model

The specification presented hereabove is based on the assumption that the observed values of the disturbance function H_t are either larger or smaller than one for any time-to-maturity u. What has to be done if the observed disturbance functions come across the value one as observed on Figure 1 ?

As before, we may add some flexibility to the initial model to take into account this information provided by the data. In Section I, this was done by taking time varying values of the random function while maintaining the two regime setting. We now consider a further extension by introducing several regimes.

There are a priori 2^N observable regimes depending on "highs" and "lows" across the maturities. We denote by $\epsilon_t(u)$ the regime-by-maturity indicators :

$$\begin{cases} \epsilon_t(u) = 1, & \text{if } H_t(u) > 1, \\ \epsilon_t(u) = 0, & \text{if } H_t(u) < 1, \qquad u = 1, \ldots, N. \end{cases}$$

A global regime corresponds to a sequence of 0 and 1 :

$$A = \{i_1, \ldots, i_N\} \subset \{0,1\}^N,$$

and the indicator of such global regime is $\epsilon_t(A)$:

$$\epsilon_t(A) = 1 \iff \{\epsilon_t(1) = i_1, \ldots, \epsilon_t(N) = i_N\}.$$

The only effective regimes are the ones with strictly positive probabilities : $p(A) > 0$. We do not introduce any constraints on these probabilities except the usual condition : $\sum_A p(A) = 1$. In particular we neither assume the independence of the regime-by-maturity indicators, nor their equidistribution.

Finally we introduce the levels in each regime $h_t^A(u)$ such that :

$$H_t(u) = \sum_A \epsilon_t(A) h_t^A(u). \tag{13}$$

It remains to specify the distributions of the latent processes : $\left(\epsilon_t(A), h_t^A, A \,\text{varying}\right)$. We assume :

A' 1 $\left(\epsilon_t(A), h_t^A, A\right.$ *varying*$\left.\right)$ *are independent and identically distributed.*

A' 2 $(\epsilon_t(A), A$ *varying*$)$, $\left(h_t^A, A\right.$ *varying*$\left.\right)$ *are independent.*

A' 3 $(\epsilon_t(A), A$ *varying*$)$ *is a sequence of independent variables with multinomial distribution of parameter* $p(A), A$ *varying.*

B No arbitrage restrictions

In this multinomial framework, we can parallel the discussion on the role of information presented in the two regime model (Section III). As discussed in the Appendix, when information on the future regime and the future levels are not available to anybody, the no arbitrage assumption has no additional incidence. In the other two cases we can still derive implications from no arbitrage using standard arguments. Indeed, we know (see e.g. Ekeland (1979) p. 70, Duffie (1988) chapter II) that no arbitrage is equivalent to the existence of state prices q_{t+1}^A such that :

$$B(t, t+u) = \sum_A q_{t+1}^A B^A(t+1, t+u),$$

where $B^A(t+1, t+u)$ denotes the value at date $t+1$ in regime A of the discount bond which matures at date $t+u$. Since $B(t, t+1)$ is the price of the asset delivering one money unit at date $t+1$ (i.e. the price of the riskless asset), we deduce that there exist $\pi_{t+1}^A = q_{t+1}^A / B(t, t+1) \geq 0$ such that $\sum_A \pi_{t+1}^A = 1$, and :

$$B(t, t+u) = B(t, t+1) \sum_A \pi_{t+1}^A B^A(t+1, t+u), \ \forall u.$$

The probability π_{t+1}^A is the risk-neutral probability to be in regime A at date $t+1$ conditional to the information available at t. Taking into account the dynamics of the discount bonds : $B^A(t+1, t+u) = h_{t+1}^A(u-1) B(t, t+u)/B(t, t+1)$, we immediately get the no arbitrage constraints :

$$\exists \pi_t^A \in [0,1] : \sum_A \pi_t^A = 1, \ \sum_A \pi_t^A h_t^A(u) = 1, \ \forall u, \tag{14}$$

which extend the restrictions (4).

The coefficients π_t^A are not necessarily unique. They are uniquely determined as soon as the number of effective regimes is less or equal to the number N of maturities. Such condition corresponds to market completeness. In the case of completeness and asymmetric information, the risk neutral probability will be unique and will be perceived by the econometrician as a stochastic variable.

C Inference procedures

It is important in practice to determine the number and types of existing regimes. For instance, if we observe only two regimes with significant probability weights, we are in a binomial framework. Indeed if we have for example six maturities ($N = 6$) and the regimes are $\{1,1,1,0,1,0\}, \{0,0,0,1,0,1\}$ (exactly opposite because of no arbitrage), we have :

Regime 1 :

$$H(1) = h(1) > 1, \quad H(2) = h(2) > 1, \quad H(3) = h(3) > 1,$$
$$H(4) = h^*(4) < 1, \quad H(5) = h(5) > 1, \quad H(6) = h^*(6) < 1.$$

Regime 2 :

$$H(1) = h^*(1) < 1, \quad H(2) = h^*(2) < 1, \quad H(3) = h^*(3) < 1,$$
$$H(4) = h(4) > 1, \quad H(5) = h^*(5) < 1, \quad H(6) = h(6) > 1.$$

In Section II, we have assumed that $h_t(u) > h_t^*(u)$ holds for all u. These inequalities are not a consequence of no arbitrage and we can get $h_t(u) < h_t^*(u)$ for some u, as shown in the previous example. In the multiregime case, the only constraint on the location of the disturbance function with respect to the one implied by restrictions (14) is the following :

For *any* time-to-maturity u, there exists at least one regime with positive probability such that $h_t^A(u) > 1$, and one regime with positive probability such that $h_t^A(u) < 1$.

All regimes are observable by construction, and the probability of occurrence of regime $A = \{i_1, \ldots, i_N\}$ can be estimated by :

$$\hat{p}_T(A) = \frac{1}{T} \sum_{t=1}^{T} \mathbf{I}_{\epsilon_t(1)=i_1} \ldots \mathbf{I}_{\epsilon_t(N)=i_N}. \tag{15}$$

Similarly the first and second order moments of the levels of h_t^A can be estimated consistently from their empirical counterparts regime by regime as discussed in Section II.
This nonparametric approach allows to examine if the number of regimes is indeed equal to two, i.e. if the binomial model is adequate, and to test if the two values of the disturbance function satisfy the constraints induced by no arbitrage. When the binomial model is rejected this approach provides insights on the type of dynamics to model and the direction for further theoretical development of term structure models.

D Test of the no arbitrage restrictions

Is it possible to test for consequences (14) of no arbitrage :

$$\exists \pi_t^A \in [0,1] : \sum_A \pi_t^A = 1, \ \sum_A \pi_t^A h_t^A(u) = 1, \ \forall u,$$

in this general framework ? As in the two regime case, we have to remember that we only observe one regime at one date; for this reason, we can only identify the marginal distributions of the disturbance functions $(h_t^A(u), u = 1, \dots, N)$, A varying, and not their joint distribution.

Property 2 *: The no arbitrage restrictions (14) only imply constraints on the identifiable marginal distributions in the two regime case. In particular, they are not identifiable in more than two regimes.*

This result is easy to understand from the examples of two and three regimes.
In the two regime case, the no arbitrage restriction writes :

$$\pi_t^1 h_t^1(u) + \pi_t^2 h_t^2(u) = 1, \ \forall u,$$

or equivalently :

$$h_t^1(u) - 1 = \frac{\pi_t^2}{\pi_t^1}(1 - h_t^2(u)), \ \forall u.$$

Therefore, by taking logarithms of both sides, it is possible to derive some constraints between the marginal moments : $E \log |h_t^1(u) - 1|$, $E \log |h_t^2(u) - 1|$ (as already seen in Section II).
In the three regime case, we get similarly :

$$h_t^1(u) - 1 = \frac{\pi_t^2}{\pi_t^1}(1 - h_t^2(u)) + \frac{\pi_t^3}{\pi_t^1}(1 - h_t^3(u)), \ \forall u.$$

It is no more possible to find a nonlinear transformation of $h_t^1(u) - 1$, whose expectation admits a decomposition as a function of marginal moments of $1 - h_t^2(u)$, $1 - h_t^3(u)$.
We may only consider relation (14) in a model a priori submitted to identification constraints. For instance, if we assume that, under no arbitrage, the risk neutral probabilities and the disturbance functions are uncorrelated :

A' 4 *:* $Cov[\pi_t^A, h_t^A(u)] = 0, \forall A, u,$

then we deduce from (14) :

$$\sum_A E[\pi_t^A] E[h_t^A(u)] = 1, \ \forall u,$$

$$\iff \exists \bar{\pi}^A \in [0,1] : \sum_A \bar{\pi}^A = 1, \ \sum_A \bar{\pi}^A E[h_t^A(u)] = 1, \ \forall u.$$

Such a constraint under a mixed form may be tested by standard methods (Gouriéroux and Monfort (1995) chapter 18).

IV Empirical Results

In this section, we examine how multiregime models perform on US term structure data. The data are drawn from the file of McCulloch and Kwon (1993)[1] corresponding to 465 months from 12/1947 to 08/1985. We consider time-to-maturity u from one to 17 months.

A The observed regimes

Figure 1 represents the observed values of the disturbance functions $(H_t(u),\ u = 1, \ldots, N)$, $t = 1, \ldots, T$, computed from equation (1) ($T = 464$, $N = 17$).

Using the multiregime model described in Section III, we find 55 different regimes with at least one observation in each regime. We show their estimated probabilities in Figure 2. The two largest values are 44.61 % and 12.93 % (207 dates and 60 dates divided by 464 dates). All other values are below 4 %. In particular, the values ranked third, fourth and fifth are equal to 3.88 %, 3.88 % and 3.66 %, respectively. We give in Table I the structure of the observed regimes for a probability of occurrence larger than 1 %.

The two more frequent regimes correspond exactly to the binomial model of Section I with a disturbance function uniformly larger than 1 in the first regime, uniformly less than 1 in the second one. These regimes account for 57.54 % of the total probability mass. The observed disturbance functions of the next three regimes are greater than 1 for short maturities and less than 1 for longer maturities; they cross only once the value 1 after four, two, and six months, respectively.

To have further insight into the effects of the regimes on the discount bond prices, it is worth looking at the average behavior of the disturbance functions. Indeed, according to the model, if we start from a flat term structure of interest rates at date $t-1$, i.e. $\forall u,\ B(t-1, t+u) = \exp(-(u+1)c)$ for some constant c, the next period bond with the same maturity will take value : $B(t, t+u) = \exp(-uc)H_t(u)$, and the corresponding interest rate : $r(t, t+u) = c - \frac{1}{u}\log H_t(u)$. Hence by examining the means of the function $\beta_t(u) = -\frac{1}{u}\log H_t(u)$ regime by regime, we can deduce which effects the regimes have on average on the term structure. For example, a constant beta corresponds to a parallel shift of the term structure, a linear effect to a modification of the slope, ... It can be seen from

Figure 3 that the first regime has a negative effect (almost linear at the long end of the term structure) while the second regime has a nonlinear positive effect. Other regime effects are irregular and rather mixed by maturity. Besides the means of the beta function are increasing with the maturity for most of the regimes.

B Constraints implied by no arbitrage

Let us now consider the test of the constraints on the moments implied by no arbitrage. This test cannot be applied to more than two regimes since we have too few observations for the regimes other than regimes 1 and 2, and because of the identification problems discussed in Section III.C. Therefore we focus on a two regime setting applied to the regimes with the two highest estimated probabilities.

The SURE routine yields first the estimates $\widehat{m}_T$, $\widehat{m}_T^*$, $\widehat{\Sigma}_T$, $\widehat{\Sigma}_T^*$. Next we compute the two statistics ξ_{1T} and ξ_{2T}. The residual sum of squares are respectively equal to 3.363 and 9.195 which leads to a very strong rejection of the null hypothesis H_0 (as these sums are multiplied by $T = 464$ to derive the statistics) and hence of the no arbitrage restrictions. This is not surprising since the difference between the estimated means $\widehat{m}_T(u)$ and $\widehat{m}_T^*(u)$, is clearly nonlinear (Figure 4) and in particular not constant. The same remark applies to the difference between the estimated variances $\widehat{\sigma}_{uu,T}$ and $\widehat{\sigma}^*_{uu,T}$ (Figure 5).

V Conclusion

The statistical inference tools and visual inspection of actual empirical disturbance functions for US term structure data reveal that the binomial Ho and Lee model is misspecified. We have proposed in this paper two extensions of this basic model by allowing the disturbance function to be random in each regime, and by increasing the number of regimes. We have also discussed the constraints implied by no arbitrage in this more general framework.

Despite the misspecification of the Ho and Lee model, the empirical analysis shows that two regimes in the sample are much more frequent than others. This would suggest that the basic assumptions of the model on the number of regimes were correct. However the restrictions implied by no arbitrage are clearly not fulfilled when these two regimes are considered.

References

Bliss, R., and E. Ronn (1989) : "Arbitrage-Based Estimation of Nonstationary Shifts in the Term Structure of Interest Rates", *Journal of Finance*, 44, 591-610.

Clément, E., Gouriéroux, C., and A. Monfort (1997) : "Econometric Specification of the Risk Neutral Valuation Model", CEPREMAP DP. Forthcoming in *Journal of Econometrics*.

Cox, J., Ross, S., and M. Rubinstein (1979) : "Option Pricing : A Simplified Approach", *Journal of Financial Economics*, 7, 229-263.

Duffie, D. (1988) : *Security Markets : Stochastic Models*, Academic Press Inc., London

Ekeland, I. (1979) : *Eléments d'économie mathématique*, Hermann, Paris.

Gouriéroux, C., and A. Monfort (1995) : *Statistics and Econometric Models*, Vol. II, Cambridge University Press, Cambridge.

Gouriéroux, C., Scaillet, O., and A. Szafarz (1997) : *Econométrie de la Finance : Analyses Historiques*, Economica, Paris.

Ho, T., and S. Lee (1986) : "Term Structure Movements and Pricing of Interest Rate Contingent Claims", *Journal of Finance*, 41, 1011-1029.

McCulloch, J., and H.-C. Kwon (1993) : "US Term Structure Data, 1947-1991", Ohio State University, Working Paper 93-6.

Turnbull, S., and F. Milne (1991) : "A Simple Approach to The Pricing of Interest Rate Options", *Review of Financial Studies*, 4, 87-120.

Appendix

We derive in this Appendix the implications of the no arbitrage assumption when neither information on the future levels, nor on the future regimes are available.

Let us take a portfolio of N discount bonds and denote by δ_u, $u = 1, \ldots, N$, the asset allocation in the portfolio. The no arbitrage assumption means that a portfolio with a non negative value at the next date and a positive probability of a positive price at that date has necessarily a positive price. Formally :

$$\begin{aligned} & \sum_{u=1}^{N} \delta_u B(t, t + (u-1)) \geq 0 \qquad \text{a.s.,} \\ \text{and} \qquad & P\left[\sum_{u=1}^{N} \delta_u B(t, t + (u-1)) > 0\right] > 0, \\ \text{implies} \quad & \sum_{u=1}^{N} \delta_u B(t-1, t-1+u) > 0. \end{aligned}$$

By taking into account the evolution of the term structure, we get :

$$\begin{aligned} & \sum_{u=1}^{N} \delta_u B(t-1, t-1+u) H_t(u-1) \geq 0 \qquad \text{a.s.,} \\ \text{and} \qquad & P\left[\sum_{u=1}^{N} \delta_u B(t-1, t-1+u) H_t(u-1) > 0\right] > 0, \\ \text{implies} \quad & \sum_{u=1}^{N} \delta_u B(t-1, t-1+u) > 0. \end{aligned}$$

Let us first consider the two regime model of Section II. Because of assumption A 4, the condition :

$$\sum_{u=1}^{N} \delta_u B(t-1, t-1+u) H_t(u-1) \geq 0 \qquad \text{a.s.}$$

is equivalent to :

$$\sum_{u=1}^{N} \delta_u B(t-1, t-1+u) h(u-1) \geq 0, \quad \forall h, \text{ with } h(0) = 1, h(u) \geq 1, \forall u, \tag{16}$$

and :

$$\sum_{u=1}^{N} \delta_u B(t-1, t-1+u) h^*(u-1) \geq 0, \quad \forall h, \text{ with } h^*(0) = 1, h^*(u) \leq 1, \forall u. \tag{17}$$

Since $B(t-1, t-1+u)$ and $H_t(u-1)$ are non negative, we only have to verify that all δ_u are non negative. In order to do so, we first take a sequence of functions h such that, for a given $u_0 \geq 1$, $h(u_0)$ tends to infinity, and the other components are fixed. We deduce that δ_{u_0+1} must be greater or equal to zero in order to keep the non negativity in (16). Therefore we obtain $\delta_u \geq 0$, $\forall u \geq 2$. Besides if we take a sequence of functions h^* such that, for any $u \geq 1$, $h^*(u)$ tends to zero, we get the non negativity of δ_1, from condition (17).

Finally thanks to the condition : $P\left[\sum_{u=1}^{N} \delta_u B(t, t+(u-1)) > 0\right] > 0$, at least one of the components δ_u of the asset allocation is positive and therefore the condition $\sum_{u=1}^{N} \delta_u B(t, t+(u-1)) > 0$ is automatically satisfied.

We conclude that there is no additional constraints implied by the no arbitrage assumption. The same conclusion holds for the multiregime case where the degree of market incompleteness is still greater.

Tables and Table legends :

maturity	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	prob. (%)	cumul (%)
regime 1	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	44.61	44.61
regime 2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	12.93	57.54
regime 3	+	+	+	+	-	-	-	-	-	-	-	-	-	-	-	-	-	3.88	61.42
regime 4	+	+	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3.88	65.30
regime 5	+	+	+	+	+	+	-	-	-	-	-	-	-	-	-	-	-	3.66	68.96
regime 6	+	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3.45	72.41
regime 7	+	+	+	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3.02	75.43
regime 8	+	+	+	+	+	-	-	-	-	-	-	-	-	-	-	-	-	2.37	77.80
regime 9	+	+	+	+	+	+	+	+	+	+	-	-	-	-	-	-	-	2.16	79.96
regime 10	-	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	1.72	81.68
regime 11	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-	-	-	1.51	83.19
regime 12	+	+	+	+	+	+	+	+	+	-	-	-	-	-	-	-	-	1.08	84.27
regime 13	+	+	+	+	+	+	+	+	+	+	+	+	+	-	-	-	-	1.08	85.35
regime 14	+	+	+	+	+	+	+	+	-	-	-	-	-	-	-	-	-	1.08	86.43

Table I : Structure of observed regimes and estimated probabilities

The listed regimes correspond to regimes with an estimated probability of occurrence (expressed in percents) larger than 1 %. The column labelled cumul gives the probability of occurrence cumulated over the regimes, while maturity refers to time-to-maturity from 1 to 17 months. A plus indicates that the value of the observed disturbance function $H_t(u)$ is above one (i.e. the regime-by-maturity indicator $\epsilon_t(u)$ is equal to one) and a minus below one (i.e. the regime-by-maturity indicator $\epsilon_t(u)$ is equal to zero).

Figure legends :

Figure 1 : Observed values of the disturbance function

The observed values of the disturbance function $H_t(u)$ are plotted for monthly US term structure data (dates $t = 1, \ldots, 464$ from 01/1948 to 08/1985) and for time-to-maturity u from 1 to 17 months.

Figure 2 : Estimated probabilities

This figure shows the estimated probabilities for the 55 observed regimes ranked in a decreasing order.

Figure 3 : Estimated means of the beta function

Estimated means of the beta function $\beta_t(u) = -\frac{1}{u} \log H_t(u)$ are displayed for the five observed regimes with the highest probabilities of occurrence.

Figure 4 : Estimated means and difference

The estimated means $\widehat{m}_T(u)$, $\widehat{m}_T^*(u)$ of the reparametrized disturbance functions $\log \alpha_t(u)$ (high regime), $\log \alpha_t^*(u)$ (low regime), and their difference $\widehat{m}_T^*(u) - \widehat{m}_T(u)$ are plotted against time-to-maturity u. A test of the no arbitrage restrictions is obtained by examining whether the difference is statistically constant through time-to-maturity.

Figure 5 : Estimated variances and difference

This figure plots the estimated variances $\widehat{\sigma}_{uu,T}$, $\widehat{\sigma}_{uu,T}^*$ of the reparametrized disturbance functions $\log \alpha_t(u)$ (high regime), $\log \alpha_t^*(u)$ (low regime), and their difference $\widehat{\sigma}_{uu,T}^* - \widehat{\sigma}_{uu,T}$ against time-to-maturity u. If the no arbitrage restrictions hold, the difference should not be statistically different from a straight horizontal line.

Notes :

1. The data file ZEROYLD1 can be downloaded from the web site : http://www.cob.ohio-state.edu/dept/fin/osudown.htm. The Gauss programs developped for this paper are available on request.