The Hotelling model with Capacity Precommitment

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Abstract

We consider the two-stage game proposed by Kreps & Scheinkman [83] in the address model of horizontal differentiation developed by Hotelling. Firms choose capacities in the first stage and then compete in price. We show that price competition is drastically soften since in almost all subgame perfect equilibrium, firms behave as if they were an integrated monopoly i.e., choose capacities which exactly cover the market, so that there is no room for price competition. If furthermore the installation cost for capacity is one fourth of the transportation cost, then this result stands for all SPE. Like Kreps & Scheinkman, we show that the Cournot allocations (quantity competition) coincide with the SPE allocations of our game form. Finally our analysis provides an interesting treatment of mixed strategies equilibria which is quite new in this literature.

JEL Classification : D43, F13, L13
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1) Introduction

The intuition according to which price competition yields lower equilibrium mark-ups than quantity competition has been definitely illustrated by the Bertrand [1883] critique of Cournot [1838] analysis. The comparison of these two models conveys a very simple message : equilibrium outcomes in oligopolistic industries crucially depend on the nature of the strategic variable. It is however fair to say that both models suffer from important limitations. The main drawback of quantity setting models is that no explicit price mechanism is stipulated. On the other hand, allowing firms to set prices leads to the Bertrand result, if and only if firms face constant marginal costs. As shown by Edgeworth [25], no pure strategy equilibrium exists in the Bertrand model with increasing marginal costs. An important by-product of Edgeworth's analysis is that firms' payoffs in a mixed strategy equilibrium are positive, so that firms could find it profitable to voluntarily limit their production capacities, in order to depart from the Bertrand outcomes. This provides the intuition underlying the result established by Kreps & Scheinkman [83]. They showed that in a two-stage game where firms precommit to capacities and then compete in prices, Cournot outcomes could prevail as the unique SPE outcome. In addition to reconciling Bertrand and Cournot competition into a single framework, this result essentially suggests that quantity precommitment is a natural way out of a too fierce price competition.

In order to depart from the Bertrand outcome, Hotelling [29] followed a completely different route. Sticking to a price competition framework, he showed that firms could secure positive profits in equilibrium by differentiating their products. Interestingly enough, he thought that product differentiation would solve both the Bertrand and the Edgeworth problem. To quote Hotelling [29], p 471 : "The assumption, implicit in Cournot, Amoroso and Edgeworth's work that all buyers deal with the cheapest seller leads to a type of instability which disappears when the quantity sold is considered as a continuous function of the differences in prices". Although he was clearly right in arguing that continuous demand would solve the Bertrand paradox, he was wrong on the Edgeworth's front. As shown by Shapley & Shubik [69] or McLeod [85], product differentiation is not sufficient to restore the existence of a pure strategy equilibrium in a pricing game with increasing marginal costs. Indeed, the presence of quantitative restrictions in the pricing game typically yields non quasi-concave payoffs and these migth preclude the existence of a pure strategy equilibrium. Existence conditions have then been investigated by Friedman [88], Benassy [89] and Canoy [96] for price setting models of product differentiation.

However, the incentives to voluntarily limit production capacities have been completely neglected in a context of product differentiation. This is remarkable because it is clear that firms' incentives to relax price competition through some form of quantitative restrictions do not disappear due to product differentiation. For instance, at the equilibrium prices in the Hotelling model, all consumers enjoy a strictly positive surplus. Therefore, firms could sell exactly the same amount at higher prices, thereby enjoying higher profits. In this sense, prices are too low in the Hotelling equilibrium and there is room for relaxing price competition further through capacity limitations.

Addressing this question is the aim of the present paper. To this end, we apply the Kreps & Scheinkman framework to the Hotelling model i.e., firms precommit to capacities and then compete in prices in an horizontally differentiated market.

Our main results are the following. In a subgame perfect equilibrium (SPE), capacity precommitment softens price competition, as in Kreps & Scheinkman but more drastically : the capacity choices exactly cover the market, so that there is no room for price competition at all, the only degree of freedom being the sharing of the market. The intuition behind this result is that if capacity choices overlap, at least one firm will reduce its capacity to the complement of its competitor, the reason being that its second stage payoff does not depend on its first stage strategy (capacity). The foundation of this result is that *capacity precommitment enables firms to take advantage of the local monopoly structure inherent to the Hotelling model*. Note that our result is not driven by the existence of costs for capacity installation ; this cost must be positive but can be arbitrarily small relative to the other parameters of the game. We must mention however the existence of other SPE's for a small set of values of the reservation price involving excess capacities and mixed strategies in the pricing game ; however, they appear only only for low installation cost. Finally, we show that the SPE's involving exact market coverage are the outcomes of Cournot competition, thus we provide an analog of Kreps & Scheinkman's statement to horizontally differentiated market.

The paper is organized as follows. In section 2 we present the basic model and the equilibrium when firms do not face limited capacities. Section 3 presents the main intuitions underlying the analysis of price competition with capacity constraints in the Hotelling model. In section 4, we study the various kind of price subgames. The capacity game is then solved in section 5. Section 6 concludes.

2) The Hotelling model without capacity constraints

We introduce the address model of Hotelling with fixed locations (this point shall never change) and capacities and analyse the resulting price competition.

An indivisible homogeneous good is sold by two shops located at the boundaries[1] of the [0;1] segment along which consumers are uniformly distributed. Each consumer is identified by its address **x** in the street. An agent buy at most one unit of the good, the common reservation price is **S**. When buying one of the products, the consumer goes to a shop and bears a transportation cost linear in the distance to the shop. Since we can normalise prices, we set the transportation cost between the two shops at 1$. The utility derived by a consumer located at **x** in the interval [0,1] is

[1] We choose maximum differentiation to relax as much as possible price competition . If firms find it profitable to further relax price competition through capacity precommitment, it is likely that they would face even greater incentives if they were less horizontally differentiated.

$$\begin{cases} S - x - p_1 & \text{if product bought at firm 1} \\ S - (1-x) - p_2 & \text{if product bought at firm 2} \end{cases}$$

Refraining from consuming any of the two products yields a nil level of utility[2]. Although being a fairly standard result, we first characterise the Hotelling equilibrium in full length. Indeed, this will provide a useful benchmark for the analysis to follow.

Proposition 1 (Hotelling)

If S > 3/2 and firms face no capacity constraints, the only Nash equilibrium of the pricing game is (1,1) and the market is covered.

Proof As one can see with the plain lines of figure 1, if prices are not too large, all agents buy the good at one of the shops. Those living in the segment $[0;\tilde{x}(p_1,p_2)]$ will buy at firm 1 whose demand is thus $\tilde{x}(p_1,p_2)$ as consumers are uniformly distributed on [0;1]. Likewise, the demand addressed to firm 2 is $1-\tilde{x}(p_1,p_2)$.

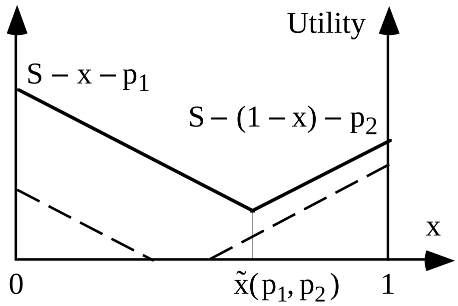

Figure 1

The address of the indifferent consumer is $\tilde{x}(p_1,p_2) \equiv \frac{1-p_1+p_2}{2}$ i.e., the solution of $S - x - p_1 = S - (1-x) - p_2$. It is also clear from the dashed lines of figure 1 that the market is not covered if prices are too large. In that case, firm[3] **i** is a local monopoly and her demand is $\min\{1, S-p_i\}$, this happens if $S - \tilde{x}(p_i,p_j) - p_i < 0 \Leftrightarrow p_i > 2S - 1 - p_j$.

The demand function of firm **i** is $D_i(p_i,p_j) = \begin{cases} \frac{1-p_i+p_j}{2} & \text{if } p_i \le 2S-1-p_j \\ \min\{1, S-p_i\} & \text{if } p_i > 2S-1-p_j \end{cases}$. The respective maximisers are $H(p_j) \equiv \frac{1+p_j}{2}$ and the monopoly price $p^m \equiv \min\left\{S-1, \frac{S}{2}\right\}$.

Since $D_i(p_i,p_j)$ is piecewise linear and decreasing in p_j, the profit function is concave in p_j so that the best reply to a mixed strategy is the best reply to its expectation which is a pure strategy. Therefore, the unique Nash equilibrium of this pricing game is pure.

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[2] In Hotelling's original model, this possibility is not considered, formally, this correspond to an infinite S.

[3] In the remainder of the text, **i** stands for either of the firm and **j** for her competitor.

The best reply of the domestic firm, BR_d, is displayed in bold and dashed on figure 2 while that of the foreign firm is displayed in bold and plain. BR_d follows the classical best reply $H(p_f)$ for low p_f's until it leads to an uncovered market. From that point on, BR_d decreases along the frontier until it reaches the monopoly price p^m. The best reply lines intersect at the unit price for both firms as soon as S > 3/2. Otherwise, there is a continüm of equilibria on the frontier which entail no "real" price competition.♦

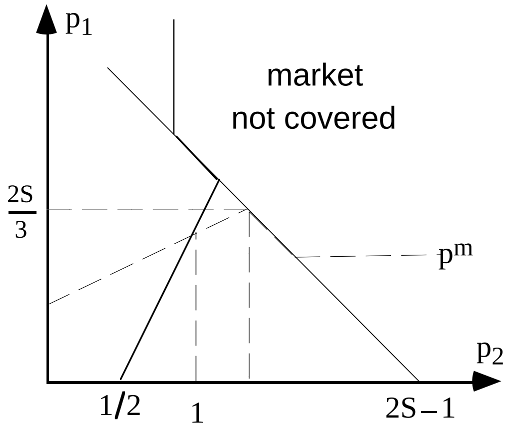


Figure 2

Observe that the equilibrium prices do not depend on S and are "too low" in the sense that all consumers enjoy a strictly positive surplus. Clearly, firms could benefit from using capacity pre-commitment in order to relax the price competition.

3) Capacity pre-commitment

We add a preliminary step to the game of the previous section : firms choose simultaneously sales capacity k_1 and k_2 and then simultaneously choose prices p_1 and p_2 knowing the chosen capacity of their competitor. We are considering a two stage game G that is analysed using subgame perfect Nash equilibrium. The subgame after the choices of k_1 and k_2 is denoted $G(k_1,k_2)$ and called the pricing game.

Definition : A strategy of firm **i** in G is a capacity k_i and a function σ_i assigning to any couple of capacities (k_i,k_j) a pricing strategy $\sigma_i(k_i,k_j)$ which is a probability measure over the positive prices. A subgame perfect equilibrium is a quadruple $(m_1, m_2, \sigma_1(.,.), \sigma_2(.,.))$ such that in every subgame $G(k_i,k_j)$, including the equilibrium path $G(m_i,m_j)$, the pricing strategy $\sigma_i(k_i,k_j)$ of firm **i** is a best reply to $\sigma_j(k_i,k_j)$. Secondly, m_i is a best reply to m_j, knowing that σ_i and σ_j are used in any pricing game.

The introduction of capacities constraints k_1 and k_2 complicates the analysis. Indeed, the presence of limited capacities considerably affects firms' incentives in the pricing game. First, a limited capacity may decrease the incentive of a firm to reply to the other's price with a low price. Consider for instance that firm 1 has chosen a capacity k_1. When firm 1 is aggressive and sets a price p_1 low relative to p_2, she receives a large demand but she is not able to serve all of it as soon as her capacity k_1 is reached ; thus her incentives to price competition are lowered.

A second observation induced by limited capacities is that some consumers might be rationed at the prevailing prices. This possibility is the cornerstone of the price competition analysis as it may reverse firms' incentives in the price game. More precisely, one firm could find it profitable to quote a high price, anticipating the fact that some consumers will be rationed by the other firm and could be willing to report their purchase to her. This was the original intuition of Edgeworth. The incentives for that behaviour basically depend on the willingness of consumers to switch to the high price firm in case of rationing. The extent to which rationed consumers will be recovered by this firm directly depends on who the rationed consumers are. Therefore, the organisation of rationing in the market is of central importance.

We will assume that the efficient rationing rule is at work in the market, as in Kreps & Scheinkman [83]. Under this rule, rationed consumers are those exhibiting the lowest reservation price for the good. Consider the example depicted on figure 3.

Some consumers willing to buy at firm 1 are rationed. Under efficient rationing, they are located in the interval $\left[k_1;\tilde{x}(p_1,p_2)\right]$ and thus are precisely the most inclined to switch to firm 2. Despite firm 2 has a potentially low demand (p_2 is high relative to p_1), the fact that firm 1 is constrained by her capacity k_1, could give firm 2 an effective demand of $1-k_1$.

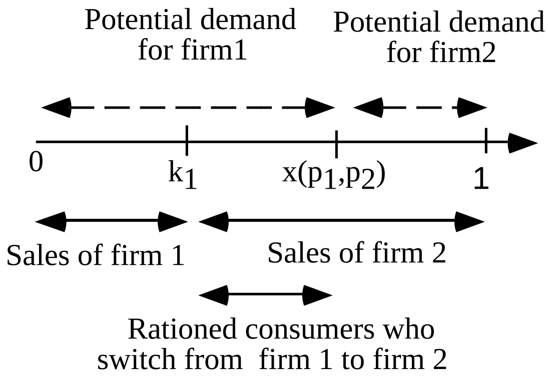


Figure 3

More precisely, as long as p_2 is less than $S-1+k_1$, the net reservation price of the consumer located in k_1, the effective demand of firm 2 is $1-k_1$. This feature of the market allocation rule also lowers firm 2's incentives to enter a price competition "à la Bertrand" since her demand is locally independent of her own price p_2. Note thus that within our framework, efficient rationing defines the largest residual demand for firm 2, so that, contrarily to Kreps & Scheinkman, the incentives to use rationing strategically are maximised. This phenomenon will have a strong feedback on the choices of capacities.

Nevertheless, our study will be meaningful if firms are always lead to choose capacities whose total exceeds the market size. This way, they will enter into a price competition at the second stage. Recalling that the total transportation cost from one shop to the other is unity, Proposition 2 clarifies this point.

Proposition 2

If the unit cost of capacity installation ε is larger than S – 1, the unique SPE entails monopoly pricing by both firms.

Proof If firm **i** has a monopoly over the market and has installed a capacity k_i, the demand addressed to her is $f_i \equiv \min\{k_i, S - p_i\}$. The second period profit $p_i(S - p_i)$ is maximum for $p_i = \min\left\{S - k_i, \frac{S}{2}\right\}$. We denote by ε the unit cost of capacity installation so that the first period profit is $\Pi_i(k_i) = \begin{cases} k_i[S - k_i - \varepsilon] & \text{if} \quad k_i < S/2 \\ \frac{S^2}{4} - \varepsilon k_i & \text{if} \quad k_i \geq S/2 \end{cases}$. As the second part is decreasing, only the first matters for the optimal capacity choice which is $k^m \equiv \min\left\{1, \frac{S-\varepsilon}{2}\right\}$. Now, it clear that $\varepsilon > S - 1$ implies $k^m < 1/2$; thus both firms are able to achieve their full monopoly profit without interacting which means that k^m is a dominant strategy and thus characterise a unique SPE allocation. ♦

Corollary

If ε < S – 1, then a SPE involves total capacity exceeding the market size.

It is clear that when $\varepsilon < S - 1$, capacity choices "interior" to the market i.e., such that $k_1 + k_2 < 1$, are not stable since one of the firm (may be both) has an incentive to choose at least a capacity completary to her competitor. However, exact market coverage pairs (k_1,k_2) subject to the constraint $\max\{k_1, k_2\} \leq k^m$, are candidates to be subgame perfect equilibrium of the overall game. The next section studies the pricing game when the choice of capacities exceed the market size.

4) The price subgame

4.1) The Demand functions

We now proceed to the derivation of demands in the pricing game $G(k_1,k_2)$ when $k_1 + k_2 > 1$ and, without loss of generality, $k_1 > k_2$. The intuition underlying the analysis is the following one : demands are piecewise linear, thus continuous in prices (contrarily to the case of homogenous goods) but exhibit outward kinks i.e., non concavity. Therefore, the best reply functions are discontinuous which can preclude the existence of pure strategy equilibria. Nevertheless, payoff functions are "piecewise" concave, thus firms have only a finite number of best replies and equilibria are *atomic*. By this term, we mean that firms use a mixed strategy with a finite support ; contrarily to the case studied by Kreps & Scheinkman [1983] and Osborne & Pitchik [1986], there is no density of prices in any equilibrium of the second stage game. This is a property of product differentiation. The following paragraph which is devoted to the derivation of demand functions is quite heavy but things become clear on the figure that summarises our findings.

The rationing rule that we use is the efficient one which means that if the demand addressed to firm **i** exceeds k_i, she serves the segment $[0;k_i]$. Thus, if we let D_i be the sales of firm **i**, the demand addressed to firm **j** is bounded by $1 - D_i$. Since D_i is bounded by the capacity k_i and by the monopoly sales $S - p_i$, we have $D_i = \min\{S - p_i, k_i, 1 - D_j\} = \min\{f_i, 1 - D_j\}$ where $f_i = \min\{k_i, S - p_i\}$.

Observe now that $\left.\begin{array}{ll}\tilde{x}(p_1,p_2) < f_1 & (E1)\\ 1-\tilde{x}(p_1,p_2) < f_2 & (E2)\end{array}\right\} \Rightarrow \begin{cases} D_1 = \tilde{x}(p_1,p_2) \\ D_2 = 1-\tilde{x}(p_1,p_2)\end{cases}$ because the demands addressed to the firms can be served by both. The reverse implication is also true as one can see from the definition of D_i. We now investigate the prices that enables the conditions (E1) and (E2) to hold.

- $p_1 \le S - k_1 \Rightarrow f_1 = k_1$ and (E1) is $\tilde{x} = \frac{1-p_1+p_2}{2} \le k_1 \Leftrightarrow p_1 \ge a(k_1,p_2) \equiv p_2 + 1 - 2k_1$
- $p_1 > S - k_1 \Rightarrow f_1 = S - p_1$ and (E1) is $\tilde{x} = \frac{1-p_1+p_2}{2} \le S - p_1 \Leftrightarrow p_1 \le c(p_2) \equiv 2S - p_2 - 1$
- $p_2 \le S - k_2 \Rightarrow f_2 = k_2$ and (E2) is $1-\tilde{x} = \frac{1-p_2+p_1}{2} \le k_2 \Leftrightarrow p_1 \le b(k_2,p_2) \equiv p_2 - 1 + 2k_2$
- $p_2 > S - k_2 \Rightarrow f_2 = S - p_2$ and (E2) is $1-\tilde{x} = \frac{1-p_2+p_1}{2} \le S - p_2 \Leftrightarrow p_1 \le c(p_2)$

The conditions (E1) and (E2) are satisfied for prices such that $a(k_1,p_2) \le p_1 \le \min\{b(k_2,p_2), c(p_2)\}$. The benchmark $\delta(k_i) \equiv S - k_i$ is the maximum price compatible with sales of k_i while $\rho(k_j) \equiv S - 1 + k_j$ is called the default option as it ensures firm **i**, a demand of at least $1 - k_j$ whatever his opponent's price.

As shown on figure 4 below, $k_1 + k_2 > 1$ implies $a(k_1,.) < b(k_2,.)$ and $\delta(k_j) < \rho(k_i)$. Lastly, $a(k_1,.) = c(.) = \delta(k_1)$ for $p_2 = \rho(k_1)$ and $b(k_2,.) = c(.) = \rho(k_2)$ for $p_2 = \delta(k_2)$. The area delimited by the functions **a**, **b** and **c** will be called “the band“.

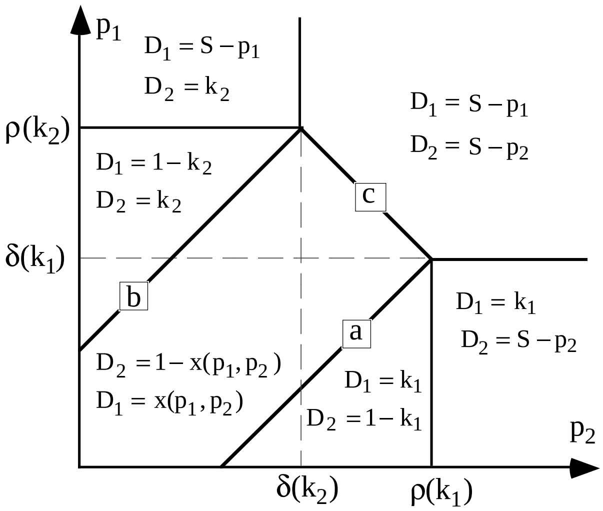


Figure 4

Thanks to figure 4, the demands addressed to the firms are easy to understand. When prices are similar, they give rise to a point in the band where we have the classical Hotelling demands. Now, if p_1 increases, firm 1 looses sales until firm 2 is constrained by her capacity i.e., we reach the upper triangle. From that point on, if p_1 increases further, D_1 remains constant at $1 - k_2$. This conduct last until p_1 is so large that the market is not covered anymore and firm 1 obtains a monopoly demand.

4.2) The best reply functions

We now perform the derivation of the best reply of firm 1 to a price charged by firm 2. We already assumed $S > 1+ \varepsilon$ and we go a step further[4] by assuming $S > 2$ to create a fierce price competition between the duopolists. Indeed, it basically implies that a lonely monopolist with full capacity would choose to cover the market (his unconstrained choice would be S/2). Since there are two firms and only one market, it cannot be the case that the market is left partially uncovered at equilibrium prices. Whatever their capacity choices, firms will always engage into a price competition. Technically, it implies $S/2 < \delta(k_i)$ for all **i** and all capacities k_i and thus $S/2 < \rho(k_i)$ because $k_1 + k_2 > 1$.

Lemma 1

*Firm **i** never charge prices above* $\rho(k_j) \equiv S - 1 + k_j$, *thus strategy spaces are bounded.*

<u>Proof</u> Let F_1 be the cumulative function of the mixed strategy used by firm **1** in equilibrium. Recall that $\forall p_2, \forall p_1 \geq \rho(k_2), D_1(p_1, p_2) = S - p_1$. Now, since the monopoly price S/2 is less than $\rho(k_2)$, $\Pi_1(p_2,.)$ must be decreasing over $[\rho(k_2);+\infty[$ and the same is true for the average $\Pi_1(F_2,.) \equiv \int \Pi_1(p_2,.) dF_2(p_2)$. Therefore F_1 as a best reply to F_2, has no mass above $\rho(k_2)$ and symmetrically the support of F_2 is included in $[0;\rho(k_1)]$. ♦

We now study the best reply of firm 1 to a price p_2 lesser than $\rho(k_1)$ knowing that p_1 is itself lesser than $\rho(k_2)$. There are two radically different patterns of behaviour. Either, firm 1 act in a classical fashion (à la Hotelling) with an agressive price in order to gain market shares or she hides behind the quota. We mean that she contents herself to serve the part of market that is out of reach for her opponent i.e., the $[0;1 - k_2]$ interval. On this residual demand, firm 1 acts as a monopolist (this is the key feature of the Hotelling framework) and the optimal price is $\rho(k_2)$; we call it the security strategy.

[4] Note that the standard analysis of the Hotelling model generally assumes S *large enough* to ensure market-covering in equilibrium. A close look at the proofs shows that it is unnecessary but simplifies the exposition of the price equilibrium.

On the domain of monopoly demand, $S/2 < \delta(k_1)$ implies that profit is decreasing with p_1 over the domain, thus the best choice is first $\rho(k_2)$ and then $c(p_2)$. However this latter value is itself dominated by the argmax of the band. In the triangle above **b** and in the triangle under **a**, demand is constant so that profit is increasing and the best choices are respectively $\rho(k_2)$ and $a(k_1,p_2)$. Again, this latter value is itself dominated by the argmax of the band.

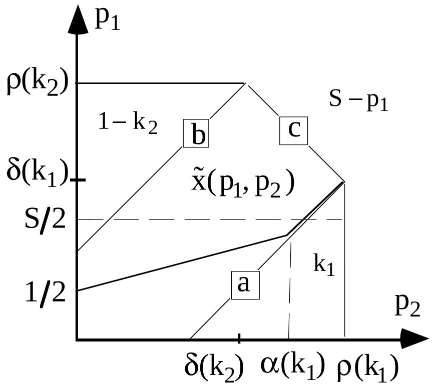


Figure 5

In the band where $D_1 = \tilde{x}(p_1,p_2)$, profit is quadratic, hence the best choice is either H(.), $a(k_1,.)$ or $b(k_2,.)$. As seen on figure 5, H(.) is in the band if $b(k_2,p_2) > H(p_2) > a(k_1,p_2)$. The first inequality leads to $p_2 > \beta(k_2) \equiv 3 - 4k_2$ (it is always satisfied if $k_2 > 3/4$ as on figure 5). The second inequality leads to $p_2 < \alpha(k_1) \equiv 4k_1 - 1$ (it is never satisfied if $k_1 < 1/4$). Since $b(k_2,p_2)$ is dominated by $\rho(k_2)$, we can, without loss of generality, take $\mathrm{Max}\{H(.),a(k_1,.)\}$ to be the best choice in the band because we will then choose between this candidate and the security strategy $\rho(k_2)$. The profit function associated with this "Hotelling" pattern is $\hat{\Pi}_1(p_2) \equiv \begin{cases} \dfrac{(1+p_2)^2}{8} & \text{if } p_2 < \alpha(k_1) \\ k_1[p_2 + 1 - 2k_1] & \text{if } p_2 \geq \alpha(k_1) \end{cases}$.

The meaning of the modified Hotelling best reply is that firm 1 responds in an aggressive manner to any increase of p_2 in order to gain market shares. This conduct lasts until her sales reach her capacity ; above that threshold, she can only sell her full capacity at a maximum price.

The security strategy $\rho(k_2)$ generates the profit $\Pi^d(k_2) \equiv [S - 1 + k_2](1 - k_2)$ which might dominate $\hat{\Pi}_1(p_2)$ against a low price p_2 ; more specifically, lemma 3 in the appendix[5] shows that $BR(p_2) \equiv \begin{cases} \rho(k_2) & \text{if } \quad p_2 \leq \gamma(k_1,k_2) \\ \mathrm{Max}\{H(p_2),a(k_1,p_2)\} & \text{if } \quad p_2 > \gamma(k_1,k_2) \end{cases}$.

Firm 1's best reply function is plotted in bold on figure 6 (the other lines are used in the proof of lemma 2 below) ; its particular shape is intuitive : when the price p_2 is low, firm 1 tends to reply with a high price, thereby benefiting from the resulting rationing at firm 2. Against a high price p_2, firm 1 fights for market shares.

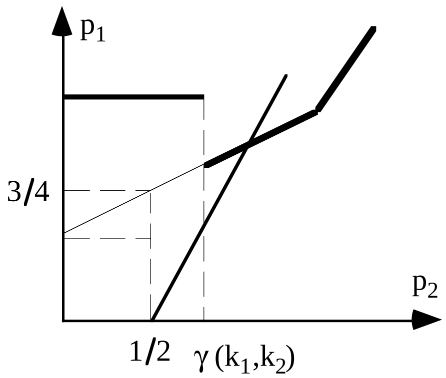


Figure 6

An intuitve consequence of the previous calculations is the following lemma.

[5] The cut-off function $\gamma(k_1,k_2)$ having no economic meaning, it is not reproduced in the text.

Lemma 2

In any equilibrium of the pricing game, firms name prices larger than the unit price characterising the Hotelling equilibrium.

Proof Observe on figure 6 above, that the best reply of firm 1 is always larger than 1/2 because both H(.) and $\rho(k_2)$ are larger than 1/2. This is due to the fact that $\Pi_1(p_2,.)$ is increasing over [0;1/2] for any price p_2. This in turn implies that the average $\Pi_1(F_2,.)$ is also increasing over [0;1/2]. Therefore F_1, which is a best reply to F_2, puts no mass under 1/2 and by symmetry, this result is also true for F_2.

Now, when looking again at $\Pi_1(p_2,.)$, we can restrict ourselves to prices $p_2 \geq 1/2$ so that $\Pi_1(p_2,.)$ is increasing over [0;3/4] (cf. figure 6) and so is the average $\Pi_1(F_2,.)$. The same optimality argument now implies that F_1 puts no mass under 3/4. Repetition implies that neither F_1 nor F_2 put mass under the unit. This argument is also used in lemma 4 of the appendix where it is more completely developed. ♦

Combining lemmas 1 and 2, we obtain :

Proposition 3

In equilibrium, the support of the mixed strategy F_i is included in $[1;\rho(k_j)]$.

4.3) Equilibria of the pricing game

In this sub-section, we characterise the equilibria of the price subgame ; however, analytical developments are relegated to the appendix.

Proposition 4

The price equilibrium can take either of the following forms :
- *A) both firms use the pure strategy at the Hotelling unit price*
- *B) one firm plays a pure strategy and the other mixes over two atoms*
- *C) both firms use a mixed strategy involving the same number of "atoms"*

Proof *A) Equilibria involving only pure strategies*

Assuming that both capacities are arbitrarily close to 1, one could expect that the standard Hotelling equilibrium of proposition 1 is preserved.

We show in lemma 3 of the appendix, the existence of an area A_1 of the (k_1,k_2) plane (cf. figure 7) where $\gamma(k_1,k_2) < 1$; it consist of large values of k_2 while the symmetrical area A_2 where $\gamma(k_2,k_1) < 1$ consists of large values for k_1. In $A_1 \cap A_2$ which is a square, best replies curves do indeed cross at (1,1) which is an equilibrium. We cannot claim that it is the only one because profit functions are not concave.

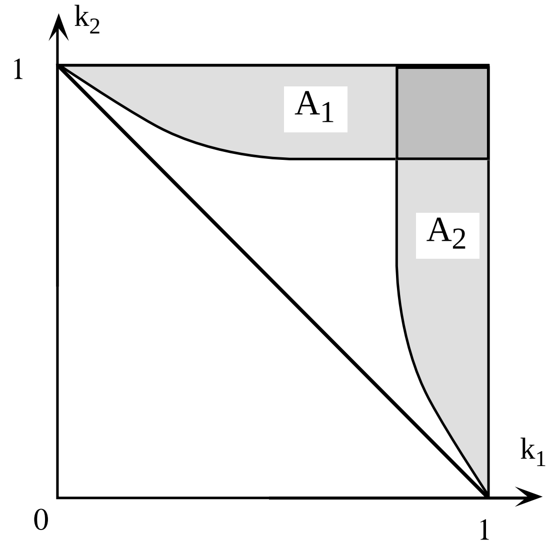


Figure 7

B) Equilibria involving a pure strategy and a mixed one

In the area $A_1 \backslash A_2$ (large k_2 and not so large k_1) of figure 7 above, the pure strategy equilibrium ceases to exists because $\gamma(k_1,k_2) > 1$. To understand the characterisation of type B equilibrium, it is useful to give the intuition of this result by presenting the Edgeworth cycle in a market for differentiated products.

If $p_1 = 1$, firm 2 uses the fact that k_1 is not very large to enjoy the market share $1 - k_1$ at the maximum price $\rho(k_1)$ rather than fighting against $p_1 = 1$ with its "Hotelling" best reply $H(1) = 1$. Since there is actually no competition, the best reaction of firm 1 is to increase its price to $\delta(k_1)$ which is the maximum price compatible with sales of k_1.

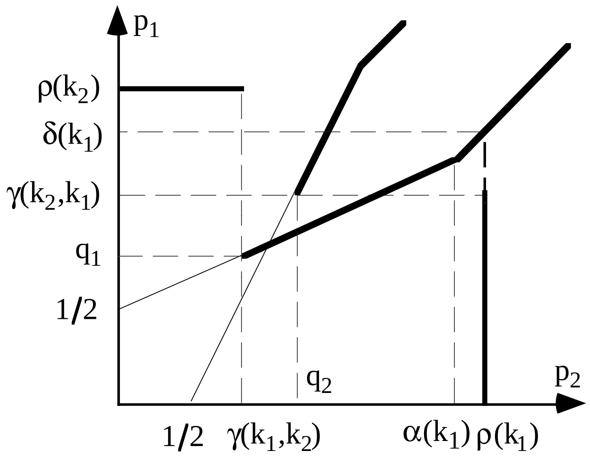


Figure 8

Now, both prices are at their peak and the only way to increase profit is to capture new market shares by undercutting one's opponent price. The next best move of firm 2 is $p_2 = H(p_1)$, followed by a low value $p_1 = H(p_2)$; at this moment we are back to the beginning of the story : it is better for firm 2 to retreat over its protected share $1 - k_1$.

According to the Nash definition in this context, the equilibrium sees firm 1 playing the pure strategy $\gamma(k_2,k_1)$ while firm 2 mixes between $\rho(k_1)$ and a lower price $q_2 \equiv \frac{1+\gamma(k_2,k_1)}{2}$ (cf. figure 8). The symmetric vector of strategies is not an equilibrium because to make firm 1 indifferent between $\rho(k_2)$ and q_1, firm 2 would have to play the pure strategy $\gamma(k_1,k_2)$ which is strictly less than

1 by definition of the area $A_1\backslash A_2$, but this contradicts the fact that equilibrium prices are larger than 1. As k_2 is "large", the default option is never relevant for firm 1 because it involves nearly zero profits. Area $A_2\backslash A_1$ is entirely symmetric.

Comments

The preceding story should not be criticised for its dynamic presentation of the static concept of Nash equilibrium. Beyond showing why there is no equilibrium in pure strategies, it helps to understand the nature of the new equilibrium. When firm 1 plays the pure strategy $\gamma(k_2,k_1)$, if firm 2 perceives a slightly larger price, she replies aggressively for sure while if she perceives a slightly lower price, she plays her default option for sure. Clearly, firm 2 is not throwing a coin to decide on her pricing strategy, she plays a pure strategy that depends (crucially) on her perception of firm 1's price.

This interpretation of mixed strategy equilibria is the purification argument of Harsanyi [73] ; our setting is an example where it fully makes sense. Moreover, the experimental study of Brown-Kruse & al. [94] suggest that disequilibrium adjustment process (called Edgeworth cycle in their paper) or mixed strategy equilibria are the most robust theoretical explanation of the observed pricing pattern in a Bertrand-Edgeworth oligopoly game.

C) Equilibria involving completely mixed strategies

In our economic model, there exists "Edgeworth" cycles but they need not involve a pure strategy for one of the firms. Therefore, we should expect the existence of other equilibria where both firms use mixed strategies ; they are fully characterised in lemmas 4 and 5 of the appendix. Note that their existence is not related to areas A_1 and A_2.

The most important technical characteristic of our model is that, because of the piecewise linearity of the demand functions, firms do not use densities and furthermore the support of their mixed strategies contain the same number of atoms. Therefore an **n** atom equilibrium is a quadruple $\left(p_1^m, p_2^m, \mu_1^m, \mu_2^m\right)^{m\leq n}$ where μ_i^m is the weight put by firm **i** on her m^{th} atom p_i^m (prices are ranked by increasing order). To derive an **n** atom equilibrium of the pricing game $G(k_1,k_2)$, we first solve numerically[6] a system of $2n - 2$ polynomial equations in $2n - 2$ variables and then check two conditions on the vector of prices derived from the system in relation to k_1, k_2 and S i.e., we eliminate some capacity points whose associated candidate equilibria violate one of those conditions.

The symmetry of the problem enables us to limit ourselves to the case where $k_1 > k_2$. The first necessary condition (cf. lemma 5 in appendix) states that $2k_1 - 1 > p_1^m - p_2^m > 2k_2 - 1$ for every atom **m** ; it disqualifies capacity points (k_1,k_2) exhibiting a too large differential. The reduced form of the condition reads $k_2 > g^n(k_1)$ where each g^n is an increasing and convex function. Figure

[6] The skeptical reader will be convinced of the necessity to rely on numerical computations by noting that for a 5 atoms equilibrium, a system of 8 equations involving polynomials of degree 7 with more than 1500 monomials has to be solved.

9 displays those functions for n = 2, 3, 4 and 5, k_1 varying between 0.5 and 0.95 (the limit at $k_1 = 1$ is studied below). As **n** increases, more inequalities have to be satisfied, more capacity points are eliminated and the area where atomic equilibrium exist shrinks ; hence those functions satisfy $g^2 < g^3 < g^4 < g^5$.

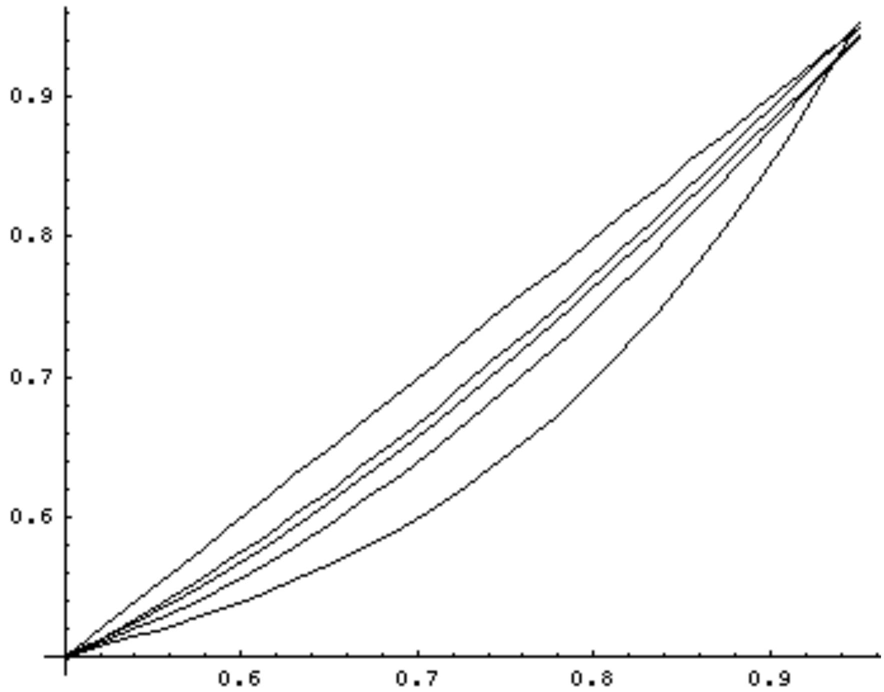


Figure 9

The second necessary condition is related to the upper bound on prices ; it links the upper prices of the distributions to the reservation price by $p_1^n + p_2^n < 2S - 1$. Since the equilibrium prices do not critically depend on the capacity differential but on the total capacity, we study this condition on the diagonal. For a given symmetric capacity choice (k,k), we compute the symmetric candidate equilibrium $\left(p^m(k)\right)^{m \le n}$ and the minimal reservation price for which the condition is satisfied i.e., $S_{min}^n(k) \equiv \frac{2p^n(k)+1}{2}$. The inverse of this function gives us the maximal capacity $K_{max}^n(S)$ such that points (k_1,k_2) with $k_1+k_2 \le 2.K_{max}^n(S)$ have an **n** atoms price equilibrium at the given S. Those functions will be useful in the subsequent section.

As one can expect, the larger the capacities, the larger the prices in a candidate equilibrium. In fact, our computations shows that the upper prices of a candidate equilibrium tend to infinity as capacities tend to (1,1). Now, since lemma 1 showed that prices are bounded by $\rho(k_j) \equiv S - 1 + k_j$, capacity choices around the corner (1,1) have no atomic price equilibria ; for that reason the plot of figure 9 can safely stop at $k_1 = 0.95$. ♦

5) The capacity game

Going backward is difficult in the game G because $G(k_1,k_2)$ has often several price equilibria as shown in proposition 4. The focal subgame perfect equilibrium of this model involves symmetric choices by the firms which happen to be identical to that of a monopoly owner of both firms : the market is shared evenly, there is no excess of capacity, global surplus is maximised and consumer surplus is minimised.

Theorem 1

For almost nil capacity cost, two kinds of SPE coexists
***i)** The total capacity exactly covers the market and each firm enjoys a minimum share; furthermore, the price equilibrium is in pure strategies.*
***ii)** The total capacity exceed the market size, the difference in the capacity choices is limited and the price equilibrium is in completely mixed strategies.*

Proof ***i)*** *Equilibria with exact market coverage*

The following characterisation is valid for any capacity cost ε. Using the symmetry of the game, we take attention to SPE where the choices of capacities are (m,1–m) with m ≥ 1/2. The profit function of firm 1 and 2 is $(S - \varepsilon - k)k$ which apply on the intervals $[0;m]$ and $[0;1-m]$ respectively. We have shown in proposition 2, that this function is increasing up to $\frac{S-\varepsilon}{2}$ which is therefore an upper limit to **m** in order to deter downward deviations. Note that for large S, this limit is in fact not binding.

An upwards deviation $k_1 > m$ by firm 1 can only lead to a price equilibrium of type B or C. There is no possibility for a type A equilibrium because it requires both capacities to be large and since m ≥ 1/2, the capacity choice of firm 2 is smaller than the required limit $\Phi(S,1/2)$ (cf. figure 7 for a graphical explanation and lemma 3 in appendix for the computation of this limit). To deter this upward deviation of firm 1, we define the continuation price equilibrium of $G(k_1,1-m)$ to be of type B in order for the net profit of firm 1 to be $(S-m)m - \varepsilon k_1$; this is a non profitable deviation because of the supplementary cost of capacity. As mixed strategies enable large prices, type C equilibria provide too large payoffs and cannot be used to sustain our candidate SPE.

If $m < \Phi(S,1/2)$, an upwards deviation $k_2 > 1 - m$ by firm 2 can only be followed by a type B or C equilibrium and we apply the same trick as for firm 1 to deter this upward deviation. A problem appears for very large **m**'s because firm 2's payoff is almost nil, hence she has an incentive to deviate to the large capacity $\Phi(S,1/2)$ in order to play the Hotelling price equilibrium and earn a net profit of $1/2 - \varepsilon\Phi(S,1/2)$. Whenever ε is less than half (the relevant condition for the original Hotelling model), the solution of the equation $1/2 - \varepsilon\Phi(S,1/2) = [S - \varepsilon - 1 + m](1-m)$ give us a bound on **m** which is obviously less than one. We may conclude that any sharing of the market is a SPE allocation as long as each firm obtains its "Hotelling profit".

ii) Equilibria involving overlapping capacities when capacity cost is almost nil

Consider a candidate SPE outcome (m_1,m_2). To prove that it is not a SPE, we must consider deviation to (m_1,k_2) or (k_1,m_2) and look at the worst price equilibrium for the deviant ; if the deviation is still profitable then (m_1,m_2) is not a SPE.

Claim If (m_1,m_2) is such that no type C equilibria exists in $G(m_1,m_2)$, then this choice is not part of a SPE.

Proof If the price equilibrium in $G(m_1,m_2)$ is of type A, firms earn a profit independant of their capacity choices. Therefore, each has an incentive to reduce capacity since the cost ε is positive (almost nil is exactly what is needed). If the price equilibrium is of type B, the payoff of one firm, say **i**, in the pricing game is $\Pi^d(m_j)$; by choosing $k_i = 1 - m_j$, firm **i** sets herself in a non overlapping situation and achieves $(S - 1 + m_j)(1 - m_j) = \Pi^d(m_j)$ with a lower cost of capacity installation, thus she will deviate.

This artifice is our main instrument to rule out "unwanted" equilibria ; we also obtain a first result : there are no SPE of type **ii)** where the capacity point (k_1,k_2) fall outside of the lens displayed on figure 9 above.

We now build an SPE with capacity choices (m_1,m_2) such that the equilibrium of $G(m_1,m_2)$ is an **n** atom one. This couple must satisfy $m_2 > g^n(m_1)$ (by symmetry of G, we can always assume $m_1 < m_2$) and $m_1 + m_2 < 2K^n_{max}(S)$; those conditions give an upper bound on capacities. We now define the strategies out of the equilibrium path : at (k_i,m_j), we define pricing strategies to be the pure strategy $\gamma(k_i,m_j)$ for firm **j** while firm **i** mixes between $\rho(m_j)$ and the lower price $\frac{1+\gamma(k_i,m_j)}{2}$ (type B equilibrium). Firm **i** obtains $\Pi^d(m_j)$ and to deter the deviation k_i, it must be less than $\Pi^n(m_1,m_2)$, the profit accruing to firm **i** at the **n** atom equilibrium.

Since this latter function mostly depends on the total capacity, we may study this condition on the diagonal. We thus solve $\Pi^n(k,k) - \varepsilon k \geq [S - (1-k) - \varepsilon](1-k)$ in the variable S to get $S \leq S^n_{max}(k,\varepsilon) \equiv \frac{\Pi^n(k,k) - \varepsilon k}{1-k} + 1 - k + \varepsilon$. The numerical computation is performed for ε = 0 as we are studying the case of almost nil capacity cost. Then, we can invert $S^n_{max}(k,0)$ to obtain a lower bound $K^n_{min}(S)$ on capacities which is compatible with the upper bound $K^n_{max}(S)$ derived in proposition 4 (the above simplification is therefore valid up to small numerical errors).

Contrarily to type **i)** SPE, the capacity combinations that appear as SPE of type **ii)** depend on S. Figure 10 summarises our result : the various $K^n_{min}(S)$ and $K^n_{max}(S)$ functions are plotted for n = 2, 3, 4 and 5. Consider for example S = 4. There exists a symmetrical[7] SPE with a 2 atoms price equilibrium if the capacity is between .81 and .85 and an SPE with a 3 atoms price equilibrium if the

[7] Whenever a symmetrical n-atom equilibrium exists, there also exists asymmetrical ones for all capcity choices with the same mean and satisfying $m_i > g^n(m_j)$ i.e., the capacity point must lie in the lens of figure 9.

common capacity is between .6 and .61. For S = 6, there exists SPE with 2, 3, 4 or 5 atoms price continuation equilibria. ♦

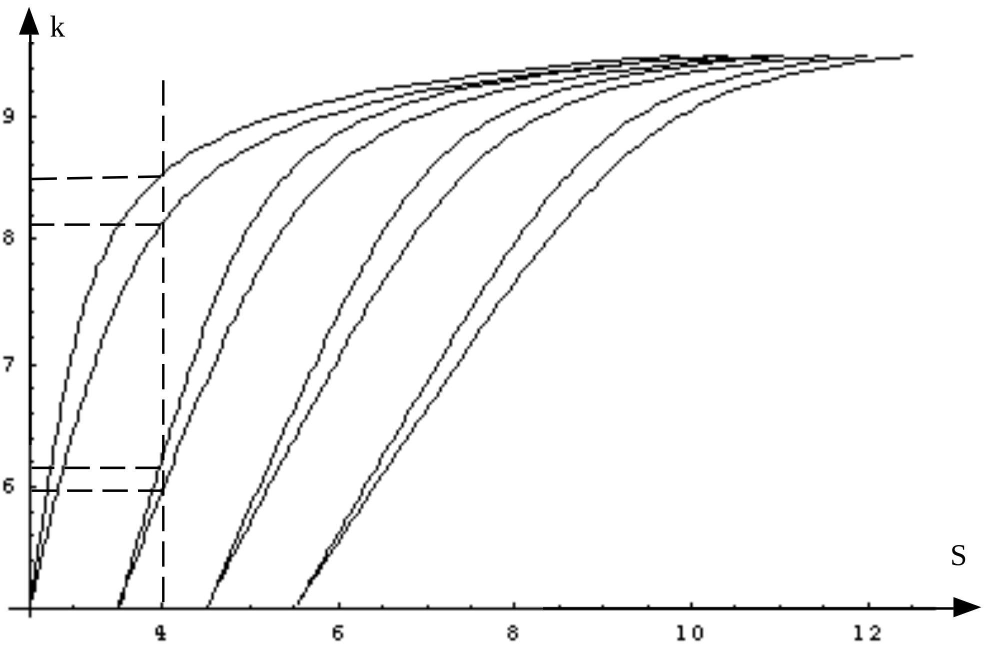


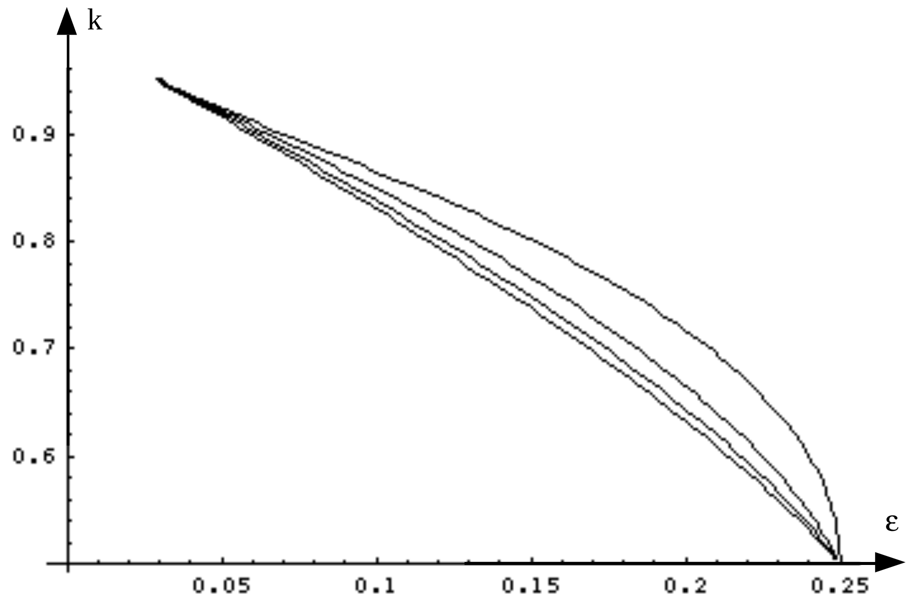
Figure 10

Theorem 2

If the cost for capacity is larger than 1/4 of the transportation cost, then in all SPE, the market is exactly covered by the capacity choices of the firms.

Proof We have shown that a type **ii)** SPE exists for the symmetric capacity **k** only if $S^n_{min}(k) \le S \le S^n_{max}(k,\varepsilon)$. As the latter function is decreasing in ε, $S^n_{min}(k) = S^n_{max}(k,\varepsilon)$ has a solution $\varepsilon^n(k)$ and for any $\varepsilon > \varepsilon^n(k)$, the candidate SPE is removed. The equation to solve is : $S^n_{min}(k) = \dfrac{\Pi^n(k,k) - \varepsilon k}{1-k} + 1 - k + \varepsilon \Rightarrow \varepsilon^n(k) \equiv \dfrac{\Pi^n(k,k) - \left[S^n_{min}(k) - (1-k)\right](1-k)}{2k-1}$.

The ε^n functions satisfy $\varepsilon^2 > \varepsilon^3 > \varepsilon^4 > \varepsilon^5$ and are plotted with reversed axes on figure 11 below. It is clear that for every ε > 1/4, no type **ii)** equilibria remains. ♦



<u>Figure 11</u>

Note that in the SPE's involving exact market coverage, both firms are on their monopolist's profit curve. This perfectly illustrates how firms may benefit from capacity precommitment. Indeed, the basic feature of the Hotelling model lies in the fact that firms enjoy local monopolies around their locations. However, in the absence of capacity constraints, they cannot prevent price competition to take place. Although positive mark-ups are preserved in equilibrium, price competition is damaging to the firms. This is clearly seen by observing that in the Hotelling equilibrium, prices do not depend on S. In other words, firms fail to capture a large part of the consumers' surplus.

The main virtue of capacity precommitment is precisely to avoid this failure. Indeed, through capacity precommitment, firms are now able to capture the greatest part of the consumers' surplus. In the most "natural" equilibrium in which both firms commit to a capacity of 1/2, they sell exactly the same quantity than in the original Hotelling equilibrium, but at much higher prices. In particular, their payoffs now depend positively on S. This is so because they move along their local monopoly profit curve which is increasing in the reservation price S.

The existence of equilibria involving excess capacities is mainly due to the existence of multiple equilibria in the price subgame where firms fight for market shares. However, it remains true that in these equilibria, prices are always above the Hotelling prices and increase with S. The corresponding payoffs are also positively linked to the reservation price. Finally, theorem 2 shows that excess capacity is profitable only if installation costs are low.

Thus, *whatever the equilibrium considered, we are led to conclude that capacity precommitment enables firms to take advantage of the most profitable feature of the industry which is its local monopoly structure.*

6) Cournot Competition

Kreps & Scheinkman [83] established that firms tend to avoid destructive price competition through capacity precommitment, in the market for an homogenous product. In theorem 1 and 2, we have extended their result to the case of horizontal differentiation. Although product differentiation by itself relaxes price competition, we have shown that firms still have an incentive to relax it further through capacity precommitment. The nicest feature of the Kreps & Scheinkman [83] result is that it provides a theoretical foundation for Cournot competition which allows for an explicit price mechanism. We now show that the same is true for horizontally differentiated products.

Theorem 3

The equilibrium allocations of the Cournot competition in the Hotelling model are those of the capacity precommitment game for $\varepsilon > 1/4$.

Proof In the Cournot game, firm supply quantities q_1 and q_2 to an otherwise competitive market. If the proposed quantities q_1 and q_2 do no cover the market, there is excess demand and the prices increase until supply equal demand on each side of the market i.e., $q_1 = S - p_1$ and $q_2 = S - p_2$. As already studied in proposition 2, this situation is unstable since at least one firm has an incentive to increase its quantity above the complement of the other. If now the proposed quantities q_1 and q_2 exceed the market size, there is excess supply and at least one of the price, say p_1, must be nil on this competitive market. Therefore firm 1 has a profitable deviation by offering a quantity slightly less than $1 - q_2$ to be on its monopoly profit curve.

The remaining candidates for a Cournot equilibrium are $(q, 1 - q)$ with $q \leq 1/2$. The competitive[8] prices are $S - q$ and $S - 1 + q$. Without loss of generality, firm 1 offers **q**, thus sells less than 1/2 in equilibrium. Hence, p_1 cannot be nil because it would attract at least one half of the consumers, thereby implying an excess demand. Firm 2 cannot profitably deviate to a larger quantity than $1 - q$ because she would face a zero price (one price is nil and by the preceding argument, it must be her's).

Firm 1 may however profitably deviate to some Q larger than q but still less than 1/2. Since there is excess supply, p_2 is nil, thus firm 1 sells all of Q and the consumer located at $x = Q$ must indifferent in equilibrium which means that $p_1 = 1 - 2Q$. The profit $Q(1 - 2Q - \varepsilon)$ reaches a maximum of $\frac{(1-\varepsilon)^2}{8}$ at $\frac{1-\varepsilon}{4}$ to be compared with $q(S - q - \varepsilon)$. Since $q \leq 1/2$, the only relevant root is $q^* \equiv \frac{2S-2\varepsilon-\sqrt{2(S-\varepsilon)^2-(1-\varepsilon)^2}}{4} > 0$. Finally, the Cournot equilibria feature exact market coverage $(q, 1 - q)$ with q larger than this lower bound[9] q^*.

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[8] For exact market coverage, there exists a continüm of prices which clear the market and they need not be the highest possible ones (cf. the forthcoming Grilo & Mertens [97] for a foundation of our price selection).

[9] This lower bound is different from that derived in theorem 1 but both are small so that our equivalence applies for the most likely sharings of the market.

When there exists a demand operator $\begin{cases} D_1(p_1,p_2)=\tilde{x}(p_1,p_2) \\ D_2(p_1,p_2)=1-\tilde{x}(p_1,p_2) \end{cases}$, d'Aspremont & Motta [94] define a Cournot equilibrium as a price and a quantity vectors (p^c,q^c) such that for each firm **i**, $(p^c,q^c)\in \arg\max(p_i-\varepsilon)q_i$ over $\left\{p\geq 0, 0\leq q_i\leq D_i(p), q_j=D_j(p)\right\}$. In our setting, the constraint $q_j=D_j(p)$ means that firm **i** optimises over the complement to firm **j**'s capacity. Again, according to this alternative definition, all SPE's involving exact market coverage are Cournot equilibria. ♦

7) Final comments

Let us discuss now the main assumptions we have made.

i) It is well known that the nature of the rationing rule plays a central role in pricing models with capacity constraints. For instance, Davidson & Deneckere [86] show that the result of Kreps & Scheinkman entirely rests on their assumption of efficient rationing. It is however intuitive that this is not the case in the present analysis. Indeed, any alternative to the efficient rule would result in a lower residual demand addressed to the "high" price firm. However, the local monopoly structure of the model does not depend on the rationing rule. Therefore, it is intuitive that other rationing rules would yield more stability into price competition and would make exact market coverage equilibria more likely. Moreover, in our setting, the efficient rationing rule may be considered as a rather natural one if one views the Hotelling model as a spatial model. In this case, it basically amounts to organise rationing on a "first arrived-first served" basis.

ii) We consider a market in which the location of firms are fixed at the extremities of the market. As mentioned previously, this assumption was motivated by its implications for price competition. Since product differentiation is maximised, one could think that this is the case where the firms have the lowest incentives to further relax price competition. The robustness of our result to alternative location patterns is not easy to trace. In particular because no pure strategy equilibrium exists in the Hotelling model without capacity constraints when firms are located inside the first and third quartiles. (see Osborne & Pitchick [87] for a characterisation of mixed strategies equilibria). Two remarks are in order here.

First, it should be noted that the presence of capacity constraints may help to restore the Hotelling equilibrium for locations inside the first and third quartiles (see Wauthy [96]). At the same time, inside locations will tend to make upward deviations less profitable, because it could imply less favourable residual demands. We may therefore suspect that with inside locations, there is less scope for excess capacity choices whereas exact market coverage remains most attractive.

Second, under quadratic transportation costs, maximal differentiation has been shown to be optimal for the firms. Although our results has been derived using linear transportation costs, it is clear their qualitative features do not depend on this particular assumption.

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Appendices

Lemma 3

The cut-off between aggressive pricing and monopoly behaviour.

Proof The profit associated with the modified Hotelling best reply $\text{Max}\{H(.), a(k_1,.)\}$, is $\begin{cases} \underline{\Pi}_1(p_2) \equiv \dfrac{(1+p_2)^2}{8} & \text{if } p_2 < \alpha(k_1) \\ \overline{\Pi}_1(p_2) \equiv k_1[p_2 + 1 - 2k_1] & \text{if } p_2 \geq \alpha(k_1) \end{cases}$ while that associated with the security strategy $\rho(k_2)$ is $\Pi^d(k_2) \equiv [S - 1 + k_2](1 - k_2)$.

Since this latter function is decreasing, we have $k_2 > k'_2$ on figure 11 below.

The best reply of firm 1 to a low price p_2 is the default option $\rho(k_2)$ and above the threshold **x**, the optimal play becomes H(.). For the smaller capacity k'_2, the cut-off value is **y** and the optimal play above **y** is $a(k_1,.)$.

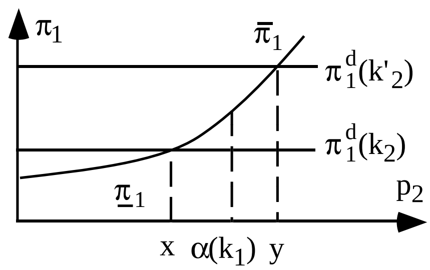

Figure 11

The solution of $\Pi^d(k_2) = \underline{\Pi}_1(p_2)$ is $x \equiv \sqrt{8[S - 1 + k_2](1 - k_2)} - 1$, while that of $\Pi^d(k'_2) = \overline{\Pi}_1(p_2)$ is $y \equiv \dfrac{[S - 1 + k'_2](1 - k'_2)}{k_1} - 1 + 2k_1$.

Observe first that if S is too small then **x** is negative and the security strategy is never used ; second the bound **x** is useful for large values of k_2 as Π^d is decreasing. Thus we have to solve in k_2 the equation $x \leq \alpha(k_1) \Leftrightarrow \sqrt{8[S - 1 + k_2](1 - k_2)} - 1 \leq 4k_1 - 1$ which leads to $k_2 \geq \Lambda(S, k_1) \equiv \dfrac{2 - S + \sqrt{S^2 - 8k_1^2}}{2}$ having eliminated the negative solution. Letting $\gamma(k_1, k_2) \equiv \begin{cases} y & \text{if } k_2 < \Lambda(S, k_1) \\ \max\{0, x\} & \text{otherwise} \end{cases}$, we obtain the best reply function of firm 1 as $BR(p_2) \equiv \begin{cases} \rho(k_2) & \text{if } p_2 \leq \gamma(k_1, k_2) \\ \text{Max}\{H(p_2), a(k_1, p_2)\} & \text{if } p_2 > \gamma(k_1, k_2) \end{cases}$.

For the complete characterisation of the pricing game, we need to solve $\gamma(k_1,k_2) < 1$ i.e., $\{k_2 < \Lambda(S,k_1) \text{ and } y < 1\}$ or $\{k_2 \geq \Lambda(S,k_1) \text{ and } x < 1\}$.

When $\gamma(k_1,k_2)$ = y, we solve 1 > $2k_1 - 1 + \dfrac{[S-1+k_2](1-k_2)}{k_1}$ and we get $k_2 > \Phi(S,k_1) \equiv \dfrac{2-S+\sqrt{S^2-8k_1(1-k_1)}}{2}$.

Note that $\Phi(S,k_1) < k_2 < \Lambda(S,k_1)$ make sense for $k_1 < 1/2$.

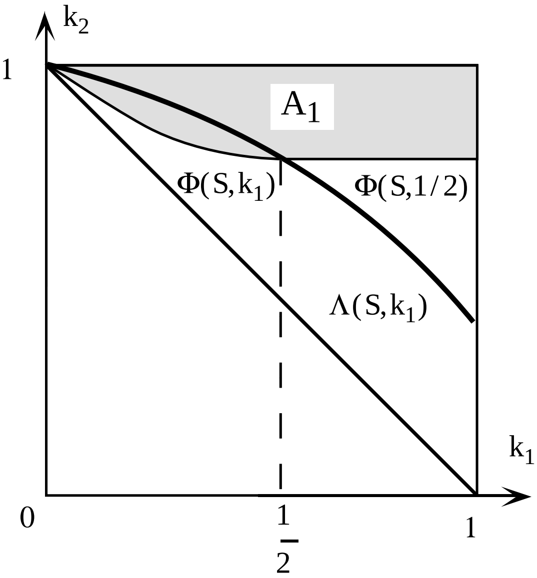

Figure 12

When γ = x, we solve $\sqrt{8[S-1+k_2](1-k_2)} - 1 < 1$ and we obtain $k_2 > \Phi(S,1/2)$. The combination of the two sets of solutions leads to the shaded area of figure 12. Note also that $S > \frac{3}{2} \Rightarrow S^2 - 2 \geq (S-1)^2 \Rightarrow 2 - S + \sqrt{S^2-2} \geq 1 \Rightarrow \Phi(S,1/2) \geq \frac{1}{2}$ which is used in theorem 1 part **i)**.

♦

Lemma 4

In a mixed equilibrium firms use the same number of atoms and the diagonal points lie in the band.

Proof Let F_i be the cumulative function of the mixed strategy used by firm **i** in equilibrium. Its support is included in $[1;\rho(k_j)]$. The choice of α on figure 13 is made in order that $D_2(p_1,.)$ is decreasing and concave over $[0;\alpha]$ for any price p_1 (we want to avoid the lower triangle where D_2 is again constant which precludes the concavity of Π_2). Over $[0;\alpha]$, every profit function $\Pi_2(p_1,.)$ is concave, so is their average $\Pi_2(F_1,.)$ and thus the best reply of firm 2 to F_1 in the interval $[0;\alpha]$ is unique which means that F_2 possesses at most one atom $\underline{p}_2$ in $[0;\alpha]$.

Now, ω is chosen so that $D_1(p_2,.)$ is non-increasing over $[0;\omega]$ for any price p_2 larger than α (this time, we want to avoid the upper triangle). Thus the average of the profit function Π_1 conditional on $p_2 \neq \underline{p}_2$ is concave.

$\Pi_1(\underline{p}_2,.)$ may be non-concave over $[0;\omega]$ since the shape of the demand function changes at $p_1 = \upsilon$, however it is concave over $[0;\upsilon]$ and over $[\upsilon;\omega]$.

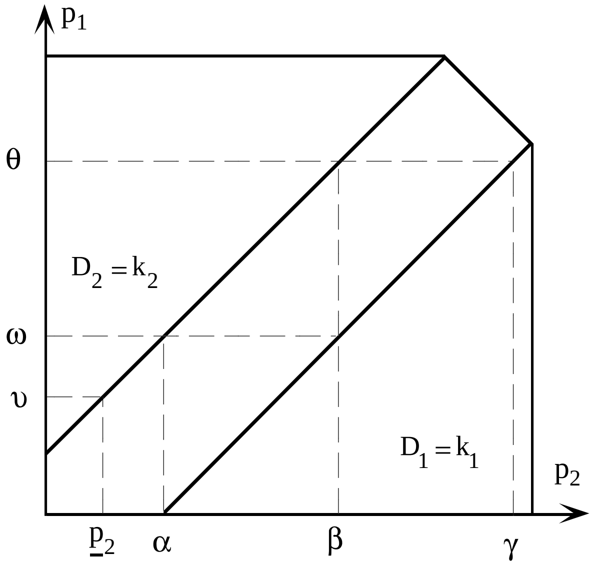

Figure 13

Thus, $\Pi_1(F_2,.)$ as the average of the two previous profit functions, is concave over $[0;\upsilon]$ and over $[\upsilon;\omega]$ and has a unique maximum on each interval. We conclude that the best reply of firm 1 over $[0;\omega]$ possesses at most two atoms $\underline{p}_1$ and $\overline{p}_1$.

We will now repeat this argument. Observe that β is such that $D_2(p_1,.)$ is non-increasing over $[0;\beta]$ for any price p_1 larger than ω, thus the average of the profit function Π_2 conditional on $p_1 \neq \underline{p}_1$ and $\overline{p}_1$ is concave. The existence of those two atoms means that $\Pi_2(F_1,.)$ has at most 3 maximisers on $[0;\beta]$ so that F_2 possesses at most 3 atoms over $[0;\beta]$. Proceeding in the same way, we see that F_1 has at most 4 atoms over $[0;\theta]$, that F_2 has at most 5 atoms over $[0;\gamma]$. This process eventually[10] reaches the limits $\rho(k_1)$ and $\rho(k_2)$. Hence, we have proven that a mixed strategy equilibrium involves a finite number of atoms for each firm.

To prove the second part of the lemma, we use the table of points formed by the distributions $\left(p_1^m\right)_{m\geq 1}$ and $\left(p_2^n\right)_{n\geq 1}$ (cf. figure 14) ; we speak of lines when p_1 is fixed, of columns when p_2 is fixed and of the "diagonal" for the pairs $\left(p_1^m, p_2^m\right)_{m\geq 1}$.

We claim that the pair $\left(\underline{p}_1, \underline{p}_2\right)$ of minimal atoms lie in the band (cf. point α on figure 14). Indeed, if $\underline{p}_1 \leq \underline{p}_2 - 1 + 2k_1$, the whole line $\left(\underline{p}_1, p_2^m\right)_{m\geq 1}$ lies under the band and we get $D_1(F_2,.) = k_1$, thus $\Pi_1(F_2,.)$ is locally increasing which means that $\underline{p}_1$ cannot be part of an equilibrium. Likewise, if $\underline{p}_1 \geq \underline{p}_2 + 1 - 2k_2$, the column $\left(p_1^m, \underline{p}_2\right)_{m\geq 1}$ would lie in the area where $D_2 = k_2$ and $\Pi_2(F_1,.)$ would be locally increasing at $\underline{p}_2$.

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[10] Indeed, we are in the domain where $k_1 + k_2 > 1$ which means that the band is non void.

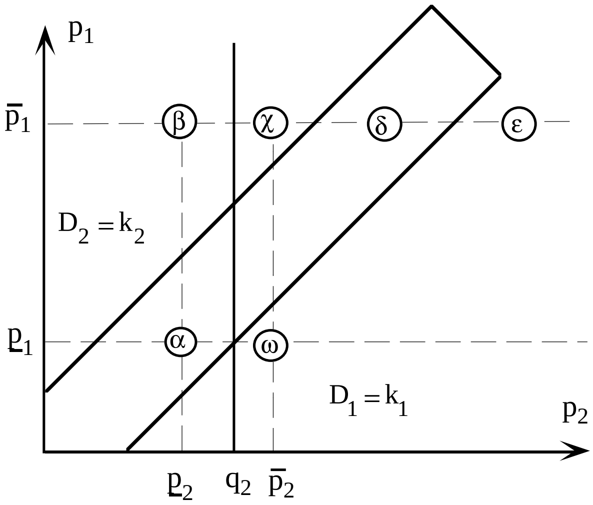

Figure 14

The same reasoning enables to show that any line and any column has a point in the band. Whatever the position of the atoms above α in the α–β column, $\Pi_2(F_1,.)$ is concave up to $q_2 \equiv \underline{p}_1 + 1 - 2k_1$. Knowing that this function has a unique maximiser over $[0;q_2]$ and that it is precisely $\underline{p}_2$, the second atom of F_2, $\bar{p}_2$, must be larger than q_2 so that $\omega = \left(\underline{p}_1, \bar{p}_2\right)$ is under the band. Likewise $\beta = \left(\bar{p}_1, \underline{p}_2\right)$ is above the band, however, $\left(\bar{p}_1, \bar{p}_2\right)$ could be either χ, δ or ε. If it were χ, then the ω–χ column would have no point in the band and $\Pi_2(F_1,.)$ would be locally increasing at $\bar{p}_2$. Likewise, if it were ε, the β–ε line would have no point in the band. Repetition of this argument shows that for every integer **m**, $\left(p_1^m, p_2^m\right)$ lies in the band. As a corollary, there must be the same number of atoms in each distribution. ♦

Lemma 5

*The nature of the equilibria with **n** atoms in the pricing game.*

Proof Consider the distributions $\left(p_1^m, \mu_1^m\right)_{m \le n}$ and $\left(p_2^m, \mu_2^m\right)_{m \le n}$ of an equilibrium with **n** atoms on each side. By the preceding lemma, $\Pi_1(F_2, p_1^j) = p_1^j \left[(1-k_2) \sum_{m<j} \mu_2^m + \mu_2^j \frac{1 - p_1^j + p_2^j}{2} + k_1 \sum_{m>j} \mu_2^m \right]$ and the FOC of local optimality is

$\frac{\partial \Pi_1(F_2, p_1^j)}{\partial p_1^j} = 0 \quad \Leftrightarrow \quad 2(1-k_2) \sum_{m<j} \mu_2^m + \mu_2^j (1 - 2p_1^j + p_2^j) + 2k_1 \sum_{m>j} \mu_2^m = 0$. Remark that at the optimum, $\Pi_1(F_2, p_1^j) = \frac{\mu_2^j (p_1^j)^2}{2}$. The same first order condition for firm **2** at her j^{th} atom leads to the following system with the constants M_1 and M_2 adequately defined :

$$\begin{cases}\mu_2^j - 2\mu_2^j p_1^j + \mu_2^j p_2^j + M_1 = 0 & \text{multiply by } 2\mu_1^j \\ \mu_1^j - 2\mu_1^j p_2^j + \mu_1^j p_1^j + M_2 = 0 & \text{multiply by } \mu_2^j\end{cases}$$

$$\Leftrightarrow \quad \begin{cases}2\mu_1^j\mu_2^j - 4\mu_1^j\mu_2^j p_1^j + 2\mu_1^j\mu_2^j p_2^j + 2\mu_1^j M_1 = 0 \\ \mu_2^j\mu_1^j - 2\mu_1^j\mu_2^j p_2^j + \mu_2^j\mu_1^j p_1^j + \mu_2^j M_2 = 0\end{cases}$$

$$\Rightarrow \quad 3\mu_1^j\mu_2^j + 2\mu_1^j M_1 + \mu_2^j M_2 = 3\mu_1^j\mu_2^j p_1^j$$

$$\Rightarrow \quad p_1^j = 1 + \frac{2M_1}{3\mu_2^j} + \frac{M_2}{3\mu_1^j}$$

$$\Rightarrow \quad p_1^j = 1 + \frac{4}{3}\frac{(1-k_2)\sum\limits_{m<j}\mu_2^m + k_1\sum\limits_{m>j}\mu_2^m}{\mu_2^j} + \frac{2}{3}\frac{(1-k_1)\sum\limits_{m<j}\mu_1^m + k_2\sum\limits_{m>j}\mu_1^m}{\mu_1^j}$$

We also have the **n** corresponding equations for the prices charged by firm 2. Having eliminated the prices, the number of unknowns is reduced from 4n to 2n. Since $\mu_1^n = 1 - \sum\limits_{i<n}\mu_1^i$ and $\mu_2^n = 1 - \sum\limits_{i<n}\mu_2^i$, we can use the vectors of 2n − 2 unknowns $u \equiv \begin{pmatrix}\mu_1^i \\ \mu_2^j\end{pmatrix}_{\substack{i<n \\ j<n}}$ and the 2n − 2 equations system is obtained by equating profit for each firm at each of the atoms she plays in equilibrium i.e. $0 = X(u) \equiv \begin{pmatrix}\mu_2^n(p_1^n)^2 - \mu_2^i(p_1^i)^2 \\ \mu_1^n(p_2^n)^2 - \mu_1^j(p_2^j)^2\end{pmatrix}_{\substack{i<n \\ j<n}}$. The equality between the number of unknowns and the number of equations tells us that there is a finite number of solutions. Except for the 2-atoms case (cf. lemma 4), we have not been able to check unicity, thus the following algorithm is only an equilibrium selection, not the equilibrium correspondence.

It must be noted that all equations are fractional and can thus be reduced to polynomial equations with a maximum exponent of 7 (independently of **n**). Furthermore, if we count k_1 and k_2 as variables, each equation contains 60 monomials for n = 2, 247 for n = 3, 686 for n = 4 and 1533 for n = 5. It must be noted that even for n = 2, the *Mathematica* software is not able to solve this polynomial system. We have therefore programmed an algorithm for this purpose. Since p_1^n is proportional to $\frac{1}{\mu_2^n}$, profit $\mu_2^n(p_1^n)^2$ decreases with μ_2^n, thus by choosing μ_1^n and μ_2^n nearby 1 i.e. **u** near 0, we obtain X(u) << 0. The first order Taylor expansion of the differentiable function X is X(u+du) = X(u) + dX.du where dX is the jacobian of X evaluated at **u**. We approach a solution by following the path of optimal growth i.e., we choose **du** = - $\delta.(dX)^{-1}.X(u)$ where δ is chosen to enable a rapid but certain convergence of the numerical computation. ♦

Lemma 6

A more thorough analysis of the two atoms price equilibrium

Proof For a two atoms equilibrium with prices $\left(\underline{p}_i, \overline{p}_i\right)$ and probability distribution $(\mu_i, 1-\mu_i)$, the system of the previous lemma is $3\,\underline{p}_i = 3 + 4\dfrac{1-\mu_j}{\mu_j}k_j + 2\dfrac{1-\mu_i}{\mu_i}k_i$ (E3)

and $3\,\overline{p}_i = 3 + 4\dfrac{\mu_j(1-k_i)}{1-\mu_j} + 2\dfrac{\mu_i(1-k_j)}{1-\mu_i}$ (E4)

Letting $\beta_i \equiv \dfrac{1-\mu_i}{\mu_i}$, (E3) reads $3\,\underline{p}_i = 3 + 4\beta_j k_j + 2\beta_i k_i$ and defines a function $G(\beta_i, \beta_j, k_i, k_j)$ and while (E4) becomes $3\,\overline{p}_i = G\left(\dfrac{1}{\beta_i}, \dfrac{1}{\beta_j}, 1-k_j, 1-k_i\right)$. The necessary equality of the profits $\Pi_i(F_j, \underline{p}_i) = \dfrac{\mu_j(\underline{p}_i)^2}{2}$ and $\Pi_i(F_j, \overline{p}_i) = \dfrac{(1-\mu_j)(\overline{p}_i)^2}{2}$ simplifies to

$$\underline{p}_i = \overline{p}_i\sqrt{\beta_j}$$

$$\Leftrightarrow \qquad 3 + 4\beta_j k_j + 2\beta_i k_i = \sqrt{\beta_j}\left(3 + 4\frac{1-k_i}{\beta_j} + 2\frac{1-k_j}{\beta_i}\right)$$

$$\Leftrightarrow \qquad 2k_i[\beta_i]^2 + \left[3 + 4\beta_j k_j - 3\sqrt{\beta_j} - 4\frac{1-k_i}{\sqrt{\beta_j}}\right]\beta_i - 2\sqrt{\beta_j}(1-k_j) = 0$$

$$\Leftrightarrow \qquad A[\beta_i]^2 + B[\beta_i] + C = 0$$

$$\Rightarrow \qquad \beta_i = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \equiv f(\beta_j, k_i, k_j),\ C < 0 \Rightarrow \text{only one positive solution}$$

By symmetry for firm **j**, we get $\beta_j = f(\beta_i, k_j, k_i)$. It is now clear that an equilibrium of the pricing game is a fixed point of $f\left(f(., k_i, k_j), k_j, k_i\right)$. Since $\underline{p}_i < \overline{p}_i$ and profits are equal, it must be true that $\mu_i > 1/2$, thus $\beta_j < 1$ and symmetrically $\beta_i < 1$. Those supplementary conditions are helpful to analyse the large capacity case. Observe that, independently of the capacities, if β_j tends to 0, the second degree equation tends to $2k_i[\beta_i]^2 - 4\dfrac{1-k_i}{\sqrt{\beta_j}}[\beta_i] = 0$. Its positive solution diverges and by symmetry, we obtain $f(\beta_i, k_j, k_i) \xrightarrow[\beta_i \to 0]{} +\infty$. Now, since β_j is bounded, $C = -4(1-k_j)\sqrt{\beta_j} \xrightarrow[k_j \to 1]{} 0 \quad \Rightarrow \beta_i = f(\beta_j, k_i, k_j) = \dfrac{-B + \sqrt{B^2 - 4AC}}{2A} \xrightarrow[k_j \to 1]{} 0$. Plugging this in the preceding result, we see that $\beta_j = f(\beta_i, k_j, k_i)$ diverges, a contradiction with the constraint $\beta_j < 1$.

We can therefore undertake numerical computations without worrying about the behaviour at the corner (1,1). As **f** is analytically known, we have been able to compute the roots of $f\left(f(., k_i, k_j), k_j, k_i\right)$ for a lattice of capacities such that $k_i + k_j > 1$; it appears that this function is always decreasing, thus there is at most one equilibrium. Moreover, the conditions provided by lemma 4 enable us to eliminate couples with high capacity differential ; figure 15 plots the lower contour curve of the capacities area where the equilibrium exists.

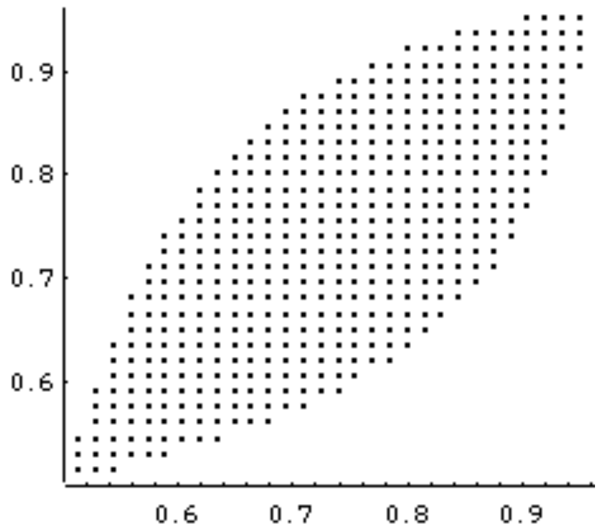

Figure 15

Our computations show that the upper prices $\overline{p}_i$ and $\overline{p}_j$ increase with capacities so that the condition $\overline{p}_i \leq \rho(k_j) = S - 1 + k_j$ is violated for large capacities i.e., the point $\left(\overline{p}_i, \overline{p}_j\right)$ leaves the "band" as described in lemma 2. Consequently, atomic equilibrium will never exist for capacities around 1. ♦