DEFAULT RISK IN ASSET PRICING

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Abstract

This paper provides an analytical solution for the impact of default risk on the valuation of realistically intricate claims on time dependent uncertain income streams. Its modular structure allows us to adjust the set of assumptions concerning the event of default to the specificity of the environment which surrounds the asset. The importance of such a flexibility is illustrated in the context of corporate debt, examining the simplest case of finite lived coupon paying corporate bonds with principal repayment at maturity. The magnitude of risk premia, as well as the term structure of credit spreads, are not surprisingly largely determined by the assumed default scenario.

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1 Introduction

In recent years, a contingent-claims-based approach to asset valuation has become an important tool in incorporating the effects of real options affecting the life of assets. These options are typically decisions that an agent is expected to take as the state of the world evolves. Such open decisions largely affect the value of the assets held by the different principals. The most natural example, substantially discussed in the literature, is the decision to default on contracted obligations. Although this risk is very influential, it has proved difficult, for sufficiently realistic contracts, to assess the impact of default risk on asset valuation.

This paper develops, within a continuous time framework, closed form solutions for the value of a wide range of contracts, taking into account the possibility of default. The exercise is relatively simple for a perpetual entitlement to a time homogeneous payoff function of the state variable.[1] However, most claims are not infinitely lived. Furthermore, attached payoffs are not just state dependent, but they contractually (or are expected to) depend on time.

Our contribution is to solve for the value of the asset not only when it has an overall finite maturity, but also when the payoffs are intricate functions of an economic fundamental and time. The contracts we consider can possibly involve (i) a series of state dependent streams of income flow, each applicable for specified lengths of time. This encompasses a share on an earnings flow, a complex interest payment scheme, or simply a fixed coupon. The claim can additionally include (ii) series of lump sum payments expected to occur at different dates, such as expected dividend payments, or principal repayments.

Our framework gives a rich tool that may be applied to many financial assets. The methodology employed is fairly simple. The influence of the different terms of the contract remains easily separable, which enables the intuition to be developed. Furthermore, the model flexibly allows for different descriptions of the environment surrounding the asset.

The modular structure of the framework allows us to separately adjust the set of assumptions which describe the expected triggering of default to the specificity of the environment which surrounds the asset. Although the finance literature is very silent on this issue, we ob-

[1] Black and Cox (1976) derived the pricing of perpetual interest paying bonds. Leland (1994) extended the results in a capital Bartolini and Dixit (1991) employed similar models to price the secondary market value of perpetual sovereign debt when the country has a limited ability to pay. Fries, Mella-Barral and Perraudin (1997) applied these techniques in banking, to determine the fair pricing of deposit guarantees.

tain that, not surprisingly, the conjecture concerning the expected scenario triggering default drastically modifies the asset value.

We apply the method in the context of corporate finance considering the simplest possible contract. We discuss the implementation of the different descriptions of the event of bankruptcy encountered in the literature. Here, our purpose is to illustrate how differences in parametrisation of the event of bankruptcy affect not only the valuation of bonds, but also the implied term structure of credit spreads.

The simple contract we examine is a debt contract with a fixed coupon, hence a time homogeneous payoff constant payoff, which is promised until the lump-sum full repayment of the principal at maturity. In this base case, some terms in our solution somewhat resembles the expression of Leland and Toft (1996) who considered this type of contract for the purpose of an optimal capital structure model.[2] The bottom line result of our analysis is that the magnitude of risk premia, as well as the term structure of credit spreads, are largely determined by the initially expected bankruptcy scenario. Adjusting the expected bankruptcy scenario to the specificity of the firm and its economic context is therefore of crucial importance.

The paper is organised as follows. Section 2 defines the general class of assets we consider and provides the pricing formula. Section 3 applies the framework to the case of fixed coupon paying corporate bonds with principal repayment at maturity, and Section 4 illustrates the flexibility of the method, implementing two different possible descriptions of the same firm. We then examine the impact on the resulting term structure of credit spreads and carry out a sensitivity analysis. Section 5 concludes suggesting extensions of the model.

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[2] Brennan and Schwartz (1978) use numerical methods to derive the value of coupon paying corporate bonds and optimal capital structures. Similarly, Kim, Ramaswamy and Sundaresan (1993) and Longstaff and Schwartz (1995) have examined risky finite-lived debt with both a stochastic process for the total value of the firm and the short term riskless interest rate.

2 Pricing the Risk of Default

Let x_t designate a state variable which reflects economic fundamentals and captures all the uncertainty affecting the valuation of the asset we consider. We assume that the dynamics of x_t follow a geometric Brownian motion

$$dx_t \quad = \quad x_t\,\mu\,dt \;+\; x_t\,\sigma\,dB_t, \qquad \text{where } \mu,\, \sigma \text{ are constants.} \tag{1}$$

Consider an asset $\mathcal{A}$, which can be defined in terms of

1. An entitlement to a series of K lump sum payments $\{p_k(x)\}$, expected to respectively occur at dates $\{T_1, ...T_K\}$. This can be a series of expected dividend payments, or principal repayments. These payments are function of the economic fundamental, in order to reflect, for example, the influence of an expected dividend policy.

2. An entitlement to a unit-period cash flow function $a(x,t)$. This can be a share on an earnings flow, a complex interest payment scheme or simply a fixed coupon. This cash flow function possibly encompasses a series of state dependent streams of income flow, each applicable for specified lenghts of time. That is there exists J "starting" dates $\{T_1', ...T_J'\}$, and J "finishing" dates $\{T_1'', ...T_J''\}$, such that the $a(x,t)$ takes a different form of payoff in each one of the J intervals: $a(x,t) = a_j(x)\ ,\ \ \forall t \in (T_j'; T_j'']$.

3. Default corresponds to an interruption of these payments. This event is expected to be triggered the first time the economic fundamental hits a threshold level $\underline{x}$. The value of the asset in default is denoted $\underline{V}(x)$ which is a function of the state of the world.

The contract has an overall maturity date T, which is just the largest of all previously cited payment dates: $T = \max\{T_J''; T_K\}$. Figure 1 provides a graphical interpretation of the time dependency of expected payoffs in such contracts.

First Step: To start with, denote $\overline{A}_j(x_t|t : +\infty)$, the value of a default riskless perpetual flow of income $a_j(x)$, discounting at the short-term riskless interest rate r,[3]

$$\overline{A}_j(x_t|t : +\infty) \quad \equiv \quad E_t \int_t^{+\infty} e^{-r\tau}\ a_j(x_\tau)\ d\tau\ . \tag{2}$$

[3] The results of the paper are derived implicitly assuming a risk neutral world. However, they can be extended to a non risk neutral world, for risk adjusted probabilities i.e under an equivalent martingale measure. See Harrison and Kreps (1979).

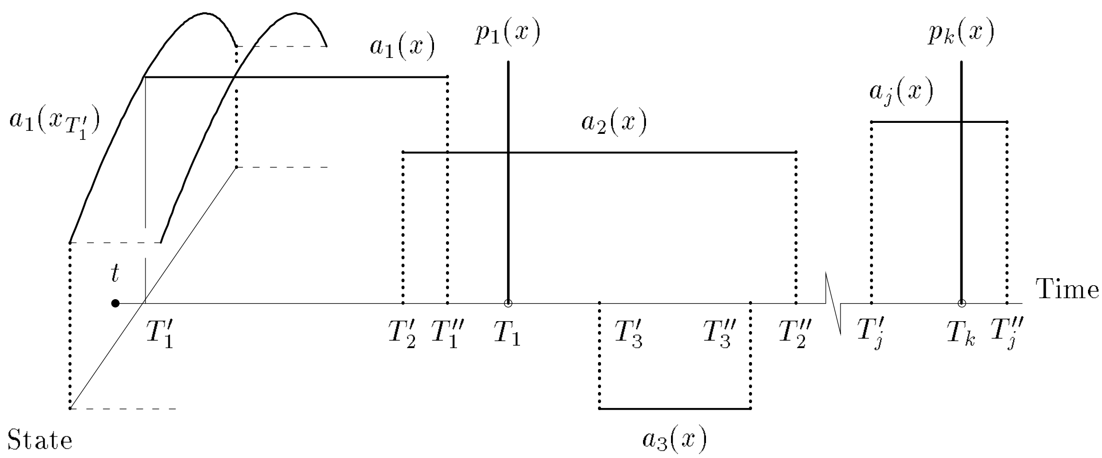

Figure 1: Time Structure of Expected Payoffs

We can infer the infinite maturity value of this payoff stream, taking into account the risk of interruption of such payments

$$A_j(x_t|t:+\infty) \quad = \quad \overline{A}_j(x_t|t:+\infty) \; - \; \overline{A}_j(\underline{x}|t:+\infty) \; (x_t/\underline{x})^{\lambda_0} \; .$$

This is the value of a riskless and perpetual flow of income $a_j(x)$, minus the change in asset value intervening the first time $\underline{x}$ is hit, multiplied by the probability-weighted discount factor attached to this event. This expression is determined as follows: The Laplace transform of the first passage time to a single barrier for a geometric Brownian motion yields $\int_t^\infty e^{-r(\Gamma-t)} f_t(\Gamma) d\Gamma = (x_t/\underline{x})^{\lambda_0}$ where $f_t(\Gamma)$ is the density of Γ conditional on information at t, and where Γ is the first time at which the price process hits $\underline{x}$ (see Karlin and Taylor (1975) page 363). The constant λ_0 is the negative root of the quadratic equation $\rho(\lambda) = 0$, where $\rho(\lambda) \equiv r - \lambda[\mu + (\lambda - 1)\sigma^2/2]$.

We now derive the value, $A_j(x_t|t:T_j'')$, of the payoff stream $a_j(x)$ from now to the date T_j'', with risk of default. To do this we express it as the difference between its value if the maturity was infinite, $A_j(x_t|t:+\infty)$, and the discounted expected value of an infinite maturity claim written at the date T_j'' against the same income flow, conditional on the process not being absorbed in the meanwhile.

$$A_j(x_t|t:T_j'') \quad = \quad A_j(x_t|t:+\infty) \; - \; e^{-r(T_j''-t)} \; \underline{E}_{T_j''} \left[A_j(x_{T_j''}|T_j'', +\infty) \right] \; .$$

$\underline{E}_T$ is the expected value operator conditional on the state variable not hitting the absorbing barrier $\underline{x}$ before T. Finally the value, $A_j(x_t|T'_j : T''_j)$, of a stream of payoff $a_j(x)$, starting at date T'_j and finishing at date T''_j, is obtained substracting $A_j(x_t|t : T'_j)$ to $A_j(x_t|t : T''_j)$.

$$A_j(x_t|T'_j : T''_j) \quad = \quad e^{-r(T'_j - t)} \, \underline{E}_{T'_j} \left[A_j(x_{T'_j}|T'_j, +\infty) \right] \; - \; e^{-r(T''_j - t)} \, \underline{E}_{T''_j} \left[A_j(x_{T''_j}|T''_j, +\infty) \right] \; .$$

• Similarly, $P_k(x_t|T_k)$, the present value of a lump-sum payment $p_k(x)$ expected at date T_k if default does not occur in the meantime is also easily expressed using the conditional expected value operator $\underline{E}_T$,

$$P_k(x_t|T_k) \quad = \quad e^{-r(T_k - t)} \, \underline{E}_{T'_k} \left[p_k(x_{T'_k}) \right] \; . \tag{3}$$

• Finally, the current value of an infinite maturity guarantee to receive $\underline{V}(\underline{x})$ in the event of default is $\underline{V}(\underline{x})\ (x_t/\underline{x})^{\lambda_0}$. Given that the maturity of the contract T is finite, $\underline{V}(\underline{x})$, contributes to the overall value of the asset by

$$\underline{V}(x_t|t : T) \quad = \quad e^{-r(T-t)} \, \underline{E}_T \left[\underline{V}(\underline{x})\ (x_t/\underline{x})^{\lambda_0} \right] \; . \tag{4}$$

Second Step: Let us develop separately the central pricing operator we employ, which is applicable to any elementary value function $\alpha {x_T}^{\lambda}$ of the realisation at time T of the economic fundamental. Define $I(x_t|\alpha, \lambda, \underline{x}, T)$ as the expected discounted value of a promise to receive $\alpha {x_T}^{\lambda}$ at time T, if and only if the underlying process, whose current level is x_t, does not reach $\underline{x}$ in the meantime.

$$I(x_t|\alpha, \lambda, \underline{x}, T) \quad \equiv \quad e^{-r(T-t)} \, \underline{E}_T \left[\alpha \, {x_T}^{\lambda} \right] \; , \tag{5}$$

We show in the Appendix

$$I(x_t|\alpha, \lambda, \underline{x}, T) \quad = \quad \alpha \, {x_t}^{\lambda} \, e^{-\rho(\lambda)\ (T-t)} \left[\; \Phi\left[d_1\right] - \Phi\left[d_2\right] (x_t/\underline{x})^{\gamma(\lambda)} \; \right] \; , \tag{6}$$

$$\text{where} \quad \rho(\lambda) \quad \equiv \quad r - \lambda[\mu + (\lambda - 1)\sigma^2/2] \; , \qquad \gamma(\lambda) \; \equiv \; \frac{\mu + (\lambda - 1/2)\sigma^2}{-\sigma^2/2} \; ,$$

$$d_1 \quad \equiv \quad \frac{ln(x_t/\underline{x}) \; - \; \gamma(\lambda)\, \sigma^2 (T-t)/2}{\sigma\sqrt{T-t}} \; , \qquad d_2 \; \equiv \; d_1 - \frac{2ln(x_t/\underline{x})}{\sigma\sqrt{T-t}} \; ,$$

and $\Phi[.]$ is the cumulative normal distribution.

Third Step: With a very weak restriction on the functional forms of $a_j(x)$ and $\{p_k(x)\}$, we can use the versatile operator $I(x_t|\alpha, \lambda, \underline{x}, T)$ to obtain closed-form solutions for the value of the asset $\mathcal{A}$, as a sum of such operators. We simply need the arguments inside the expectations operators in equations (4) to (6) to be sums of power functions of x.

Assumption 1 Each $a_j(x)$ and $\{p_k(x)\}$ can be expressed in terms of

$$a_j(x) \quad = \quad \sum_{i=1}^{I} \alpha_{ij} x^{\lambda_i} \ , \qquad p_k(x) \ = \ \sum_{l=1}^{L} \alpha_{lk} x^{\lambda_{lk}} \ ,$$

for all $j \in \{1, ...J\}$, and $k \in \{1, ...K\}$, where all α_{ij}, λ_{ij}, α_{lk}, and λ_{lk} are constants.

Proposition 1 *Under Assumption 1, the asset $\mathcal{A}$ is currently worth*

$$\begin{aligned} V(x_t|t) \quad = \quad & \sum_{j=1}^{J}\sum_{i=1}^{I} I(x_t|\frac{\alpha_{ij}}{r-\lambda_{ij}\mu}, \lambda_{ij}, \underline{x}, T'_j) \ - \ \sum_{j=1}^{J} I(x_t|\sum_{i=1}^{I} \frac{\alpha_{ij}\,\underline{x}^{\lambda_{ij}-\lambda_0}}{r-\lambda_{ij}\mu}, \lambda_0, \underline{x}, T'_j) \\ - \ & \sum_{j=1}^{J}\sum_{i=1}^{I} I(x_t|\frac{\alpha_{ij}}{r-\lambda_{ij}\mu}, \lambda_{ij}, \underline{x}, T''_j) \ + \ \sum_{j=1}^{J} I(x_t|\sum_{i=1}^{I} \frac{\alpha_{ij}\,\underline{x}^{\lambda_{ij}-\lambda_0}}{r-\lambda_{ij}\mu}, \lambda_0, \underline{x}, T''_j) \\ + \ & \sum_{k=1}^{K}\sum_{l=1}^{L} I(x_t|\alpha_{lk}, \lambda_{lk}, \underline{x}, T_k) \ + \ I(x_t|\underline{V}(\underline{x})\,\underline{x}^{-\lambda_0}, \lambda_0, \underline{x}, T) \ . \end{aligned} \tag{7}$$

To obtain the formula, simply notice that with Assumption 1,

$$\overline{A}_j(x_t|t:+\infty) \quad \equiv \quad E_t \int_t^{+\infty} e^{-r\tau} \ a_j(x_\tau) \ d\tau \ . \ = \ \sum_{i=1}^{I} \frac{\alpha_{ij}\, x_t^{\ \lambda_{ij}}}{r-\lambda_{ij}\mu} \ .$$

This pricing is flexible because it is modular: In this framework, the set of parameter functions describing (i) the choice and dynamics of the economic fundamental $\{\mu, \sigma\}$, (ii) the expected default trigger $\underline{x}$, and (iii) the value of the asset in default $\underline{V}(x)$, are considered as exogenous inputs. This choice can therefore be adjusted and justified separately. This not only enables an understanding of the relative impact of alternative sets of assumptions on the pricing of default risk, but also provides great latitude to accommodate for the specificity of each asset.

It is hard for any particular description of the environment to prove universally acceptable. However adjusting such assumptions to the particularities of the environment surrounding the claim is very important for pricing purposes. We now show how strikingly true this is in the context of corporate debt. In the following sections, we illustrate the use of such a pricing formula in the simplest case. We provide a pricing formula for finite maturity coupon paying bonds with default risk. We then examine how substantially differences in parametrisation of bankruptcy affect not only the valuation of bonds, but also the implied term structure of credit spreads.

3 Corporate Debt

A coupon paying bond with default risk and finite maturity entitles its holder to a stream of constant interest payments, the coupon c, each period until the lump sum repayment of the principal p, at the date of maturity T. Default corresponds to an interruption of the debt service. Using the previous section's notation, this case corresponds to $J = 1$ and $a_1(x) = c$ ($I = 1$); $K = 1$ and $p_1(x) = p$ ($L = 1$); Debt service obligation is not defered, hence $T_1' < t$; The fact that the principal is repaid at maturity implies $T_1 = T_1'' = T$. The value of this asset is therefore

$$\begin{aligned} V(x_t \mid t) \quad &= \quad \frac{c}{r} - \left[\frac{c}{r} - \underline{V}(\underline{x})\right]\left(\frac{x_t}{\underline{x}}\right)^{\lambda_0} \\ &\quad - I(x_t, [c/r - p], 0, \underline{x}, T) \;+\; I(x_t, [c/r - \underline{V}(\underline{x})]\underline{x}^{-\lambda_0}, \lambda_0, \underline{x}, T) \end{aligned} \tag{8}$$

In corporate finance, the parameters of the environment are interpreted as characteristics of the firm in its economic context. Parametrisations we have encountered in the finance literature are as follows:

(i) DYNAMICS OF THE ECONOMIC FUNDAMENTAL: $\{\mu, \sigma\}$. Studies by Merton (1974), Black and Cox (1976), Kim Ramaswamy and Sundaresan (1993), Longstaff and Schwartz (1995), Leland (1994) and Leland and Toft (1996) all employ the *total value of the firm's assets* as the economic fundamental.[4] In the absence of arbitrage and if investors are assumed risk neutral, the expected drift rate μ of this process must equal r, the discount factor.

Alternatively, cash flow based models such as Mella-Barral and Perraudin (1997) consider the firm's *operating earnings*, hence the selling price of production as the driving process. This choice is particularly suitable when liquidity problems are an issue. Furthermore, using data concerning the cash flows of a firm is easier than assessing the total value of its assets.

(ii) DEFAULT TRIGGER: $\underline{x}$. Brennan and Schwartz (1978) consider debt with a positive *net-worth covenant* written in the contract. The existence of the covenant implies that bankruptcy is triggered when the market value of specific assets falls below a pre-established level. For example, using Leland's (1994) terminology, protected debt corresponds to the case where contractually the overall value of the firm cannot fall below the debt principal.

[4] Even though requiring numerical methods, the models of Kim, Ramaswamy and Sundaresan (1993) and Longstaff and Schwartz (1995) have the ability to handle a stochastic riskless interest rate following an Ornstein-Uhlenbeck process.

Alternatively, Kim Ramaswamy and Sundaresan (1993) assume that bankruptcy occurs as soon as current earnings do not suffice to cover the firm's contractual debt service obligations. This *strict liquidity constraint* probably yields an excessively early closure rule. It is nevertheless attractive because liquidity problems often determine the timing of bankruptcy.

In Leland and Toft (1996), bankruptcy is declared at the *equity holders' optimal abandonment point*. That is, the optimal exercise time of their limited liability option. The equity value being the total market value of the firm minus the value of all outstanding debt, $\underline{x}$ is such that the derivative of the equity value with respect to x_t equals zero at the closure trigger level $\underline{x}$. The endogenous nature of this closure rule is attractive.[5] Equity holders are assumed to have deep-pockets: they are able, when willing, to cover operating losses. Therefore as far as liquidity problems are concerned, this model corresponds to the softest possible constraint, i.e. the opposite of Kim, Ramaswamy and Sundaresan (1993).

(iii) Value of the Asset in Default: $\underline{V}(x)$. Most of the corporate finance pricing literature assumes that defaulting is sanctioned by a liquidation of the firm, and that the bankruptcy procedure involves some exogenously given costs. The total value of the firm if default is triggered is calculated subtracting these costs to the total value if bankruptcy was avoided. In the papers previously cited, the exogenous parameter quantifying bankruptcy costs consists either of a proportional reduction factor θ, or of a fixed lump sum.

If the absolute priority rule is respected, debt holders are paid first out of the proceeds of a piece-meal liquidation sale. This is because debt is senior to equity. The equity has only a non zero residual value if debt holders fully recover their principal. Departures from absolute priority are easily handled assuming instead that equity holders are expected to get hold of a fraction of the total value of the firm in default.

[5] It can be applied not only when the driving economic fundamental is the total value of the firm's assets but also with a cash flow model. It is nevertheless only justifiable considering a stationary debt structure environment, where the firm continuously sells a constant amount of new debt, and always with the same maturity. The coupon written on newly issued debt is furthermore assumed independent of expected changes in the level of the state variable. Equity holders are basically guaranteed to be able to roll over the debt with the same maturity, always paying the same coupon. This assumption generates a debt service burden which is constant throughout the existence of the firm, hence creates a time-independent willingness to service it. This is necessary, given the requirement of a time-independent $\underline{x}$.

4 Term structure of credit spreads

With a finite maturity pricing, it becomes possible to study the influence of default risk on the term structure of yields. In this section we illustrate the impact of a particular choice of (i) underlying economic fundamental and (ii) default trigger rule, on the value of debt, and on the term structure of credit spreads. We assume that the whole firm is financed with a single debt contract, and contrast two different sets of modelling assumptions:

1. (i) The economic fundamental x_t is the total value of the firm's assets, therefore $\mu = r$, and (ii) the debt is *protected*, hence bankruptcy is triggered the first time x_t hits the level p. (iii) There are no departures from the absolute priority rule. The debt residual value in bankruptcy $\underline{V}(x)$ is therefore equal to $\min\{(1-\theta)x \ ; \ p\}$, where θ represents a proportional bankruptcy cost.

2. (i) The economic fundamental x_t is the firm's operating earnings. The drift rate μ is related to the expected inflation rate. We set μ equal to zero, assuming no inflation. (ii) Bankruptcy is triggered by a strict *liquidity constraint*, hence $\underline{x} = c$. (iii) Debt value in bankruptcy $\underline{V}(x)$ is $\min\{(1-\theta) \ x/r \ ; \ p\}$, corresponding to the same scenario.

Figures 2(a) and 3(a) exhibit the value of corporate debt $V(x|0)$, at the date of entry $t = 0$, for the two descriptions of the environment.[6] The level of the state variable at entry, x_0, corresponds in both cases to the same total value of the firm. The value of the bonds $V(x|0)$ is an increasing function of the state variable x. The higher the value of x, the further the firm is from the default threshold $\underline{x}$ and the smaller the discount for default risk. As x increases, the bond value is asymptotic to the value of riskless debt. Conversely, when x decreases, debt value tends to the value of the firm if default occurs $\underline{V}(\underline{x})$.

We then successively assume that the contract is initially written with a different maturity. To isolate the influence of our assumptions, we make sure that although contracts differ in their maturity, they represent comparable entities. That is, they all have the same face value and are sold at par, regardless of their maturity. The coupon written at entry is then a function

[6] For our simulations we chose the following baseline parameters: a riskless interest rate r of 6 % reflects approximately the current US prime rate. The asset price volatility σ is set to 0.15 and we adopt a figure of 30% for bankruptcy costs. Altman (1984) reports an average writedown rate of 40% for secured debt considering a sample of defaulted bonds and Alderson and Betker (1995) report a 36.5% mean liquidation cost calculated as the ratio of going-concern value less liquidation value to going-concern value.

of the maturity: It is such that the initial value of the bonds equals the principal, p. In other words, for a given maturity date T, the coupon contracted upon at entry, $c(T)$, is such that $V(x_0|0) = p$.

We see in Figures 2(a) and 3(a) that the shape of the value function $V(x|0)$, strongly depends on the maturity of the debt. For $x < x_0$, debt value is decreasing with maturity, and conversely when $x > x_0$. This is the result of two competing effects:

1. For all levels of x , the probability of reaching a given $\underline{x}$ before the contract ends is lower for shorter maturities. This induces a lower weighted probability discount factor associated with the event of bankruptcy. Shorter maturities therefore correspond to lower probabilities of default.
2. For a high x, the debt value asymptotes to its riskless value $c(T)/r - [c(T)/r - p]e^{-rT}$, which is increasing in maturity (as bonds are sold at par at entry). Shorter maturities are therefore associated with lower riskless values.

Notice that with the strict liquidity constraint closure rule, bankruptcy points depend on the maturity of the contract (figure 3(a)). This is because the coupon written at entry is a function of the maturity and $\underline{x} = c(T)$. Clearly, this is not the case with a net-worth covenant.

We can now examine the implied term structures of credit spreads at entry, i.e. $c/p - r$ function of the maturity. The term structure corresponding to each set of assumption is plotted in figures 2(b) and 3(b). The second one is much more hump-shaped. The spreads increase until an intermediate maturity of either 8 or 4 years and then decrease as the maturity increases. This is the result of a trade-off between the exposure to credit risk and the expected present value of the coupon payoff stream, which both increase with maturity.

Observe that the magnitude of credit spreads is substantially different depending on the description of the environment. Not surprisingly the first set of assumptions generates much smaller credit spreads.[7] When the total value of the firm's assets is the economic fundamental, the chances that the process hits a barrier from above are lower than when the dynamics are

[7] Notice that the magnitudes of credit spreads obtained are consistent with the average levels observed in debt markets. Kim, Ramaswamy and Sundaresan (1993) report 77 basis points as the average spread for investment-grade corporate bonds, and Litterman and Iben (1991) report historical ranges for par spreads between 20 and 130 basis points.

driven by the operating earnings. This is because the expected drift rate μ, is equal to r instead of 0, thus the expected time of default is further.

We also carried out a sensitivity analysis of the term structure of credit spreads with respect to a 10% upwards or downwards movement in (i) initial face value of the debt contract, (ii) volatility, (iii) interest rate and (iv) bankruptcy costs. This is plotted in figure 4 when the driving process is the total value of the firm. All graphs share the same central curve, which corresponds to the baseline parameters.

A rise in face value, volatility, and bankruptcy costs increases credit spreads. Changes in the safe interest rate do not affect much the magnitudes of credit spreads for short-term contracts. Credit spreads are only marginally increasing in the riskless rate for long maturities[8].

5 Concluding Remarks and Extensions

The analysis can be pursued assuming that the firm is financed solely through a combination of shares and identical bonds. With a costly bankruptcy and introducing a tax advantage of debt, we can derive the optimal capital structure of this firm. Equity is just an infinite maturity claim on the profits from operation. We have optimised the capital structure controlling the maturity of debt contracts. The bottom line results favour shorter maturities. We do not report our analysis, because we would like to stress instead that in all the models previously cited, as well as with this current set-up, the results of such an optimal capital structure are strongly biased in favour of short term debt.

We can see in figures 2(a) and 3(a) that for levels of the state variable in the interval just above $\underline{x}$, the value of the bonds exceeds the value of the firm in default. Whenever bankruptcy is costly, there is a surplus to be renegotiated prior to triggering the procedure.[9] Most importantly, this is more the case when the maturity is short. The surplus, hence the scope for renegotiation is larger.[10] We conclude that models without renegotiation exaggerate

[8] In the operating earnings' environment, we obtained that credit spreads increase with the safe rate more rapidly.

[9] Empirical studies by Franks and Torous (1989 and 1994) suggest that bankruptcy procedures give scope for opportunistic behaviour by the parties involved. If they have substantial bargaining power, debtors are in a position to extract any such surplus. This debtors' opportunistic behaviour in financial distress is analysed in Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997).

[10] This is in accordance with the intuition: Creditors willingness to reduce debtors' debt service obligations,

more the debt value for shorter maturities, not ony because creditors are more willing to make self interested concessions to the debtors, but also because debtors can expect to strategically extract more concessions from creditors. None of the finite maturity models allows for the firm in financial distress to renegotiate its debt, and this modelling short fall yields larger errors with short maturity contracts. Therefore, as long as the debt renegotiation process is not taken into account, the conclusion coming from the analysis of the optimal capital structure is hardly convincing.

Here, the modular structure of our pricing may prove useful. Whereas default induces a permanent alteration to the agreement between the contracting parties, this event is not necessarily associated with liquidation. When contracts are renegotiated, default just corresponds to a permanent modification of the terms of the agreement. This model, although not providing a specific scenario for debt renegotiation, structurally allows for such a wider definition of default. The value function $\underline{V}(x)$, can represent the value of the asset once the contract has been renegotiated.

The next step in future research consists of incorporating sequences of debt renegotiation in a globally justified pricing model. This involves constructing a recursive pricing structure that preserves the set of renegotiation options open after a first renegotiation. $\underline{V}(x)$ is then expressed as a function of its own residual value function $\underline{V}^{(2)}(x)$, which represents the value of the asset after a second renegotiation.[11]

Even then, it will remain difficult to pretend that renegotiation will occur through the optimal control of one specific term written in the contract. Furthermore, the solution to the game played by the different contracting parties, depends on the value of the assets if default is not followed by a successful renegotiation of the contract. What we really need is a theory of the global sharing function $\{\underline{V}(x)\}$ which describes the allocation of the overall pre-renegotiation value of the assets to each individual claimant, if negotiations fails.[12]

in order to reduce the likelihood of formal bankruptcy and increase the market value of their bonds, is greater when the lump sum principal repayment is due in the near future (but will only intervene if bankruptcy is avoided today).

[11]With this structure Mella-Barral (1995) derives path-dependent closed form solution for marginal reduction in the coupon. This involves solving for a converging infinite series of such nested real options. In the context of sovereign debt, Hayri (1996) provides an algorithm when there is a fixed number of possible renegotiation rounds. Nevertheless, in both cases the debt is infinite maturity.

[12]This may be complex in the presence of multiple classes of debt seniority.

Our objective in considering the case of corporate debt, was first to illustrate the use of our pricing method. It was also mainly to show how the set of assumptions describing the event of default crucially influences the result. This final discussion further stresses the importance of being able to adjust this set of assumptions to the specificity of the problem. This is an important topic for future research, where the flexibility of such a framework could prove instrumental.

APPENDIX

To obtain $I(x_t \mid \alpha, \lambda, \underline{x}, T)$, first, apply Ito's lemma to equation (1),

$$dln(x_t) \;=\; (\mu - \sigma^2/2)dt \;+\; \sigma dB_t$$

$$ln(x_T/x_t) \;\sim\; N\left[(\mu - \sigma^2/2)(T-t)\;;\;\sigma(T-t)\right] ,$$

Cox and Miller (1965) p.221 give the probability density function of an arithmetic Brownian motion with an absorbing barrier.

$$\begin{aligned} f(ln(x_T/x_t)) &= \frac{1}{\sigma\sqrt{2\pi(T-t)}} \left\{ exp\left[\frac{-\left[ln(x_T/x_t) - (\mu-\sigma^2/2)(T-t)\right]^2}{2\sigma^2(T-t)} \right] \right. \\ &- \left. exp\left[\frac{2(\mu-\sigma^2/2)\, ln(\underline{x}/x_t)}{\sigma^2} - \frac{\left[ln(x_T/x_t) - 2ln(\underline{x}/x_t) - (\mu-\sigma^2/2)(T-t)\right]^2}{2\sigma^2(T-t)} \right] \right\}. \end{aligned}$$

Secondly change variable in the expression of $I \;\equiv\; I(x_t \mid \alpha, \lambda, \underline{x}, T)$:

$$\begin{aligned} I &= exp^{-r(T-t)} \int_{ln(\underline{x}/x_t)}^{+\infty} \alpha\, x_t^{\lambda}\, exp\left[\lambda ln(x_T/x_t)\right]\; f(ln(x_T/x_t))\; dln(x_T/x_t) \\ I &= exp^{-r(T-t)} \int_{ln(\underline{x}/x_t)}^{+\infty} \alpha\, x_t^{\lambda} \frac{1}{\sigma\sqrt{2\pi(T-t)}} \\ &\quad .exp\left[\lambda ln(x_T/x_t)\right]\, exp\left[\frac{-\left[ln(x_T/x_t) - (\mu-\sigma^2/2)(T-t)\right]^2}{2\sigma^2(T-t)} \right] dln(x_T/x_t) \\ &- exp^{-r(T-t)} \int_{ln(\underline{x}/x_t)}^{+\infty} \alpha\, x_t^{\lambda} \frac{1}{\sigma\sqrt{2\pi(T-t)}} exp\left[\frac{(\mu-\sigma^2/2)\, ln(\underline{x}/x_t)}{\sigma^2/2} \right] \\ &\quad .exp\left[\lambda ln(x_T/x_t)\right]\, exp\left[\frac{-\left[ln(x_T/x_t) - 2ln(\underline{x}/x_t) - (\mu-\sigma^2/2)(T-t)\right]^2}{2\sigma^2(T-t)} \right] dln(x_T/x_t) \\ I &= exp^{-r(T-t)} \alpha\, x_t^{\lambda} \left(\int_{ln(\underline{x}/x_t)}^{+\infty} \frac{1}{\sigma\sqrt{2\pi(T-t)}}\, exp\left[\psi(0)\right]\, dln(x_T/x_t) \right. \\ &- \left. exp\left[\frac{(\mu-\sigma^2/2)\, ln(\underline{x}/x_t)}{\sigma^2/2} \right] \int_{ln(\underline{x}/x_t)}^{+\infty} \frac{1}{\sigma\sqrt{2\pi(T-t)}}\, exp\left[\psi(ln(\underline{x}/x_t))\right]\, dln(x_T/x_t) \right). \\ &\quad \text{where} \qquad \psi(y) \;=\; -\frac{\left[ln(x_T/x_t) - \left\{2y + (\mu-\sigma^2/2)(T-t)\right\}\right]^2}{2\sigma^2(T-t)} \;+\; \lambda\, ln(x_T/x_t). \end{aligned}$$

Rearranging terms yields

$$\psi(y) \;=\; -\frac{\left[ln(x_T/x_t) - 2y[\mu + (\lambda - 1/2)\sigma^2](T-t)\right]^2}{2\sigma^2(T-t)} + 2\lambda y + \lambda[\mu + (\lambda-1)\sigma^2/2](T-t).$$

After replacing $\psi(.)$ in $I \;\equiv\; I(x_t \mid \alpha, \lambda, \underline{x}, T)$ we directly obtain the expression in equation (6).

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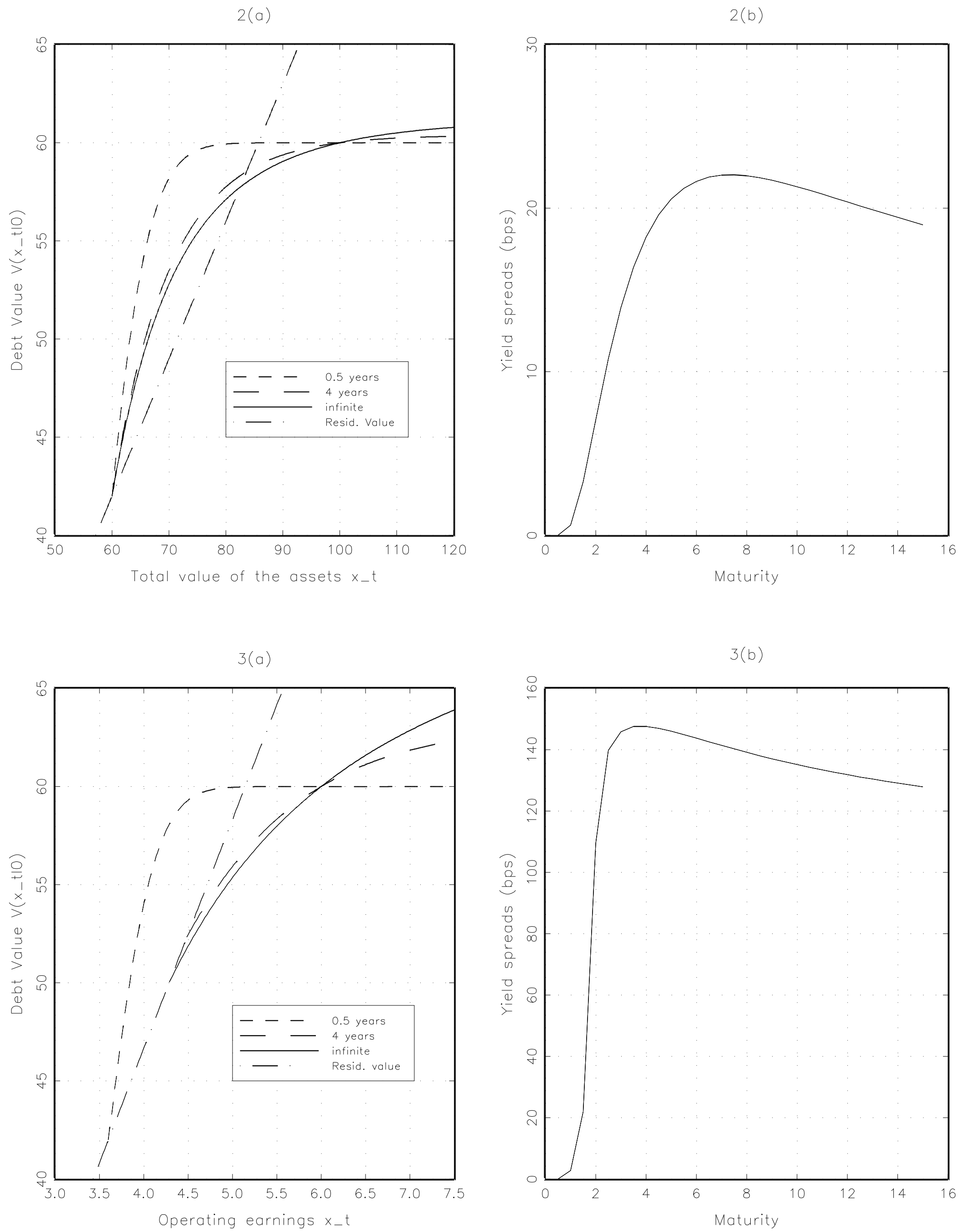

Figures 2 and 3 : **Debt Value and Term Structure of Credit Spreads at Entry**

fig. 2(a) represents debt value at entry function of the total value of the firm and fig. 3(a) is debt value function of the operating earnings. Fig. 2(b) and 3(b) are the corresponding term structures of credit spreads. $p = 60$, $r = 0.06$, $\theta = 0.3$

and $\sigma = 0.15$.

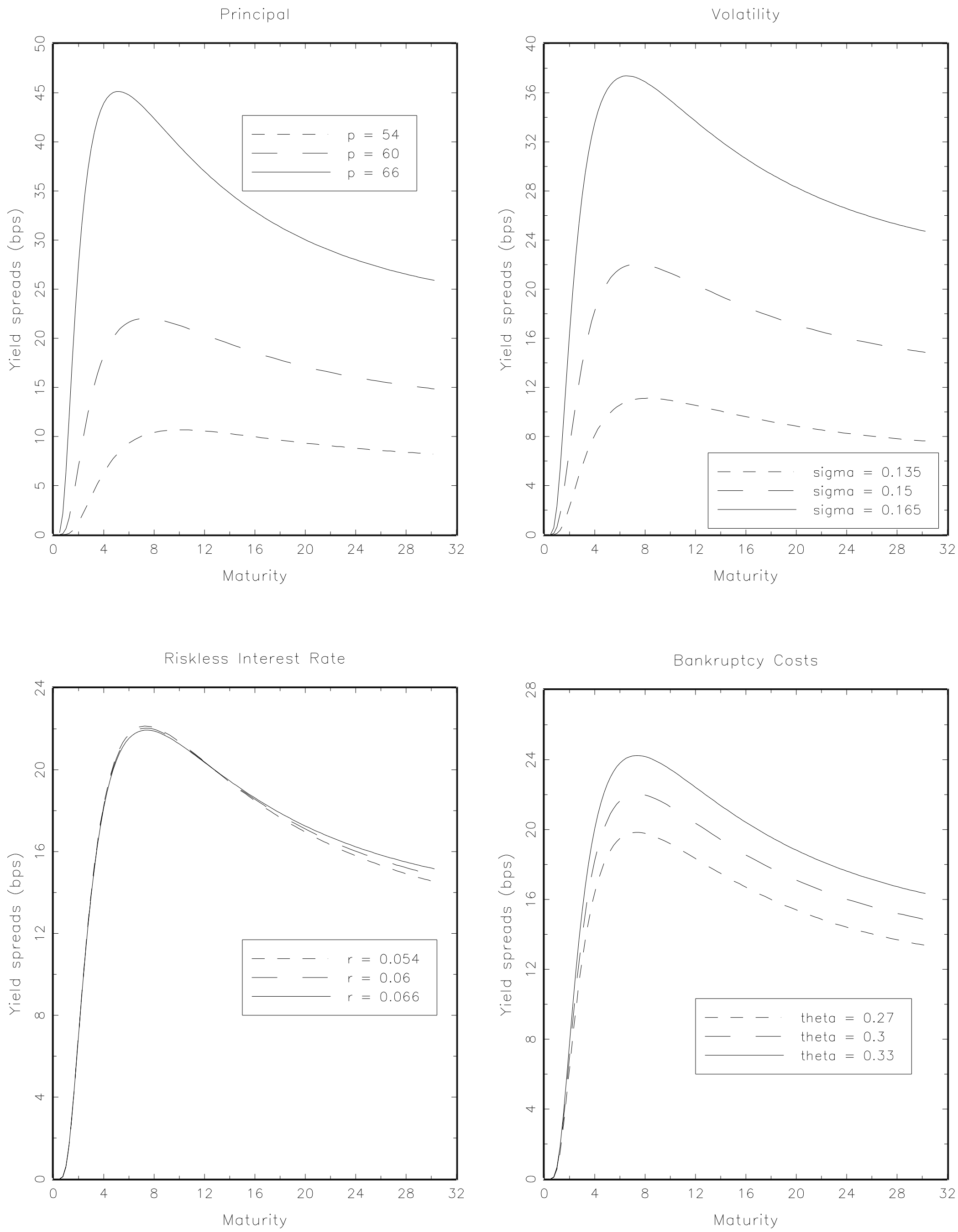

Figure 4 : **Comparative Statics of the Term structure of credit spreads**

With total value of the firm as the driving process and a net-worth covenant closure rule.