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GIORGIO FABBRI*, SILVIA FAGGIAN†, AND GIUSEPPE FRENI‡

Abstract. This study examines the dynamics of capital stocks distributed among several nodes, representing different sites of production and connected via a weighted, directed network. The network represents the externalities or spillovers that the production in each node generates on the capital stock of other nodes. A regulator decides to designate some of the nodes for the production of a consumption good to maximize a cumulative utility from consumption. It is demonstrated how the optimal strategies and stocks depend on the productivity of the resource sites and the structure of the connections between the sites. The best locations to host production of the consumption good are identified per the model's parameters and correspond to the least central (in the sense of eigenvector centrality) nodes of a suitably redefined network that combines both flows between nodes and the nodes' productivity.

Keywords: Capital allocation, Production externalities, Network spillovers, Economic centrality measures.

JEL Classification: C61, D62, O41, R12.

1. Introduction

There is a growing interest in the literature in the study of the economic effects of heterogeneous interactions of different entities. Multisector growth models with externalities (e.g., Benhabib et al., 2000), metapopulation models of interconnected natural resources (e.g., Sanchirico and Wilen, 2005), and network models of various kinds (e.g., Ballester et al., 2006, Elliott and Golub, 2019) are examples of this trend. In this paper, we take a network perspective in studying a multisector growth model with externalities.

We consider a growth model where production is distributed among several locations, different for productivity, and connected by the fact that production in one engenders positive externalities on the production of the others. A single agent aims to localize the production of a consumption good, to maximize the sum of the sites' utility from consumption.

This work develops a simple dynamic model where the n nodes of a weighted, directed network represent the n sites where the capital stock is accumulated, and the weights on the edges between two nodes represent the externalities of production in one node on the production in the other. Specifically, it aims to show how the structure of the network and other parameters of the system affect the agent's decision in the choice of one or more nodes/locations for the production of the consumption good.

As the primary contribution to the literature, this study reveals that when the network is strongly connected, and the agent is sufficiently "patient" - which in the context of generalized growth theory, means their rate of discount is close to a critical discount rate[1] - the optimal closed-loop strategies exhibit linearity in the stock. This is accompanied by the presentation of an analytical formula for such strategies, as detailed in Theorems 1 and 3.

At optimum, independently of the assignment, the different site stocks are evaluated via a constant common vector of relative prices that proves to be the eigenvector centrality of another related network that combines the spillover flows and the sites' net rates of growth. The effect of these two forces is jointly captured by the adjacency matrix of such a modified network, that is the sum of the adjacency matrix of the original network and the diagonal matrix of net productivities of sites.[2] It is proven that the best allocation of the production of the consumption good is at the most peripheral nodes, namely those with the least centrality. For initial stocks in a cone contained in the positive orthant (and characterized through eventually exponentially positive matrices, as in Noutsos and Tsatsomeros, 2008), such allocation is placed immediately at the most peripheral node. For other initial capital stocks, the allocation in the most peripheral node is best in the long run, and initially may have to be placed otherwise, notably when the initial capital stock in the most peripheral nodes is small. Furthermore, if the least eigencentrality is unique, the optimal control is unique, at least when the initial stock belongs to such cone.

The model is further extended to enclose transportation costs (Theorem 3), whose effect is to modify the eigencentrality and the associated hierarchy of nodes.

[1]See e.g., McFadden, 1973 for a discussion of critical discount rates in optimal growth theory.

[2]The resulting matrix may well have negative terms on the principal diagonal, although the other entries remain nonnegative - i.e. it is a Metzler matrix - hence the associated network could be referred to as a "signed" network. Nonetheless, the fact that its adjacency matrix is Metzler helps preserve several properties, such as the Perron-Frobenius property.

The network structure is similar to the one used by Fabbri et al. (2024), with a notable distinction: here, a single decision-maker selects an optimal policy, whereas in Fabbri et al. (2024), multiple players, each occupying a node, engage in a competitive differential game (see also Remark 5). The model is also related to the continuous space-time growth models introduced by Boucekkine et al. (2013); Fabbri (2016); Boucekkine et al. (2019) and the discrete space version developed by Calvia et al. (2023). While some of the techniques employed overlap with those found in the aforementioned papers, the economic models feature notable distinctions. In Boucekkine et al. (2013) the model involves strategic iterations for natural resource extraction, in Fabbri (2016) consumption takes place independently at each node[3] (at every node, a portion of the production is not invested, resulting in consumption that takes place exclusively on-site). Furthermore, in the latter cases, explicit results are provided exclusively for the symmetric scenario, involving the Laplacian in continuous time and symmetric matrices in the discrete case. These disparities also manifest themselves in the behavioral aspects of the system when examining its asymptotic state in response to variations in the agents' preference parameters (as discussed in Section 3.5).

The article is structured as follows: in Section 2 we introduce the model and discuss its mathematical formulation. In Section 3, we present the main results of the article: the introduction of the candidate optimal strategy (Theorem 1) and its admissibility (Theorem 2), followed by a discussion (Subsection 3.5) on the asymptotic properties of the system as the parameters of the agents' preferences vary. Section 4.1 introduces the extension of the model with transportation costs. Section 5 concludes. Appendix A contains the proofs of the statements.

2. The model

We consider capital stocks available in different but interconnected areas that are considered to be sufficiently different from each other (and sufficiently homogeneous in their interior) so as to be described by different parameters. We then analyze a growth model in which the production in one area generates non-negative spillovers on stocks in the others.

[3]This hypothesis can be conceived as an economic model featuring infinitely high transportation costs. The findings presented in Section 4.1, illustrating the extension of our results to include explicit (iceberg) transportation costs, may be viewed as an intermediary model bridging the gap between the two extremes: one characterized by infinite transportation costs and the other by negligible transportation costs.

Mathematically speaking, we consider a network $\mathcal{G}$ with n nodes – as many as the number of areas – that we assume to be directed and weighted. We also set $N = \{1, 2, \cdots, n\}$.

We denote by $K_i(t)$ the capital stock at node i at time t and by y_i the local productions that we assume to linearly depend on the used capital: $y_i = \Gamma_i K_i$, where $\Gamma_i > 0$ is a productivity coefficient. We suppose that production at a node j generates the spillover $b_{ji} y_j = b_{ji} \Gamma_j K_j$ at node i, where b_{ij} are given nonnegative coefficients. They are the weight of the links of our network so that $\mathcal{G}$ will represent the spillover network of our economy that we will suppose to be strongly connected. $B = (b_{ij})$ is the adjacency matrix of $\mathcal{G}$.

The budget constraint at each location i imposes that the augmented production $\Gamma_i K_i(t) + \sum_{j \neq i} b_{ji} \Gamma_j K_j(t)$ is split at each time between the consumption $c_i(t)$ and the investment in the location-specific investment $I_i(t)$. We assume that investments are reversible (i.e., each $I_i(t)$ can be negative). If we suppose that the capital at the location i decays at rates δ_i we get the evolution of the capital stock at node i:

$$\dot{K}_i(t) = I_i(t) - \delta_i K_i(t) = (\Gamma_i - \delta_i) K_i(t) + \sum_{j \neq i} b_{ji} \Gamma_j K_j(t) - c_i(t),$$

where $i, j \in N$, and in matricial form the system dynamics is given by

$$\begin{cases} \dot{K}(t) = [B^\top \Gamma + \Gamma - D] K(t) - c(t), \quad t \geq 0 \\ K(0) = k \end{cases} \tag{1}$$

where Γ is the diagonal matrix of productivities Γ_i, $B = (b_{ij})$, D is the (also diagonal) matrix of decay rates δ_i, $c(t) = (c_1(t), \cdots, c_n(t))^\top$

We require the capital stocks in every node to be nonnegative, that is

$$K_i(t) \geq 0, \quad \forall i \in N, \forall t \geq 0. \tag{2}$$

Our goal is to identify the nodes where a certain agent, having free access to all, deems it more profitable to produce the consumption good. The total consumption of the agent is

$$\sum_{i=1}^{n} c_i(t) = \langle c(t), \mathbf{e} \rangle$$

where $\mathbf{e} = \sum_{i=1}^{n} e_i = (1, 1, \cdots, 1)^\top$, with e_i the i-th vector of the canonical base in $\mathbb{R}^n$. We assume the agent maximizes the functional

$$J(c(\cdot)) = \int_0^{+\infty} e^{-\rho t} u \left(\sum_{i=1}^{n} c_i(t) \right) dt = \int_0^{+\infty} e^{-\rho t} u \left(\langle \mathbf{e}, c(t) \rangle \right) dt, \tag{3}$$

with u the utility function

$$u(c) = \ln(c) \quad or \quad u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \qquad \sigma > 0,\ \sigma \neq 1$$

(the case of a logarithmic u stands for the case $\sigma = 1$), and $\rho \in \mathbb{R}$ is the discount rate.[4]

Remark 1 In the above model, all externalities are non-negative. However, extensions of the Perron-Frobenius theory to matrices with some negative entries (for example, eventually positive or eventually exponentially positive matrices, see e.g. Noutsos and Tsatsomeros, 2008) can be used to extend the analysis to cases in which positive and negative externalities coexist.

2.1. **Primitives of the Network.** Eigenvalues and eigenvectors of the matrix of the system with null extraction, namely $B^\top\Gamma + \Gamma - D$, and of its transpose $\Gamma B + \Gamma - D$, play a crucial role in the present study. Since by assumption the network $\mathcal{G}$ is strongly connected, the matrix B is irreducible. Moreover, by hypothesis, B is non-negative. Since Γ is diagonal with strictly positive values on the diagonal, the same properties hold for ΓB, so that the matrix $\Gamma B + \Gamma - D$ is again irreducible and has non-negative values out of the diagonal (it is a Metzler matrix).

Lemma 1 *The matrix $\Gamma B + \Gamma - D$ has a simple (not necessarily positive) real eigenvalue λ, strictly greater than the real parts of the other eigenvalues, and with a unique* positive *associated normalized eigenvector η.*

This fact is a direct consequence of the Perron-Frobenius theorem in its strong form (see Bapat and Raghavan, 1997). The eigenvalue λ enjoying the above properties is called *dominant*.

Remark 2 Since $\Gamma B + \Gamma - D$ is irreducible and has non-negative non-diagonal entries, $B^\top\Gamma + \Gamma - D$ enjoys the same properties and then has a unique positive eigenvector ζ associated to the same dominant eigenvalue λ.

3. Explicit Solutions

We here identify the set of parameters for which there exists an explicit solution, and we do so employing Bellman's Dynamic Programming. We denote by $K(t; k, c(\cdot))$ the trajectory of system (1), and by $J(k, c(\cdot))$ the objective functional (3), to point

[4] The results hold regardless of the sign of ρ. Although a negative discount rate is uncommon in applications, a stream of literature considers "upcounting" (see e.g., Le Van and Vailakis, 2005, Dolmas, 1996, and Rebelo, 1991).

out their dependence from the initial stock k and the control $c(\cdot)$. We define the value function

$$V(k) = \sup_{c(\cdot),\, K(t;k,c(\cdot))\geq 0} J(c(\cdot)).$$

Indeed the positivity constraints (2) on the stock will be checked *a posteriori*. The associated Hamilton-Jacobi-Bellman (briefly, HJB) equation, of which $V(k)$ is the candidate solution

$$\rho v(k) = H(\nabla v(k)) + \langle \nabla v(k), [(I + B^\top)\Gamma - D]k\rangle \tag{4}$$

where $H(p)$, with p in the positive orthant $\mathbb{R}^n_+ = [0, +\infty)^n$ given by

$$H(p) = \sup_{c\geq 0} \left\{ u\left(\langle \mathbf{e}, c\rangle\right) - \langle c, p\rangle \right\}. \tag{5}$$

is (convex and) possibly infinite valued. An explicit formula for H is provided in the following Lemma.

Lemma 2 *When $p \in (0, +\infty)^n$, then*

$$H(p) = \max_{c\geq 0} \left\{ u\left(\langle \mathbf{e}, c\rangle\right) - \langle c, p\rangle \right\} = \begin{cases} \dfrac{\sigma}{1-\sigma}\left(\min_i p_i\right)^{1-\frac{1}{\sigma}}, & \sigma \neq 1, \\ -\left[\ln\left(\min_i p_i\right) + 1\right] & \sigma = 1. \end{cases} \tag{6}$$

The above formula holds also for $p \in \partial\mathbb{R}^n_+$, when $\sigma > 1$, whereas

$$H(p) = +\infty, \quad p \in \partial\mathbb{R}^n_+ \quad \sigma \leq 1.$$

Proof. See Appendix A. □

3.1. **Optimal Strategies.** We now provide an explicit solution to the model problem in the assumption that θ, defined as

$$\theta := \begin{cases} \dfrac{\rho - \lambda(1-\sigma)}{\sigma}, & \sigma \neq 1 \\ \rho, & \sigma = 1 \end{cases} \tag{7}$$

is (positive and) small enough, as specified later.

Theorem 1 (Optimal Strategies) *Assume $\sigma \neq 1$. Let η be the dominant eigenvector defined in Lemma 1, $N^* = argmin_i \{\eta_i\}$, and $\theta > 0$. Then:*

(*i*) *when admissible, the closed-loop optimal controls c^* are all of type*

$$c^*_j = 0, \quad \forall j \notin N^* \quad \textit{and} \quad \sum_{i\in N^*} c^*_i(t) = \frac{\theta}{\min_i \eta_i}\langle K(t), \eta\rangle. \tag{8}$$

(*ii*) *moreover, for all $k \neq 0$, the value function of the problem in Section 2 is*

$$V(k) = \frac{\theta^{-\sigma}}{1-\sigma}(\min_i \eta_i)^{\sigma-1}\langle k, \eta\rangle^{1-\sigma}; \tag{9}$$

(*iii*) *the associated optimal trajectory $K^*(t)$ satisfies*

$$\langle K^*(t), \eta\rangle = \langle k, \eta\rangle e^{(\lambda-\theta)t} \tag{10}$$

Proof. See Appendix A. □

REMARK 3 Observe that when there is a unique i for which η_i is minimal then the described optimal control is unique. □

REMARK 4 One can prove that in the case of logarithmic utility $\sigma = 1$ the optimal control is obtained by setting $\theta = \rho$ in (8), while the value function is

$$V(k) = \frac{1}{\rho}\left[\ln\left(\frac{\langle k, \eta\rangle\rho}{\min_i \eta_i}\right) - 1\right].$$

The sketch of the proof is contained in Appendix A. □

REMARK 5 Although the network structure is similar to the one in Fabbri et al. (2024), the two models differ in several respects:

(a) The model outlined in Fabbri et al. (2024) addresses a distinct problem, that is, the exploitation of a shared resource that traverses through nodes (where the weights on the edges signify the intensity of the flow between connected nodes), such as fish moving across territorial waters. This has significant implications for the structure of the network: the underlying interaction dynamic (i.e., the dynamic abstracting from new production and extraction) maintains the total quantity of the resource (thus accurately reflecting the fact that the resource moves from one node to another) and is therefore represented by a Laplacian matrix. This is not the case here. Qualitatively, we can also observe that the spillovers in this model enhance the node's capital stock (positive spillovers/flows from neighboring nodes are possible, but no negatives towards them), whereas in Fabbri et al. (2024), inflows and outflows of the resource at a node can occur simultaneously.

(b) More importantly, in the present study a single decision-maker optimally chooses actions across all nodes, while in Fabbri et al. (2024), multiple players each occupy a node and engage in a competitive differential game. Consequently, in the case of a single planner, it is possible to choose where to allocate the production of the consumption good (all nodes are available), and the crux of the problem is deciding in which nodes it occurs. On the contrary, in the game the nodes where consumption/extraction takes place are given (they correspond to each of the nodes occupied by a player), production takes place necessarily at each occupied node because players maximize separate utilities depending on local consumptions c_i's.

Nevertheless, the results of the two studies are consistent under the following perspective: in a game where a supervising planner has the sole authority to choose which nodes are to be left consumption-free (see section 4.1 in Fabbri et al. (2024)), they choose those with maximal (eigen)centrality, meaning the allocate players and their consumption in the least central ones; this is consistent with the case of a single planner, in which production is best allocated at nodes with minimal centrality.

Finally, it's important to note that the model developed here can be considered as one with null transportation costs and centralized consumption. In contrast, the game can be interpreted as a model with infinite transportation costs and compulsory local consumption. An intermediate framework, incorporating iceberg transportation costs, is described in Section 4.1.

3.2. **Examples.** [5]

The examples in this section are meant to explain how both the productivity of a node and the strength of the connection with neighboring nodes can impact its eigencentrality. We assume a network with three nodes, with the same depreciation rate $\delta = 0.03$, productivity rates $\Gamma_1 = 1$, $\Gamma_2 = \gamma$, $\Gamma_3 = 1.03$, externalities $b_{12} = b_{32} = b_{23} = b_{31} = 1$, $b_{13} = 1.1$, $b_{21} = \alpha$. The parameters α and γ are either fixed or variable in each example. The network is represented in Figure 1.

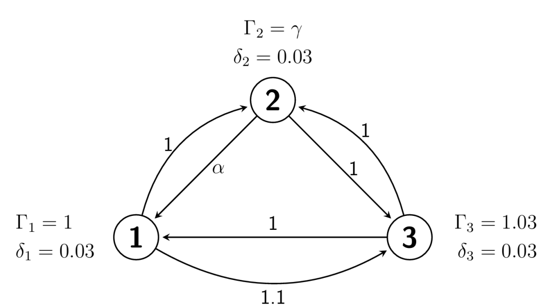

FIGURE 1

[5]The code of the simulation can be found at `https://sites.google.com/site/giorgiofabbri1979/code?authuser=0`

3.2.1. *Example 1.* We here consider a fixed productivity rate $\Gamma_2 = \gamma = 1.1$ (that implies that site 1 is slightly less productive than site 3, which is less productive than site 2), and a varying externality $b_{21} = \alpha \in [0, 3]$. We intend to show how the choice of the consumption node changes with the externality rate α, accordingly with the results in Theorem 1.

The optimal node of choice as a function of α is depicted in Figure 2. For low values of α, the preference is for node 2, despite its higher intrinsic productivity Γ_2 compared to the other nodes. This is because the production at node 2 has a small (with α) positive impact on neighboring nodes' production. Consequently, depleting the capital stock at node 2 results in less overall utility loss than depleting the capital of another node.

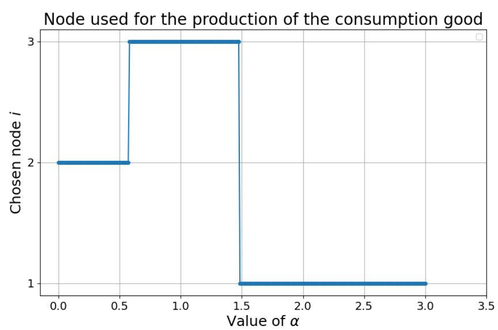

FIGURE 2. The node chosen for consumption for a varying α.

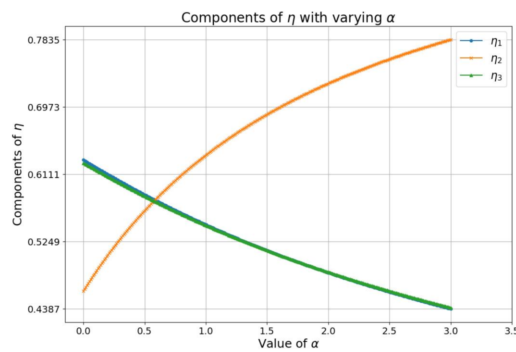

FIGURE 3. The components of the vector η for a varying α.

As α increases, the centrality of node 2 increases until it becomes more advantageous to shift production to node 3. This shift occurs because the direct externality of node 1 on node 3 surpasses that of node 3 on node 1, making node 3 more central. With further increases in α, the indirect externality of node 3 on node 1—meaning the impact of node 3 on the production of node 1 through its influence on node 2—grows due to the strengthening link between nodes 2 and 1. This results in node 3 becoming more central than node 1, making node 1 the best choice for the production of the consumption good.

The thresholds for α at which a change takes place correspond to changes in the corresponding minimal (eigen)centrality η_i, as depicted in Figure 3

3.2.2. *Example 2.* In this second example we keep a fixed $b_{21} = \alpha = 1$, and let the productivity $\Gamma_2 = \gamma$ vary in $[0, 3]$.

The effect of the variation in productivity γ on the choice of the node where to produce the consumption good is represented in Figure 4. For small values of γ, node 2 is less productive than the other two, and this feature dominates: the lesser impact on the system's profitability occurs when consumption takes place at node 2.

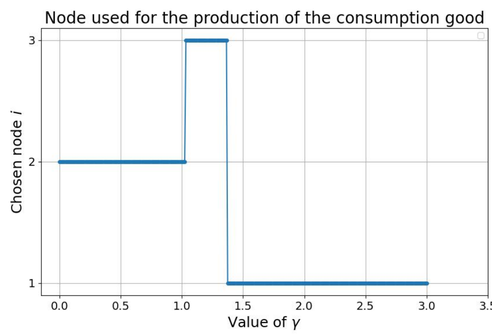

FIGURE 4. The node chosen for consumption for a varying γ.

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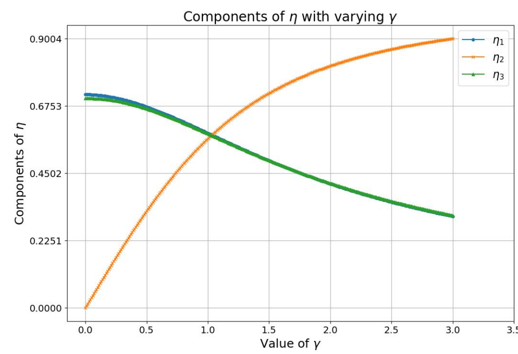

FIGURE 5. The components of the vector η for a varying γ.

As γ increases, the productivity of node 2 increases, along with its centrality, making it advantageous to move production to another node. The choice between the two remaining nodes is determined by two factors: the higher productivity of node 3 ($\Gamma_3 > \Gamma_1$) and the greater externalities of node 1 on node 3 compared to the reverse of node 3 on node 1. For intermediate values of γ, the second effect prevails, while for sufficiently large values of γ, the first effect dominates: for large γ, the externality resulting from production in node 2 outweighs the remaining externalities, making it optimal to establish production at node 1. The thresholds for γ at which a change takes place correspond to changes in the corresponding minimal (eigen)centrality η_i, as depicted in Figure 5.

3.3. **Admissibility.** From now on, we assume that there is a unique minimal (positive) coordinate of the eigenvector η.[6] By possibly renaming the nodes, we can

[6]Note that the assumption is not extremely restrictive, as the vectors η for which minima are multiple constitute a subset of $\mathbb{R}^n$ of zero Lebesgue measure.

assume

$$\eta_1 = \min_i \eta_i.$$

Next, we discuss the admissibility of the optimal control described by (8). The following remarks come in handy:

(a) The closed-loop equation (briefly, CLE), namely the evolution system associated with the optimal control (8), has the form

$$\dot{K}(t) = AK(t)$$

where A is the matrix

$$A = \Gamma - D + B^\top \Gamma - \frac{\theta}{\eta_1} e_1 \eta^\top. \tag{11}$$

so that the optimal trajectory is given by

$$K^*(t) = e^{tA}k.$$

(b) It is also useful to recall that a matrix A is said *eventually exponentially non negative (resp., positive)*, with exponential index t_0 if

$$e^{tA} \geq 0 \quad (\text{resp., } e^{tA} > 0), \quad \forall t \geq t_0. \tag{12}$$

All Metzler matrices are eventually exponentially non-negative with exponential index 0, and vice versa (see Lemma 3.1 in Noutsos and Tsatsomeros, 2008). Thus, condition (14) in Theorem 2 below is equivalent to requiring A eventually exponentially positive with exponential index 0, from which non-negativity of the trajectory is inferred, for every initial nonnegative stock k.

In the next Theorem, we discuss under which assumptions the control described by (8) is admissible in terms of eventual exponential positivity of the matrix A.

THEOREM 2 (Admissibility) *Assume $\sigma \neq 1$, $\theta > 0$. The optimal control c^* described in (8) is admissible (and then optimal) in the following two sets of assumptions:*

(i) *if A given by (11) is eventually exponentially positive, with index $t_0 = t_0(A)$ and the initial stock k lies in the cone K, defined by*

$$K := e^{t_0 A}\left(\mathbb{R}^n_+\right). \tag{13}$$

(ii) *if θ satisfies*

$$0 < \theta < \eta_1 \frac{\Gamma_j b_{j1}}{\eta_j}, \qquad \text{for all } j \in N, \text{ with } j \neq 1. \tag{14}$$

and the initial stock k is in the positive orthant $\mathbb{R}^n_+$.

Proof. See Appendix A. □

Remark 6 The assumption of A being eventually exponentially positive yields an implicit bound on the magnitude of θ. We refer the reader to Section 3.5 where we discuss how such implicit condition can be further made explicit.

3.4. **Long-run Stocks.** We now analyze the long-term behavior of the stock, establishing whether the stock tends to stabilize over time around certain values at different nodes. Note that for a null extraction, the convergence is toward the direction of the eigenvector ζ associated with the dominant eigenvalue λ, defined in Lemma 1 and Remark 2. Here, we will explain how the equilibrium extraction reduces the growth rate to $\lambda - \theta$ and modifies the direction of the associated eigenvector to $\hat{\zeta}$.

In the following lemma we establish a relationship between the eigenvectors and eigenvalues of $A^\top$ to those of $\Gamma - D + B^\top\Gamma$ (regardless of whether condition (14) is met).

Lemma 3 *Let η and ζ be respectively the (real) eigenvectors of the matrices $\Gamma B + \Gamma - D$ and its transpose, both associated to the dominant eigenvalue λ, as described in Lemma 1.*

(i) *The vector η is an eigenvector of $A^\top$ associated with the eigenvalue $\lambda - \theta$; hence, there exists a real eigenvector $\hat{\zeta}$ of A associated with $\lambda - \theta$. If $\theta > 0$ is small enough, then $\lambda - \theta$ is the dominant eigenvalue of both the matrices and $\hat{\zeta}$ is a positive vector.*

(ii) *Consider a basis $\{\zeta, v_2, \dots, v_n\}$ of generalized eigenvectors of $\Gamma - D + B^\top\Gamma$, associated with the eigenvalues $\{\lambda, \lambda_2, \dots, \lambda_n\}$. Then $\{\hat{\zeta}, v_2, \dots, v_n\}$ is a basis of generalized eigenvectors for A associated with eigenvalues $\{\lambda - \theta, \lambda_2, \dots, \lambda_n\}$. In particular, the eigenspace associated with $\lambda - \theta$ has dimension 1.*

Proof. See Appendix A. □

We now establish that, in the long run, the optimal trajectory K^* converges towards the direction of the eigenvector $\hat{\zeta}$ of A or, more precisely, that the *detrended optimal trajectory*

$$Y(t) = e^{-(\lambda-\theta)t} K^*(t)$$

converges towards a multiple of $\hat{\zeta}$, provided that θ is small enough.

Proposition 1 *Assume that either (i) or (ii) in Theorem 2 are satisfied. Assume also*

$$0 < \theta < \lambda - \operatorname{Re} \lambda_2,$$

where λ_2 is the eigenvalue with the highest real part among $\{\lambda_2, ..., \lambda_n\}$. Then the detrended optimal trajectory $Y(t)$ satisfies

$$\lim_{t \to +\infty} Y(t) = \frac{\langle k, \eta \rangle}{\langle \hat{\zeta}, \eta \rangle} \hat{\zeta} \tag{15}$$

Proof. See Appendix A. □

3.5. **Bounds on Impatience of the Decision Maker.** We have already noted that the assumption of A being eventually exponentially positive, appearing in Theorem 2(i) and in Proposition 1, conceals an implicit bound on the magnitude of θ, embodying "impatience" of the decision maker. Such interpretation is straightforward for a logarithmic utility where $\theta = \rho$ and a small enough θ can be seen as the decision maker being sufficiently patient.

We intend to provide a more explicit bound and try to verify the robustness of the results of Theorem 2(i) for changes in agent preferences in terms of impatience.

Remark 7 It will not be restrictive to limit the analysis to the case of a non-negative matrix $\Gamma B + \Gamma - D$. Indeed, this is a Metzler matrix. If there are negative elements on the main diagonal, we can add to $\Gamma B + \Gamma - D$ the matrix aI, and for a big enough the resulting matrix $\Gamma B + \Gamma - D + aI$ has the same eigenspaces (and generalized eigenspaces) as $\Gamma B + \Gamma - D$ but a spectrum which is shifted by a in the complex plane. For our purposes, which primarily involve understanding the relative ranking of the real parts of the eigenvalues of $\Gamma B + \Gamma - D$ and its submatrices, as well as the behaviors of the associated eigenvectors, there is no loss of generality in assuming that $\Gamma B + \Gamma - D$ is non-negative.

Remark 8 We recall the following facts:

(i) a matrix M is said to have *strong Perron–Frobenius property* if its spectral ray $\rho(M)$ is a (positive, real) simple eigenvalue, strictly larger than the norm of all other eigenvalues, and associated to a positive eigenvector; for nonnegative matrices, the condition on maximal norm can be replaced by $\rho(M)$ larger than *the real part* of all other eigenvalues;

(ii) the property (12) is equivalent to the following fact (Theorem 3.3 in Noutsos and Tsatsomeros, 2008):
There exists $a \geq 0$ such that $A + aI$ and $A^\top + aI$ both have the strong Perron–Frobenius property.

Now we analyze what happens when θ grows, starting from a positive level close to zero. Preliminarily, we observe that the feedback described in equation (8) naturally extends to the limiting case of $\theta = 0$ (even though the optimization problem is ill-posed in this scenario), with $c^* = 0$. In this situation, the system evolves with a matrix A defined in (11) coinciding with that of the system without extraction $\Gamma B + \Gamma - D$, implying $\zeta = \hat{\zeta}$. As noted in Section 2.1, A is irreducible, and thus (since it is also non-negative) it satisfies the strong Perron-Frobenius property described in Remark 8. Consequently, every detrended trajectory converges to a multiple of ζ, in view of Proposition 1.

When instead θ is strictly positive, Lemma 3 indicates the two phenomena at play for increasing values of θ:

(1) the eigenvector $\hat{\zeta}$ (which is ζ modified by the effect of consumption) may cease to be contained inside the positive orthant; in this case, the trajectory associated with A may bear negative or null components; possibly, this fact takes place for θ surpassing a first threshold θ_1 (yet to be determined, see also Lemma 4);

(2) if θ surpasses the threshold $\theta_2 := \lambda - Re(\lambda_2)$, then $\lambda - \theta$ is no longer the greatest eigenvalue of A; in this case, the trajectories of the system ruled by A no longer converge towards the direction of $\hat{\zeta}$, the system becomes unstable, and condition (12) fails to hold (although a trajectory starting on the direction of $\hat{\zeta}$ may still be optimal).

In what follows we will prove that, at least in the case in which λ_2 is real, the threshold θ_1 exists, we provide a characterization of θ_1, and we show that

$$0 < \theta_1 \leq \theta_2$$

meaning that, for increasing values of θ, the stability is lost after the modified dominant eigenvector $\hat{\zeta}$ leaves the positive orthant.

LEMMA 4 *We define λ_{22} as the dominant eigenvalue*[7] *of the $(n-1) \times (n-1)$ matrix A_{22}, obtained from A by removing the first row and the first column, and we set $\theta_1 = \lambda - \lambda_{22}$. If $0 < \theta < \theta_1$, then $\hat{\zeta}$ is a positive vector. If $\theta = \theta_1$ then $\hat{\zeta}$ is non-negative with null first component $\hat{\zeta}_1$.*

Proof. See Appendix A. □

[7] The matrix A_{22} is non-negative (see Remark 7), we can then apply the weak form of the Perron-Frobenius Theorem and obtain that A_{22} has a non-negative real eigenvalue, larger (or equal) than the real part of any other eigenvalue of A_{22}.

The next proposition orders the thresholds θ_1, θ_2 when λ_2 is real.

PROPOSITION 2 *Suppose that* λ_2*, the second (ordered in terms of the highest real part) eigenvector of* $\Gamma - D + B^\top \Gamma$ *is real. Then, as long as* $0 < \theta < \theta_1$*, we have* $\lambda_2 < \lambda - \theta$.

Proof. See Appendix A. □

The above proposition implies $0 < \theta_1 \leq \theta_2$, where $\theta_1 = \lambda - \lambda_{22}$ and $\theta_2 = \lambda - \lambda_2$ Therefore, Theorems 2 and Proposition 1 hold for $0 < \theta < \theta_1$.

4. Extensions and Games

4.1. **An Extension of the Model with Transportation Costs.** We now assume that the intertemporal utility takes into account iceberg-type transportation costs $\beta_i \in [0, 1)$, namely

$$J(c) = \int_0^{+\infty} e^{-\rho t} u \left(\sum_{i=1}^{n} (1 - \beta_i) c_i(t) \right) dt, \tag{16}$$

where $\beta = (\beta_1, \cdots, \beta_n)^\top$. A $\beta_i = 0$ means that there would be no loss of consumption goods during transportation, while $\beta_i = 1$ would mean a complete loss. In this case, the Hamiltonian function becomes

$$\tilde{H}(p) = \max_{c \geq 0} \left\{ u \left(\langle \mathbf{e}, (I - \mathcal{B}) c \rangle \right) - \langle c, p \rangle \right\},$$

where

$$I - \mathcal{B} = \begin{pmatrix} 1 - \beta_1 & 0 & \cdots & 0 \\ 0 & 1 - \beta_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 - \beta_n \end{pmatrix}$$

and since $\langle c, p \rangle = \langle (I - \mathcal{B})^{-1} p, (I - \mathcal{B}) c \rangle$, one has

$$\tilde{H}(p) = H((I - \mathcal{B})^{-1} p) = \begin{cases} \dfrac{\sigma}{1 - \sigma} \left(\min\limits_i \dfrac{p_i}{1 - \beta_i} \right)^{1 - \frac{1}{\sigma}}, & \sigma \neq 1, \\ - \left[\ln \left(\min\limits_i \dfrac{p_i}{1 - \beta_i} \right) + 1 \right], & \sigma = 1. \end{cases} \tag{17}$$

In this more general context, the results obtained for null iceberg costs can be replicated. In particular the analog of Theorem 1 reads as follows.

THEOREM 3 *Assume* $\sigma \neq 1$, $\theta > 0$, $k \in \mathbb{R}^n_+$*, and either set of assumptions:*

(*i*) θ *satisfies* (14)*;*

(ii) $k \in K$, *where* K *is defined by* (13), *and* (12) *is satisfied.*

Then, the Value Function of the problem of maximizing (16) , *subject to* (1) *is*

$$v(k) = \frac{\theta^{-\sigma}}{1-\sigma} \left(\min_i \frac{\eta_i}{1-\beta_i} \right)^{\sigma-1} \langle k, \eta \rangle^{1-\sigma}. \tag{18}$$

Moreover, if we define

$$N^*_\beta = argmin_{i \in N} \left\{ \frac{\eta_i}{1-\beta_i} \right\}$$

the closed-loop optimal controls are the vectors $c^* \in \mathbb{R}^n_+$ *such that*

$$c^*_j = 0, \quad \forall j \notin N^*_\beta \quad \textit{and} \quad \sum_{i \in N^*_\beta} c^*_i = \frac{\theta}{\min_i \frac{\eta_i}{1-\beta_i}} \langle k, \eta \rangle. \tag{19}$$

Proof. See Appendix A. □

5. Conclusions

This study delves into a model of distributed capital stocks across nodes, each representing distinct production sites interconnected through a directed, weighted network. The network is used to represent the spillover effects originating from production at one site, impacting the capital stock of neighboring sites.

A primary aspect of the investigation revolves around a regulatory decision to earmark specific nodes for consumption goods production to maximize cumulative utility from consumption. Results highlight the complex relationship between optimal strategies, capital stocks, and both the productivity of resource nodes and the structure of their interconnections. The optimal locations to draw resources for consumption corresponds to the nodes with the least eigenvector centrality in a redefined network that merges node productivities with inter-node flows. The study emphasizes the critical role of network structure and node productivity in shaping production, consumption, and resource allocation decisions.

Some unresolved questions remain. The primary and most immediate one is what transpires when the conditions outlined in Theorem 2 are not satisfied, leading to the inadmissibility of the proposed optimal control. There are two potential reasons why the theorem's assumptions may not hold. Firstly, even if the parameter constraints are adhered to, it is possible for the system to initiate from a state outside the cone specified in the statement. One would be willing to understand, in this scenario, how the dynamics is characterized and whether it is possible to determine whether the system still converges to the steady state described in Proposition 1. The second reason pertains to situations where the assumptions on the parameters, particularly concerning the agent's level of impatience, are not fulfilled. This second scenario gains more significance in the context of an extension that encompasses transportation costs, particularly when these costs carry substantial weight.

References

Ballester, C., Calvó-Armengol, A., and Zenou, Y. (2006). Who's who in networks. wanted: The key player. *Econometrica*, 74(5):1403–17.

Bapat, R. B. and Raghavan, T. E. S. (1997). *Nonnegative matrices and applications*. Cambridge university press.

Benhabib, J., Meng, Q., and Nishimura, K. (2000). Indeterminacy under constant returns to scale in multisector economies. *Econometrica*, 68(6):1541–8.

Boucekkine, R., Camacho, C., and Fabbri, G. (2013). Spatial dynamics and convergence: The spatial ak model. *Journal of Economic Theory*, 148(6):2719–2736.

Boucekkine, R., Fabbri, G., Federico, S., and Gozzi, F. (2019). Growth and agglomeration in the heterogeneous space: a generalized ak approach. *Journal of Economic Geography*, 19(6):1287–1318.

Calvia, A., Gozzi, F., Leocata, M., Papayiannis, G. I., Xepapadeas, A., and Tannacopoulos, A. N. (2023). An optimal control problem with state constraints in a spatio-temporal economic growth model on networks. *arXiv preprint arXiv:2304.11568*.

Dolmas, J. (1996). Endogenous growth in multisector ramsey models. *International Economic Review*, 37(2):403–421.

Elliott, M. and Golub, B. (2019). A network approach to public goods. *Journal of Political Economy*, 127(2):730–76.

Fabbri, G. (2016). Geographical structure and convergence: A note on geometry in spatial growth models. *Journal of Economic Theory*, 162:114–136.

Fabbri, G., Faggian, S., and Freni, G. (2024). On competition for spatially distributed resources in networks. *Theoretical Economics*.

Fleming, W. H. and Rishel, R. W. (2012). *Deterministic and stochastic optimal control*, volume 1. Springer Science & Business Media.

Le Van, C. and Vailakis, Y. (2005). Recursive utility and optimal growth with bounded or unbounded returns. *Journal of Economic Theory*, 123(2):187–209.

McFadden, D. (1973). On the existence of optimal development programmes in infinite-horizon economies. In Mirrlees, J. A. and Stern, N. H., editors, *Models of Economic Growth*, pages 260–282. Macmillan: Great Britain.

Noutsos, D. and Tsatsomeros, M. J. (2008). Reachability and holdability of nonnegative states. *Journal on Matrix Analysis and Applications*, 30(2):700–712.

Rebelo, S. (1991). Long-run policy analysis and long-run growth. *Journal of Political Economy*, 99(3):500 –21.

Sanchirico, J. N. and Wilen, J. E. (2005). Optimal spatial management of renewable resources: matching policy scope to ecosystem scale. *Journal of Environmental Economics and Management*, 50:23–46.

Appendix A. Appendix: Proofs

Proof of Lemma 2. Assume $p \in (0,+\infty)^n$. We show that the maximum in (6) is obtained when consumption takes place only at the node(s) where prices p_i are at their minimum. More specifically, we set

$$N(p) = \mathrm{argmin}_i\{p_i\} \subseteq N, \quad \hat{C}(p) = \{\hat{c} \in \mathbb{R}^n_+ : \hat{c}_i = 0, \forall i \notin N(p)\}.$$

We show that for every $c \in \mathbb{R}^n_+$ there exists a $\hat{c} \in \hat{C}(p)$ such that

$$u(\langle \mathbf{e}, c\rangle) - \langle c, p\rangle \le u(\langle \mathbf{e}, \hat{c}\rangle) - \langle \hat{c}, p\rangle.$$

Indeed, we can choose $\hat{c}$ such that $q := \sum_i \hat{c}_i = \sum_i c_i$, and note that

$$\sum_i p_i c_i \ge (\min_i p_i) \sum_i c_i = (\min_i p_i) \sum_{i\in N(p)} \hat{c}_i = \sum_i p_i \hat{c}_i,$$

where the last equality holds as $p_i = \min_j p_j$ for all $i \in N(p)$. As a consequence

$$\max_{c\ge 0} \{u(\langle \mathbf{e}, c\rangle) - \langle c, p\rangle\} = \max_{c\in \hat{C}(p)} \{u(\langle \mathbf{e}, c\rangle) - \langle c, p\rangle\} = \max_q \left\{u(q) - (\min_i p_i) q\right\},$$

and, since $p \in (0,+\infty)^n$, implies $\min_i p_i > 0$

$$q^* = \mathrm{argmax}_q\{u(q) - q(\min_i p_i)\} = (u')^{-1}(\min_i p_i),$$

so that the maximum in (6) is attained at

$$c_i^* = 0, \quad \forall i \notin N(p) \quad \text{and} \quad \sum_{i\in N(p)} c_i^* = q^*, \tag{20}$$

providing the characterization of maximizers, and (6) readily follows.

Now we assume $p \in \partial\mathbb{R}^n_+$, so that $\min_i p_i = 0$. Then for $\sigma > 1$, the utility u is bounded above by 0, and formula (6) extends to this case, with $H(p) = 0$. On the other hand, when $\sigma \le 1$, and $p \in \partial\mathbb{R}^n_+$, at least one coordinate of p is null, say, $p_j = 0$. Consider now the controls of type $c = \gamma e_j$ with $\gamma \ge 0$. One has

$$u(\langle \mathbf{e}, \gamma e_j\rangle) - \langle \gamma e_j, p\rangle = u(\gamma) - \gamma p_j = u(\gamma),$$

so that

$$H(p) = \sup_{c\ge 0} \{u(\langle \mathbf{e}, c\rangle) - \langle c, p\rangle\} \ge \sup_{\gamma\ge 0} \{u(\gamma)\} = +\infty.$$

□

Proof of Theorem 1. We search for a solution of HJB of type $v(k) = \frac{b}{1-\sigma}\langle k, \eta\rangle^{1-\sigma}$, with $\nabla v(k) = b\langle k, \eta\rangle^{-\sigma}\eta$ so that HJB would imply

$$\frac{\rho b}{1-\sigma}\langle k, \eta\rangle^{1-\sigma} = \frac{\sigma}{1-\sigma}\left(\min_i \frac{\partial v}{\partial k_i}\right)^{1-\frac{1}{\sigma}} + \langle \nabla v(k), [(I + B^{\top})\Gamma - D]k\rangle \tag{21}$$

$$= \frac{\sigma}{1-\sigma}\left(\min_i \eta_i\right)^{1-\frac{1}{\sigma}} b^{1-\frac{1}{\sigma}}\langle k, \eta\rangle^{1-\sigma} + \lambda b\langle k, \eta\rangle^{1-\sigma} \tag{22}$$

that is

$$b = \left(\frac{\sigma}{\rho - \lambda(1-\sigma)}\right)^{\sigma}\left(\min_i \eta_i\right)^{\sigma-1} = \theta^{-\sigma}\left(\min_i \eta_i\right)^{\sigma-1}$$

so that by (20) we derive

$$\sum_{i\in N^*} c_i^* = \left(b\langle k, \eta\rangle^{-\sigma}(\min_i \eta_i)\right)^{-\frac{1}{\sigma}} = \frac{\theta}{\min_i \eta_i}\langle k, \eta\rangle, \quad c_i^* = 0,\ i \notin N^*.$$

We can then apply a rather standard verification technique (see for instance Fleming and Rishel, 2012) to prove that the obtained controls are optimal and that v is indeed the value function of the problem. The uniqueness of the optimal control follows from the concavity of the value function along the directions which do not belong in $span_{i\in N^*}\{e_i\}$. □

Proof of Remark 4. The proof of the analog of Theorem 1 for the case of logarithmic utility is obtained by replicating the argument of the proof for a candidate value function of type $v(k) = a\ln(k) + b$ and deriving the coefficients a and b from HJB equation. □

Proof of Theorem 2. (i) The only property to check is

$$e^{tA}(K) \subset \mathbb{R}^n_+.$$

By (12), we have that $e^{tA} > 0$ for all $t \geq t_0$, thus $K \subseteq \mathbb{R}^n_+$. Moreover, by definition $k \in K$ implies $k = e^{t_0 A}k_1$ for some $k_1 \in \mathbb{R}^n_+$, which implies that, for every $s \geq 0$,

$$e^{sA}k = e^{(s+t_0)A}k_1 \geq 0.$$

(ii) When instead the stronger assumption (14) holds, it is immediate to check that the matrix A is a Metzler matrix, so that the trajectory $X^*(t) = e^{tA}k$ remains positive at all times, for every initial condition k.

□

Proof of Lemma 3. The validity of (i) is straightforward. For the proof of (ii), we observe first that any generalized eigenvector v corresponding to an eigenvalue $\lambda_i \neq \lambda$ is orthogonal to η. Let us consider an eigenvalue $\lambda_i \neq \lambda$ (which implies $i \geq 2$) and let v_i be an element of the generalized eigenspace V_i. There exists a positive integer m such that $(\Gamma - D + B^\top \Gamma - \lambda_i I)^m v_i = 0$. This leads to:

$$0 = \eta^\top \left[(\Gamma - D + B^\top \Gamma - \lambda_i)^m v_i \right] = \left[\eta^\top (\Gamma - D + B^\top \Gamma - \lambda_i)^m \right] v_i = (\lambda - \lambda_i)^m \eta^\top \, v_i.$$

Since $\lambda \neq \lambda_i$, it follows that $\eta^\top \, v_i = 0$, which means that η is orthogonal to v_i. Given the definition of A this fact ensures that v_i is also a generalized eigenvector of A with the eigenvalue λ_i.

Knowing that $\hat{\zeta}$ is an eigenvector for A with the eigenvalue $\lambda - \theta$, it remains to note that the set $\{\hat{\zeta}, v_2, \dots, v_n\}$ consists of linearly independent vectors. This is evident since $\{v_2, \dots, v_n\}$ are linearly independent (forming a subset of a basis) and $\hat{\zeta}$ belongs to a distinct generalized eigenspace (of A) from all the V_i for $i \geq 2$, ensuring it cannot be expressed as a linear combination of the v_i for $i \geq 2$. □

Proof of Proposition 1. Since, thanks to Lemma 3 we know that the eigenspace associated with $\lambda - \theta$ has dimension 1 and all other eigenvectors have a lower real part than $\lambda - \theta$, it is straightforward to prove that the detrended trajectory converges to some real multiple of $\hat{\zeta}$. So there exists $\alpha > 0$ such that

$$\lim_{t \to +\infty} Y(t) = \alpha \hat{\zeta}.$$

On the other hand, applying first the definition of $Y(t)$ and then (10), we derive

$$\langle Y(t), \eta \rangle = \langle e^{-\lambda t} X(t), \eta \rangle \equiv \langle k, \eta \rangle.$$

so that the left-hand side is constant in time. Consequently

$$\langle k, \eta \rangle = \lim_{t \to +\infty} \langle Y(t), \eta \rangle = \langle \lim_{t \to +\infty} Y(t), \eta \rangle = \langle \alpha \hat{\zeta}, \eta \rangle,$$

so that

$$\alpha = \frac{\langle k, \eta \rangle}{\langle \hat{\zeta}, \eta \rangle}.$$

□

Proof of Lemma 4 . We show first that, for an increasing value of θ, the first component $\hat{\zeta}_1$ of $\hat{\zeta}$ is the first (or among the first) to become non-positive. By contradiction, assume that $\hat{\zeta}$ is non-negative, with $\hat{\zeta}_1 > 0$ and $\hat{\zeta}_\ell = 0$ for some $\ell \neq 1$. Since $\hat{\zeta}$ is an eigenvector of A of eigenvalue $\lambda - \theta$, along the optimal evolution of the system starting at $\hat{\zeta}$, we have $K(t) = e^{tA}\hat{\zeta} = e^{t(\lambda-\theta)}\hat{\zeta}$. In particular, if $\hat{\zeta}_\ell = 0$ then $K_\ell(t) = \dot{K}_\ell(t) \equiv 0$ for all $t \geq 0$. Hence

$$\dot{K}_\ell(t) = (\Gamma_\ell - \delta_\ell)K_\ell(t) + \sum_{j\neq\ell} b_{j\ell}\Gamma_j K_j(t) \Rightarrow \sum_{j\neq\ell} b_{j\ell}\Gamma_j K_j(t) = 0 \tag{23}$$

at all times t. Since all terms in the last sum are nonnegative and $\Gamma_j > 0$, either $b_{j\ell}$ or $K_j(t) = e^{t(\lambda-\theta)}\hat{\zeta}_j$ is null. That implies $\hat{\zeta}_j = 0$ for all j's such that a positive flow $b_{j\ell}$ exists from j to ℓ. Iterating the argument and using the fact that the network is strongly connected, we obtain that $\hat{\zeta}_1$ needs to be equal to zero as well. This contradicts our initial assumption of a strictly positive first component[8].

So, as we increase the value of θ, the first component to become non-positive is necessarily the first one. When the first component is zero, the eigenvector has the form $\hat{\zeta} = (0, \hat{\zeta}_2)$ where $\hat{\zeta}_2$ is a non-negative vector in $\mathbb{R}^{n-1}$ and

$$(\lambda - \theta)(0, \hat{\zeta}_2) = A(0, \hat{\zeta}_2) = (a_1, A_{22}\hat{\zeta}_2)$$

so that $a_1 = 0$ and $\lambda - \theta$ is an eigenvalue of A_{22}. Increasing θ this condition is satisfied for the first time when $\hat{\zeta}_2$ is an eigenvector for the dominant eigenvalue λ_{22}. This proves the claim. □

Proof of Proposition 2 . We rewrite the matrix $B^\top\Gamma + \Gamma - D$ as composed of four blocks

$$B^\top\Gamma + \Gamma - D = \begin{pmatrix} f_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$$

where f_{11} is a 1×1 matrix, F_{22} is a $(n-1) \times (n-1)$ matrix, and F_{12} and F_{21} are respectively $1 \times (n-1)$ and $(n-1) \times 1$ matrices. Consequently, A can be written as

$$A = \begin{pmatrix} f_{11} - \theta & \left(F_{12} - \theta\left(\frac{\eta_2}{\eta_1}, \frac{\eta_3}{\eta_1}, \frac{\eta_4}{\eta_1}, \ldots, \frac{\eta_n}{\eta_1}\right)\right) \\ F_{21} & F_{22} \end{pmatrix}.$$

If $v = (v_1, \ldots, v_n)$ is an eigenvector associated to λ_2 we have, restricting to the last $n-1$ components,

$$v_1 F_{21} + F_{22}(v_2, .., v_n)^\top = \lambda_2 (v_2, .., v_n)^\top.$$

[8]Note that the evolution of $K_1(\cdot)$, differently from those of the other k_j's, bears an additional non-positive term

$$\dot{K}_1(t) = (\Gamma_1 - \delta_1)K_1(t) + \textstyle\sum_{j\neq 1} b_{j1}\Gamma_j K_j(t) - \frac{\theta}{\eta_1} e_1 \eta^\top K(t)$$

so that $K_1(\cdot)$ can be zero, together with its time-derivative, without implying all terms $b_{j\ell}\Gamma_j K_j(t)$ are zero.

Now two cases may occur:

(i) if $v_1 = 0$ then $(v_2, .., v_n)$ is an eigenvector of F_{22} but, since λ_{22} is the dominant eigenvalue of F_{22} we have $\lambda_2 < \lambda_{22} < \lambda - \theta$ and we get the claim;

(ii) if $v_1 \neq 0$ we can suppose (up torescaling) that $v_1 = 1$ and we get

$$F_{21} = (\lambda_2 I - F_{22})(v_2, .., v_n)^\top.$$

We observe that, since v is an eigenvector for $\Gamma - D + B^\top \Gamma$, which is irreducible, but not the one associated with the dominant eigenvalue, then it necessarily has negative coordinates among $v_2, \dots, v_n$.

If by contradiction, $\lambda_2 > \lambda_{22}$ then $\lambda_2 I - F_{22}$ is invertible and its inverse can be written as

$$\frac{1}{\lambda_2}\left(1 - \frac{F_{22}}{\lambda_2}\right)^{-1} = \frac{1}{\lambda_2}\sum_{k=0}^{\infty}\left(\frac{F_{22}}{\lambda_2}\right)^k$$

so that

$$(v_2, \dots, v_n)^\top = \frac{1}{\lambda_2}\sum_{k=0}^{\infty}\left(\frac{F_{22}}{\lambda_2}\right)^k F_{21}$$

but, since F_{21} is non-negative and each term of the sum $\sum_{k=0}^{\infty}\left(\frac{F_{22}}{\lambda_2}\right)^k$ is non-negative, that would imply that $(v_2, \dots, v_n)^\top$ is non-negative, a contradiction. □

Proof of Theorem 3 . The proof is very similar to that of Theorem 1 so that here we point out only the differences.

We search for a solution of HJB of type $v(k) = \frac{b}{1-\sigma}\langle k, \eta\rangle^{1-\sigma}$, with $\nabla v(k) = b\langle k, \eta\rangle^{-\sigma}\eta$, so that HJB would imply

$$\frac{\rho b}{1-\sigma}\langle k, \eta\rangle^{1-\sigma} = \frac{\sigma}{1-\sigma}\left(\min_i \frac{1}{1-\beta_i}\frac{\partial v}{\partial k_i}\right)^{1-\frac{1}{\sigma}} + \langle \nabla v(k), [(I + H^\top)\Gamma - D]k\rangle \tag{24}$$

$$= \frac{\sigma}{1-\sigma}\left(\min_i \frac{\eta_i}{1-\beta_i}\right)^{1-\frac{1}{\sigma}} b^{1-\frac{1}{\sigma}}\langle k, \eta\rangle^{1-\sigma} + \lambda b\langle k, \eta\rangle^{1-\sigma} \tag{25}$$

that is

$$b = \left(\frac{\sigma}{\rho - \lambda(1-\sigma)}\right)^{\sigma}\left(\min_i \frac{\eta_i}{1-\beta_i}\right)^{\sigma-1} = \theta^{-\sigma}\left(\min_i \frac{\eta_i}{1-\beta_i}\right)^{\sigma-1}$$

so that

$$q^* = \sum_{i\in N^*} c_i^* = \left(b\langle k, \eta\rangle^{-\sigma}\left(\min_i \frac{\eta_i}{1-\beta_i}\right)\right)^{-\frac{1}{\sigma}} = \frac{\theta}{\min_i \frac{\eta_i}{1-\beta_i}}\langle k, \eta\rangle$$

The remainder of the proof proceeds as in the case of Theorem 1. □

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