LONG-TERM CARE EXPENDITURES AND INVESTMENT DECISIONS UNDER UNCERTAINTY

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ABSTRACT. Long-term care (LTC) expenditures of the elderly are high in developed countries and will grow further with population aging. In addition, LTC costs are heterogeneous across individuals and unknown early in life. In this paper, we add uncertainty over the arrival and magnitude of future LTC costs into a life-cycle model with endogenous aging, and we analyze how this affects the optimal behavior of agents. We show that uncertainty boosts precautionary savings, lowers investment in preventive care, and weakens the effectiveness of subsidies to encourage prevention. Our results therefore suggest that uncertainty should not be ignored in models that study positive or normative aspects of health investment.

JEL Codes: C60, D15, D81, I12, I18.

Keywords: health; long-term care costs; uncertainty; stochastic model.

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1. Introduction

Health care costs surge at old age. In the US, for instance, average medical expenditures for a 65+ individual are 2.6 times higher than the national average (De Nardi et al., 2016). Long-term care (LTC), which involves the services designed to meet a person's everyday needs, is a key component of this increase in health care costs (OECD, 2020). With the threat of rising medical bills looming on the horizon, young adults are often encouraged to save and to invest in preventive health care. Preventive health care lowers the burden of disease and associated risk factors. Examples include medical screening, exercise and a healthy diet. Though recent research has studied this intertemporal link between choices by young adults and old-age LTC costs (Marchiori and Pierrard, 2022; Schünemann et al., 2022), an essential element is missing: uncertainty.[1]

Indeed, medical bills of elderly people are not only high, but also wildly uncertain. According to De Nardi et al. (2018), the average 70-year-old US household pays $122 000 in medical-related activities over remaining life. At the right tail of the distribution, however, the outlook is rather grimmer: 5% of households pay more than $300 000 and 1% more than $600 000. Moreover, much of the risk is uninsured, for the private insurance market for LTC is almost nonexistent (Brown and Finkelstein, 2011).[2] Our paper therefore draws attention to how uncertainty shapes the interactions between preventive care, savings, and LTC costs. Using a simple theoretical model, we will argue that uncertainty about LTC costs *boosts* savings, *lowers* investment in preventive care, and *weakens* the effectiveness of preventive care subsidies.

Our baseline model follows the deterministic life-cycle setup of Dalgaard and Strulik (2014) and Schünemann et al. (2022). A simple law of motion for human frailty is at the core of the model: as individuals age, they become weaker. This aging process is unstoppable, but individuals can slow it down by investing in preventing care. In its deterministic version, the model assumes that upon reaching a fixed level of human frailty, individuals begin to pay LTC costs proportional to their frailty levels. Therefore, individuals know when they will begin to pay LTC costs (*switch time* henceforth), and also know their overall LTC bills.

This is not true in our stochastic setup: the switch time is no longer given by a fixed known threshold, but by a random variable with a hazard rate increasing in the level of human frailty.

[1]Risk and uncertainty are different notions: risk can be measured precisely whereas uncertainty cannot be expressed by specific probabilities of occurrences (Knight, 1921). Strictly speaking, our analysis relates to risk but, by abuse of language, we will use the two terms interchangeably, as in the related literature (see below).

[2]Estimations from De Nardi et al. (2018) include out-of-pocket as well as Medicaid expenditures, which reduce the 'net' medical spending risks, especially of the poorest households. These considerations are beyond the scope of our paper.

Our modeling approach implies that preventive health care delays the *expected* switch time, thus introducing risk: however large an individual's health investment, LTC costs can bite at any moment.[3] Modeling the switch time as a random variable generates both income risk and return risk. Income risk because total LTC costs, and hence net income, can no longer be perfectly predicted. Return risk because preventive care is now a risky investment with an uncertain return. These two risk sources are behind all of our insights.

We obtain three main results. First, uncertainty over the switch time makes it optimal to reduce investment in preventive care. While in the deterministic setup every extra unit of preventive care pushes the switch time into the future, in the stochastic setup preventive care only pushes the expected switch time into the future. In other words, the payoff on health investment is certain in the deterministic setup but uncertain in the stochastic one. Under this return risk, two well-known opposite effects are at play (see for instance Sandmo, 1970): an income effect which stimulates individuals to increase health investment in face of uncertainty, and a substitution effect which encourages individuals to hold fewer risky assets (here health investment). Jouini et al. (2013) show that under standard assumptions on the utility function, the second effect dominates. This is what we also obtain in our model.

Our second result is that uncertainty reduces the effectiveness of subsidies for investment in preventive care. Put differently, deterministic setups overestimate the effectiveness of health policy. This result can be understood as a portfolio choice problem. Subsidies on health investment will increase its return. If health investment is risk-free, an investor will augment its portfolio share, but if the return to investment is stochastic, a second – opposite – effect arises, more health investment also makes the portfolio riskier. As a result, a risk-averse investor will invest less in health (see for instance Chang, 1996, for a similar discussion).

Third, uncertainty over the switch time increases precautionary savings. That is, risk-averse individuals build a capital buffer to meet sooner-than-expected medical bills. This finding echoes the unambiguous link between income uncertainty and precautionary savings put forward by Sandmo (1970), Skinner (1988), Caballero (1991) and many others.

Our work relates to the literature extending the seminal Grossman (1972) model with uncertainty; see Dardanoni and Wagstaff (1990) for a static setup, and Nocetti and Smith (2010) and Picone et al. (1998) for discrete-time versions. This body of work, where medical care enhances health, which enters utility directly, conveys conflicting messages about the effects

[3]Throughout the paper, uncertainty only relates to the switch time, i.e. to the moment an individual starts paying LTC costs. An alternative would be a fixed switch time but uncertain LTC costs. This would leave our key insights unchanged. We also assume an exogenous end time (death). However, a similar kind of uncertainty could apply to death: healthy people die later on average, but there exists individual randomness. We leave this extension to 'double uncertainty' (switch time and end time) for future research.

of uncertainty on health investment. Results typically depend on the way uncertainty is introduced and health enters the utility function. We differ from this literature in at least four ways: (i) ours is a continuous-time model with an endogenous regime switch; (ii) physical frailty entails costs – or equivalently affects disposable income – rather than disutility flows; (iii) uncertainty blurs the effectiveness of preventive care (return risk on health investment); (iv) uncertainty also generates a net income risk. Our approach is therefore in line with the empirical observations in De Nardi et al. (2016, 2018) that medical expenses of the elderly are high and uncertain, come mostly from nursing home spending, and once incurred, tend to be persistent over time.

Our paper also connects to the literature modeling health externalities in deterministic frameworks (see e.g. Kuhn et al., 2011; Leroux et al., 2011). This class of models exhibit under-investment in health, since individuals do not fully internalize its marginal benefits. In our setup, however, low health investment does not originate from a market failure, but from an intrinsic feature of the economy: uncertainty.

Section 2 presents the deterministic model. Section 3 adds uncertainty (stochastic model). Section 4 studies the effectiveness of a health subsidy policy. Section 5 extends the previous model with savings. Section 6 concludes.

2. Deterministic Model

Our benchmark model is a simplified version of the deterministic setup due to Dalgaard and Strulik (2014) and Schünemann et al. (2022). As humans age, they experience a growing number of disorders, often referred to as 'health deficits'. When this deficit reaches a certain level, individuals incur persistent medical expenditures (or equivalently, LTC costs). Agents thus face an intertemporal tradeoff: boost current utility by raising consumption or slow down the aging process, and the future LTC costs, by investing in preventive care. To solve the model analytically, we assume that (i) agents cannot save; and that (ii) preventive care has a linear impact on health deficits. To simplify discussion, we do not discount the future. As shown in Section 5, our findings are robust to all these assumptions.

2.1. **Setup.**

An individual lives from age $t = 0$ to $t = T > 0$. Her overall welfare is thus given by

$$\int_0^T \ln c(t)\ \mathrm{d}t - \phi\ d(T) \tag{1}$$

with $\phi > 0$. As usual, the first term represents utility flows from consumption c, while the second term captures the salvage cost of health deficit $d(T)$. Without this salvage cost, there is no reason to invest in health towards the end of life, and h would tend to $-\infty$ when

$t \to T$.[4] This health deficit, which embodies the link between aging and frailty, evolves as

$$\dot{d}(t) = \mu(d(t) - Ah(t)) \tag{2}$$

where $\mu \in (0, 1/T)$ is the natural growth rate of d, that is, its growth rate in the absence of preventive care h. $A > 0$ governs the effectiveness of preventive care at slowing health deficit accumulation. At each instant $t \in (0, T)$, individuals receive an exogenous income $y > 0$. the budget constraint is then

$$c(t) = \begin{cases} y - h(t) & \text{if} \quad d(t) < \bar{d} \\ y - h(t) - Bd(t) & \text{if} \quad d(t) \geq \bar{d} \end{cases} \tag{3}$$

where $B > 0$. The rationale for investing in preventive care is now clear: as soon as the health deficit $d(t)$ crosses the known threshold $\bar{d}$, individuals face LTC costs $Bd(t)$.

Agents choose the sequence $\{h(t)\}_{t=0}^{T}$ that maximizes (1) subject to (2) and (3). This optimization program has two boundary conditions: $d(0) = d_0$ with $0 < d_0 < \bar{d}$, and $q(T) = -\phi$, where q is the shadow price (i.e. intrinsic value) of d. The initial condition ensures that individuals do not face LTC costs when they are born. The final condition balances the shadow price of $d(T)$ and the marginal benefit that an extra unit of $d(T)$ would have on welfare. Since the marginal benefit is in fact negative in our setup, $q(T)$ is also negative.

2.2. **Solution.** We need to solve a two-stage optimal control problem featuring one control $h(t)$, one state $d(t)$, and one co-state $q(t)$. Solving the model means finding the optimal switch time $\tau \in (0, T)$ such that $d(\tau) = \bar{d}$, and the optimal piecewise sequences for $\{h(t), q(t), d(t)\}_{t=0}^{T}$. Below we directly report the model's optimality conditions, but Appendix A provides mathematical details and intuitions. Importantly, we implicitly assume when deriving the solution below that $c(t) > 0$ and that the regime switch happens once and only once during the lifetime. In Section 2.3, we provide sufficient conditions on the parameters to satisfy these assumptions.

(i) Solution paths when $0 \leq t < \tau$

$$\begin{cases} q(t) &= -\frac{\tau \exp(-\mu t)}{(\bar{d} - Ay)\exp(-\mu\tau) - (d_0 - Ay)} \\ d(t) &= Ay + (d_0 - Ay)\frac{\tau - t}{\tau}\exp(\mu t) + (\bar{d} - Ay)\frac{t}{\tau}\exp(-\mu(\tau - t)) \\ h(t) &= y + \frac{1}{\mu A q(t)} \end{cases}$$

[4] An alternative solution is to impose an upper bound $\tilde{d}$ on $d(T)$ and use the Kuhn-Tucker conditions, but we have no information on how to choose the upper bound. We may interpret the salvage cost in two ways. One can see it as utility obtained from a 'decent' end of life (or equivalently disutility from a 'painful' end of life). Alternatively, one can consider that the final period (death) $T' > T$ is endogenous. The salvage value encompasses all extra income from the non-modeled period $(T, T']$. The higher is health deficit, the lower is T' and hence the extra income.

(ii) Solution paths when $\tau \leq t \leq T$

$$\begin{cases} q(t) &= -\phi \exp(\eta(T-t)) \\ d(t) &= (\bar{d}-\theta)\exp(\eta(t-\tau)) + \frac{t-\tau}{\phi}\exp(-\eta(T-t)) + \theta \\ h(t) &= y - Bd(t) + \frac{1}{\mu A q(t)} \end{cases}$$

Here $\eta \equiv \mu(1+AB)$ and $\theta \equiv Ay/(1+AB)$. Though $d(t)$ is continuous at τ, neither its time derivative nor $q(t)$ nor $h(t)$ are. The logic is straightforward: once the threshold is crossed, agents' budget constraints change, and so do shadow prices, investment in preventive care, and the speed at which health deficits accumulate.

(i) Switch time τ

Let $H^-(t) = \ln(y - h(t)) + q(t)\mu(d(t) - Ah(t))$ be the Hamiltonian function for $t < \tau$. Likewise, let $H^+(t) = \ln(y - h(t) - Bd(t)) + q(t)\mu(d(t) - Ah(t))$ be the Hamiltonian function for $t \geq \tau$. The switch time $\tau \in (0, T)$ is implicitly defined by $H^-(\tau) = H^+(\tau)$. Using the piecewise solutions above, $H^-(\tau)$ and $H^+(\tau)$ are given by

$$\begin{cases} H^-(\tau) &= -\frac{\tau\mu(d_0-Ay)}{(\bar{d}-Ay)\exp(-\mu\tau)-(d_0-Ay)} - \ln\frac{\tau}{(\bar{d}-Ay)\exp(-\mu\tau)-(d_0-Ay)} \\ H^+(\tau) &= -\eta(T-\tau) - \ln\phi - \eta\phi(\bar{d}-\theta)\exp(\eta(T-\tau)) \end{cases}$$

In words, the Hamiltonian function is continuous at τ, for at the optimal switch time agents must be indifferent between remaining in the first regime or moving to the second one.

The next subsection provides parameter restrictions ensuring a well-defined and meaningful equilibrium. The analysis not only enhances our understanding of the deterministic model, but also helps us to calibrate it and set the stage for the introduction of uncertainty.

2.3. **Parameter Restrictions.** We want to make sure our model satisfies three features: (i) $\dot{d}(t) > 0\ \forall t$; (ii) $\exists \tau \in (0, T)$ such that $d(\tau) = \bar{d}$; and (iii) $h(t) > 0\ \forall t$.[5] Feature (i) makes health deficits irreversible and rules out negative deficits, since $d(0) = d_0 > 0$. Feature (ii) ensures the existence of an optimal switch time τ. When combined with the first feature ($\dot{d} > 0$), the existence of τ also guarantees its uniqueness, which implies that agents cannot go back and forth between regimes. Lastly, feature (iii) imposes strictly positive investment in preventive care. The following parameter restrictions guarantee these three features.

Proposition 1. $\min[(\bar{d} - Ay) - (d_0 - Ay)\exp(\mu T)\,,\, (\bar{d} - Ay) - (d_0 - Ay)(1 - \mu T)] > 0$ *is a sufficient condition to have $\dot{d}(t) > 0$ in the regime without LTC* ($0 \leq t < \tau$).

Proof. See Appendix B. □

[5] By abuse of language, we define the time derivative on all $t \in (0, T)$. Strictly speaking, it is only defined on $(0, \tau) \cup (\tau, T)$.

Proposition 2. *$(\bar{d}-\theta)\eta\phi\exp(\eta T)+\exp(-\eta T)>0$ is a sufficient condition to have $\dot{d}(t)>0$ in the regime with LTC ($\tau\leq t\leq T$).*

Proof. See Appendix B. □

The conditions in Propositions 1 and 2 ensure that $\dot{d}(t)>0$ $\forall t\in[0,T]$. Crucially, Propositions 1 and 2 also guarantee that $q(t)<0$ and $\dot{q}(t)>0$ $\forall t\in[0,T]$. This makes sense: health deficits hinder welfare, so their shadow prices must be negative. In addition, health deficits are more costly at an early age, because they naturally accumulate at rate μ ($\dot{d}/d=\mu$ if there is no health investment). Therefore, their shadow prices must converge to their final value ($-\phi$) from below. Since the optimality condition for consumption reads $c(t)=-1/(\mu Aq(t))$, Propositions 1 and 2 also ensure that $c(t)>0$ $\forall t$. Lastly, Propositions 1 and 2 imply the continuity of $H^-(\tau)$ and $H^+(\tau)$, $\forall\tau\in[0,T]$.

Moving to feature (ii), it can be shown that $H^-(\tau=0)\to\infty$ and $H^+(\tau=0)<\infty$. Therefore, if $H^+(\tau=T)>H^-(\tau=T)$, the continuity of $H^-(\tau)$ and $H^+(\tau)$ ensures there exists at least one τ such that $d(\tau)=\bar{d}$. Proposition 3 provides a necessary and sufficient condition.

Proposition 3. *$\frac{\mu T(\bar{d}-Ay)}{(\bar{d}-Ay)-(d_0-Ay)\exp(\mu T)}-\ln\big((\bar{d}-Ay)-(d_0-Ay)\exp(\mu T)\big)+\ln T-\ln\phi-\eta\phi(\bar{d}-\theta)>0$ is a necessary and sufficient condition to have $H^+(T)>H^-(T)$.*

Proof. See Appendix B. □

Furthermore, since d is strictly increasing everywhere, the existence of a τ such that $d(\tau)=\bar{d}$ immediately implies its uniqueness.[6]

Finally, the last two propositions ensure that investment in preventive care does not take negative values.

Proposition 4. *$y-\frac{(\bar{d}-Ay)-(d_0-Ay)\exp(\mu T)}{\mu AT}>0$ is a sufficient condition to have $h(t)>0$ in the regime without LTC ($0\leq t<\tau$).*

Proof. See Appendix B. □

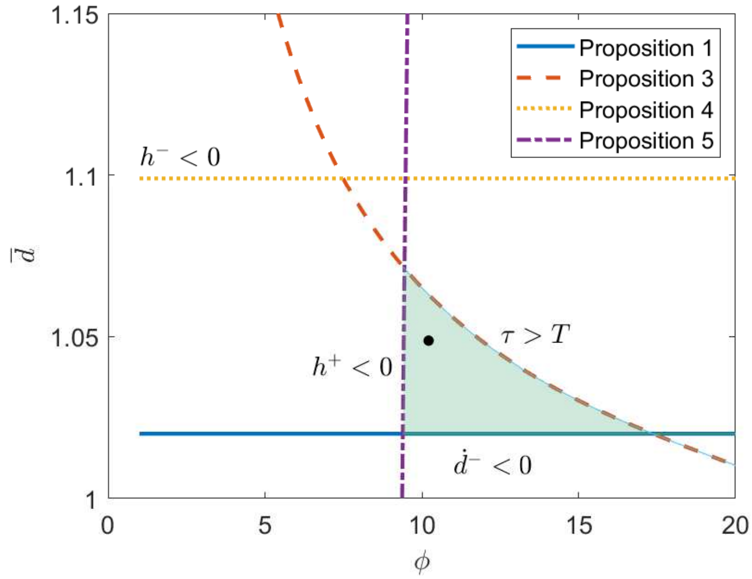
Proposition 5. *$\min\left[y-B\left(\bar{d}+\frac{T}{\phi}\right)-\frac{1}{\phi\mu A}\,,\,y-B\left(\bar{d}+\frac{T}{\phi}-\theta(\exp(\eta T)-1)\right)-\frac{1}{\phi\mu A}\right]>0$ is a sufficient condition to have $h(t)>0$ in the regime with LTC ($\tau\leq t\leq T$).*

Proof. See Appendix B. □

To better understand all these propositions, consider a simple numerical example. Let us normalize T, y, and d_0 to 1, and set $\mu=0.1$, $A=1.2$ and $B=0.1$. Figure 1 shows where

[6] More formally, Proposition 1 implies that $H^-_\tau(\tau)<0$ and Proposition 2 implies that $H^+_\tau(\tau)>0$ (see Appendix B). Therefore, conditional on Propositions 1 and 2, Proposition 3 ensures not only the existence but also the uniqueness of τ.

Figure 1. Sufficient Conditions on Parameters ϕ and $\bar{d}$



Notes. Sufficient conditions in the graph are based on the calibration $T = y = d_0 = 1$, $A = 1.2$, $B = 0.1$ and $\mu = 0.1$. The parameter ϕ (x-axis) represents the salvage cost of health deficit and the parameter $\bar{d}$ (y-axis) represents the health deficit threshold. The sufficient condition from Proposition 2 is not binding in the chart area and therefore does not appear. Propositions 1 to 5 are only respected in the shaded area. The dot represents the $\{\phi, \bar{d}\}$ combination we use for the numerical simulations in Sections 3 and 4. Superscripts $-$ and $+$ indicate before and after the switch time, respectively.

our propositions are satisfied when we vary parameters ϕ and $\bar{d}$.[7] As $\bar{d}$ falls, for instance, individuals boost their investment in preventive care to avoid paying LTC costs while young. Below a certain level of $\bar{d}$ (i.e. below the solid blue line), these large investments actually reduce health deficits, thus violating Proposition 1. Likewise, Proposition 3, which ensures the existence of a τ such that $d(\tau) = \bar{d}$, does not hold for large values of ϕ and $\bar{d}$ (i.e. area above the dashed red line). This is intuitive: too large a ϕ makes health deficits too costly. Individuals therefore invest more in preventive care, always keeping health deficits below the threshold $\bar{d}$. Similarly, too large a $\bar{d}$ simply makes the threshold unreachable. In turn, Proposition 4 and 5, under which $h(t) > 0 \ \forall t$, do not hold for large $\bar{d}$'s and low ϕ's. As mentioned above, too large a $\bar{d}$ (i.e. dotted yellow line) makes the LTC regime out of reach even if individuals were to 'dis-invest' in preventive care. In addition, too low a ϕ (i.e. purple dotted line) kills the incentives to invest in preventive care once the threshold $\bar{d}$ is crossed, for the negative welfare effects of health deficits become negligible.

[7]Proposition 2 is too weak to appear in Figure 1.

All told, only the shaded area ensures that Propositions 1 to 5 hold. Section 3.3 will use the calibration just discussed with the admissible pair $\phi = 10$ and $\bar{d} = 1.05$.[8]

3. Stochastic Model

In our deterministic model, there was no uncertainty over the impact of preventive care on the switch time τ. Put differently, τ was determined by agents' optimal choices. This is no longer the case in our stochastic setup. Here τ is a random variable with hazard rate $\lambda(d) \in [0, \infty)$. This hazard rate is a C^1-function of d with $\lambda_d \geq 0$, implying that higher health deficits increase the likelihood of switching to the LTC regime. τ therefore becomes partly random, which limits agents' ability to control it.

Crucially for our analysis, the deterministic model is a limit case of its stochastic counterpart when $\lambda(d)$ approximates a step function equaling 0 when $d < \bar{d}$ and L when $d \geq \bar{d}$, with L sufficiently large. At the other end of the spectrum, $\lambda_d = 0$ breaks the link between τ and d, making the switch time purely random.

3.1. **Setup and Solution.** The stochastic setup is similar to (1) to (3), but now the switch time is not known in advance, so individuals maximize *expected* life-time utility. Formally, the optimization problem is

$$E\left[\int_0^T \ln c(t)\,\mathrm{d}t - \phi\, d(T)\right] \tag{4}$$

under

$$\dot{d}(t) = \mu(d(t) - Ah(t)) \tag{5}$$

with

$$c(t) = \begin{cases} y - h(t) & \text{if} \quad t < \tau \\ y - h(t) - Bd(t) & \text{if} \quad t \geq \tau \end{cases} \tag{6}$$

τ is a random variable and its density is

$$\lambda(d(t)) \exp\left(-\int_0^t \lambda(d(u))\mathrm{d}u\right) \tag{7}$$

The associated hazard rate is therefore $\lambda\left(d(t)\right)$, which must respect the above-mentioned restrictions.

We rely on the Hamilton-Jacobi-Bellman equation to solve the stochastic model. As before, the model features one control $h(t)$, one state $d(t)$, and one co-state $q(t)$. There is an additional auxiliary variable $\gamma(t)$ measuring the expected flow of remaining utility along the

[8]Though this calibration is certainly arbitrary, any alternative satisfying Propositions 1 to 5 generates similar qualitative outcomes.

optimal path. As shown in Appendix C, the solution in the non-LTC regime is governed by the system of three differential equations

$$\begin{cases} \dot{q}(t) &=& (\lambda(d(t)) - \mu)q(t) + \phi\exp(\eta(T-t))\left[\lambda(d(t)) - \lambda_d(d(t))(d(t)-\theta)\right] \\ & & +\lambda_d(d(t))\left[\gamma(t) + \frac{\eta}{2}(T-t)^2 + (1+\ln(\mu A\phi))(T-t) + \phi\,\theta\right] \\ \dot{\gamma}(t) &=& \ln(-\mu A q(t)) + \lambda(d(t))\left[\gamma(t) + \frac{\eta}{2}(T-t)^2 + (1+\ln(\mu A\phi))(T-t)\right. \\ & & \left. +\phi(d(t)-\theta)\exp(\eta(T-t)) + \phi\,\theta\right] \\ \dot{d}(t) &=& \mu(d(t) - Ay) - \frac{1}{q(t)} \end{cases}$$

with three boundary conditions $d(0) = d_0$, $q(T) = -\phi$ and $\gamma(T) = -\phi d(T)$, as well as

$$h(t) = y + \frac{1}{\mu A q(t)}$$

The first two boundary conditions are as before. The third one ensures that the only utility remaining at T is the salvage value of the health deficit. This non-linear boundary value problem has no closed-form solution. Below, we will solve it numerically.

However, after the switch to the LTC regime, the model accepts the following closed-form solution

$$\begin{cases} q(t) &=& -\phi\exp(\eta(T-t)) \\ \gamma(t) &=& -\frac{\eta}{2}(T-t)^2 - \ln(\mu A\phi)(T-t) - \phi(d(\tau)-\theta)\exp(\eta(T-\tau)) - (T-\tau) - \phi\theta \\ d(t) &=& (d(\tau)-\theta)\exp(\eta(t-\tau)) + \frac{t-\tau}{\phi}\exp(-\eta(T-t)) + \theta \\ h(t) &=& y - Bd(t) + \frac{1}{\mu A q(t)} \end{cases}$$

After the regime switch, the paths for $\{q(t), d(t), h(t)\}$ are identical to the ones from the deterministic setup. This makes sense, for all uncertainty vanishes after τ. The closed-form solution for the new auxiliary variable $\gamma(t)$ is helpful to derive the solution before the regime switch (see Appendix C).

3.2. **Uncertainty and Health Investment.** We assume that the hazard rate follows a logistic function

$$\lambda(d) = \frac{L}{1 + \exp(-k(d - x_0))}$$

with L and $x_0 \in (0, \infty)$, and $k \in [0, \infty)$. Let us consider two limiting cases for which we can provide analytical insights.

The first case is $k \to \infty$, $x_0 = \bar{d}$ and L very large. $\lambda(d)$ becomes a continuous approximation of a step function with $\lambda(d) \to 0$ when $d < \bar{d}$ and $\lambda(d) \to L$ when $d \geq \bar{d}$. As a result, the probability of switching from the non-LTC regime to the LTC regime tends to zero when d is below $\bar{d}$ and to 1 when d reaches $\bar{d}$. That is, uncertainty disappears and thus the stochastic model nests its deterministic counterpart.[9]

The second limiting case is $k = 0$. The logistic function becomes $\lambda(d) = \lambda = L/2$, making the switch time entirely exogenous. The closed-form solution for the co-state variable $q(t) \in [0, \tau)$ is now

$$q(t) = -\frac{\phi}{\eta + \lambda - \mu} \left((\eta - \mu) \exp(-(\lambda - \mu)(T - t)) + \lambda \exp(\eta \, (T - t)) \right)$$

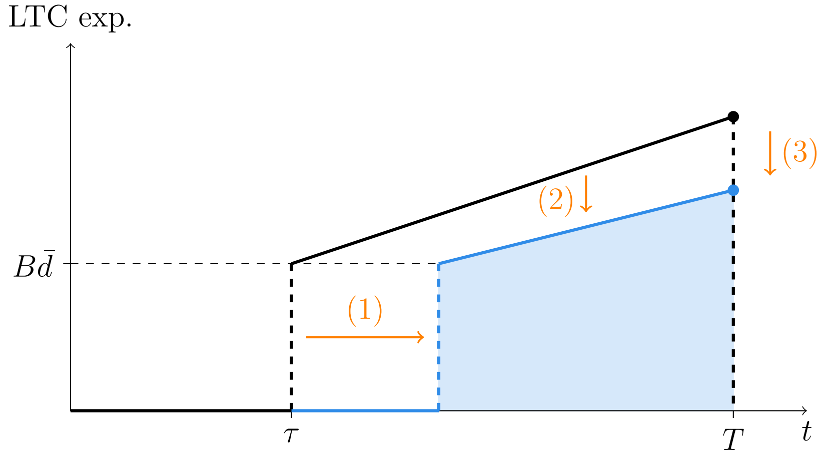
Appendix D shows that $q(t) < 0$ and $\dot{q}(t) > 0$. As before, health deficits lower utility, and are more harmful early in life, because they naturally increase over time. In addition, Appendix D proves that $q_\lambda < 0$. The intuition is as follows. Higher values of λ will lower the expected value of τ. For lower expected values of τ, the expected LTC costs of accumulating health deficits will be higher and hence $q(t)$ will become more negative.

Comparing these limiting cases yields our first key insight: uncertainty lowers investment in preventive care. In the absence of uncertainty ($k \to \infty$), rising $h(t)$ decreases $d(t)$, which in turn entails three positive effects: (1) a higher value of τ, (2) lower per-period LTC costs, and (3) lower health deficits at T. Figure 2 illustrates these three channels. Channels (1) and (2) reduce total LTC expenditures (shaded area) and channel (3) reduces the salvage cost (proportional to the dots at time T). However, under maximal uncertainty ($k = 0$), the first channel disappears, as τ becomes exogenous. There is therefore less incentive to invest in preventive care. Mathematically, $k = 0$ raises the shadow price of health deficits (i.e. decreases the absolute value of $q(t)$), calling for less investment in preventive care. We show numerically in the next section that the transition between these two extremes is monotone.

That uncertainty lowers preventive care echoes the literature on investment under return risk. This body of work (see for instance Sandmo, 1970) reveals that return risk leads to two competing forces. On the one hand, high risk makes it necessary to protect oneself against the future. This is an income effect calling for *greater* health investments. On the other hand, high risk makes an agent less inclined to expose her resources to the probability of loss. This is a substitution effect calling for *lower* health investments. As shown by Jouini et al. (2013), for standard utility functions the substitution effect always dominates the income effect. This is exactly reflected in our first key finding.

[9]This can be easily shown mathematically: taking $\lambda(d) \to 0$ and $\lambda_d(d) \to 0$ when $t < \tau$, the differential equation for q simplifies to $\dot{q} = -\mu q$ as in the deterministic setup (see Appendix A). With the same boundary conditions and using $d(\tau) = \bar{d}$, we obtain the deterministic solution paths for all $t \in [0, T]$.

FIGURE 2. Effects of health investment on LTC costs, without uncertainty



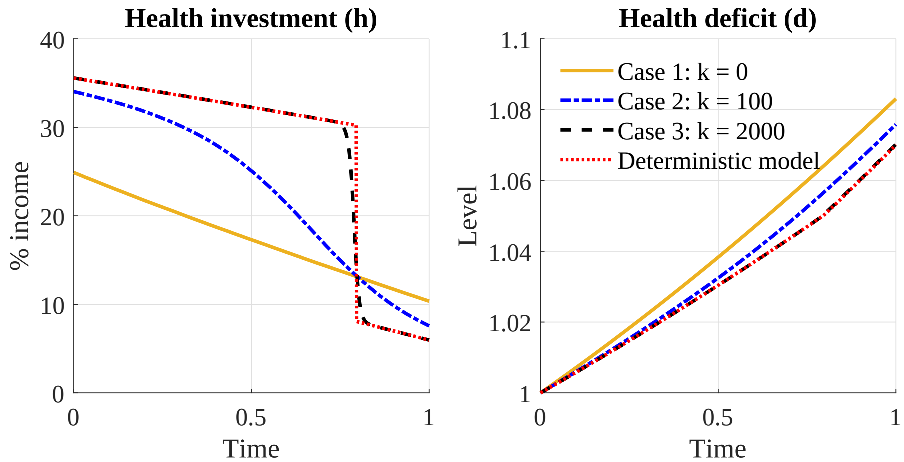
Notes. The black line shows a stylized evolution of LTC costs over time. Higher investment in health lowers health deficit. Without uncertainty, this affects LTC expenditures and the salvage cost through three channels (numbered arrows). Proposition 2 ensures that the blue line is always below the black line, i.e. that the direction of channels (2) and (3) is unambiguous.

3.3. **Numerical Illustration.** As mentioned earlier, the general boundary value problem presented in this section has no closed-form solution. Therefore, we solve it using the collocation method proposed by Shampine et al. (2003). We will study the two limiting cases considered above with an intermediate case taking $0 < k << \infty$.

We calibrate parameters $\{T, y, d_0, A, B, \mu, \phi\}$ as in the deterministic model in Section 2.3. However, parameters $\{x_0, k, L\}$, which govern $\lambda(d)$, are specific to the stochastic model. Our choice for the curve's midpoint is $x_0 = \bar{d}$, thus allowing the stochastic model to nest its deterministic counterpart (see Section 3.2). Regarding the curve's steepness, we consider three cases: $k = \{0, 100, 2000\}$. The first case breaks the link between d and $\lambda(d)$. In the second case, $\lambda(d)$ increases smoothly with d, taking meaningful positive values over a long period.[10] The third case approximates a step function: around x_0, $\lambda(d)$ jumps from 0 to L . Lastly, we choose the curve's maximum value L so that if health deficits followed the same path as in the deterministic model, the median switch time in a large panel of agents

[10] In a large panel of agents, $k = 100$ (together with $L = 6.65$) implies that 3% of the population faces long term costs at the age of $T/4$, 14% at the age of $T/2$ and 85% at the age of T.

FIGURE 3. Health dynamics



Notes. Mean paths of health investment (h) and health deficit (d) in a stochastic time series simulation of 50,000 agents.

would equal the switch time in such a model. The resulting values are $L = 1.74$ when $k = 0$, $L = 6.65$ when $k = 100$, and $L = 128$ when $k = 2000$.[11]

Figure 3 plots the mean paths of health investment and health deficit in a stochastic time series simulation of 50,000 agents under $k = \{0, 100, 2000\}$. As a benchmark, we also show the paths from the deterministic model. The figure illustrates our key result: uncertainty reduces investment in preventive care.

In addition, the figure reveals five important properties. First, $k = 2000$ effectively eliminates uncertainty, because individuals know when the switch time will occur. Therefore, the stochastic economy converges to the deterministic equilibrium. Second, by lowering investment in preventive care, uncertainty makes health deficits grow faster and reach higher levels. Third, preventive care falls over time, as health deficits are more harmful early in life (see Section 3.2). The second and third points apply both at the individual level and for the population as a whole. Fourth, LTC costs begin to bite at τ, leading agents to abruptly reduce investment in preventive care at that point. This is clear in the deterministic case and when $k = 2000$, since all individuals change regimes simultaneously. However, in the other two cases, switch times are distributed more heterogeneously across the population, smoothing the mean path of preventive care. Lastly, the wider distribution of switch times means that part of the population dies before they face LTC costs (i.e. $\tau > T$). This is

[11]We take the median switch time instead of the mean switch time because we cannot compute the latter. Indeed, computing the mean switch time requires knowing the switch time of all individuals. However, in the stochastic setup we do not know the switch time of part of the population, because it would happen after time T (in other words, at the end of their lives, some individuals are still in the regime without LTC).

why when k is low, the mean health investment at T is above the level in the deterministic model.[12]

4. Uncertainty and Policy Transmission

Financial incentives may encourage healthier lifestyles, leading to better health outcomes and lower costs of care (see for instance Mantzari et al., 2015, or Vlaev et al., 2019, for a review of the empirical literature). Could uncertainty over the switch time τ affect this link?

To tackle this question, let us consider a simple policy: the government reimburses part of private investment in preventive care.[13] Formally, the budget constraint becomes

$$c(t) \;=\; \begin{cases} y - (1-S)h(t) & \text{if} \quad t < \tau \\ y - (1-S)h(t) - Bd(t) & \text{if} \quad t \geq \tau \end{cases}$$

where $S \in [0, 1)$ is the subsidy rate. As before, we rely on the Hamilton-Jacobi-Bellman equation to solve the model (see Appendix E for more details).

We assess the performance of policy S on a sample of 50,000 agents. First, we calculate investment in preventive care as a share of income over the life cycle of the average agent

$$\mathcal{H}(S) = \mathbb{E}\ \frac{\int_0^T h(t \mid S)\ \mathrm{d}t}{\int_0^T y\ \mathrm{d}t}$$

Here $h(t \mid S)$ is preventive care investment at time t conditional on policy S. Therefore, the cost of the subsidy as a fraction of total income is $S\,\mathcal{H}(S)$. Next, we compute LTC costs conditional on policy S for the average agent

$$LTC(S) = \mathbb{E}\ \frac{\int_\tau^T B\ d(t \mid S)\ \mathrm{d}t}{\int_0^T y\ \mathrm{d}t}$$

Hence, cost-saving due to S is $\Delta LTC(S) = -(LTC(S) - LTC(0))$. Our measure of the efficiency of the policy – or equivalently the policy return – is then

$$R = \frac{\Delta LTC(S)}{S\,\mathcal{H}(S)}$$

Table 1 presents this policy return R as a function of the uncertainty parameter k when $S = 0.05$.[14] The key message is clear: uncertainty (low values for k) reduces the return of S, making it a less attractive policy.

[12] In the left panel of Figure 3, curves cross at different points, although this is not visible under our calibration.

[13] In our partial equilibrium model, we do not consider how the government finances the subsidy.

[14] Although $S = 0.05$ is an arbitrary choice, the logic underpinning Table 1 holds $\forall S \in (0, 1)$.

Table 1. Effects of a subsidy rate S on health investment

	$\mathcal{H}(0)$	$\mathcal{H}(S)$	$S\,\mathcal{H}(S)$	$LTC(0)$	$LTC(S)$	$\Delta LTC(S)$	R
$k = 0$	17.4	22.5	1.1	3.5	3.5	+0.0	1.2%
$k = 100$	23.0	31.3	1.6	2.8	2.1	+0.7	43%
$k = 2000$	27.5	41.9	2.1	2.2	0.2	+1.9	92%

Notes. Mean values of health investment $\mathcal{H}$ and LTC costs (both as a percentage of income) in a stochastic time series simulation of 50,000 agents. We assume a health investment subsidy rate $S = 0.05$. $\Delta LTC(S) = -(LTC(S) - LTC(0))$ and $R = \Delta LTC(S)/(S\,\mathcal{H}(S))$ represents the return on policy. For instance a return of $R = 50\%$ means that for a policy subsidy of \$1, LTC costs are reduced by \$0.5. The level of uncertainty is inversely related to k (see the discussion in Sections 3.2 and 3.3).

In Table 1, columns 1 and 2 reveal that uncertainty limits agents' response to policy S. When uncertainty is almost nil (i.e. $k = 2000$), the policy raises preventive care investment by 14.4 percentage points. However, when the switch time is fully random (i.e. $k = 0$), it only does so by 5.1 percentage points. Policy S thus leads to lower health deficits under mild uncertainty, postponing the switch time and reducing LTC costs. Our cost-saving measure, $\Delta LTC(S)$, and hence the policy return R, then increase as uncertainty falls. Again, when uncertainty tends to zero, a \$1 subsidy cuts LTC costs by \$0.92, making it almost self-financed. In contrast, when the switch time is fully random, a \$1 subsidy has almost no effect on LTC costs.

The channel through which uncertainty limits agents' response to policy S relates to Arrow (1965)'s portfolio selection under uncertainty (see also Chang, 1996, for an application to health). Specifically, policy S lowers the relative price of preventive care, increasing its expected return. This mean-effect raises the share of income allocated to investment in preventive care. If the return to investment is risk-free, or close to, this mean-effect fully explains how policy S shapes agents' choices. However, if the return to investment is risky, then a competing variance-effect arises: larger investments expose agents to more risk. Since agents are risk-averse, the variance-effect partly offsets the mean-effect, hindering the effectiveness of policy S in encouraging investment in preventive care.

That uncertainty reduces the effectiveness of a subsidy to investment in health has a clear corollary. Evaluating health policy through the lens of deterministic setups could lead to conclusions that are too optimistic (e.g. Marchiori and Pierrard, 2022, for a deterministic OLG model with optimal subsidy rate). On a related note, uncertainty could also explain the weak pass-through of some health policies documented in the empirical literature (e.g. Finkelstein et al., 2019).

5. Uncertainty and Savings

The stochastic model from Section 3 presents agents with a standard inter-temporal trade-off between consumption, which raises short-term utility flows, and investment in preventive care with a risky return, which raises expected long-term welfare. We now expand this trade-off by introducing risk-free saving. We also adopt a more general setup with a CRRA utility function, a strictly positive time preference rate, and decreasing returns to investment in preventive care. While confirming the robustness of previous findings, we also unveil a new insight: uncertainty about the date when LTC costs appear boosts precautionary savings. That is, individuals build savings buffers to prepare for potential LTC costs early in their lives.

Formally, the welfare function is now

$$E\left[\int_0^T \exp(-\rho t)\,\frac{c(t)^{1-\sigma}-1}{1-\sigma}\,\mathrm{d}t-\exp(-\rho T)\,\phi d(T)+\exp(-\rho T)\,\psi s(T)\right] \tag{8}$$

where $\rho \geq 0$ is the time preference rate, $\sigma > 0$ is the coefficient of relative risk aversion, $\psi > 0$ is a preference parameter, and s is risk-free savings. The last term captures the salvage value of $s(T)$.[15] Health deficits evolve according to

$$\dot{d}(t) = \mu(d(t) - Ah(t)^{\nu} - a) \tag{9}$$

where $a \in \mathbb{R}$ captures the effects of exogenous forces on the aging process, and $\nu \in (0, 1)$ represents decreasing returns to investment in preventive care.[16] As for risk-free savings, the law of motion is

$$\dot{s}(t) \;=\; \begin{cases} y + r\,s(t) - c(t) - h(t) & \text{if} \quad t < \tau \\ y + r\,s(t) - c(t) - h(t) - Bd(t) & \text{if} \quad t \geq \tau \end{cases} \tag{10}$$

where $r > 0$ is the risk-free rate, and τ is still a random variable with a density function given by (7) and a state dependent hazard rate $\lambda(d(t))$ (see Section 3).

Individuals choose the sequence $\{c(t), h(t)\}_{t=0}^{T}$ that maximizes (8) subject to (9) and (10). The optimization problem has two state variables, and hence four boundary conditions: $d(0) = d_0 \geq 0$, $s(0) = s_0 \in \mathbb{R}$, $q(T) = -e^{-\rho T}\phi$, and $p(T) = -e^{-\rho T}\psi$. Here q and p are the co-state variables associated with d and s, respectively. To solve this stochastic optimal control model, we rely on the Hamilton-Jacobi-Bellman equation (see Appendix F for more

[15] As mentioned in Footnote 4, salvage values prevent state variables exploding at T. An alternative solution is to impose bounds on these state variables and work with the Kuhn-Tucker slackness conditions.

[16] $\nu < 1$ is necessary to get an interior solution to the portfolio allocation choice. Should ν be equal to 1, we would get a corner solution where agents either save on the riskless asset or invest in preventive care (see also Dalgaard and Strulik, 2014, for a similar discussion). Ruling out this corner solution while setting $\nu = 1$ would require non-linear salvage functions.

details). We also develop and solve a deterministic version, which we use as a yardstick to compare to the stochastic version.

A technical point here deserves further comment. The baseline stochastic model studied in Section 3 accepted a closed-form solution in $[\tau, T]$. Thus, we could plug these analytical decision rules into the Boundary Value Problem determining the model dynamics in $[0, \tau)$, and solve the latter numerically (see Appendix C). The extended stochastic model, however, never accepts a closed-form solution. Instead of working with the exact optimal solution in $[\tau, T]$, we shall therefore work with a set of parametric decision rules that, as shown in Appendix F, approximate it very accurately (see for instance Fernandez-Villaverde et al., 2016, for an overview of projection methods).

Table 2 lists the values assigned to the model parameters. Without loss of generality, we normalize the initial state variables to 0, and the income stream together with the end-of-life period to 1. In addition, we choose conventional values for the utility and savings parameters: the relative risk aversion parameter σ is set to 1.5 (Chetty, 2006), while the time preference rate ρ and the risk-free return r are set to 0.05 (Kaplan et al., 2018). In the deterministic version, we fix the health deficit threshold $\bar{d} > d_0$ such that agents start paying LTC costs at $t = T/1.2$ (i.e. 67 years assuming a lifetime of 80 years). In the stochastic version, we continue to model the state dependent hazard rate $\lambda(d)$ as a logistic function and proceed as in Section 3. The curve's midpoint x_0 is set to $\bar{d}$. We consider three levels of uncertainty k while calibrating L to ensure that, if health deficits followed the same paths as in the deterministic model, the median switch time in a large panel of agents would remain equal to the switch time in the deterministic version. The medium level of uncertainty $\{k, L\} = \{400, 10.4\}$ implies that 84% of the population at age T (end of life) faces LTC costs and that 73% of all LTC recipients are above 65 years old (assuming a lifetime of 80 years). This last number is very close to the average OECD number (75%, see OECD, 2021).

The remaining parameters $\{B, \mu, A, \nu, a, \phi, \psi\}$ are chosen to be consistent with the following outcomes. First, investment in preventive care and investment in savings, as a percentage of income, are both close to 10% in $[0, \tau)$ (World Health Organization, 2019). Second, health deficits are irreversible (i.e. $\dot{d}(t) > 0 \; \forall t$) meaning that LTC costs increase in $[\tau, T]$. Moreover, savings decrease in $[\tau, T]$. Third, LTC recipients face costs averaging 55% of their income (European Commission, 2021). Lastly, as in Dalgaard and Strulik (2014), investment in preventive care features strong decreasing returns to scale.[17]

[17] Unfortunately, in this extended setup we can no longer rely on the propositions in Section 2.3 to impose parameter restrictions.

Table 2. Calibrated parameters

Parameter	Value	Description	Parameter	Value	Description
Normalizations			*Utility and savings*		
d_0, s_0	0	Initial state	σ	1.5	Relative risk aversion
y	1	Income	ρ	0.05	Time preference
T	1	Final period	r	0.05	Risk-free return
LTC costs			*Salvage values*		
$\bar{d}$	0.02	Health deficit threshold	ϕ	40	Deficit cost
B	25	LTC cost level	ψ	15	Savings value
Health deficit accumulation			*Hazard rate*		
μ	0.02	Natural growth rate	x_0	0.02	Midpoint
A	3	Effectiveness of investments	$\{k, L\}$	$\{0, 1.74\}$	Full uncertainty
ν	0.05	Decreasing returns		$\{400, 10.4\}$	Medium uncertainty
a	-4	Exogenous shocks		$\{2000, 50.8\}$	Low uncertainty

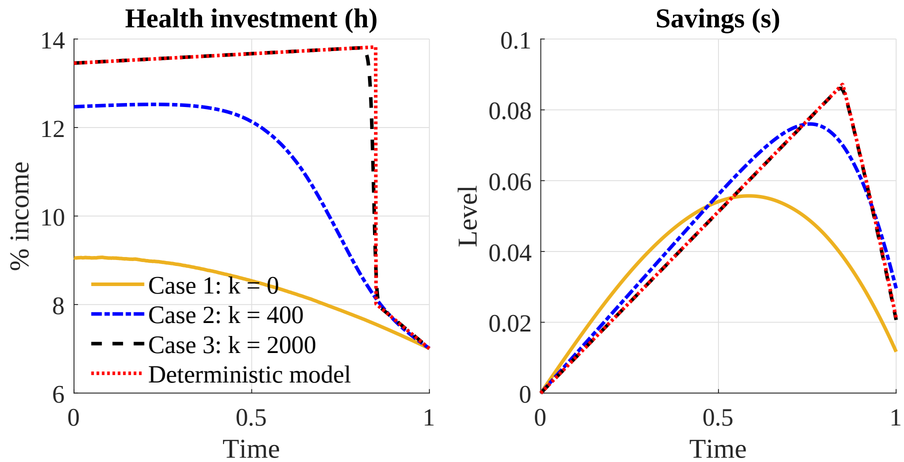
We are finally in a position to show that, as claimed earlier, uncertainty over the appearance of LTC costs boosts precautionary savings. The right panel in Figure 4, which plots the mean path of savings in a large set of agents, shows that individuals in the fully random economy (solid yellow line) accumulate more savings early in life than their counterparts in the deterministic economy (dotted red line). The logic is straightforward: risk-averse individuals build precautionary savings to meet unexpected medical bills, allowing them to smooth consumption over time. This finding is consistent with the strong link between income uncertainty and precautionary savings put forward by Sandmo (1970), Skinner (1988), Caballero (1991) and many others.

Finally, the left panel in the figure confirms that our main insight from Section 3 carries through in the more realistic setting, namely that uncertainty over the switch time lowers investment in preventive care. Therefore, it follows naturally that in such a framework uncertainty also limits agents' response to financial incentives, just as it did in Section 4.

6. Conclusion

Long term care (LTC) costs faced by elderly individuals are not only high, but also wildly uncertain. In this paper, we develop a stochastic version of an otherwise standard deterministic life-cycle setup with endogenous aging. We show that uncertainty about future LTC costs boosts precautionary savings, lowers investment in preventive care, and weakens the

FIGURE 4. Health and savings dynamics



Notes. Mean paths of health investment (h) and savings (s) in a stochastic time series simulation of 50,000 agents.

effectiveness of subsidies for preventive care. This leads us to believe that evaluating health policy through the lens of deterministic models could be misleading.

Further research is necessary in the following areas. First, stochastic lifetimes would provide another channel through which health-related uncertainty distorts the intertemporal trade-off between consumption and investment in preventive care. Second, a general equilibrium analysis of different health policies (e.g. subsidies on prevention vs. reimbursement of LTC costs) would provide a normative assessment. This is important since market outcomes in life-cycle models tend to be Pareto inefficient. Lastly, extending our analysis to environmental economics would help us tackle the question: does our utter failure to fight climate change relate to the uncertainty surrounding its potential economic effects?

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Appendix A. Solution of the Deterministic Model

The system of equations (1) to (3) is a two-stage optimal control problem. Optimal regime switch can typically be of two types. In the first type, the switch time is an explicit decision variable (see for instance Tomiyama, 1985). In the second type, the regime switch happens when the state variable crosses a threshold (see for instance Boucekkine et al., 2013). Our problem relates to the second type. Below, we consider the two-stage optimal control problem of type two, and provide the first order optimality conditions (all mathematical proofs can be found in Boucekkine et al., 2013). This problem can be generally written as

$$\max\left\{\int_0^\tau g^-(t,h(t),d(t))\,\mathrm{d}t+\int_\tau^T g^+(t,h(t),d(t))\,\mathrm{d}t+v(d(T))\right\}$$

under

$$\begin{cases}\dot{d}(t) &= f(t,h(t),d(t))\\ d(0) &= d_0\\ d(\tau) &= \bar{d}\end{cases}$$

For simplicity, we drop the time index unless needed for the understanding. We define an Hamiltonian function for each stage

$$\begin{cases}H^- &= g^- + q\,f\\ H^+ &= g^+ + q\,f\end{cases}$$

The necessary optimality conditions are

$$\begin{cases}H_h^- = 0, & H_d^- = -\dot{q}, & H_q^- = \dot{d}\\ d(0) = d_0, & d(\tau) = \bar{d} & \end{cases}$$

$$\begin{cases}H_h^+ = 0, & H_d^+ = -\dot{q}, & H_q^+ = \dot{d}\\ d(\tau) = \bar{d}, & q(T) = v_d(T) & \end{cases}$$

$$H^-(\tau)-H^+(\tau)=0$$

h is a control variable, d a state variable and q the associated co-state variable. The first set of necessary conditions relates to the first stage and the second set to the second stage. The last condition is a matching condition meaning that the Hamiltonian must be continuous at the junction time. To understand why, we observe that the left-hand side represents the marginal gain from extending the first stage regime at the expense of the second stage regime, whereas the right-hand represents the marginal switch cost (which is nil in our setup). At the optimal equilibrium, the marginal gain must be equal to the marginal cost. A last observation: in this problem of type two, the state variable at the switch time τ is not free but equal to the fixed threshold. This generally implies the discontinuity of the co-state variable at time τ.[18] We can apply this general formulation to our model in Section 2 by posing

$$\begin{cases} g^- &= \ln(y-h) \\ g^+ &= \ln(y-h-Bd) \\ f &= \mu(d-Ah) \\ v &= -\phi d \end{cases}$$

The concavity of g^- and g^+ guarantees that the necessary conditions are also sufficient. The solution in Section 2 directly follows from applying the general necessary and sufficient conditions to our specific functional forms.

Appendix B. Proofs of Propositions

Throughout all proofs, we take strict inequalities to simplify the notation, without prejudice for the results.

B.1. **Proof of Proposition 1.** In the first stage (non-LTC regime when $0 < t < \tau$), the time derivative of deficit is

$$\dot{d} = \frac{d_0 - Ay}{\tau}((\tau - t)\mu - 1)\exp(\mu t) + \frac{\bar{d} - Ay}{\tau}(1 + \mu t)\exp(-\mu(\tau - t))$$

Therefore

$$\dot{d} > 0 \Leftrightarrow (\bar{d} - Ay)\underbrace{(1+\mu t)\exp(-\mu\tau)}_{\in(0,1)} - (d_0 - Ay)\underbrace{(1-(\tau-t)\mu)}_{\in(0,1)} > 0$$

We distinguish three cases: (i) when $d_0 < Ay < \bar{d}$, $\dot{d}$ is always positive; (ii) when $d_0 < \bar{d} < Ay$, $(\bar{d} - Ay) - (d_0 - Ay)(1 - \mu T) > 0 \Rightarrow \dot{d} > 0$ for all $t \in (0, \tau)$ and $\tau \in (0, T)$; (iii) when $Ay < d_0 < \bar{d}$, $(\bar{d} - Ay) - (d_0 - Ay)\exp(\mu T) > 0 \Rightarrow \dot{d} > 0$ for all $t \in (0, \tau)$ and $\tau \in (0, T)$. Moreover, when $d_0 < \bar{d} < Ay$, $(\bar{d} - Ay) - (d_0 - Ay)(1 - \mu T) > 0 \Rightarrow (\bar{d} - Ay) - (d_0 - Ay)\exp(\mu T) > 0$; and when $Ay < d_0 < \bar{d}$, $(\bar{d} - Ay) - (d_0 - Ay)\exp(\mu T) > 0 \Rightarrow (\bar{d} - Ay) - (d_0 - Ay)(1 - \mu T) > 0$. As a result,

$$\min[(\bar{d} - Ay) - (d_0 - Ay)\exp(\mu T)\,,\,(\bar{d} - Ay) - (d_0 - Ay)(1 - \mu T)] > 0 \Rightarrow \dot{d} > 0$$

This proves Proposition 1.

[18] This is the key difference with respect to a two-stage optimal control problem of type one: the state variable is free and the optimality conditions impose not only the continuity of the Hamiltonian but also of the co-state variable (see Boucekkine et al., 2013, for a in-depth discussion).

B.2. **Proof of Proposition 2.** In the second stage (LTC regime when $\tau < t < T$), the time derivative of deficit is

$$\dot{d} = (\bar{d} - \theta) \underbrace{\eta \exp(\eta(t - \tau))}_{>0} + \underbrace{\frac{\exp(-\eta(T - t))}{\phi}(1 + \eta(t - \tau))}_{>0}$$

We distinguish two cases: (i) when $\bar{d} > \theta$, $\dot{d}$ is always positive; (ii) when $\bar{d} < \theta$, $(\bar{d} - \theta)\eta \exp(\eta T) + \exp(-\eta T) > 0 \Rightarrow \dot{d} > 0$ for all $t \in (0, \tau)$ and $\tau \in (0, T)$. This proves Proposition 2.

B.3. **Proof of Proposition 3.** The first part of the proof shows that $H^-(0) = \infty$ and that $H^-_\tau < 0$, under the sufficient condition from Proposition 1. The second part of the proof shows that $H^+(0) < \infty$ and that $H^+_\tau > 0$, under the sufficient condition from Proposition 2. The third part of the proof gives a necessary and sufficient condition to get $H^+(T) > H^-(T)$. All these conditions therefore imply that $\tau \in (0, T)$.

First, we compute $H^-(\tau) = -\mu(d_0 - Ay)x - \ln x$, with $x \equiv \tau/((\bar{d} - Ay)\exp(-\mu\tau) - (d_0 - Ay))$. The sufficient condition from Proposition 1 ensures that $x > 0$ and $x_\tau > 0$ for all $\tau \in (0, T)$. We see that $H^-(0) = \infty$ and compute $H^-_\tau = -\mu(d_0 - Ay)x_\tau - x_\tau/x$. Therefore

$$\begin{aligned} H^-_\tau < 0 \quad &\Leftrightarrow \quad \mu(d_0 - Ay) + 1/x > 0 \\ &\Leftrightarrow \quad (\bar{d} - Ay)\exp(-\mu\tau) - (d_0 - Ay)(1 - \mu\tau) > 0 \end{aligned}$$

The sufficient condition from Proposition 1 ensures that $H^-_\tau < 0$ for all $\tau \in (0, T)$.

Second, we compute $H^+(\tau) = \eta(\tau - T) - \ln\phi - \eta\phi(\bar{d} - \theta)\exp(\eta(T - \tau))$. We see that $H^+(0) < \infty$ and compute $H^+_\tau = \eta(1 + \eta\phi(\bar{d} - \theta)\exp(\eta(T - \tau)))$. The sufficient condition from Proposition 2 ensures that $H^+_\tau > 0$ for all $\tau \in (0, T)$.

Third, from the above expressions for $H^-(\tau)$ and $H^+(\tau)$, we immediately get

$$\begin{aligned} H^+(T) > H^-(T) \quad \Leftrightarrow \quad & \tfrac{\mu T(\bar{d} - Ay)}{(\bar{d} - Ay) - (d_0 - Ay)\exp(\mu T)} - \ln\big((\bar{d} - Ay) - (d_0 - Ay)\exp(\mu T)\big) + \ln T \\ & > \ln\phi + \eta\phi(\bar{d} - \theta) \end{aligned}$$

This proves Proposition 3.

B.4. **Proof of Proposition 4.** In the first stage (non-LTC regime when $0 < t < \tau$)

$$h > 0 \Leftrightarrow y > -\frac{1}{q\mu A}$$

Because $q < 0$ and $\dot{q} > 0$, a sufficient condition to get $h > 0$ for all $t \in (0, \tau)$ and $\tau \in (0, T)$ is

$$y > -\frac{1}{q(T)\mu A} = \frac{(\bar{d} - Ay) - (d_0 - Ay)\exp(\mu T)}{\mu A T}$$

This proves Proposition 4.

B.5. **Proof of Proposition 5.** In the second stage (LTC regime when $\tau < t < T$)

$$h > 0 \Leftrightarrow y > Bd - \frac{1}{q\mu A}$$

Because $d > 0$, $\dot{d} > 0$, $q < 0$ and $\dot{q} > 0$, a sufficient condition to get $h > 0$ for all $t \in (\tau, \infty)$ is

$$y > Bd(T) - \frac{1}{q(T)\mu A} = B\left((\bar{d} - \theta)\exp(\eta(T-\tau)) + \frac{T-\tau}{\phi} + \theta\right) + \frac{1}{\phi\mu A}$$

We distinguish two cases: (i) when $\bar{d} < \theta$, $y > B(\bar{d} + T/\phi) + 1/(\phi\mu A) \Rightarrow h > 0$ for all $\tau \in (0, T)$; (ii) when $\bar{d} > \theta$, $y > B(\bar{d} + T/\phi - \theta(\exp(\eta T) - 1)) + 1/(\phi\mu A) \Rightarrow h > 0$ for all $\tau \in (0, T)$. Moreover, when $\bar{d} < \theta$, $y > B(\bar{d} + T/\phi) + 1/(\phi\mu A) \Rightarrow y > B(\bar{d} + T/\phi - \theta(\exp(\eta T) - 1)) + 1/(\phi\mu A)$; and when $\bar{d} > \theta$, $y > B(\bar{d} + T/\phi - \theta(\exp(\eta T) - 1)) + 1/(\phi\mu A) \Rightarrow y > B(\bar{d} + T/\phi) + 1/(\phi\mu A)$. As a result

$$\min\left[y - B\left(\bar{d} + \frac{T}{\phi}\right) - \frac{1}{\phi\mu A}\,,\; y - B\left(\bar{d} + \frac{T}{\phi} - \theta(\exp(\eta T) - 1)\right) - \frac{1}{\phi\mu A}\right] > 0 \Rightarrow h(t) > 0$$

This proves Proposition 5.

Appendix C. Solution of the Stochastic Model

The problem (4) to (6) is a piecewise deterministic control problem. Let $\tau \in (0, \infty)$ be the stochastic time at which the regime change happens. The optimization problem is

$$\max E\left\{\int_0^{\min[\tau,\,T]} g^-(t, h(t), d(t))\ \mathrm{d}t + \int_{\min[\tau,\,T]}^{T} g^+(t, h(t), d(t))\ \mathrm{d}t + v(d(T))\right\}$$

under

$$\begin{cases} \dot{d}(t) &=& f(t, h(t), d(t)) \\ d(0) &=& d_0 \end{cases}$$

The density of τ is

$$\lambda(d(t))\exp\left(-\int_0^t \lambda(d(u))\mathrm{d}u\right)$$

For simplicity, we now drop the time index unless needed for the understanding. We solve this problem using the backward procedure method proposed by Naevdal (2006) or Seierstad (2009).

We first solve the problem *after* the regime switch occurs. Uncertainty has hence vanished and we therefore define the usual Hamiltonian function $H^+ = g^+ + q^+ f$. The necessary optimality conditions are

$$\begin{cases} H^+_h &=& 0, & H^+_d &=& -\dot{q}^+, & H^+_{q^+} &=& \dot{d}, \\ d(\tau) & \text{given,} & & q^+(T) &=& v_d(T) \end{cases}$$

h is a control variable, d a state variable, q^+ the associated co-state variable. We moreover define a new auxiliary state variable γ^+, measuring the expected flow of remaining welfare along the optimal path:

$$\begin{cases} \dot{\gamma}^+ & = & -g^+ \\ \gamma^+(T) & = & v(d(T)) \end{cases}$$

The first equation means that the remaining welfare decreases by g^+ every period. The second equation is a terminal condition stating that the remaining welfare at time T is only the salvage value. The switch time τ matters in above boundary value problem, for it determines the initial conditions. The solution is therefore conditional to τ and, in particular, we will get $q^+(t \mid \tau)$ and $\gamma^+(t \mid \tau)$.

We then solve the problem *before* the regime switch occurs. Because of the uncertainty, we now define a risk-augmented Hamiltonian function

$$\tilde{H}^- = \underbrace{g^- + q^- f}_{\equiv H^-} + \lambda(d)\left(\gamma^+(t \mid \tau = t) - \gamma^-\right)$$

The risk-augmented Hamiltonian is the deterministic one H^-, plus the net change in remaining welfare $(\gamma^+(t \mid \tau = t) - \gamma^-)$ times the hazard rate $\lambda(d)$ of the random process. q^- and γ^- are the associated co-state variable and auxiliary state variable, respectively. The necessary optimality conditions are the usual ones but applied to the risk-augmented Hamiltonian

$$\begin{cases} \tilde{H}_h^- & = & 0, & \tilde{H}_d^- & = & -\dot{q}^-, & \tilde{H}_{q^-}^- & = & \dot{d}, \\ d(0) & = & d_0, & q^-(T) & = & v_d(T) \end{cases}$$

Using the definition of the risk-augmented Hamiltonian, as well as the fact that $\partial\gamma^-/\partial d = q^-$ and $\partial\gamma^+/\partial d = q^+$ (see Seierstad, 2003, for a formal proof), we can easily transform the above necessary optimality conditions into

$$\begin{cases} H_h^- & = & 0, & H_d^- & = & -\dot{q}^- + \lambda(d)(q^- - q^+(t \mid \tau = t)) + \lambda_d(d)(\gamma^- - \gamma^+(t \mid \tau = t)) \\ H_{q^-}^- & = & \dot{d}, & d(0) & = & d_0, \qquad q^-(T) \; = \; v_d(T) \end{cases}$$

Finally, we need to define the differential equation and give the boundary value for γ^-

$$\begin{cases} -g^- & = & \dot{\gamma}^- + \lambda(d)\left(\gamma^+(t \mid \tau = t) - \gamma^-\right) \\ \gamma^-(T) & = & v(d(T)) \end{cases}$$

The only difference with the similar equations after the switch is that $\dot{\gamma}^-$ is here augmented by its net gain if the shock occurs at time $\tau = t$, which takes into account the uncertainty.

We apply this general formulation to our model in Section 3 by posing

$$\begin{cases} g^- &= \ln(y-h) \\ g^+ &= \ln(y-h-Bd) \\ f &= \mu(d-Ah) \\ v &= -\phi d \end{cases}$$

The concavity of the problems guarantees that the necessary conditions are also sufficient. The solution in Section 2 directly follows from applying the general necessary conditions to our specific functional forms. Note that with our functional forms (in particular the linearity of the salvage function), q^+ does not depend on τ.

Appendix D. Proofs when $\lambda(d) = \lambda$

We here successively prove $q < 0$, $\dot{q} > 0$ and $q_\lambda < 0$.

First, from the definition $\eta \equiv \mu(1 + AB)$, we have $\eta > \mu$, which immediately gives $q < 0$.

Second, we compute

$$\dot{q} = \underbrace{\frac{\phi}{\eta+\lambda-\mu}}_{>0}\left(\underbrace{\lambda\eta\exp(\eta\,(T-t))}_{>0} - (\lambda-\mu)\,\underbrace{(\eta-\mu)\exp(-(\lambda-\mu)(T-t))}_{>0}\right)$$

If $\lambda < \mu$, then $\dot{q} > 0$ is immediate. If $\lambda > \mu$, then we have $\lambda\eta\exp(\eta\,(T-t)) - (\lambda-\mu)(\eta-\mu)\exp(-(\lambda-\mu)(T-t)) > \lambda\eta - (\lambda-\mu)(\eta-\mu) = \lambda\mu + \mu(\eta-\mu) > 0$. Therefore $\dot{q} > 0$.

Third

$$\begin{aligned} q_\lambda < 0 \quad &\Leftrightarrow \quad -\tfrac{(\eta+\lambda-\mu)[\exp(\eta\alpha)-(\eta-\mu)\alpha\exp(-(\lambda-\mu)\alpha)]-\lambda\exp(\eta\alpha)-(\eta-\mu)\exp(-(\lambda-\mu)\alpha)}{(\eta+\lambda-\mu)^2} < 0 \\ &\Leftrightarrow \quad (\eta-\mu)\exp(\eta\alpha) - (\eta-\mu)\exp(-(\lambda-\mu)\alpha)[(\eta+\lambda-\mu)\alpha+1] > 0 \\ &\Leftrightarrow \quad \exp(\eta\alpha) > [(\eta+\lambda-\mu)\alpha+1]\exp(-(\lambda-\mu)\alpha) \\ &\Leftrightarrow \quad \exp(\alpha(\eta+\lambda-\mu)) > 1+\alpha(\eta+\lambda-\mu) \end{aligned}$$

with $\alpha \equiv (T-t) > 0$. The latter inequality is always satisfied.

Appendix E. Solution of the Stochastic Model with Subsidy Rate S

We define $\eta \equiv \mu(1 + AB/(1-S))$ and $\theta \equiv Ay/((1-S)+AB)$ The solution in $[0,\tau)$ (i.e. non-LTC regime) is governed by the system of three differential equations

$$\begin{cases} \dot{q}(t) &=& (\lambda(d(t)) - \mu)q(t) + \phi \exp(\eta(T-t)) \left[\lambda(d(t)) - \lambda_d(d(t))(d(t) - \theta)\right] \\ && +\lambda_d(d(t)) \left[\gamma(t) + \frac{\eta}{2}(T-t)^2 + \left(1 + \ln\left(\frac{\mu A \phi}{1-S}\right)\right)(T-t) + \phi\,\theta\right] \\ \dot{\gamma}(t) &=& \ln\left(\frac{-\mu A q(t)}{1-S}\right) + \lambda(d(t)) \left[\gamma(t) + \frac{\eta}{2}(T-t)^2 + \left(1 + \ln\left(\frac{\mu A \phi}{1-S}\right)\right)(T-t)\right. \\ && \left. +\phi(d(t) - \theta)\exp(\eta(T-t)) + \phi\,\theta\right] \\ \dot{d}(t) &=& \mu\left(d(t) - \frac{Ay}{1-S}\right) - \frac{1}{q(t)} \end{cases}$$

with three boundary conditions $d(0) = d_0$, $q(T) = -\phi$ and $\gamma(T) = -\phi d(T)$, as well as

$$h(t) = \frac{y}{1-S} + \frac{1}{\mu A q(t)}$$

In $[\tau, T]$, the stochastic model does accept the closed-form solution

$$\begin{cases} q(t) &=& -\phi \exp(\eta(T-t)) \\ d(t) &=& (d(\tau) - \theta)\exp(\eta(t-\tau)) + \frac{t-\tau}{\phi}\exp(-\eta(T-t)) + \theta \\ h(t) &=& \frac{y - Bd(t)}{1-S} + \frac{1}{\mu A q(t)} \end{cases}$$

Appendix F. Solution of the Stochastic Model with Savings

The problem (8) to (10) is a piecewise deterministic control problem. Let $\tau \in (0,\infty)$ be the stochastic time at which the regime change happens. The optimization problem is

$$\max E \int_0^T \exp(-\rho t)\, g(t, c(t))\, \mathrm{d}t + \exp(-\rho T)\, v(d(T), s(T))$$

under

$$\begin{cases} \dot{d}(t) &=& f(t, h(t), d(t)) \\ \dot{s}(t) &=& u^-(t, h(t), c(t), d(t), s(t)) \quad \text{if} \quad t < \tau \\ \dot{s}(t) &=& u^+(t, h(t), c(t), d(t), s(t)) \quad \text{if} \quad t \geq \tau \\ d(0) &=& d_0 \\ s(0) &=& s_0 \end{cases}$$

The density of τ is

$$\lambda(d(t)) \exp\left(-\int_0^t \lambda(d(u))\mathrm{d}u\right)$$

For simplicity, we now drop the time index unless needed for the understanding. We solve this problem using the backward procedure method proposed by Naevdal (2006) or Seierstad (2009). See also Appendix C for more intuitions.

We first solve the problem *after* the regime switch occurs. Uncertainty has hence vanished and we therefore define the usual Hamiltonian function $H^+ = g + q^+ f + p^+ u^+$. The necessary optimality conditions are

$$\begin{cases} H_h^+ & = & 0, & H_d^+ & = & -\dot{q}^+ + \rho\, q^+, & H_{q^+}^+ & = & \dot{d}, \\ H_c^+ & = & 0, & H_s^+ & = & -\dot{p}^+ + \rho\, p^+, & H_{p^+}^+ & = & \dot{s}, \\ d(\tau) & \text{given}, & & q^+(T) & = & v_d(T) \\ s(\tau) & \text{given}, & & p^+(T) & = & v_s(T) \end{cases}$$

h and c are control variables, d and s state variables, q^+ and p^+ the associated co-state variables. We moreover define a new auxiliary state variable γ^+, measuring the expected flow of remaining welfare along the optimal path:

$$\begin{cases} \dot{\gamma}^+ & = & -g^+ + \rho\, \gamma^+ \\ \gamma^+(T) & = & v(d(T), s(T)) \end{cases}$$

The switch time τ matters in above boundary value problem, for it determines the initial conditions. The solution is therefore conditional to τ and, in particular, we will get $q^+(t \mid \tau)$, $p^+(t \mid \tau)$ and $\gamma^+(t \mid \tau)$.

We then solve the problem *before* the regime switch occurs. Because of the uncertainty, we now define a risk-augmented Hamiltonian function

$$\tilde{H}^- = \underbrace{g + q^- f + p^- u^-}_{\equiv H^-} + \lambda(d) \left(\gamma^+(t \mid \tau = t) - \gamma^-\right)$$

The necessary optimality conditions are the usual ones but applied to the risk-augmented Hamiltonian. Using the fact that $\partial\gamma^-/\partial d = q^-$, $\partial\gamma^-/\partial s = p^-$, $\partial\gamma^+/\partial d = q^+$ and $\partial\gamma^+/\partial s = p^+$ (see Seierstad, 2003, for a formal proof), we get

$$\begin{cases} H_h^- & = & 0, & H_d^- & = & -\dot{q}^- + \rho\, q^- + \lambda(d)(q^- - q^+(t \mid \tau = t)) + \lambda_d(d)(\gamma^- - \gamma^+(t \mid \tau = t)) \\ H_c^- & = & 0, & H_s^- & = & -\dot{p}^- + \rho\, p^- + \lambda(d)(p^- - p^+(t \mid \tau = t)) \\ H_{q^-}^- & = & \dot{d}, & d(0) & = & d_0, \quad q^-(T) \; = \; v_d(T) \\ H_{p^-}^- & = & \dot{s}, & s(0) & = & 0, \quad p^-(T) \; = \; v_s(T) \end{cases}$$

Finally, we need to define the differential equation and give the boundary value for γ^-

$$\begin{cases} -g^- + \rho\, \gamma^- & = & \dot{\gamma}^- + \lambda(d) \left(\gamma^+(t \mid \tau = t) - \gamma^-\right) \\ \gamma^-(T) & = & v(d(T), s(T)) \end{cases}$$

We apply this general formulation to our model in Section 5 by posing

$$\begin{cases} g & = & \frac{c^{1-\sigma} - 1}{1-\sigma} \\ f & = & \mu(d - A h^\nu - a) \\ u^- & = & y + r\, s - c - h \\ u^+ & = & y + r\, s - c - h - B d \\ v & = & -\phi d + \psi s \end{cases}$$

In $[\tau, T]$, i.e. *after the regime switch*, the model accepts a closed-form solution for the co-state variables (to simplify notations, we drop the superscript $+$) and hence for the control variables

$$\begin{cases} q(t) &= \frac{B\psi}{\mu-r}\exp\left((\rho-r)(t-T)\right) - \frac{B\psi+\phi(\mu-r)}{\mu-r}\exp\left((\rho-\mu)(t-T)\right) \\ p(t) &= \psi\exp\left((\rho-r)(t-T)\right) \\ h(t) &= \left(-\frac{p(t)}{\nu\mu A q(t)}\right)^{\frac{1}{\nu-1}} \\ c(t) &= p(t)^{-\frac{1}{\sigma}} \end{cases}$$

Note that q and p do not depend on τ because of the linearity and separability of the salvage function. Obtaining a closed-form solution for the state variables, however, is not possible, for they are governed by a system of two nonlinear differential equations

$$\begin{cases} \dot{d}(t) &= \mu(d(t) - Ah(t)^{\nu} - a) \\ \dot{s}(t) &= y + r\,s(t) - c(t) - h(t) - Bd(t) \end{cases}$$

where the two initial conditions $d(\tau)$ and $s(\tau)$ are given. As a result, $d(T)$, $s(T)$, and hence $\gamma^{+}(T) = -\phi d(T) + \psi s(T)$ as well as $\gamma^{+}(t \mid \tau)$, cannot be computed analytically.

In $[0, \tau)$, i.e. *before the regime switch*, the model solution is governed by the system of five differential equations (to simplify notations, we drop the superscript $-$)

$$\begin{cases} \dot{q}(t) &= (\lambda(d(t)) + \rho - \mu)q(t) + \lambda_d(d(t))\left[\gamma(t) - \gamma^{+}(t \mid \tau = t)\right] \\ & \quad -\lambda(d(t))\left[\frac{B\psi}{\mu-r}\exp\left((\rho-r)(t-T)\right) - \frac{B\psi+\phi(\mu-r)}{\mu-r}\exp\left((\rho-\mu)(t-T)\right)\right] \\ \dot{p}(t) &= (\lambda(d(t)) + \rho - r)p(t) - \lambda(d(t))\psi\exp\left((\rho-r)(t-T)\right) \\ \dot{\gamma}(t) &= (\lambda(d(t)) + \rho)\gamma(t) - \frac{p(t)^{-\frac{1-\sigma}{\sigma}} - 1}{1-\sigma} - \lambda(d(t))\gamma^{+}(t \mid \tau = t) \\ \dot{d}(t) &= \mu\left(d(t) - A\left(-\frac{p(t)}{\nu\mu A q(t)}\right)^{\frac{\nu}{\nu-1}} - a\right) \\ \dot{s}(t) &= y + r\,s(t) - p(t)^{-\frac{1}{\sigma}} - \left(-\frac{p(t)}{\nu\mu A q(t)}\right)^{\frac{1}{\nu-1}} \end{cases}$$

with five boundary conditions $d(0) = d_0$, $s(0) = 0$, $q(T) = -\phi$, $p(T) = \psi$ and $\gamma(T) = -\phi d(T) + \psi s(T)$.

Solving this system requires an analytical expression for $\gamma^{+}(t \mid \tau = t)$. However, as mentioned earlier, we do not have such an expression, for the dynamics in $[\tau, T]$ are themselves governed by a nonlinear system of differential equations. We overcome this problem by working with a parametric decision rule $\psi(\tau, d(\tau), s(\tau)|\boldsymbol{\theta})$ instead of with $\gamma^{+}(t \mid \tau = t)$. The vector of parameters $\boldsymbol{\theta}$ is chosen such that $\psi(\cdot|\boldsymbol{\theta})$ provides an accurate approximation of $\gamma^{+}(t \mid \tau = t)$ in the relevant domain.

Specifically, we proceed as follows:

(1) Let $\psi(\cdot|\boldsymbol{\theta})$ be a linear combination of monomials in $\{\tau, d(\tau), s(\tau)\}$ parameterized by vector $\boldsymbol{\theta}$.

(2) Generate a 3 dimensional grid on $\{\tau, d(\tau), s(\tau)\}$ covering the relevant domain where the model solution lives.
(3) At each grid point, solve the boundary value problem governing the model in $[\tau, T]$ numerically. Compute and store $\gamma^+(t \mid \tau = t)$.
(4) Estimate $\boldsymbol{\theta}$ by least squares

$$\boldsymbol{\theta}^* = \text{argmin} \sum_i^n (\gamma^+(t \mid \tau = t)_i - \boldsymbol{x}_i \boldsymbol{\theta})$$

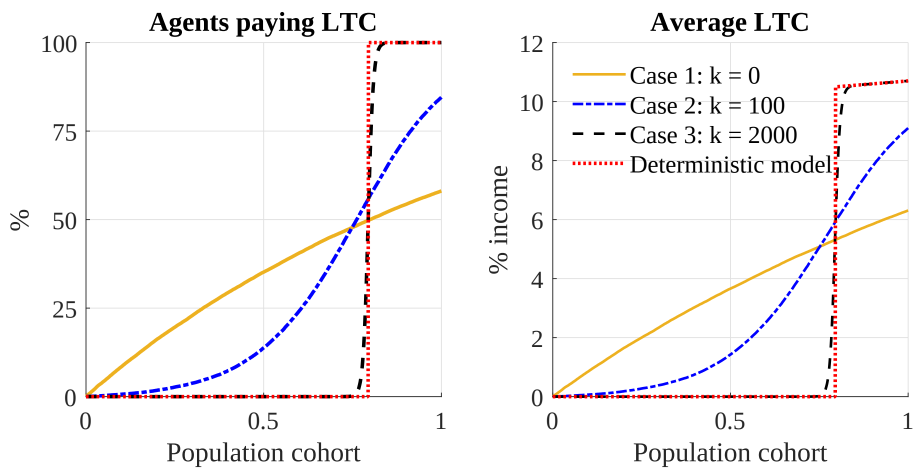
where n is the number of nodes in the grid and $\boldsymbol{x}$ are the variables of function $\psi(\cdot|\boldsymbol{\theta})$.
(5) Test the goodness of fit using R^2.

Therefore, $\psi(\tau, d(\tau), s(\tau)|\boldsymbol{\theta}^*)$ gives the value of $\gamma^+(t \mid \tau = t)$ for all possible combinations of $\{\tau, d(\tau), s(\tau)\}$. Inserting this parametric decision rule into the boundary value problem dictating the model dynamics in $[0, \tau)$ allows us to solve the stochastic model with savings.

Two exercises confirm the validity of the approach just described. In the first exercise (not shown), we solve the baseline model presented in Section 3 using this numerical approach instead of the true closed-form decision rule. We then check that both solutions coincide. The second exercise, in turn, compares the solution to the stochastic model when $k \to \infty$ with the solution of the deterministic model. Figure 4 in the main text confirms that both solutions converge, which is exactly what one should expect.

Appendix G. Additional Charts - Not for Publication

Figure 5. Long term costs – Small model



Notes. LTC stands for long term costs.

Figure 6. Initial health investment $h(0)$ as a function of income y – Small model

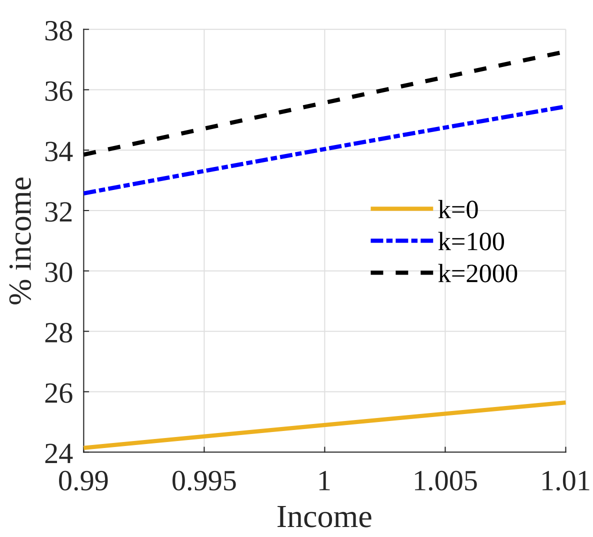
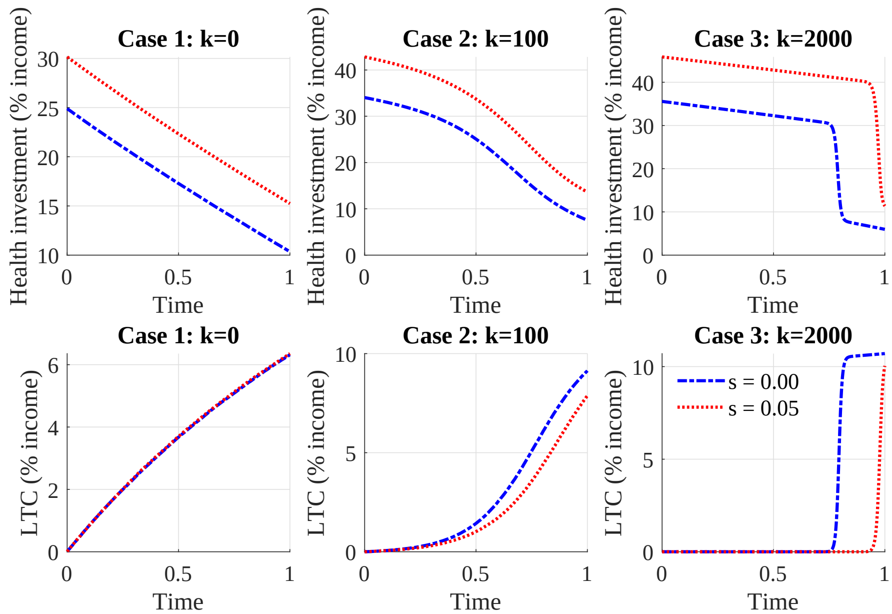
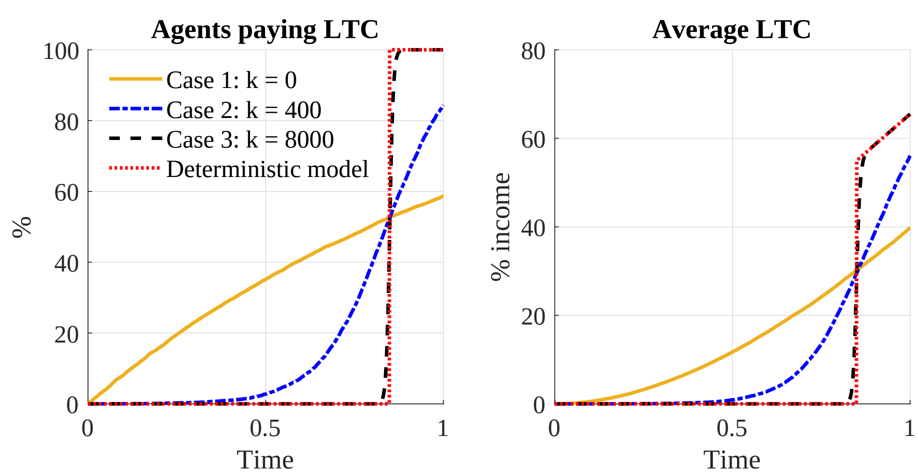


FIGURE 7. Health subsidy rate – Small model



Notes. LTC stands for long term costs.

FIGURE 8. Long term costs – Large model



Notes. LTC stands for long term costs.

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