A DYNAMIC PROGRAMMING APPROACH TO OPTIMAL POLLUTION CONTROL UNDER UNCERTAIN IRREVERSIBILITY: THE POISSON CASE

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A dynamic programming approach to optimal pollution control under uncertain irreversibility: The Poisson case*

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Abstract

We solve a bimodal optimal control problem with a non-concavity and uncertainty through a Poisson process underlying the transition from a mode to another. We use a dynamic programming approch and are able to uncover the global optimal dynamics (including optimal non-monotonic paths) under a few linear-quadratic assumptions, which do not get rid of the non-concavity of the problem. This is in contrast to the related literature on pollution control under irreversibility which usually explores local dynamics along monotonic solution paths to first-order Pontryagin conditions.

Keywords: Multi-stage optimal control, Poisson process, HJB equations, irreversible pollution
JEL classification: C61, Q53

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*This paper is dedicated to the memory of Professor Jack Warga on the occasion of his 100th birthday. Boucekkine is corresponding member of IRES, UCLouvain (Belgium), on leave from Aix-Marseille School of Economics. The usual disclaimer applies.

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1 The problem and related literature

Pollution irreversibility is being a crucial aspect of the current debate on sustainable development, in particular in relation with global warming. Pollution is irreversible when its impact on Nature and Humanity can no longer be reverted. There are strong reasons to believe that irreversible change due to global warming is under way as documented by Boucekkine et al. (2013). The problem is however under scrutiny since the 70s in hard sciences (for example, see Holling, 1973, for an example in ecology). It has become the object of deep investigation since the mid-90s in mathematical economics and operation research (see for example, Tsur and Zemel, 1995, and, in particular, Tahvonen and Withagen, 1996).

A key complication in the mathematical treatment of optimization problems involving irreversibile pollution is the induced non-concavity of the problem. For example, in Tahvonen and Withagen (1996), pollution may turn irreversible because above a certain threshold level of pollution (the state variable), Nature self-cleaning capacity drops suddenly to zero, which makes the problem non-concave. As a result, beside multiple stationary states and potential complex dynamics, establishing the optimality of solution paths derived from first-order conditions may not be easy. Nonetheless, the related literature uses the standard hamiltonian-based Pontryagin method to tackle the optimization problems involved, at the cost of burdensome posterior elaborations in the best case (see in particular, the seminal work of Tahvonen and Withagen, 1996). Together with the multi-stage (or multi-modal) intrinsic nature of the optimal control problems under study (as the associated state equations will feature two modes: irreversible vs irreversible pollution), this makes the analysis highly tricky. Quite often, the analysis provided falls however short to identify the global optimal dynamics and to provide with deep non-local analysis.

In this paper, we propose a dynamic programming (DP) approach, which in our view fits better the structure of irreversible pollution control problems. While it will not of course eliminate the complexity of the problem, we will show that it does allow to provide with the full picture for (optimal) global dynamics. To this end, we consider the basic deterministic model studied in Tahvonen and Withagen (1996). We extend it by introducing uncertainty in the following way: we assume that the move from the reversible to irreversible pollution mode occurs through a Poisson process with constant arrival rate. This is probably the simplest stochastic extension of the basic model, it allows us to show in a way the flexibility of the DP approach followed.

There are a few papers studying irreversibility in stochastic environmental problems. The most known is due to Tsur and Zemel (1995) who studied the optimal pace of underground water extraction in the context where there exists a threshold of water reserves under which further extraction is no longer feasible. The threshold is unknown and it's assumed that it follows a random process with given distribution. Another interesting work taking this avenue has been proposed by Le Kama et al. (2014), it's closer to our frame though both the mathematical and economic modelling are different. Here the state variable under pressure is the environmental quality: the irreversibility threshold for this variable is assumed to be reached at an uncertain time with given distribution.

As argued above, both papers do not study optimal global dynamics. In the case of Le Kama et al. (2014), the results are derived for the steady state equilibria and their respective neighborhood. Moreover, both papers specialise in nondecreasing state variable paths. While this restriction makes economic sense, it also hides part of the complexity of the problem. By applying the DP approach to a generic irreversibility problem with Poisson arrival rates for the irreversible mode, we are able to produce the big picture of the optimal dynamics under a few linear-quadratic specifications (which do not remove the non-concave nature of our optimization problem). With respect to the literature quoted above, our main contributions are twofold: (1) we present the complete possible dynamics under different modes, which are essentially attracted or repelled by the two potential long-run steady states of the two modes, showing the potential emergence of optimal non-monotonic dynamics, and (2) we investigate when the irreversible regime, under Poisson process, can be triggered or not.

The rest of the paper is organized as following. Section 2 describes the model and Section 3 provides solutions to the optimal control problems in two difference modes– reversible and irreversible environmental regimes via dynamic programming. Section 4 presents possible outcomes from the solutions and draws the main contributions.

2 The model

Following Tahvonen and Withagen (1996), we investigate a situation where the decision maker faces irreversible pollution accumulation. For simplicity, the pollution emission, $y(t)$, is used to measure the output level. The objective of the decision maker is to maximize social welfare:

$$\max_{y} W = \int_0^{+\infty} (U(y) - D(z))e^{-rt}dt = \int_0^{+\infty} \left[\left(ay - \frac{y^2}{2} \right) - \frac{c}{2}z^2 \right] e^{-rt}dt, \tag{1}$$

where r is time preference, $z(t)$ is accumulated pollution, $U(y)$ is the utility from enjoying final output generated with pollution $y(t)$, and $D(z)$ is damaging function from aggregate pollution stock z. For simplicity, we take linear-quadratic functional forms which can yield closed-form solutions.

Pollution stock $z(t)$ may decay at rate $\delta(z)$. However, the decay rate may spontaneously and irreversibly drop to zero. In other words, the pollution accumulation is given by the following:

$$\dot{z} = y - \delta(z), \quad z(0) = z_0 \text{ given}, \tag{2}$$

where $\delta(z)$ is the decay function before the abrupt drop of the decay rate. After the drop, no decay is possible. Hence, there are two modes, with and without decay, denoted by $m = 1$ and 0, respectively. The jump from mode 1 to 0 occurs with the constant rate

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \Pr \left\{ m\left(t + \Delta t\right) = 0 | m\left(t\right) = 1 \right\} = \lambda.$$

In other words, the probability of the mode change during the interval $(t, t+\Delta t]$, given that the mode at t is 1, is proportional to Δt, that is, the arrival of the irreversible regime follows Poisson process with intensity parameter $\lambda \geq 0$. Obviously, when $\lambda = 0$, no regime change happens.

As a result, the planner's optimal control problem are divided in periods I and II, corresponding to modes 1 and 0, respectively, as follows.

Period I

$$\dot{z} = y - \delta(z), \quad z(0) = z_0,$$

where T is the time of mode switching.

Period II

$$\dot{z} = y, \quad z(T) = z\left(T^{-}\right).$$

A special case: In the following in order to obtain explicit solution and equilibrium, we consider a special case:

$$\delta(z) = \begin{cases} \alpha - \beta z & \text{for } z \leq \bar{z}, \\ 0 & \text{for } z > \bar{z}, \end{cases} \tag{3}$$

where $\bar{z} = \alpha/\beta$ and the reaching time to $\bar{z}$ could be random as mentioned above.

3 The optimal choices

3.1 Hamiltonians

We construct the value functions $V_m(z)$ in mode $m(=0,1)$ as follows. Define functions f_m and g by

$$f_1(z, y) = y - \delta(z), \qquad f_0(z, y) = y. \tag{4}$$

The Hamiltonians $H_m(z, p)$ in mode m is

$$H_m(z, p) = U(y_m^*) - D(z) + p f_m(z, y_m^*),$$

where y^* is the maximizer

$$y_m^* = \arg\max_y \left\{U(y) - D(z) + pf_m(z,y)\right\}. \tag{5}$$

Denote random variable $\bar{z}$ as the threshold stock of pollution when the mode change happens. Then, for $z \neq \bar{z}$ the value functions V_0 and V_1 satisfy the following HJB equations:

$$\begin{aligned} rV_0(z) &= H_0(z, V_0'(z)), \\ (r+\lambda) V_1(z) &= H_1(z, V_1'(z)) + \lambda V_0(z) \end{aligned} \tag{6}$$

where

$$\begin{aligned} H_0(z,p) &= ay_0^* - \frac{1}{2}(y_0^*)^2 - \frac{c}{2}z^2 + py_0^* \\ H_1(z,p) &= ay_1^* - \frac{1}{2}(y_1^*)^2 - \frac{c}{2}z^2 + p\left[y_1^* - \delta(z)\right]. \end{aligned}$$

It is easy to see

$$y_m^* = a + p \qquad \text{for } m = 0, 1.$$

Hence, HJB equations (6) becomes

$$\begin{aligned} &2rV_0(z) = (a + V_0'(z))^2 - cz^2, \\ &2(r+\lambda) V_1(z) = (a + V_1'(z))^2 - 2\delta(z) V_1'(z) - cz^2 + 2\lambda V_0(z). \end{aligned} \tag{7}$$

3.2 Solution in Mode 0

The value function V_0 in mode 0 satisfies the first equation in (7). To solve the equation, we differentiate the both sides with respect to z to obtain

$$rV_0'(z) = \big(a + V_0'(z)\big) V_0''(z) - cz.$$

In terms of $f_0(z) \equiv a + V_0'(z)$, the equation becomes

$$f_0(z) f_0'(z) = rf_0 + cz - ra. \tag{8}$$

It is easy to see that there is a steady state, $\bar{z}_0$ that satisfies $f_0(\bar{z}_0) = 0$. From the above equation it follows that

$$\bar{z}_0 = \frac{ra}{c}. \tag{9}$$

Let $x = z - \bar{z}_0$ and $W_0(x) = f_0(z) = f_0(x + \bar{z}_0)$. The equation becomes

$$W_0 \frac{dW_0}{dx} = rW_0 + cx.$$

Using the new unknown $Y_0 = W_0/x$, we derive

$$x \frac{dY_0}{dx} = r - Y_0 + \frac{c}{Y_0}.$$

The separable equation has the general solution in the implicit form

$$|Y_0 - y_1|^{k_1} |Y_0 - y_2|^{k_2} |x|^{-1} = C$$

where

$$y_1 = \frac{r - \sqrt{r^2 + 4c}}{2}, \qquad y_2 = \frac{r + \sqrt{r^2 + 4c}}{2}, \qquad k_1 = \frac{y_1}{y_2 - y_1}, \qquad k_2 = \frac{-y_2}{y_2 - y_1} \tag{10}$$

and C is a constant. Therefore, W_0 satisfies the equation

$$|W_0 - xy_1|^{k_1} |W_0 - xy_2|^{k_2} = C,$$

which leads to

$$|f_0(z) - (z - \bar{z}_0) y_1|^{k_1} |f_0(z) - (z - \bar{z}_0) y_2|^{k_2} = C. \tag{11}$$

The value function V_0 is constructed from f_0 by the first equation in (7). As a result,

$$V_0(z) = \frac{1}{2r} \left[f_0(z)^2 - cz^2 \right]. \tag{12}$$

Note that for special case where constant $C = 0$ there are two quadratic solutions, $V_{0,i}(z)$, corresponding to

$$f_{0,i}(z) = (z - \bar{z}_0) y_i \qquad \text{for } i = 1, 2.$$

The corresponding value functions are

$$V_{0,i}(z) = \frac{1}{2r} \left[(z - \bar{z}_0)^2 y_i^2 - cz^2 \right], \qquad i = 1, 2.$$

These are ones that lead to the steady state $\bar{z}_0$. Only $V_{0,1}(z)$ is concave, thus qualifies as a valid value function for the above optimal control problem if there is no Mode 1. It is straightforward this value function, among others, is affine-quadratic in term of state variable z. The other value functions depend on C, which is determined by maximizing the value function in Mode 1. In general, the graph for $f_0(z)$ is hyperbolic shaped. (See Figs. 1–6.)

3.3 Solution in Mode 1

For $z \geq \bar{z}$, there is no difference whether mode changes or not. So $V_1(z) = V_0(z)$ for such z. Therefore, one only needs solve $V_1(z)$ for $z < \bar{z}$. To solve the second equation in (7), we differentiate the both sides with respect to z to obtain

$$(r + \lambda - \beta) V_1'(z) = \big(a + V_1'(z) - \delta(z)\big) V_1''(z) - cz + \lambda V_0'(z).$$

In terms of

$$f_0(z) = V_0'(z) + a, \qquad f_1(z) \equiv V_1'(z) + a - \delta(z),$$

the equation becomes

$$f_1(z) f_1'(z) = (r + \lambda) f_1(z) + (r - \beta + \lambda) \delta(z) + (\beta - r) a + cz - \lambda f_0(z) \quad (13)$$

for $0 < z < \bar{z}$

In the case $\lambda f_0(z) = \lambda(z - \bar{z}_0) y_1$ is zero or linear, the equation can be written as

$$f_1(z) f_1'(z) = (r + \lambda) f_1(z) + B_1 z + C_1$$

where

$$B_1 = \beta(\beta - r - \lambda) + c - \lambda y_1, \qquad C_1 = (r - \beta + \lambda)\alpha + (\beta - r) a + \lambda y_1 \bar{z}_0. \quad (14)$$

Using a change of variable $x = z + C_1/B_1$, the equation becomes

$$W_1(x) W_1'(x) = (r + \lambda) W_1(x) + B_1 x.$$

where $W_1(x) = f_1(z)$. The general solution in implicit form is

$$|W_1 - xY_1|^{p_1} |W_1 - xY_2|^{p_2} = C$$

where C is a constant,

$$Y_1 = \frac{1}{2}\left[r + \lambda - \sqrt{(r+\lambda)^2 + 4B_1}\right], \qquad Y_2 = \frac{1}{2}\left[r + \lambda + \sqrt{(r+\lambda)^2 + 4B_1}\right], \tag{15}$$

and

$$p_1 = \frac{Y_1}{Y_2 - Y_1}, \qquad p_2 = \frac{-Y_2}{Y_2 - Y_1}.$$

This leads to

$$\left|f_1(z) - (z - \bar{z}_1^*)\, Y_1\right|^{p_1} \left|f_1(z) - (z - \bar{z}_1^*)\, Y_2\right|^{p_2} = C,$$

where

$$\bar{z}_1^* = -\frac{C_1}{B_1} = -\frac{(r - \beta + \lambda)\,\alpha + (\beta - r)\,a + \lambda y_1 \bar{z}_0}{\beta(\beta - r - \lambda) + c - \lambda y_1}. \tag{16}$$

Obviously if there were no mode change, $\bar{z}_1^*$ would be the "potential" long-run steady state in the reversible environmental regime. Within the current framework there is either uncertain Poisson process or pollution accumulation across the threshold $\bar{z}$, nevertheless, $\bar{z}_1^*$ plays an important role in determining the trajectory of the dynamics. The detail results will be presented in the following Theorem 1 and 2.

With $f_1(z)$ solved, one can find V_1 from the second equation of (7) as

$$V_1(z) = \frac{1}{2(r+\lambda)}\left[f_1(z)^2 - \delta(z)^2 + 2a\delta(z) - cz^2 + 2\lambda V_0(z)\right]. \tag{17}$$

In the special case where $C = 0$, there are two value functions $V_{1,i}(z)$ with

$$f_{1,i}(z) = (z - \bar{z}_1^*)\, Y_i \qquad \text{for } i = 1, 2.$$

These value functions may not match the value functions $V_0(z)$ in Mode 0 at $\bar{z}$. In other words, the linear-quadratic autonomous system may not generate linear state strategy in Mode 1, even in the Mode 0, the strategy is linear in state variable. More generally, $C > 0$, and so, the solution is hyperbolic shaped. More precisely, due to the transversality condition at $\bar{z}$, $V_0(\bar{z}) = V_1(\bar{z})$, linear strategy, thus linear-quadratic value functions, may not hold in both regime at the same time.

4 Possible outcomes

As shown above, in Mode 0, the only possible steady state is $\bar{z}_0$ given by (9). The outcome depends on whether the threshold value $\bar{z}$ is lower or above $\bar{z}_0$.

4.1 Case 1: $\bar{z} < \bar{z}_0$

4.1.1 Steady states and dynamics

By (13), f_1 vanishes if

$$(r - \beta + \lambda)\,\delta(z) + (\beta - r)\,a + cz - \lambda f_0(z) = 0. \tag{18}$$

Theorem 1 *Suppose $\bar{z} < \bar{z}_0$ and $V_1(z)$ is differentiable. Let $\bar{z}_1$ be the maximum nonnegative solution of Eq. (18) if it exists, or $\bar{z}_1 = 0$ if nonnegative solution does not exist. Then there are three possibilities.*

1. *If*
$$\bar{z}_1^* < \bar{z}, \qquad (\bar{z} - \bar{z}_1^*)\,Y_1 < (\bar{z} - \bar{z}_0)\,y_1 < (\bar{z} - \bar{z}_1^*)\,Y_2, \tag{19}$$
then $z(t)$ decreases for $z_0 < \bar{z}_1^*$ and it increases for $\bar{z}_1^* < z_0 < \bar{z}$ in Period I. After entering Period II (by passing through $\bar{z}$ or by spontaneous mode change), $z(t)$ is increasing for all t.

2. *If either*
$$\bar{z}_1^* < \bar{z}, \qquad Y_1 < 0, \qquad (\bar{z} - \bar{z}_1^*)\,Y_2 < (\bar{z} - \bar{z}_0)\,y_1 \tag{20}$$
or
$$\bar{z}_1^* \geq \bar{z}, \qquad Y_1 < 0 \tag{21}$$
then $z(t)$ is increasing for all $0 \leq z_0 < \bar{z}$ in Period I, and after entering Period II, $z(t)$ converges to $\bar{z}_0$.

3. *In all other cases, $\bar{z}_1 < \bar{z}$. Furthermore, $z(t)$ decreases for any $z_0 < \bar{z}_1$ and it increases for any $z_0 > \bar{z}_1$ in Period I. After entering Period II, $z(t)$ converges to $\bar{z}_0$ in infinite time.*

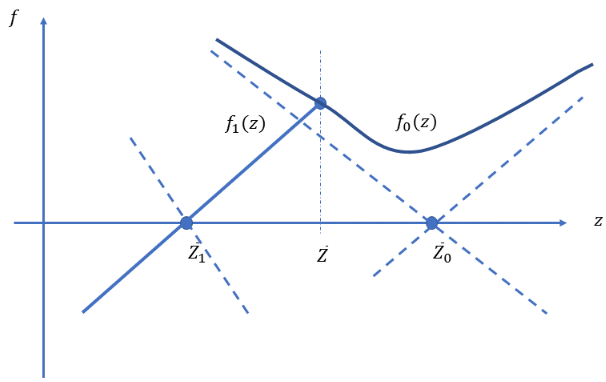

Figure 1: Case 1: $\bar{z}_1^*$ is a repeller in Period I, and $\dot{z} > 0$ in Period II.

Proof. Since $\bar{z} < \bar{z}_0$, it follows that $f_0(\bar{z}) > 0$. Since $V_1(z) = V_0(z)$ for $z \geq \bar{z}$, by continuity and differentiability, we have

$$V_1(\bar{z}) = V_0(\bar{z}), \qquad V_1'(\bar{z}) = V_0'(\bar{z}).$$

Furthermore, since $\delta(\bar{z}) = 0$, we find

$$f_1(\bar{z}) = V_1'(\bar{z}) + a - \delta(\bar{z}) = V_0'(\bar{z}) + a = f_0(\bar{z}) > 0.$$

By continuity, $f_1(z) > 0$ for some $z < \bar{z}$. Let $\bar{z}_1$ be the maximum of z such that $f_1(z) \leq 0$. Then either $\bar{z}_1 = 0$ or $\bar{z}_1 > 0$ and $f_1(\bar{z}_1) = 0$. In the latter case, by (13), $\bar{z}_1$ is a positive solution of Eq. (18).

Suppose $\bar{z}_1 > 0$. Eq. (13) is equivalent to the differential equations

$$\begin{aligned} &\dot{x} = (r+\lambda)x + (r-\beta+\lambda)\delta(z) + (\beta-r)a + cz - \lambda f_0(z), \\ &\dot{z} = x \end{aligned} \tag{22}$$

and $(0, \bar{z}_1)$ is an equilibrium. Its Jacobian matrix takes the form

$$J = \begin{pmatrix} r+\lambda & \beta(\beta-r-\lambda) + c - \lambda f_0'(\bar{z}_1) \\ 1 & 0 \end{pmatrix}. \tag{23}$$

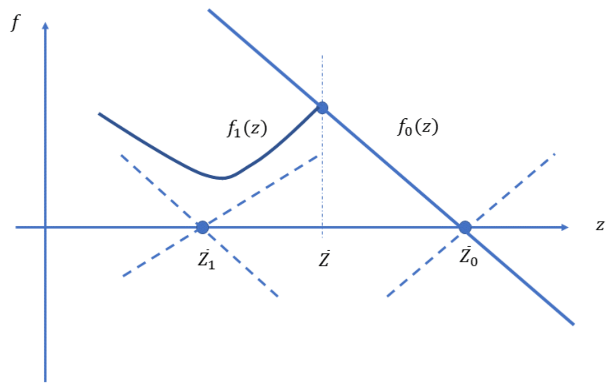

Figure 2: Case 2: $\dot{z} > 0$ in Period I and $\bar{z}_0$ is an attractor in Period II.

It has eigenvalues

$$\mu_1 = \frac{1}{2}\left[r + \lambda - \sqrt{(r+\lambda)^2 + 4B(\bar{z}_1)}\right], \qquad \mu_2 = \frac{1}{2}\left[r + \lambda + \sqrt{(r+\lambda)^2 + 4B(\bar{z}_1)}\right], \tag{24}$$

where

$$B(z) = \beta(\beta - r - \lambda) + c - \lambda f_0'(z). \tag{25}$$

Since $f_0'(z) < 0$,

$$\begin{aligned} (r+\lambda)^2 + 4B(z) &> (r+\lambda)^2 + 4\beta(\beta - r - \lambda) + 4c \\ &= (r+\lambda)^2 - 4\beta(r+\lambda) + 4\left(\beta^2 + c\right) > 0 \end{aligned}$$

for any z. Hence, μ_1 and μ_2 are real and $\mu_2 > 0$. Therefore, the equilibrium is either a repeller or a saddle point. Every trajectory $(x(t), z(t))$ associate a value function with $f_1(z)$ that satisfies $f_1(z(t)) = x(t)$. Since value functions are defined for all z, only trajectories whose range include all $z \geq 0$ are acceptable. These include stable and unstable manifolds.

We also notice that since

$$B_1 = \beta(\beta - r - \lambda) + c - \lambda f_{0,1}'(z)$$

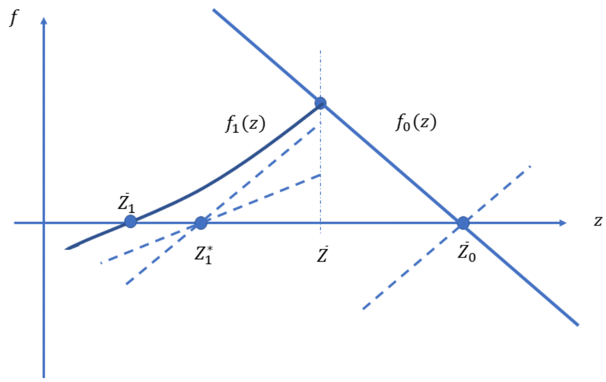

Figure 3: Case 3: $\bar{z}_1$ is a repeller in Period I, and $\bar{z}_0$ is an attractor in Period II.

is a special case of $B(z)$, it follows that $(r+\lambda)^2 + 4B_1 > 0$. Hence Y_1 and Y_2 are real and $Y_2 > 0$. In this special case, $\bar{z}_1 = \bar{z}_1^*$ that is defined in (16).

Similarly, Eq. (8) is equivalent to the dynamical system

$$\begin{aligned} \dot{x} &= rx + cz - ra, \\ \dot{z} &= x \end{aligned} \tag{26}$$

and $(0, \bar{z}_0)$ is an equilibrium. The Jacobian matrix,

$$J_0 = \begin{pmatrix} r & c \\ 1 & 0 \end{pmatrix}$$

has eigenvalues $\left(r \pm \sqrt{r^2+4c}\right)/2$. Hence $(0, \bar{z}_0)$ is a saddle point.

It can be seen that along a stable or unstable manifold, $f_1(\bar{z}_1) = 0$ and along any other trajectory, $f_1(\bar{z}_1)$ is either positive or negative. Furthermore, the eigenvectors corresponding the eigenvalue μ_i are parallel to the vector $\langle 1, \mu_i \rangle$ for $i = 1, 2$. In particular, at least one unstable manifold emanating from the equilibrium with a positive angle to the z-axis. On the other hand, $f_0(z)$ exists for all $z \geq \bar{z}$ and $f_0(z) > 0$ for $\bar{z} \leq z < \bar{z}_0$. By optimality, at least one of $f_0(\bar{z}_0)$ and $f_1(\bar{z}_1)$ must be zero, and the other is nonnegative. Which is zero depends on whether $f_0(\bar{z})$ is greater than, equal to, or less than $f_1(\bar{z})$.

There are two cases, either $\bar{z}_1^* < \bar{z}$ or $\bar{z}_1^* \geq \bar{z}$. Suppose $\bar{z}_1^* < \bar{z}$. Then $(\bar{z} - z_1^*) Y_1 < (\bar{z} - z_1^*) Y_2$ and $(\bar{z} - z_1^*) Y_2 > 0$. If (19) holds, then $f_1(z) = f_{1,2}(z)$ and $f_0(z) > f_{0,1}(z)$. (See Fig. 1.) Hence, $\bar{z}_1^*$ is a repeller in Period I and $\dot{z} = f_0(z) > 0$ in Period II. If (20) holds, then $f_1(z) > f_{1,2}(z)$ in Period I and $\bar{z}_0$ is an attractor in Period II. As a result, $\dot{z} = f_1(z) > 0$ for $0 \leq z \leq \bar{z}$. (See Fig. 2.) In the remaining case, either

$$Y_1 \geq 0, \qquad (\bar{z} - \bar{z}_1^*) Y_2 < (\bar{z} - \bar{z}_0) y_1$$

or

$$(\bar{z} - \bar{z}_1^*) Y_1 > (\bar{z} - \bar{z}_0) y_1.$$

In both cases $f_0(z) = f_{0,1}(z)$ and $f_1(z) \neq f_{1,i}(z)$. (See Fig. 3 that illustrate the former case.) In both cases the trajectories connect $(\bar{z}, f_0(\bar{z}))$ to $(0, f_1(0))$ for some $f_1(0)$. In case $f_1(0) < 0$, then there is $\bar{z}_1$ such that $0 < \bar{z}_1 < \bar{z}$ and $f_1(\bar{z}_1) = 0$. So $\bar{z}_1$ is a solution to (18) and is a repeller in Period I.

Suppose $\bar{z}_1^* \geq \bar{z}$. If (21) holds, then $(\bar{z} - \bar{z}_1^*) Y_1 \geq 0$ and $\mu_1 < 0$. So trajectories of the dynamical system (22) have negative slopes for $z \leq \bar{z}$. Therefore $f_1(z) > 0$ for $z \leq \bar{z}$. (See Fig. 4.)

It is clear that $\dot{z} = f_1(z) > 0$ in Period I, and $\bar{z}_0$ is an attractor in Period II. In the remaining case where $Y_1 \geq 0$, trajectories of (22) for $z \leq \bar{z}$ have negative slopes. So, either $f_1(z) \geq 0$ for all $0 \leq z \leq \bar{z}$ or there is $\bar{z}_1 > 0$ such that $\bar{z}_1 < \bar{z}$ and $f_1(\bar{z}_1) = 0$. (See Fig. 5.)

Hence $\bar{z}_1$ satisfies (18) and is a repeller in Period I. Obviously, $\bar{z}_0$ is an attractor in Period II.

This completes the proof.

To close this case, recall mentioned above that generally under multistage optimal control (and differential game) problems with endogenous stage changes, even with linear-quadratic autonomous framework, there is no guarantee that linear-state optimal control, thus linear-quadratic value

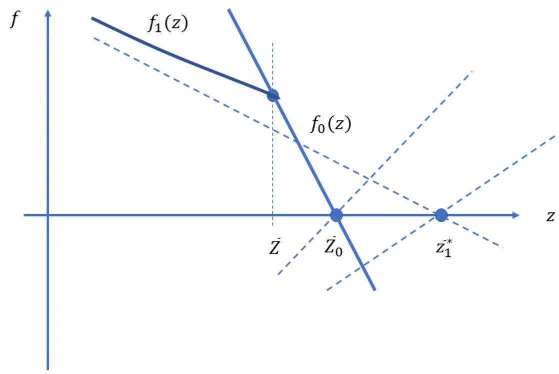

Figure 4: Case 2 with (21) holds. $\dot{z} > 0$ in Period I, and $\bar{z}_0$ is an attractor in Period II.

functions, are possible in both modes because of the transversality condition between the two modes. Nonetheless, the above analysis shows that under some special situation, such as the above special case

$$B_1 = \beta (\beta - r - \lambda) + c - \lambda f'_{0,1} (z) \tag{27}$$

when $(\bar{z} - \bar{z}_0) \, y_1 = (\bar{z} - \bar{z}_1) \, Y_2$, there exists at least one group of linear-state dependent optimal choices, and thus linear-quadratic value functions, in both mode 0 and 1. The method provided above could be applied to other studies of multistage (or multi-mode) optimal control and differential game. We conclude the results in the following.

Corollary 1 *Under the assumptions for Theorem 1, especially, $\bar{z} < \bar{z}_0$ and if (27) and $(\bar{z} - \bar{z}_0) \, y_1 = (\bar{z} - \bar{z}_1) \, Y_2$ hold, then there exists linear-state dependent optimal pollution control in both mode $m = 0$ and $m = 1$ which is given by: for any $z \geq 0$,*

$$\begin{cases} y_0^*(z) = f_{0,1}(z) = (z - \bar{z}_0) y_1, \\ y_1^*(z) = f_{1,2}(z) + \delta(z) = (z - \bar{z}_1) Y_2 + \delta(z). \end{cases}$$

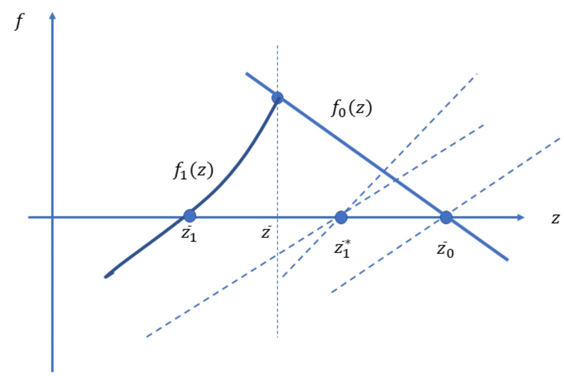

Figure 5: Case 3 with $\bar{z}_1^* \geq \bar{z}$ and $Y_1 \geq 0$. $\bar{z}_1$ is a repeller in Period I, and $\bar{z}_0$ is an attractor in Period II.

The corresponding value functions are

$$\begin{cases} V_{0,1}(z) = \dfrac{1}{2r}\left[(z-\bar{z}_0)^2 y_1^2 - cz^2\right], \\ V_{1,2}(z) = \dfrac{1}{2(r+\lambda)}\left[f_{1,2}^2(z) - \delta^2(z) + 2a\delta(z) - cz^2 + 2\lambda V_{0,1}(z)\right], \end{cases}$$

provided $V_{1,2}(z)$ *is concave in term of* z*. Furthermore, the time to reach* $\bar{z}$ *for any* z_0 *is given by*

$$T = \int_{z_0}^{\bar{z}} \frac{1}{f_{1,2}(z)} dz = \ln\left(\left|\frac{\bar{z}-\bar{z}_1}{z_0-\bar{z}_1}\right|\right).$$

Remark 1 *Assumption* $(\bar{z}-\bar{z}_0)\, y_1 = (\bar{z}-\bar{z}_1)\, Y_2$ *is not indicating one special point, rather a manifold which satisfies this equality condition.*

4.2 Case 2: $\bar{z} > \bar{z}_0$

We show that in this case the threshold is never reached, and the steady state $\bar{z}_1$ in Mode 1, if it is nonnegative, is less than $\bar{z}_0$.

Theorem 2 *Suppose* $\bar{z} > \bar{z}_0$ *and that* $V_1(z)$ *is differentiable. Then* $\bar{z}$ *is never reached. Furthermore,*

1. *If*
$$Y_1 < 0 \text{ and } (\bar{z} - \bar{z}_0)\, y_1 = (\bar{z} - \bar{z}_1^*)\, Y_1, \tag{28}$$
then $\bar{z}_1^*$ *is an attractor in Period I and* $\bar{z}_0$ *is an attractor in Period II. Thus* $z(t)$ *approaches* $\bar{z}_1^*$ *for any* z_0 *between* 0 *and* $\bar{z}$ *in Period I, and upon mode change,* $z(t)$ *turns to approach* $\bar{z}_0$*.*

2. *If*
$$Y_1 < 0, \text{ and } (\bar{z} - \bar{z}_0)\, y_1 > (\bar{z} - \bar{z}_1^*)\, Y_1, \tag{29}$$
then there is a solution $\bar{z}_1$ *to (18) with* $f_0(z) = f_{0,1}(z)$ *which is an attractor in Period I and* $z(t)$ *is decreasing in Period II. So* $z(t)$ *approaches* $\bar{z}_1$ *from any* $z_0 < \bar{z}$*. After the mode change,* $z(t)$ *decreases to zero.*

3. *In all other cases* $z(t)$ *is decreasing in Period I and* $\bar{z}_0$ *is an attractor in Period II.*

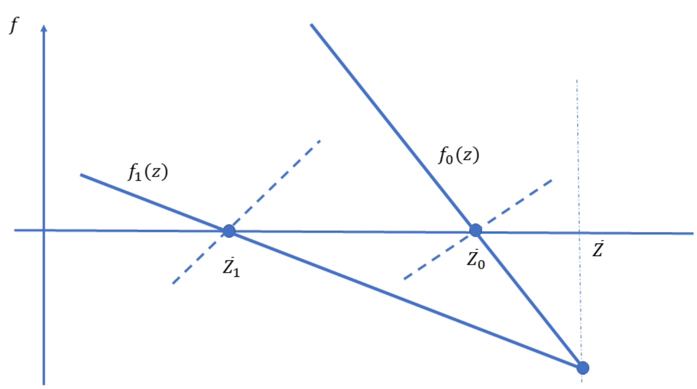

Figure 6: if (28) holds, then $\bar{z}_1^*$ and $\bar{z}_0$ are attractors in Periods I and II, respectively.

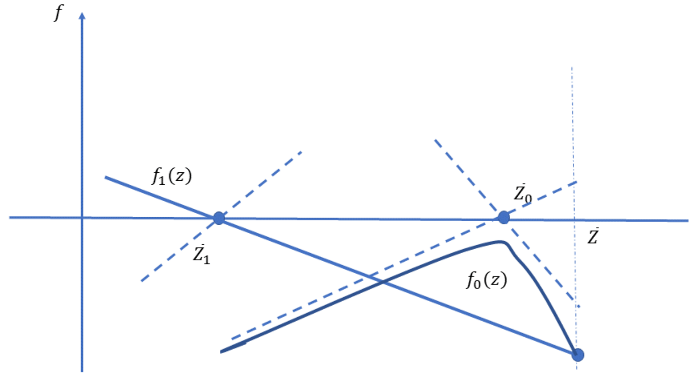

Figure 7: If (29) holds, then $\bar{z}_1$ is an attractor in Period I and $\dot{z} < 0$ in Period II.

Proof. Since $\bar{z} > \bar{z}_0$, it follows that

$$f_1(\bar{z}) = f_0(\bar{z}) < 0. \tag{30}$$

By continuity, $f_1(z) < 0$ if $\bar{z}_1 < z < \bar{z}$. Furthermore, Eqs. (13) and (8) are equivalent to the dynamical systems (22) and (26), respectively, and the solutions $\bar{z}_1$ and $\bar{z}_0$ are equivalent to equilibria $(0, \bar{z}_1)$ and $(0, \bar{z}_0)$, respectively. Furthermore, at least one of the equilibria is passed by a trajectory of the corresponding dynamical system, and $(0, \bar{z}_0)$ is a saddle point. On the other hand, $(0, \bar{z}_1^*)$ is either a saddle point, if $Y_1 < 0$, or a repeller, if $Y_1 \geq 0$.

There are four possibilities, depending on the sign of Y_1 and whether $\bar{z}_1^*$ is less than or greater than $\bar{z}_0$. If (28) holds, then $f_1(z) = f_{1,1}(z)$ and $f_0(z) = f_{0,1}(z)$ are value functions satisfying (30) and the corresponding trajectories passes through both equilibria. (See Fig. 6.) Hence both $\bar{z}_1^*$ and $\bar{z}_0$ are attractors in the respective periods.

If (29) holds, then $f_0(z) < f_{0,1}(z)$. Therefore, $f_0(z) < 0$ for all $z \geq 0$, and the trajectory corresponding to $f_1(z)$ passes through $(0, \bar{z}_1)$, where $\bar{z}_1$ is a solution to (18). (See Fig. 7.) As a result, $\bar{z}_1$ is an attractor in Period I, and $\dot{z} = f_0(z) < 0$ in Period II.

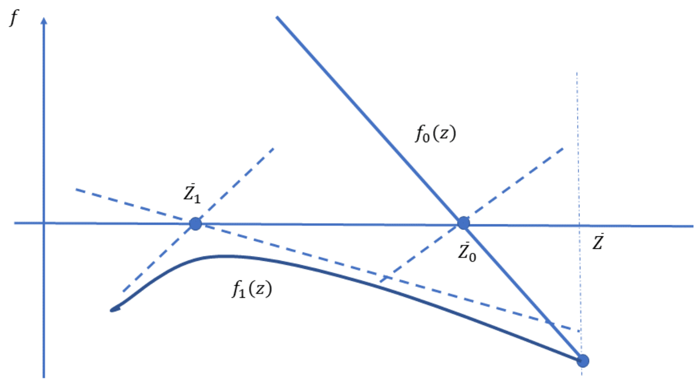

Figure 8: If (28) and (29) both fail, then $\dot{z} < 0$ in Period I and $\bar{z}_0$ is an attractor in Period II.

In the remaining cases, either $Y_1 < 0$ or $Y_1 \geq 0$. Suppose $Y_1 < 0$. For $\bar{z}_1^* < \bar{z}_0$, the only remaining case is

$$(\bar{z} - \bar{z}_0)\, y_1 < (\bar{z} - \bar{z}_1^*)\, Y_1.$$

It can be seen that $f_0(z) = f_{0,1}(z)$ is the value function in Mode 0, and $f_1(z) < f_{1,1}(z)$. (See Fig. 8.) Hence $\bar{z}_0$ is an attractor for Period II, and $\dot{z} = f_1(z) < 0$ in Period I.

Suppose $Y_1 < 0$ and $\bar{z}_1^* \geq \bar{z}_0$. Either $\bar{z}_1^* \leq \bar{z}$ or $\bar{z}_1^* > \bar{z}$. In the former case, the one not satisfying (29) must satisfy

$$(\bar{z} - \bar{z}_0)\, y_1 < (\bar{z} - \bar{z}_1^*)\, Y_1. \tag{31}$$

It follows that $f_0(z) = f_{0,1}(z)$ passes $\bar{z}_0$ and $f_1(\bar{z}_1^*) < 0$. (See Fig. 9.)

As a result, $\bar{z}_0$ is an attractor in Period II, and $\dot{z} = f_1(z) < 0$ in Period I.

Suppose $Y_1 \geq 0$. If $\bar{z}_1^* < \bar{z}_0$, then the two unstable manifolds emanating from $(0, \bar{z}_1^*)$ intersect $z = \bar{z}$ with positive z-values, thus not satisfying (30). Therefore, there is no $f_1(z)$ whose trajectory passes through $(0, \bar{z})$ intersects

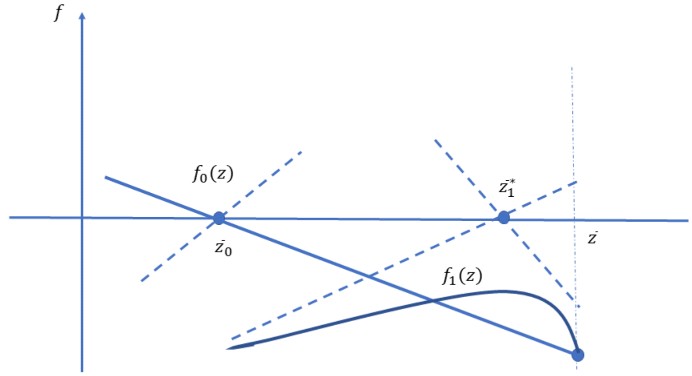

Figure 9: Case 3 with $Y_1 < 0$, $\bar{z}_1^* \geq \bar{z}_0$ and (31).

the z-axis for $z \leq \bar{z}$. Hence $f_1(z) < 0$ for $z \leq \bar{z}$. Therefore, $\dot{z} < 0$ in Period I, and $\bar{z}_0$ is an attractor. If $\bar{z}_1^* \geq \bar{z}_0$, it is necessary that $\bar{z}_1^* > \bar{z}$ for an unstable manifold exiting $(0, \bar{z}_1^*)$ to intersect the line $z = \bar{z}$ at a negative z-value. However, the corresponding value function $f_1(z)$ must be negative for all $z \leq \bar{z}$. (See. Fig. 10.) Thus $\dot{z} < 0$ in Period I and $\bar{z}_0$ is an attractor in Period II.

This completes the proof.

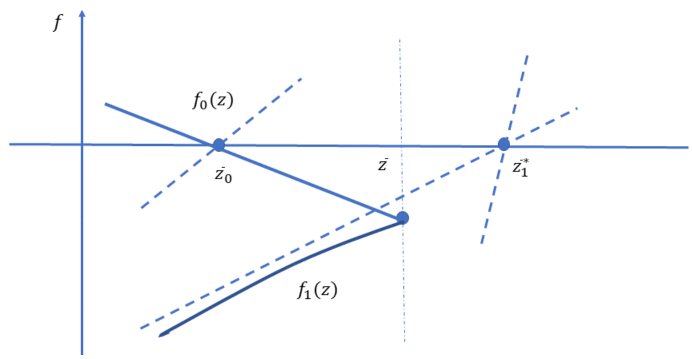

Figure 10: Case 3 with $Y_1 \geq 0$, $\bar{z}_1^* \geq \bar{z}_0$ and (31).

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