THE IRREVERSIBLE POLLUTION GAME

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Abstract

We study a 2-country differential game with irreversible pollution. Irreversibility is of a *hard* type: above a certain threshold level of pollution, the self-cleaning capacity of Nature drops to zero. Accordingly, the game includes a non-concave feature, and we characterize both the cooperative and non-cooperative versions with this general non-LQ property. We deliver full analytical results for the existence of Markov Perfect Equilibria. We first demonstrate that when pollution costs are equal across players (symmetry), irreversible pollution regimes are more frequently reached than under cooperation. Second, we study the implications of asymmetry in the pollution cost. We find far nontrivial results on the reachability of the irreversible regime. However, we unambiguously prove that, for the same total cost of pollution, provided the irreversible regime is reached in both the symmetric and asymmetric cases, long-term pollution is larger in the symmetric case, reflecting more intensive free-riding under symmetry.

Keywords: Differential games, Irreversible pollution, Non-concave pollution decay, Asymmetric pollution cost, Markov Perfect Equilibria
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*In memory of Ngo van Long.

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1 Introduction

Pollution control is among the most targeted topics in several disciplines in the last five decades. This is in particular true in economics and operational research (see an early contribution due to Bawa, 1973). Key inherent conceptual and modelling aspects are the transboundary nature of pollution and the induced spatial externality problem. In short, emission of pollutants due to the action of a given individual in a given place also affects other individuals in other neighboring places through different diffusion channels (wind, currents,...) , often giving rise to substantial free-riding problems. The related literature is still vivid on both the technical, conceptual and policy aspects around these problems. See below for a brief account.

In this paper, we tackle an important problem at the core of the current debate: the issue of irreversible pollution. Indeed, several researchers have already claimed that irreversible climate regimes have already taken place. Indeed, there is now growing evidence that oceans (the most important carbon sink) display a buffering capacity near saturation. Accordingly, the assimilation capacity of terrestrial ecosystems is thought to peak by mid-century and then decline to become a net source of carbon by the end of the century. Finally, the potential collapse of the North Atlantic meridional overturning circulation is drawing much of the attention, since it may happen for a 450 ppm CO2 concentration while we have already reached 390 ppm (Yohe et al., 2006, and Boucekkine et al., 2013, for a more comprehensive survey). While the importance of irreversibility is being reinforced by the current global warming debate, it's fair to recognize that it's a quite old topic, already outlined in ecology decades ago (for example, see Holling, 1973).

This paper's core research question is: to which extent irreversibility of pollution shapes the free-riding problems inherent in pollution (differential) games? Under which conditions irreversible pollution may be reached at the result of Nash competition? Could cooperation prevent this outcome? Does players' (countries') asymmetries trigger more or less free-riding, hence, more or less irreversibility and/or long-term pollution with respect to the symmetric case? We shall essentially differentiate between the players in terms of pollution cost.

We shall build on the seminal contribution of Tahvonen and Withagen (1996), TW hereafter, to build our game-theoretic frame. In TW, there is a single player (say a country) which faces a standard pollution control problem with the additional complication that pollution is irreversible: Nature's self-cleaning capacity may decrease with the level of pollution, and eventually drops to zero above a certain threshold level. This non-concave feature leads to a sophisticated problem potentially yielding multiple steady state equilibria and discontinuities, among other non-standard properties. Despite this additional difficulty, this model has been used in several non-game-theoretic contexts (see for example, Prieur, 2009, and Boucekkine et al., 2013). We shall study a dynamic game extension of TW: in particular, we investigate under which conditions Nash equilibria yield crossing the threshold value leading to irreversible pollution, and whether cooperation can always prevent this unpleasant outcome. Also we rigourously derive the implications of asymmetry in pollution costs in terms of frequency of reaching irreversibility and long-term pollution levels. Indeed, a key aspect already identified in the literature of pollution games and international environmental agreements is heterogenity across players (see for example Hoel, 1993, or Xepapadeas, 1995), we shall accurately study this aspect in the presence of an irreversibility threshold.

We are able to extract several significant results. First of all, we show that cooperation between players (implemented through the central planner counterpart of the game) does not always prevent the emergence of irreversible pollution regimes. This is in particular true when (perceived) pollution costs are low enough.[1]. Second, we demonstrate that when pollution costs are equal across players, irreversible regime are more frequently reached, in the sense that the range of parameters allowing for so is markedly larger than under cooperation. This incidentally reflects the extent of free-riding under symmetry. Third, we study the implications of asymmetry in the pollution cost for the reachability of the irreversible pollution regime and long-term pollution under this regime. We found far nontrivial results on the former but unambiguous ones for the latter: for the same total cost of pollution (that's summing unit pollution costs across countries), the irreversible regime may be reached under symmetry but not under asymmetry, and vice versa, but in all cases where the regime is reached in both, long-term pollution is larger in the symmetric case. This reflects primarily the more intensive free-riding under symmetry.

[1] Of course, precise reachability conditions for irreversible regimes are derived depending on the main parameters of the model, including the discounting rate, the value of the (physical) irreversibility thresholds, the payoffs' deep parameters and also those of the pollution decay function.

Relation to the literature Irreversible pollution has been much more studied in the ecological literature than in economics. Beside TW, a limited number of economic papers has been written with (hard) irreversibility as in the ecological literature and TW.[2] The differential frameworks are very much scarcer. Among the very few contributions to this line of research, one can mention Wagener and de Zeeuw (2021) and El Ouardighi et el. (2020). The former is more anchored in the tipping games literature, and has a very different and specific analytical setting, not speaking about the quite distinct research questions. The latter does explore differential games with variable self-cleaning capacity, but not in the sense of TW. By allowing the self-cleaning capacity to be directly controlled to ultimately get rid of the non-concavity inherent in TW, they end up exploring a different set of differential games with a *softer* irreversibility constraints compared to TW. We take the TW framework as it is to a differential game setting, with all the technical complications involved. [3] As to the role of asymmetry in pollution cost in the reachability and long-term outcomes of irreversible pollution regimes, it's to the best of our knowledge unexplored in the literature so far.

On the methodological side, most of the ongoing progress in pollution games has concerned the refinement of the inherent spatial modelling. New frameworks have been put forward moving from the typical discrete space setting (in particular, the so-called two-country modelling in economics, see Dockner and Van Long, 1993, Dutta and Radner, 2009, Boucekkine et al., 2011, or Bertinelli et al., 2015) to continuous space modelling of diffusion through an intensive use of partial differential equations, typically diffusion-advection equations (Camacho and Perez-Barahona, 2015, Augerau-Véron et al., 2017 and 2019, de Frutos and Martin-Herran, 2019, La Torre et al., 2021, and Boucekkine et al., 2021, 2022b), inducing quite intriguing infinite-dimensional differential games (see de Frutos et al., 2021, and Boucekkine et al., 2022a).

In this paper we stick to the traditional two-players framework. However, as we rely on the TW specifications, and given the inherent non-concave feature, the characterization

[2]For example, El Ouardighi et al. (2014) deal with irreversible pollution in a non-game theoretic frame but in a softer sense, along with the game-theoretic extension, El Ouardighi et al. (2020), cited above.

[3]Indeed, again full in line with TW, we do not consider only symmetric linear-quadratic games contrary to El Ouardighi et al. (2020): while some linear-quadratic and symmetry assumptions are made either for benchmarking or to ease the extraction of analytical results in the differential game setting, we stick to the non-concave and general specification of the pollution decay function postulated in TW and we also study departures from symmetry. Last but not least, all our results are analytical, we only use numerical examples for illustration.

of the games' equilibria is quite challenging. Clearly full analytical approaches to deal with the differential game extensions are far nontrivial. We characterize not only the existence and properties around the long-run steady state and threshold with focusing on the threshold is reachable or not, but also the total trajectories for any admissible initial conditions. This is possible due to the analytical functional forms in the first period of the game (or optimal control). Though some of our results rely largely on the implicit function theorem, our analytical characterization of equilibria is complete, and the parameter conditions are explicitly presented.

Generally, it is very difficult to solve explicitly the trajectory path of optimal control and differential game problems with multi-period, even under linear-quadratic framework. Most of the results in economic literature rely on numerical simulation (Dawid and Gezer, 2022; El Ouardighi et al. 2020, just to mention a few). The main reason is that the transversality conditions at the switching point between different periods (or modes) make it very difficult, if not impossible, to guess the functional form of Bellman value function and thus the strategies. Even with linear-quadratic functions under autonomous settings, the usually used linear-quadratic functions fail to satisfy the transversality condition. The process we present in the current work on how to look for the analytical results makes later applications, not only limited to simulation and calibration, becoming plausible for multi-mode optimal control problems and differential games.

The paper is organized as follows. Section 2 presents the basic differential game extension of TW. Section 3 solves the cooperative (central planner) case, which allows for a close comparison with the one-country case studied in TW. Section 4 and 5 are devoted to non-cooperative games under symmetric versus asymmetric pollution costs. Section 6 ranks pollution outcomes across strategic settings (central planner, Nash symmetric and Nash asymmetric) in the irreversible regime, and clarifies some aspects of free riding behavior in this regime. Comparison of reachability conditions of the irreversible regime across strategic settings is also provided. Section 7 concludes.

2 The model

We briefly present our game-theoretic extension of TW. In contrast to TW, there are two players, named as player $i = 1, 2$, both produce final consumption goods with pollution as a by-product. Ignoring differences in production, we can use their pollution emission, $y_i(t)$, to measure their output level, respectively. Player i's objective is to maximize her social welfare taking into account transboundary pollution:

$$\max_{y} W_i = \int_0^{+\infty} (U_i(y_i) - D_i(z))e^{-rt}dt, \tag{1}$$

where r is time preference, $U_i(y_i)$ is the utility from enjoying final output generated with pollution $y_i(t)$, $D_i(z)$ is the damage function from aggregate pollution stock z.

Pollution stock $z(t)$ may decay at rate $\alpha(z)$ if the pollution level is below a threshold level $\overline{z}$. Along this regime, referred to as the reversible regime in TW, pollution accumulates is following:

$$\dot{z} = y_1 + y_2 - \alpha(z), \quad z(0) = z_0 \text{ given}, \tag{2}$$

where $\alpha(z)$ is the pollution decay function. It captures Nature's self-cleaning capacity. If the threshold is attained and crossed, the economy falls into the irreversible regime where the pollution decay drops to zero. Following TW, we assume that the decay function satisfies the following properties: $\alpha(0) = 0$, $\alpha(z) > 0$ when $z \in [0, \ \overline{z})$, $\alpha(z) = 0$, $\forall z \geq \overline{z}$, and $\alpha''(z) \leq 0$ when $z \in [0, \ \overline{z})$.

Though TW do not assume that $\alpha(z)$ is decreasing in the reversible regime, they do work with the affine specification $\alpha(z) = \alpha - \beta z$, with α and β positive in their numerical example (though of course the specification is only valid locally due to positivity constraint). We shall go the same way, only using the affine specification for numerical exercises or to get some explicit results (while the positivity constraint fulfilled). Due to the differential game frame, we however need to posit quadratic utility and damage functions along the manuscript in order to bring out analytical results, for the existence of Markov Perfect Equilibria in particular. Precisely, we pose: $U_i(y_i) = a_i y_i - y_i^2$, and $D_i(z) = c_i z^2$, with all coefficients, a_i, c_i, being positive constants. Accordingly, player i's optimal control problems under the reversible and irreversible regimes, called period I and II hereafter, read as follows:

Period I

$$\max_{y_i} W_i^I = \int_0^T (U_i(y_i) - D_i(z))e^{-rt}dt = \int_0^T (a_i y_i - y_i^2 - c_i z^2)e^{-rt}dt,$$

subject to

$$\dot{z} = y_1 + y_2 - \alpha(z), \quad z(0) = z_0,$$

and

$$\begin{cases} z(T) = \overline{z}, & \text{if } T < +\infty, \\ \lim\limits_{t\to\infty} z(t) \leq \overline{z}, & \text{if } T = +\infty. \end{cases}$$

Period II

$$\max_{y_i} W_i^{II} = \int_T^{+\infty} (U_i(y_i) - D_i(z))e^{-rt}dt = \int_T^{+\infty} (a_i y_i - y_i^2 - c_i z^2)e^{-rt}dt,$$

subject to

$$\dot{z} = y_1 + y_2, \quad z(T) = \overline{z}.$$

We shall start with the benchmark cooperative game before exploring the equilibrium properties of Nash games. Different symmetry assumptions will be assumed along the way.

3 Cooperative equilibria: the central planner problem

We shall start with the cooperative game where a benevolent central planner (for example, a credible international institution or a federal state's government) enforces cooperation between the two countries. We do that to compare our results with TW. In both cases, there is a unique optimizing authority, potentially leading to similar outcomes. We shall indeed refine in some way a few results obtained by TW.

We assume that the central planner maximizes the sum of the utilities of the two players, namely:

$$\max_{y_1,y_2} W_c^I = \int_0^T \sum_{i=1,2} (U_i(y_i) - D_i(z))e^{-rt}dt = \int_0^T \sum_{i=1,2} (a_i y_i - y_i^2 - c_i z^2)e^{-rt}dt,$$

subject to

$$\dot{z} = y_1 + y_2 - \alpha(z), \quad z(0) = z_0,$$

and

$$\begin{cases} z(T) = \overline{z}, & \text{if } \ T < +\infty, \\ \lim_{t\to\infty} z(t) \le \overline{z}, & \text{if } \ T = +\infty. \end{cases}$$

Furthermore, if $T < \infty$, the system enters the situation with no decay:

$$\max_{y_1,y_2} W_c^{II} = \int_T^\infty \sum_{i=1,2} (U_i(y_i) - D_i(z))e^{-rt}dt = \int_T^\infty \sum_{i=1,2} (a_i y_i - y_i^2 - c_i z^2)e^{-rt}dt,$$

subject to

$$\dot{z} = y_1 + y_2, \quad z(T) = \overline{z}.$$

The Hamiltonian for this optimal control problem is

$$H(z,p) = \max_{y_1,y_2\ge 0} \left\{ a_1 y_1 + a_2 y_2 - y_1^2 - y_2^2 - 2cz^2 + p\left[y_1 + y_2 - \delta(z)\right] \right\},$$

where

$$\delta(z) = \begin{cases} \alpha(z), & z \le \overline{z}, \\ 0, & z > \overline{z}. \end{cases}$$

In TW, $\alpha(\bar{z}) = 0$. Here, we don't need to assume continuity of the decay function at $\bar{z}$, and consider a more general $\alpha(\bar{z}) \ge$ condition. We shall provide the analysis under this condition while singling out the TW continuous case. The maximizers y_1 and y_2 satisfy

$$y_i = \frac{a_i + p}{2} \qquad \text{for } i = 1, 2.$$

Hence

$$\begin{aligned} H(z,p) &= \frac{1}{4}\left[(a_1+p)^2 + (a_2+p)^2\right] - 2cz^2 - p\delta(z) \\ &= \frac{1}{2}\left[2ap + p^2\right] + \frac{1}{4}\left(a_1^2 + a_2^2\right) - 2cz^2 - p\delta(z). \end{aligned}$$

Hence, the HJB equation is

$$rV_c = \frac{1}{2}\left[2aV_c' + (V_c')^2\right] + \frac{1}{4}\left(a_1^2 + a_2^2\right) - 2cz^2 - \delta(z)V_c' \tag{3}$$

where $V_c(z)$ is the value function for the central planner. We now solve for the optimal solutions using a standard backward-looking model as in TW: we start with the characterization of the irreversible regime (that's for $z > \overline{z}$), then we move to the reversible regime ($z < \overline{z}$), and we finally conclude as to possible crossing of the pollution threshold $\overline{z}$ along with the optimal paths (or equilibrium paths in the Nash games later). Notice that the irreversible regime is fully linear-quadratic as pollution decay is zero in this case, which eases the computations in this regime.

3.1 Characterization of the irreversible regime ($z > \overline{z}$)

Set $\delta(z) = 0$. We assume that $V_c(z)$ is a quadratic function of z in the form

$$V_c(z) = A_c + B_c z + \frac{C_c}{2} z^2.$$

Substituting it into the equation leads to

$$r\left[A_c + B_c z + \frac{C_c}{2} z^2\right] = a\left(B_c + C_c z\right) + \frac{1}{2}\left[\left(B_c + C_c z\right)^2\right] + \frac{1}{4}\left(a_1^2 + a_2^2\right) - 2cz^2.$$

By comparing coefficients of powers of z, we find

$$\begin{aligned} rA_c &= aB_c + \frac{B_c^2}{2} + \frac{1}{4}\left(a_1^2 + a_2^2\right), \\ rB_c &= aC_c + B_c C_c. \\ r\frac{C_c}{2} &= \frac{1}{2}C_c^2 - 2c. \end{aligned}$$

Hence

$$\begin{aligned} C_c &= \frac{1}{2}\left[r - \sqrt{r^2 + 16c}\right], \qquad B_c = \frac{aC_c}{r - C_c}, \\ A_c &= \frac{1}{4r}\left[a_1^2 + a_2^2\right] + \frac{aB_c}{r} + \frac{B_c^2}{2r}. \end{aligned}$$

The state equation is

$$\dot{z} = y_1 + y_2 = a + V_c'(z) = a + B_c + C_c z.$$

We use f_c to denote the function on the right-hand side. The equilibrium is

$$z_c^* = -\frac{a + B_c}{C_c}.$$

It can be shown that

$$z_c^* = \frac{ar}{4c}. \tag{4}$$

3.2 Characterization of the reversible regime ($z < \overline{z}$) and threshold crossing conditions

The value function satisfies HJB equation and the terminal condition

$$V_c(\bar{z}) = A_c + B_c\bar{z} + \frac{C_c}{2}\bar{z}^2. \tag{5}$$

HJB equation (3) takes the form

$$rV_c = \frac{1}{4}\left[(a_1 + V_c')^2 + (a_2 + V_c')^2\right] - 2cz^2 - V_c'\delta(z). \tag{6}$$

Before looking for solution of the above HJB equation with terminal condition (5), we must mention that in this period, though the optimization system is still autonomous with free ending time, state equation and objective functions are still in the forms of affine-quadratic, due to the transversality condition (5) between the two periods, the linear-quadratic guess of value function as in the last subsection is no longer working. That is the case for central planner's optimal control problem as well for the differential games in the following sections. Thus, a different method is needed to look for solution to this kind of affine-quadratic optimal control problem with given terminal condition.

To find $V_c'(z)$, we differentiate the two sides of (6) to obtain

$$rV_c'(z) = \frac{1}{2}\left[(a_1 + V_c'(z)) + (a_2 + V_c'(z))\right]V_c''(z) - 4cz - V_c''\delta(z) - V_c'(z)\delta'(z).$$

Let $P_c(z) = V_c'(z)$. The above equation becomes

$$[P_c(z) + a - \delta(z)]\,P_c'(z) = (r + \delta'(z))\,P_c(z) + 4cz. \tag{7}$$

In addition, $P_c(\bar{z})$ satisfies the equation

$$4\left[rV_c(\bar{z})+2c\bar{z}^2\right]=(a_1+P_c(\bar{z}))^2+(a_2+P_c(\bar{z}))^2-4\delta(\bar{z})\,P_c(\bar{z})\,. \tag{8}$$

Eq. (8) has two roots in general, which are

$$P_c(\bar{z})=-(a-\delta(\bar{z}))\pm\sqrt{(a-\delta(\bar{z}))^2+\Delta^c}$$

where

$$\Delta^c=2\left(rV_c(\bar{z})+2c\bar{z}^2\right)-\frac{a_1^2+a_2^2}{2}.$$

We choose the one so that V_c is most concave for $z<\bar{z}$ and is near $\bar{z}$. Substituting the above solutions into (7) leads to

$$\pm\sqrt{(a-\delta(\bar{z}))^2+\Delta^c}P_c'(\bar{z})=(r+\delta'(\bar{z}))\left\{-(a-\delta(\bar{z}))\pm\sqrt{(a-\delta(\bar{z}))^2+\Delta^c}\right\}+4c\bar{z},$$

where $\delta'(\bar{z})=\alpha'(\bar{z})$ is the left-sided derivative. Hence,

$$P_c'(\bar{z})=r+\delta'(\bar{z})+\frac{-(r+\delta'(\bar{z}))(a-\delta(\bar{z}))+4c\bar{z}}{\pm\sqrt{(a-\delta(\bar{z}))^2+\Delta^c}}.$$

The sign is positive if

$$\frac{-(r+\delta'(\bar{z}))(a-\delta(\bar{z}))+4c\bar{z}}{\sqrt{(a-\delta(\bar{z}))^2+\Delta^c}}<\frac{-(r+\delta'(\bar{z}))(a-\delta(\bar{z}))+4c\bar{z}}{-\sqrt{(a-\delta(\bar{z}))^2+\Delta^c}},$$

which is equivalent to

$$(r+\delta'(\bar{z}))(a-\delta(\bar{z}))-4c\bar{z}>0, \tag{9}$$

and negative otherwise.

To illustrate how to solve the nonlinear differential equation (7), we consider linear decay function $\delta(z)=\alpha-\beta z$ with α,β positive constants. Denote $f_c(z)=P_c(z)+a-\delta(z)$. It follows that

$$\left|\frac{f_c(z)-u_c^-(z+a_c/b_c)}{f_c(\bar{z})-u_c^-(\bar{z}+a_c/b_c)}\right|^{p_c}\left|\frac{f_c(z)-u_c^+(z+a_c/b_c)}{f_c(\bar{z})-u_c^+(\bar{z}+a_c/b_c)}\right|^{1-p_c}=1,$$

where

$$\begin{aligned} a_c &= (\beta - r)(a - \alpha), \qquad b_c = \beta(\beta - r) + 4c, \\ u_c^- &= \frac{1}{2}\left(r - \sqrt{r^2 + 4b_c}\right), \qquad u_c^+ = \frac{1}{2}\left(r + \sqrt{r^2 + 4b_c}\right), \qquad p_c = -\frac{u_c^-}{u_c^+ - u_c^-}. \end{aligned}$$

Hence z_c' satisfies the equation

$$\left|\frac{u_c^-(z' + a_c/b_c)}{f_c(\bar{z}) - u_c^-(\bar{z} + a_c/b_c)}\right|^{p_c} \left|\frac{u_c^+(z' + a_c/b_c)}{f_c(\bar{z}) - u_c^+(\bar{z} + a_c/b_c)}\right|^{1-p_c} = 1.$$

It follows that

$$z_c' = -\frac{a_c}{b_c} + \left|\frac{u_c^-}{f_c(\bar{z}) - u_c^-(\bar{z} + a_c/b_c)}\right|^{-p_c} \left|\frac{u_c^+}{f_c(\bar{z}) - u_c^+(\bar{z} + a_c/b_c)}\right|^{p_c-1}$$

The state dynamics is

$$\dot{z} = y_1 + y_2 - \delta(z) = P_c(z) + a - \delta(z) \equiv f_c(z).$$

The following proposition give conditions under which the threshold $\bar{z}$ is or is not triggered.

Proposition 1 *If (9) holds then $\bar{z}$ is reached in finite time from some z_0. Otherwise, if the reversed inequality in (9) holds, then $\bar{z}$ is never reached and*

$$\lim_{t\to\infty} z(t) = z_c'.$$

If in addition to (9), $\delta(z) = \alpha - \beta z$ with positive α and β, and

$$4cz_0 < (r - \beta)(a - \delta(z_0)) \tag{10}$$

holds (in particular, if $a \geq \alpha$), then $\bar{z}$ is reached in finite time for any $0 \leq z_0 \leq \bar{z}$.

The proof is given in the Appendix A.1. As explained in the next sub-section, the economic interpretation of the results is perfectly in line with the general analysis provided by TW.

In particular, after rewriting condition (9) as

$$4c\bar{z} < (r + \delta'(\bar{z}))(a - \delta(\bar{z})),$$

one can observe that the larger the pollution cost as captured by parameter c, the less likely condition (9) will hold, and the more likely a permanent reversible regime will set in. The reverse occurs with the utility parameter a. That is to say condition (9) simply compares the cost of pollution and its welfare benefit.

Last but not least, it should be noted that if $\bar{z}$ is reached in finite time, $\bar{T}$, depending on whether $\bar{z} < z_c^*$ or $\bar{z} \geq z_c^*$ the state has different behavior for $t > \bar{T}$. In the first case, the state enters the irreversible regime and $z(t) \to z_c^*$ as $t \to \infty$. In the second case, $z(t) = \bar{z}$ for all $t > \bar{T}$. Therefore, the process stops at $\bar{T}$.

3.3 Comparison with TW

Clearly, our cooperative setting need not be different from TW's optimal control model. However, our more specific LQ assumptions on the utility and damage functions do allow to get more clearcut results (and there is no additional merit in that). While several key results in TW are formulated as sufficient conditions, we can provide with a full and global picture of optimal trajectories as depicted in our Proposition 1 above. This said, it's useful to relate our optimality and crossing conditions to theirs.

For example, let's consider the sufficient condition for optimal reversible paths to arise as specified in TW's Proposition 1, page 1782, that is: $D'(\bar{z}) - rU'(0) \geq 0$ with our notations.[4] To formulate accurately TW's sufficient condition in our setting, let us set $a_1 = a_2 = a$ and $c_1 = c_2 = c$. Accordingly, functions U and D turn to $U(y) = ay - (y^2)/2$ and $D(z) = 2cz^2$. Thus TW's sufficient condition $D'(\bar{z}) \geq rU'(0)$, translates into our notations as $4c\bar{z} \geq ra$. In contrast, our condition for optimal reversible regime to hold, $4c\bar{z} \geq (r + \delta'(\bar{z}))(a - \delta(\bar{z}))$ is necessary and sufficient. Of course, given that $\delta'(\bar{z}) \leq 0$ and $\delta(\bar{z}) \geq 0$, (in TW, it is zero), our condition is weaker. But this improvement is entirely due to the additional LQ assumptions made on the utility and damage functions.

[4]The discount rate r is denoted δ in TW. TW write the following about their condition : "Note that the condition $D'(\bar{z}) - \delta U'(0) \geq 0$ is sufficient only and reversible solutions may well be optimal when the reverse holds. However, in such cases the choice between reversible and irreversible solutions will be less obvious", page 1783.

Example 1 *In Tahvonen and Withagen (1996), a numerical example is given for the case of one player with*

$$U(y) = ay - by^2, \qquad D(z) = cz^2, \qquad \delta(z) = \alpha - \beta z$$

and

$$\dot{z} = y - \delta(z) \qquad \text{for } z < \bar{z}$$

using the parameter values

$$\begin{aligned} a &= 18, \qquad b = 0.5, \qquad c = 0.004, \qquad r = 0.2, \\ \alpha &= 20, \qquad \beta = 0.1. \end{aligned}$$

Their model is equivalent to our central planner model with $a_1 = a_2 = a$, $c_1 = c_2 = c/2$, *and* $y_1 = y_2 = y/2$. *We use the values of parameters. The equivalent value of* c *in our case is* $0.004/2 = 0.002$. *In TW,* $\bar{z} = \alpha/\beta = 200$. *At this value,*

$$(r + \delta'(\bar{z}))(a - \delta(\bar{z})) = (r - \beta)a = 1.8 > 1.6 = 4c\bar{z}.$$

Hence, (9) holds, and therefore, $\bar{z} = 200$ *is reached in finite time. In fact, for any*

$$c_1 = c_2 \geq \frac{a(r - \beta)}{4\bar{z}} = 0.00225$$

the reversed inequality in (9) is satisfied for $\bar{z} = 200$. *Therefore, by Proposition 1* $\bar{z}$ *is not reachable.*

Returning to the case where $c_1 = c_2 = 0.002$, *(9) is satisfied if*

$$\bar{z} > \frac{(r - \beta)(\alpha - a)}{\beta(r - \beta) - 4c} = 100.$$

Such values of $\bar{z}$ *are all reachable.*

4 Non-cooperative equilibria with symmetric pollution cost

We now move to non-cooperative (Nash) games. We first study the case where pollution costs are identical across players. This is our benchmark strategic case, asymmetric extensions are considered in Section 5. From now on, we focus on the characterization of potential Markov perfect equilibria. For simplicity, let $a = \frac{a_1+a_2}{2}$.

4.1 Characterization of the irreversible regime ($z > \overline{z}$)

Denote Bellman value function of player i as $V_i(z)$, $\forall z$. Then V_i must check the following Hamilton-Jacobin-Bellman (HJB) equation:

$$rV_i(z) = \max_{y_i} \left[a_i y_i - y_i^2 - cz^2 + V_{i,z}\,(y_1 + y_2) \right], \tag{11}$$

where $V_{i,z} = \frac{\partial V_i}{\partial z}$ is derivatives of V_i with respect to z. The right hand side's first order condition (which also check the second order condition) yields that the optimal choice of player i is

$$y_i = \frac{a_i + V_{i,z}}{2}, \quad i = 1, 2. \tag{12}$$

Thus, the HJB equation (11) becomes

$$rV_i(t,z) = a_i \frac{a_i + V_{i,z}}{2} - \frac{a_i + V_{i,z}}{2}^2 - cz^2 + V_{i,z}\left[\frac{a_1 + V_{1,z}}{2} + \frac{a_2 + V_{2,z}}{2}\right], \ \ i = 1, 2. \tag{13}$$

Given the linear-quadratic framework in the irreversible regime, we try the value function as

$$V_i(z) = A_i^m + B_i^m z + \frac{C_i^m}{2} z^2,$$

with A_i^m, B_i^m, C_i^m undetermined constants. It is easy to see

$$V_{i,z}(z) = \frac{dV_i}{dz} = B_i^m + C_i^m z \ \text{ and } \ y_i(z) = \frac{a_i + B_i^m + C_i^m z}{2}. \tag{14}$$

Substituting (14) into the right hand side of (13), rearranging terms and equating coefficients of like terms give, for $i, j = 1, 2, \;\; i \neq j$,

$$\begin{cases} rA_i^m = \frac{(a_i+B_i^m)^2}{4} + \frac{B_i^m(a_j+B_j^m)}{2}, \\ rB_i^m = \frac{C_i^m(a_i+B_i^m)}{2} + \frac{1}{2}[B_i^m C_j^m + C_i^m(a_j + B_j^m)], \\ \frac{rC_i^m}{2} = \frac{(C_i^m)^2}{4} + \frac{C_i^m C_j^m}{2} - c_i. \end{cases} \tag{15}$$

In the case where $c_1 = c_2 \equiv c$, the last equation from the above system yields unique symmetric negative root $C_1^m = C_2^m = C^m$, which is given by

$$C^m = \frac{r - \sqrt{r^2 + 12c}}{3} (< 0). \tag{16}$$

Accordingly, the other coefficients can be given by

$$\begin{cases} B_i^m = B_2^m \equiv B^m = \frac{(a_1+a_2)C^m}{2r-3C^m} (< 0), \\ A_1^m = \frac{(a_1+B^m)^2+2B^m(a_2+B^m)}{4r}, \\ A_2^m = \frac{(a_2+B^m)^2+2B^m(a_1+B^m)}{4r}. \end{cases} \tag{17}$$

Thus the optimal strategy of player $i = 1, 2$ is

$$y_i^m(z) = \frac{a_i + B^m + C^m z}{2}.$$

The state equation

$$\dot{z} = y_1^m + y_2^m = a_1 + a_2 + B^m + C^m z, \;\; t \geq T,$$

yields explicit solution

$$z^m(t) = (\overline{z} - z_s^*) e^{C^m(t-T)} + z_s^*, \tag{18}$$

where z_s^* is the asymptotically stable long-run steady state and given by

$$z_s^* = -\frac{a_1 + a_2 + 2B^m}{2C^m} = \frac{a_1 + a_2}{12c}\left(5r + \sqrt{r^2 + 12c}\right). \tag{19}$$

It is straightforward that when $z = \overline{z}$

$$V_i(\overline{z}) = A_i^m + B^m\overline{z} + \frac{C^m}{2}\overline{z}^2 \equiv \overline{V}_i, \tag{20}$$

which will be served as terminal condition for the first period under Markovian competition. In other words, this is the transversality condition between period I and II.

4.2 Characterization of the reversible regime and threshold crossing conditions $(z < \overline{z})$

In period I, before the pollution threshold is triggered, the accumulation of pollution checks

$$\dot{z} = y_1 + y_1 - \delta(z) = y_1 + y_1 - \delta(z), \quad t \leq T,$$

with initial condition $z(0) = z_0$ given.

Similar to the above subsection 3.2, in the Appendix A.2, we demonstrate the following existence of stationary Markovian subgame perfect Nash equilibrium in the first period.

Proposition 2 ***(Existence of stationary Markovian perfect Nash equilibrium)*** *Suppose $c_1 = c_2 = c$ and following equation is solvable in term of $P_s(\bar{z})$:*

$$3P_s(\bar{z})^2 + 4(a - \delta(\bar{z}))P_s(\bar{z}) + a_i^2 = 4\left(r\bar{V}_i + c\bar{z}^2\right). \tag{21}$$

Let $P_s(\bar{z})$ be the root of (21) that is close to $B^m + C^m\bar{z}$. Then, there exists stationary Markovian perfect Nash equilibrium which are given by solutions of equation

$$\left[\frac{3}{2}P_s(z) + a - \delta(z)\right]P_s'(z) = (r + \delta'(z))P_s(z) + 2cz.$$

with terminal condition $P_s(\bar{z})$.

Furthermore, for special linear decay function $\delta(z) = \alpha - \beta z$ and α, β are positive constants, the stationary Markovian perfect Nash equilibrium can be more precisely presented as:

$$y_i(z) = \frac{a_i}{2} + \frac{1}{3}\left[Q_s(z) - a + \delta(z)\right] \qquad \textit{for } z < \bar{z} \;\; i = 1, 2,$$

where $Q_s(z)$ satisfies the following equation:

$$\left|\frac{Q_s(z) - u_s^-(z + a_s/b_s)}{Q_s(\bar{z}) - u_s^-(\bar{z} + a_s/b_s)}\right|^{p_s} \left|\frac{Q_s(z) - u_s^+(z + a_s/b_s)}{Q_s(\bar{z}) - u_s^+(\bar{z} + a_s/b_s)}\right|^{1-p_s} = 1,$$

in which

$$u_s^- = \frac{1}{2}\left[r - \sqrt{r^2 + 4b_s}\right], \quad u_s^+ = \frac{1}{2}\left[r + \sqrt{r^2 + 4b_s}\right]$$

and

$$a_s = (\beta - r)(a - \alpha), \;\; b_s = \beta(\beta - r) + 3c.$$

The existence result does not depend on the explicit form of decay function $\delta(z)$, see more detailed explanation in Appendix A.2 or following the same logic as in Section 3.2.

We now study the key issue of reachability of the irreversible regime. We shall generalize to the game context the intuitive property that such an outcome depends on the position of the steady state of the pollution dynamics induced by the Markovian equilibrium and the irreversibility threshold, $\bar{z}$.

To this end, we start by substituting the above Markovian optimal strategies into the dynamic equation for $z < \bar{z}$, it follows

$$\dot{z} = f_s(z) \equiv P_s(z) + a - \delta(z) = \frac{1}{3}\left[2Q_s(z) + a - \delta(z)\right]$$

with initial condition $z(0) = z_0$ given.

Furthermore, the pollution accumulation is increasing or decreasing over time depending on the sign of $Q_s(z)$: if $Q_s(z_0) > 0$ then the pollution accumulates until the first time when $P_s + a - \delta = 0$.Let z_s' denote this root. At this point

$$Q_s(z_s') = \frac{1}{2}\left(\delta(z_s') - a\right).$$

Hence, by (60), z_s' satisfies

$$S\left(z_s', \frac{1}{2}\left(\delta(z_s') - a\right)\right) = S\left(\bar{z}, Q_s(\bar{z})\right). \tag{22}$$

We conclude the above analysis in the following

Proposition 3 *Suppose $c_1 = c_2 = c$ and Eq. (21) has a negative real root. Let z_s' satisfy (22). Then, under the Markovian perfect Nash equilibrium given by Proposition 2,*

(a) if $\bar{z} < z_s'$, the pollution decay threshold will be triggered in finite time $\bar{T}_s$, which is given by

$$\bar{T}_s = \int_{z_0}^{\bar{z}} \frac{dz}{P_s(z) + a - \delta(z)}; \tag{23}$$

(b) otherwise, if $\bar{z} \geq z_s'$, the pollution decay threshold will never be reached.

As announced above, Proposition 3 gives the counterpart of Proposition 1 (cooperative case) to the game-theoretic context under Markovian strategies in the general non-concave decay case. As in Proposition 1, the intuitions are clear but the characterization is much more nontrivial to obtain. We shall get a step further here below and express the results in terms of the deep economic parameters of the model.

4.3 Reachability of the threshold and asymptotes

We now uncover the concrete parametric implications of the proposition above to visualize better the economic and ecological determinants of reaching the irreversible regime. We also explore the resulting asymptotes. Particular attention is paid to the comparison between the cooperative and non-cooperative setting in the reachability of the irreversible regime, with concrete numerical examples to support the theoretical arguments. We start with some general reachability conditions.

Proposition 4 *The following are true.*

1. *If $a \leq \delta(\bar{z})$ and*

$$(r + \delta'(\bar{z}))(a - \delta(\bar{z})) \leq 3c\bar{z}, \tag{24}$$

then $\bar{z}$ is never reached. Furthermore,

$$\lim_{t\to\infty} z(t) = z_s' \tag{25}$$

where z_s' is given in Proposition 3.

2. If $a > \delta(\bar{z})$ and $\bar{z}$ is never reached if and only

$$c\bar{z}^2 \geq \frac{a_i^2 - (a - \delta(\bar{z}))^2}{4} - rV_i(\bar{z}). \tag{26}$$

The more economically relevant case is case 2, as the decay is typically very low (and equal to zero in TW) when we reach the irreversibility threshold. The intuition of condition (26) is straightforward: the left hand side is the direct cost of pollution accumulation, defined in the objective function, at the threshold $\overline{z}$. The right hand side is the counterpart gain at the threshold and it includes two parts: the first part is short-run net gain from emission net of decay effects (the square forms come from the functional forms defined in the objective function and optimal choices of strategies) while the second part is the long-run consequences in term of optimal value function when the threshold is crossed and the ecological system enters the second phase. The above condition provides the rather straightforward information that when the accumulated cost at the threshold is sufficiently high and dominates the gain, more efforts will be made by both players such that the threshold actually never be reached. Otherwise, when the cost is not high enough, the threshold will be crossed in finite time. Of course, this condition does not exclude the situation that the natural self-regeneration capacity is sufficiently high, i.e., $\overline{z}$ is large enough, such that the above inequality always holds. Of course, this is a rather ideal situation.

In Proposition 1 for the cooperative case, a permanent reversible regime sets in if and only if

$$4c\bar{z} \geq (r + \delta'(\bar{z}))(a - \delta(\bar{z})).$$

Here, all the results are derived under the condition (24), which is similar to the benchmark condition above. However the Nash game displays different outcomes even if we restrict ourselves to the TW case discussed in item 2 of Proposition 4 (with $\delta(\bar{z}) = 0$). A second condition is required, that is (26): in the TW continuous case, and provided condition (24) holds, we get the permanent reversible case if and only if condition (26) holds. This extra condition is due to the competition setting which is different from TW where there is only one policy maker just as the above Proposition 1. We shall come back to this point in the Subsection 6.1 when we collect all information of different kind of competitions. Nevertheless, we shall study the implications of this intricate condition in the linear decay case below and complement with numerical exercises. Before, we display some additional observations and properties.

Asymptotic behavior. If $\bar{z}$ is reached in finite time, $\bar{T}$, depending on whether $\bar{z} < z_s^*$ or $\bar{z} \geq z_s^*$ the state has different behavior for $t > \bar{T}$. In the first case, the state enters Period II and $z(t) \to z_s^*$ as $t \to \infty$. In the second case, $z(t) = \bar{z}$ for all $t > \bar{T}$. Therefore, the process stops at $\bar{T}$.

The following proposition complements Proposition 4, with in particular a clear-cut result in the case of (locally) linear decay functions.

Proposition 5 *(**Reachability of** $\bar{z}$) Suppose*

$$(r + \delta'(\bar{z}))(a - \delta(\bar{z})) > 3c\bar{z}. \tag{27}$$

Then $\bar{z}$ is reached in finite time for some z_0 if either $a > \delta(\bar{z})$, or $a \leq \delta(\bar{z})$ and

$$c\bar{z}^2 \geq \frac{a_i^2 - (a - \delta(\bar{z}))^2}{4} - rV_i(\bar{z}) \tag{28}$$

holds. If in addition, $\delta = \alpha - \beta z$ with positive α and β, and $a \geq \alpha$, then $\bar{z}$ is reached in finite time for any $0 \leq z_0 \leq \bar{z}$.

In the case where $a > \delta(\bar{z})$, the above two propositions lead to the following simple necessary and sufficient condition, while the detail proof is given in Appendix A.5.

Corollary 1 *Suppose $a > \delta(\bar{z})$. Let $\bar{C}$ be the largest solution to the equation*

$$4c\bar{z}^2 = a_i^2 - 4rV_i(\bar{z}; c) - (a - \delta(\bar{z}))^2. \tag{29}$$

Then $\bar{z}$ is reached in finite time for some z_0 if and only if $c < \bar{C}$. Furthermore,

$$\bar{C} > \frac{ar}{2\bar{z}}. \tag{30}$$

The case where $a > \delta(\bar{z})$ is quite interesting to get a sense of the implications of more or less involved conditions displayed in Propositions 4 and 5. In this case, the irreversibility threshold is crossed if

$$c < \frac{1}{4\bar{z}} \left(r + \delta'\left(\bar{z}\right)\right) \left(a - \delta\left(\bar{z}\right)\right)$$

in the cooperative game, while the Nash counterpart requires $c < \bar{C}$. Since

$$\bar{C} > \frac{1}{4\bar{z}} \left(r + \delta'\left(\bar{z}\right)\right) \left(a - \delta\left(\bar{z}\right)\right)$$

by (30), the condition for the emergence of the irreversible regime is easier to check in the Nash case. This sounds intuitive: cooperation generally allows to reach lower pollution levels in pollution games (see the survey of Van Long, 2010), and there is no particular reason the picture changes with irreversibility thresholds: absence of cooperation will lead more frequently to more polluting regimes (in this case: irreversibility). Numerical illustrations follow.

Example 2 *As explained in Example 1, the numerical example in TW is equivalent to our central planner model with the same parameter values except c is changed to* 0.002*. In the case of non-cooperative game with the same parameters as in Example 1,*

$$\left(r - \beta\right) a - 3c\bar{z} = 0.6 > 0.$$

Hence, (27) is satisfied. Since $a = 18 > 0 = \delta\left(\bar{z}\right)$*, by Proposition 5,* $\bar{z} = 200$ *is reached in finite time from some* z_0*. By (19)*

$$z_s^* = \frac{a}{12c} \left(5r + \sqrt{r^2 + 12c}\right) \approx 939.74.$$

Thus, after $\bar{z}$ *is reached, the state enters Period II and* $z\left(t\right) \to z_s^*$ *as* $t \to \infty$*.*
More generally, the above result is true for any

$$c \leq \frac{\left(r - \beta\right) a}{3\bar{z}} = 0.003.$$

In contrast, if

$$c > \frac{\left(r - \beta\right) a}{4\bar{z}} = 0.00225$$

the reversed inequality in (9) holds. By Proposition 1, $\bar{z} = 200$ *is not reached under a central planner. Therefore, for any c between* 0.00225 *and* 0.003*, the long-run pollution*

level under a central planner stays below $\bar{z} = 200$, but under competition with symmetric costs, it exceeds $\bar{z}$ and continue on to approach z_s^*. This shows that competition leads to higher level of pollution in the long run.
Returning to $c = 0.002$, for other values of $\bar{z}$, inequality (24) is satisfied if

$$\bar{z} \leq \frac{(r-\beta)(\alpha-a)}{\beta(r-\beta)-3c} = 50$$

Since $a \leq \delta(\bar{z})$ if

$$\bar{z} \leq \frac{\alpha-a}{\beta} = 20,$$

by Part 1 of Proposition 4, such $\bar{z}$ is never reached. On the other hand, it can be shown that the quadratic function

$$4cz^2 < a_i^2 - 4rV_i(z) - (a-\delta(z))^2$$

for $0 \leq z \leq 180$. By Part 2 of Proposition 4, any $\bar{z}$ that satisfies $20 < \bar{z} \leq 50$ is reached in finite time if z_0 is sufficiently close to $\bar{z}$. Furthermore, for

$$50 < \bar{z} \leq 200$$

(27) is satisfied. Since $a - \delta(\bar{z}) > 0$ for $\bar{z} > 20$, by Proposition 5, $\bar{z}$ is reached in finite time for some $z_0 < \bar{z}$. In summary, any $\bar{z} \leq 20$ is never reached, and any $\bar{z} > 20$ is reached in finite time. In the latter case, after $\bar{z}$ is reached, $z(t) \to z_s^*$ as $t \to \infty$, $z(t) \to z_s^*$ as $t \to \infty$.

5 Non-cooperative games with asymmetric pollution costs

We now come to one of the most important contributions of this paper: the role of asymmetries. We relax the assumption of identical pollution costs and study the implications in terms of emergence of the irreversible regime compared to the benchmark case. We first study the general case for any pair of pollution costs (c_1, c_2). While we are able to characterize analytically existence of Markovian equilibria and conditions for emergence of the irreversible pollution regime, the latter are largely implicit. We then move to an extreme asymmetry case to extract more explicit results.

5.1 The general case of asymmetric pollution costs

5.1.1 Characterization of the irreversible regime ($z > \bar{z}$)

For $z > \bar{z}$, $\delta(z) = 0$. We seek the value functions in the quadratic form

$$V_i(z) = A_i + B_i z + \frac{C_i}{2} z^2, \qquad i = 1, 2.$$

Substituting the quadratic functions into (20), it follows that

$$r\left[A_i + B_i z + \frac{C_i}{2} z^2\right] = \frac{1}{4}\left(a_i + B_i + C_i z\right)^2 + \frac{1}{2}\left(B_i + C_i z\right)\left(a_j + B_j + C_j z\right) - c_i z^2.$$

Comparing coefficients, we find

$$\begin{aligned}
rA_i &= \frac{1}{4}\left(a_i + B_i\right)^2 + \frac{1}{2} B_i \left(a_j + B_j\right), \\
rB_i &= \frac{1}{2} C_i \left(a_i + B_i\right) + \frac{1}{2}\left[B_i C_j + C_i \left(a_j + B_j\right)\right], \\
\frac{r}{2} C_i &= \frac{1}{4} C_i^2 + \frac{1}{2} C_i C_j - c_i.
\end{aligned}$$

We find a solution with negative C_1 and C_2. The last two equations lead to

$$\frac{r}{2} C_i - \frac{1}{4} C_i^2 + c_i = \frac{1}{2} C_1 C_2. \tag{31}$$

Let $\lambda = C_1 C_2 / 2$. Then

$$C_i = r - \sqrt{r^2 + 4\left(c_i - \lambda\right)} \qquad \text{for } i = 1, 2. \tag{32}$$

This leads to the equation for λ as

$$\left(r - \sqrt{r^2 + 4\left(c_1 - \lambda\right)}\right)\left(r - \sqrt{r^2 + 4\left(c_2 - \lambda\right)}\right) = 2\lambda. \tag{33}$$

Since $r - \sqrt{r^2 + 4(c_i - \lambda)}$ is increasing and negative for $0 \leq \lambda \leq c_i$, the left-hand is positive and decreasing for $0 \leq \lambda \leq \min\{c_1, c_2\}$. In addition, the left hand side is positive for $\lambda = 0$ and is zero at $\lambda = \min\{c_1, c_2\}$. In contrast, the right-hand side is increasing and is zero at $\lambda = 0$. Hence Eq. (33) has a unique positive solution between 0 and $\min\{c_1, c_2\}$. By (32), one solves C_1 and C_2.

Note that the equations for B_1 and B_2 are linear with the coefficient matrix

$$\begin{pmatrix} (C_1 + C_2)/2 - r & C_1/2 \\ C_2/2 & (C_1 + C_2)/2 - r \end{pmatrix}.$$

The determinant of this matrix is

$$\frac{C_1^2 + C_1 C_2 + C_2^2}{4} - (C_1 + C_2)\, r + r^2 > 0.$$

Hence, B_1 and B_2 are uniquely solved.

Finally, the equations for A_1 and A_2 are already in solved form. This completes the proof of a unique solution with negative C_1 and C_2.

5.1.2 Characterization of the reversible regime ($z < \bar{z}$) and threshold crossing conditions

Let $P_i(z) = V_i'(z)$. By differentiating the two sides of Eq. (20), we find

$$rP_i = \frac{1}{2}\left\{P_i'\left[2a + P_1 + P_2 - 2\delta(z)\right] + P_i\left[P_j' - 2\delta'(z)\right]\right\} - 2c_i z. \tag{34}$$

This is a linear system of differential equations for P_1 and P_2. Solving P_1' and P_2' from the system, we can write

$$P_i' = \frac{2\left[2a + P_1 + P_2 - 2\delta(z)\right]\left[(r + \delta'(z))\, P_i + 2c_i z\right] - 2P_i\left[(r - \beta)\, P_j + 2c_j z\right]}{\left[2a + P_1 + P_2 - 2\delta(z)\right]^2 - P_1 P_2} \tag{35}$$

for $i, j = 1, 2$, $j \neq i$. The terminal value $P_i(z)$ are obtained from solving (20) at $\bar{z}$, which takes the form

$$r\bar{V}_i = \frac{1}{4}(a_i + P_i(\bar{z}))^2 + \frac{1}{2}P_i(\bar{z})[a_j + P_j(\bar{z}) - 2\delta(\bar{z})] - c_i\bar{z}^2 \qquad \text{for } i, j = 1, 2, \quad j \neq i. \tag{36}$$

With the values $P_i(\bar{z})$ solved, we can find

$$P_i(z) = P_i(\bar{z}) - \int_z^{\bar{z}} F_i(s, P_1(s), P_2(s))\, ds \qquad \text{for } z < \bar{z}, \quad i = 1, 2$$

where $F_i(z, P_1, P_2)$ is the function on the right-hand side of (35). The value function $V_i(z)$ can then be recovered from (20) by

$$V_i(z) = \frac{1}{4r}(a_i + P_i(z))^2 + \frac{1}{2r}P_i(z)[a_j + P_j(z) - 2\delta(z)] - \frac{c_i}{r}z^2.$$

To solve (36), we write the equations as

$$P_i(\bar{z})^2 + 4(a - \delta(\bar{z}))P_i(\bar{z}) + 2P_1(\bar{z})P_2(\bar{z}) - \Delta_i = 0$$

where

$$\Delta_i = 4\left(r\bar{V}_i + c_i\bar{z}^2\right) - a_i^2.$$

Let $\mu = 2P_1(\bar{z})P_2(\bar{z})$. Then, the equation becomes

$$P_i(\bar{z})^2 + 4(a - \delta(\bar{z}))P_i(\bar{z}) + \mu - \Delta_i = 0.$$

The solution is

$$P_i(\bar{z}) = -2(a - \delta(\bar{z})) \pm \sqrt{4(a - \delta(\bar{z}))^2 + \Delta_i - \mu} \qquad \text{for } i = 1, 2.$$

We assume that $P_i(\bar{z}) \leq 0$. Then the sign in front of the square root is positive if $a \geq \delta(\bar{z})$ and $\mu \geq \Delta_i$, and it is negative if $a \geq \delta(\bar{z})$ or if $a < \delta(z)$ and $\mu < \Delta_i$.) Let $\sigma_i = 1$ if the the sign is positive and $\sigma_i = -1$ if the sign is negative. We write

$$P_i(\bar{z}) = -2(a - \delta(\bar{z})) + \sigma_i\sqrt{4(a - \delta(\bar{z}))^2 + \Delta_i - \mu} \tag{37}$$

Then, μ is a solution to the equation

$$\mu = 2\prod_{i=1}^{2}\left\{-2\left(a-\delta\left(\bar{z}\right)\right)+\sigma_i\sqrt{4\left(a-\delta\left(\bar{z}\right)\right)^2+\Delta_i-\mu}\right\}. \tag{38}$$

It follows that $f\left(\bar{z}\right)$ has the form

$$f\left(\bar{z}\right) = -\left(a-\delta\left(\bar{z}\right)\right)+\frac{1}{2}\sum_{i=1}^{2}\sigma_i\sqrt{4\left(a-\delta\left(\bar{z}\right)\right)^2+\Delta_i-\mu}. \tag{39}$$

We focus on the strategies with value functions that satisfy

$$V_i'\left(\bar{z}\right) \leq 0 \qquad \text{for} \quad i = 1, 2. \tag{40}$$

With the help of the above preparation, Appendix A.6 provides reachability conditions. The fundamental idea is using the above analysis to show that the evaluation at $\overline{z}$ of the dynamic system

$$\dot{z} = f(z)$$

is positive or non-positive.

Obviously, if $f(\overline{z}) > 0$, the threshold $\overline{z}$ is still in the pollution accumulation process, thus it would be reached in finite time given pollution in increasing over time. However, if $f(\overline{z}) < 0$, it means the dynamic system must already reach to its long-run steady state $\dot{z} = f(z) = 0$ which is asymptotically stable. In other words, the threshold $\overline{z}$ such that $f(\overline{z}) < 0$ will never be reached. The detail calculation is presented in Appendix A.6.

Proposition 6 *(Reachability condition) For the optimal strategies associated with the value functions that satisfy (40), the following are true.*

1. *If $a \leq \delta\left(\bar{z}\right)$ then $\bar{z}$ is never reached in finite time.*
2. *If $a > \delta\left(\bar{z}\right)$ and*

$$\left|\Delta_1-\Delta_2\right| \geq 6\left(a-\delta\left(\bar{z}\right)\right)^2 \tag{41}$$

then $\bar{z}$ is never reached.

3. If $a > \delta(\bar{z})$ and

$$|\Delta_1 - \Delta_2| < 2\left(2\sqrt{3} - 3\right)(a - \delta(\bar{z}))^2 \tag{42}$$

then $\bar{z}$ is never reached if

$$\max\{\Delta_1, \Delta_2\} > 0. \tag{43}$$

and it is reached from some z_0 if the reversed inequality in (43) holds.

Let's make sense of the Proposition above by relating it more closely to the results obtained in the symmetric case (Propositions 4 and 5). To ease the exposition, suppose asymmetry lies only in the pollution costs ($a_1 = a_2 = a$). In such a case, one gets:

$$\Delta_1 - \Delta_2 = 4\left(r\Delta\bar{V} + \Delta c\bar{z}^2\right),$$

where $\Delta\bar{V} = \bar{V}_1 - \bar{V}_2$ and $\Delta c = c_1 - c_2$. Clearly, if additionally $\Delta c = 0$, which corresponds to the full symmetric case, then: $\Delta\bar{V} = 0$, and therefore, $\Delta_1 = \Delta_2$. In such a case, condition (40) of Proposition 6 is immediately checked and the permanent reversible regime holds if $\Delta_1 = \Delta_2 > 0$. Let's now compare Proposition 4 and Proposition 6 when Δc goes to 0, that's when we converge to the limit symmetric case described just above. Consider $a > \delta(\bar{z})$ and compare item 2 of Proposition 4 (with $a_1 = a_2$) and item 3 of Proposition 6 in the limit case. Clearly, condition (26) for permanent reversible regimes to hold in Proposition 4 is also checked by the counterpart condition (41) in Proposition 6. While this observation makes clear the coherence of our results, it does not say something accurate on the role of asymmetry in the emergence or not of irreversible pollution regimes. We next study an extreme asymmetry case where some interesting aspects could be grasped.[5]

5.2 Extreme asymmetric pollution cost: $c_1 = 0$, $c_2 = 2c > 0$

The complete solution of the counterpart game is given in the Appendix. We just state here the main results in the next Proposition, followed by a numerical example.

Proposition 7 ***(Reachability of** $\bar{z}$**)*** *If the inequality*

[5]One would be tempted to conclude when browsing conditions (39) and (40) and the fact that as written above: $\Delta_1 - \Delta_2 = 4\left(r\Delta\bar{V} + \Delta c\bar{z}^2\right)$. Unfortunately, as one can check, $\Delta\bar{V}$ also depends (nontrivially) on the c_i.

$$(r + \delta'(\bar{z}))(a - \delta(\bar{z})) > 2c\bar{z} \tag{44}$$

holds then $\bar{z}$ is reached from some z_0. Otherwise, if the reversed inequality in (44) holds, then $\bar{z}$ is never reached and

$$\lim_{t \to \infty} z(z) = z'_e.$$

If in addition to (44) δ is linear with positive α and β, and

$$2cz_0 < (r - \beta)(a - \delta(z_0)) \tag{45}$$

holds, (in particular, if $a \geq \alpha$), then $\bar{z}$ is reached in finite time from any z_0 that satisfies $0 \leq z_0 \leq \bar{z}$.

The results are much neater in this extreme asymmetric case compared to the more general asymmetric case considered before. And they sounds as more directly comparable with those obtained on the emergence of the irreversible regime under alternative strategic setting so far. Indeed, the comparison with Proposition 5 for the symmetric case is very interesting: comparing condition (27) in Proposition 5 with condition (44) in the Proposition just above gives the immediate outcome that reaching the irreversible regime is easier under the extreme asymmetric case in the sense that the threshold $\bar{z}$ can be reached for a larger set of the pollution cost parameter, c. This is particularly apparent from (30) in Corollary 1 and (44), which imply

$$\bar{C} > \frac{1}{2}(r + \delta'(\bar{z}))(a - \delta(\bar{z})),$$

provided that $a \geq \delta(\bar{z})$ and $\delta'(\bar{z}) \leq 0$. This might not look surprising as one of the two players in the latter case does not dislike at all pollution. Unfortunately, the picture is much more complicated. In the symmetric case, reachability also depends on conditions of type (26)-(28), which may perfectly yield the opposite picture. The numerical example below illustrates the different possible outcomes, and Section 6.1 clarifies this highly nontrivial feature and gives the intuition behind.

Moreover, even though the extreme asymmetric setting allows to reach less frequently the irreversible regime, this does not mean neither that for given c leading to crossing the irreversibility threshold in both strategic settings, pollution will end up be larger in the extreme asymmetric case, in particular in the long-term. As we will see in the next section 6.2, free-riding is such a powerful mechanism that we unambiguously get the inverse ranking, that's steady state pollution may be higher in the symmetric case under the irreversible regime. We show it numerically in the TW case where pollution decay is equal to zero.

Example 3 *Using the same parameter values as in Example 1, for the extreme asymmetric case with $c_1 = 0$ and $c_2 = 2c = 0.004$, we find*

$$(r + \delta'(\bar{z}))(a - \delta(\bar{z})) = 1.8 > 0.8 = 2c\bar{z}.$$

Hence, (44) holds. By Proposition 7, $\bar{z} = 200$ is reached in finite time for some z_0.
For non-extreme asymmetric case with $c_1 = 0.0015$, $c_2 = 0.0025$ and $\bar{z} = 200$, we find that

$$\Delta_1 = -203.95, \qquad \Delta_2 = -365.28, \qquad a - \delta(\bar{z}) = 18.$$

Hence,

$$|\Delta_2 - \Delta_1| = 116.34 < 2\left(2\sqrt{3} - 3\right) 18^2.$$

Since $\max\{\Delta_1, \Delta_2\} < 0$*, by Proposition 6, $\bar{z}$ is reachable from some z_0.*
If, on the other hand, $c_1 = 0.015$, $c_2 = 0.025$, then

$$\Delta_1 = 373.75, \qquad \Delta_2 = 95.15.$$

It follows that

$$|\Delta_1 - \Delta_2| = 278.60 < 2\left(2\sqrt{3} - 3\right)(a - \delta(\bar{z}))^2.$$

Since $\max\{\Delta_1, \Delta_2\} \geq 0$*, by Proposition 6, $\bar{z}$ is never reached by optimal strategies with value functions that satisfy $V_i'(\bar{z}) \leq 0$ for $i = 1, 2$. Indeed, there are three sets of such value functions, with $(V_1'(\bar{z}), V_2'(\bar{z}))$ having the values* $(-74.58, -1.21)$*,* $(-19.31, -36.02)$*, and* $(-6.84, -59.91)$*, respectively. The corresponding values of $f(\bar{z})$ are -19.90, -9.67 and -15.38, respectively. Hence, $\bar{z}$ is never reached by either strategy.*
To compare asymmetric case with the symmetric one, we note that for

$$c \geq \frac{(r - \beta)\, a}{2\bar{z}} = 0.0045,$$

$\bar{z} = 200$ *is not reachable in finite time in the extreme asymmetric case with* $c_1 = 0$ *and* $c_2 = 2c$. *On the other hand, for the symmetric case with* $c_1 = c_2 = c$, *computation shows that the largest solution of equation (29),* $\bar{C}$, *is approximately* 0.01. *Therefore, for* $0.0045 < c \leq 0.01$, $\bar{z}$ *is not reached in the extreme asymmetric case, but is reached in the symmetric case. By continuity of solution on parameters, the same is true for cases where one* c_i *is far less than the other* c_j *and where the two are close.*

6 Reachability, irreversibility and institutional settings

This section summarizes the above analysis and makes some comparison studies. The first subsection focus on the reachability conditions of the threshold which presented in the above Proposition 1, 4, 5 and 7. This part can be considered as short-run trajectory comparison. The second subsection focus on the irreversibility regime and the corresponding long-run outcomes under different competition settings.

6.1 Reachability conditions under different competitions

Figure 1 summarizes the reachability conditions of threshold $\overline{z}$ in terms of the total cost parameter $c_1 + c_2 = c$ under the different settings, including symmetric, extreme asymmetric and general asymmetric competitions.

Before further investigation, we must remark that in the last part of the Figure 1, the position of $\overline{C}$ is not fixed above C_s, rather it depends on the combination of parameters: all ranking are possible, see the middle part of the figure. Nevertheless, the ranking of C_c, C_s and C_e are unambiguously presented in the last part of the figure.

It is straightforward that regardless the position of $\overline{C}$ (see Corollary 1), the $\overline{z}-$unreachable interval, that is $[C_c, \infty)$, is the largest. This is not surprising as central planning yields the first best scenario. The ranking between symmetric and extreme asymmetric competition is rather complicated as it depends on the location of $\overline{C}$. In the case of the last part of Figure 1, the unreachable cost interval is larger under extreme asymmetric competition than under symmetric situation: $[\overline{C}, \infty) \subset [C_e, \infty)$. But it may happen that $\overline{C} < C_e$,

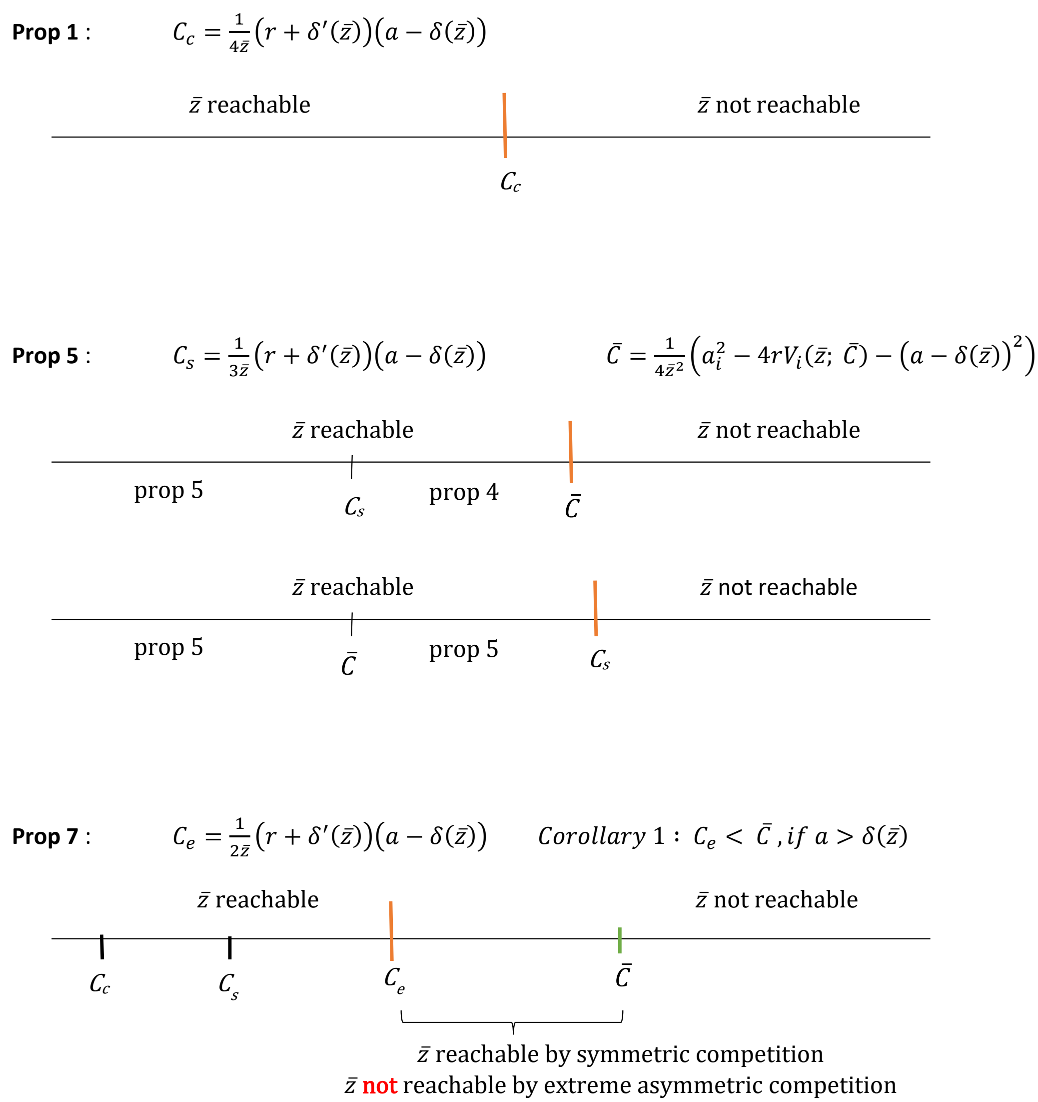

Figure 1: Comparison of reachability condition.

given condition in Corollary 1, then the opposite conclusion is true.

The extra condition $\overline{C}$, defined in Corollary 1, comes directly from the competition differences and the intuition of this condition was explained after Proposition 4. Under the case where there is only one decision maker who cares about the accumulated pollution cost, either central planner or the player 2 in the extreme asymmetric competition who suffers the most from the accumulation of pollution, the decision is made unambiguously depending on the cost-gain benefit analysis (explained after Proposition 1), which yields the conditions in terms of the threshold values C_c, C_s and C_e. However, between these two polar cases, for example under the symmetric competition, both players' efforts additionally depend on conditions of type (26), that's on the cost generated by pollution accumulation at the threshold level compared with their respective net values.

6.2 Irreversibility, institutional settings and long-term pollution outcomes

The last part of the last example in the previous section indicates that asymmetry results in lower pollution. This is also the case if $\bar{z}$ is reached in both cases. The next result gives an order of steady-state pollution levels in the four strategic settings so far considered in the irreversible regime (more precisely, with a zero decay of pollution as in TW). Concretely, it shows that pollution is lower with cooperation (compared to the three non-cooperative settings considered), and much more interestingly, asymmetric game cases deliver less pollution among the symmetric. We dig into the intuitions and the economic interpretations later.

Let the steady states of pollution without decay be denoted by z_c^*, z_s^*, z_e^* and z_a^*, for the cases with a central planner and the game-theoretic with symmetric, extreme asymmetric, and the general asymmetric pollution costs, respectively. The first three are given by (4), (19) and (76), respectively, and it can be shown, similar to the derivation of z_s^*, that

$$z_a^* = \frac{2a\left(\lambda + r^2 + r\sqrt{r^2 + 4\left(2c - \lambda\right)}\right)}{r\left(4c - \lambda\right) + \left(4c - 3\lambda\right)\sqrt{r^2 + 4\left(2c - \lambda\right)}} \tag{46}$$

with λ defined by

$$\left(r-\sqrt{r^2+4\left(c_1-\lambda\right)}\right)\left(r-\sqrt{r^2+4\left(c_2-\lambda\right)}\right)=2\lambda. \tag{47}$$

In Appendix A.7, we show the following ranking.

Proposition 8 *For any nonnegative c_1 and c_2 the steady states of pollution without decay, z_a^*, z_e^*, z_s^*, and z_c^*, are ordered as*

$$z_c^* < z_e^* \leq z_a^* \leq z_s^*.$$

We first illustrate the finding with a numerical example.

Example 4 *Using the same parameter values as in Example 1, TW shows the threshold $\overline{z}=200$ is reached in finite time. We have shown in Examples 1–3 that the same $\overline{z}$ is reached in all other cases for equivalent parameter values. In addition, the limit of $z(t)$ is $z_c^*=450$ in the central planner case (with $c=0.002$), to $z_s^*=939.74$ in the symmetric case (with $c_1=c_2=0.002$), and to $z_e^*=900$ in the extreme asymmetric case (with $c_1=0$ and $c_2=0.004$). For the asymmetric case with $c_1=0.0015$ and $c_2=0.0025$, the limit is $z_a^*=937.14$. Thus the ranking in the proposition is checked.*

Intuition behind Proposition 8 is the following. The steady states are comparable only if they lie within the same pollution regime, we focus on the irreversible regime with zero decay, which is the more original exercise in this respect. Obviously, the central planer's optimal choice yields the first best outcome with the lowest pollution accumulation. The other three cases are more intricate to compare at once but we can visualize better the results if we compare first the symmetric and extreme asymmetric configurations as the general asymmetric setting can be approached as an intermediate case between the two latter.

Consider the pollution accumulation dynamics $\dot{z}=y_1+y_2$ and the efforts of both players $y_i=\frac{a_i}{2}+\frac{B_i+C_iz}{2}$ and $y_i^m=\frac{a_i}{2}+\frac{B^m+C^mz}{2}$ under extreme asymmetric and symmetric competition respectively. Indeed given $B_i\leq 0, B^m<0$ and $C_i<0, C^m<0$, we can interpret $\frac{B_i+C_iz}{2}$ and $\frac{B^m+C^mz}{2}$ as efforts made by players to reduce the pollution accumulation in the two latter cases. Thus, it is straightforward to see that with $c_1=0$ under extreme asymmetric competition, player 1 makes no efforts to help reducing pollution (since $C_1 = 0, B_1 = 0$). In contrast, player 2, who bears the highest cost from the accumulation of pollution, will make a substantial effort to reduce pollution. It is easy to show that

$$\frac{C_2}{2} < C^m < 0 \text{ and } \frac{B_2}{2} < B^m < 0.$$

Thus, for any z, it follows at the aggregate level,

$$\frac{B_2 + C_2 z}{2} < B^m + C^m z.$$

In other words, under the extreme asymmetric competition, player 2 with $c_2 = 2c$ makes more efforts to clean-up the pollution than the sum of two players in the symmetric case ($c_1 = c_2 = c$). Under symmetric case, the well-known free-riding mechanism is at work: both players would wait for the other one to make more efforts and no one end up making enough efforts. The general asymmetric case lies in between: the player who faces higher accumulated pollution damage will make more efforts to reduce the pollution while the one who is less sensitive to accumulated pollution would free ride on the other's efforts. But the global impact of free-riding is lower than under symmetry.

7 Conclusion

In this paper, we have developed an extension of the hard pollution irreversibility model of TW to differential games. As we keep the original non-concavity originating precisely in the specification of hard irreversibility, the induced mathematical setting is nontrivial. However, we have been successful enough in providing with a full analytical handling of the game outcomes in various institutional configurations. Beside the technical contribution we have reached three meaningful results. First, we show that cooperation may not prevent irreversible pollution regimes to occur. Second, we find that under symmetry (in pollution costs), irreversible regime are more likely to emerge in the absence of cooperation. Last but not least, when studying the implications of asymmetry, we find nontrivial results on the reachability of irreversible pollution compared to the symmetric game. However, we unambiguously prove that, for the same total cost of pollution, provided where the irreversible regime is reached in both the symmetric and asymmetric cases, long-term pollution is larger in the symmetric case, reflecting more intensive free-riding under symmetry.

Needless to say, our results are worth examining in richer settings. One quite interesting extension would consider uncertainty either on the value of irreversibility thresholds ($\bar{z}$) or in the extent of irreversibility (that is, under random magnitude of the drop in pollution decay for given threshold). Obviously, it's not granted that we can keep the fully analytical approach when dealing with these natural extensions.

A Appendix

A.1 Proof of Proposition 1

From (8) we derive

$$\begin{aligned} 4\left[rV_c(\bar{z}) + 2c\bar{z}^2\right] &= \left(f_c(\bar{z}) + \frac{a_2 - a_1}{2} + \delta(\bar{z})\right)^2 \\ &+ \left(f_c(\bar{z}) + \frac{a_1 - a_2}{2} + \delta(\bar{z})\right)^2 - 4\delta(\bar{z})\left(f_c(\bar{z}) - a + \delta(\bar{z})\right). \end{aligned}$$

The equation is simplified to

$$f_c(\bar{z})^2 = 2\left[rV_c(\bar{z}) + 2c\bar{z}^2\right] + (a - \delta(\bar{z}))^2 - \frac{a_1^2 + a_2^2}{2}.$$

So

$$f_c(\bar{z}) = \sqrt{(a - \delta(\bar{z}))^2 + \Delta^c}$$

if (9) holds, and

$$f_c(\bar{z}) = -\sqrt{(a - \delta(\bar{z}))^2 + \Delta^c}$$

otherwise. In the former case, $f_c(\bar{z}) > 0$. In the latter case, $f_c(\bar{z}) \leq 0$. Hence, the first root of $f_c(z)$ is less than or equal to $\bar{z}$. Therefore, $\bar{z}$ is never reached.

Assuming that δ is linear with positive α and β and (9) and (10) both hold, we show that

$f_c(z) > 0$ for $z \leq \bar{z}$. Subsitituting $P_c(z) = f_c(z) - a + \delta(z)$ in (7), we find

$$f_c(z)\left[f_c'(z) - \beta\right] = (r - \beta)\left[f_c(z) - a + \delta(z)\right] + 4cz \qquad \text{for } z < \bar{z}.$$

It can be written as

$$\frac{1}{2}\frac{d}{dz}\left[f_c(z)\right]^2 = rf_c(z) - (r - \beta)(a - \delta(z)) + 4cz.$$

If there is $\hat{z} < \bar{z}$ such that $f_c(\hat{z}) = 0$. Then, $\left[f_c(z)\right]^2$ has a local minimum at $\hat{z}$. Thus the left-hand side of the above equation is zero. It follows that

$$-(r - \beta)(a - \delta(\hat{z})) + 4c\hat{z} = 0. \tag{48}$$

On the other hand by (9) and (10) $(r - \beta)(a - \delta(z)) > 4cz$ for $z = z_0, \bar{z}$. Since both functions are linear, it follows that

$$(r - \beta)(a - \delta(z)) > 4cz \qquad \text{for } z_0 \leq z \leq \bar{z}.$$

This contradicts (48). Hence, no such $\hat{z}$ exists. This proves that $\bar{z}$ is reached in finite time.

The proof of the proposition is complete. □

A.2 Proof of Proposition 2

It is easy to check that the Bellman value function must check the following HJB equation:

$$rV_i(z) = \max_{y_i}\left[a_i y_i - y_i^2 - cz^2 + V_{i,z}\left(y_1 + y_2 - \delta(z)\right)\right]. \tag{49}$$

The right hand side's first order condition yields that player i's optimal choice is

$$y_i = \frac{a_i + V_{i,z}}{2}, \quad i = 1, 2. \tag{50}$$

Substituting into (49), it follows

$$rV_i(z) = a_i \frac{a_i + V_{i,z}}{2} - \frac{a_i + V_{i,z}}{2}^2 - cz^2 + V_{i,z} \left[\frac{a_1 + V_{1,z}}{2} + \frac{a_2 + V_{2,z}}{2} - \delta(z) \right], \quad i = 1, 2. \tag{51}$$

For simplicity, denote $P_i(z) = V_i(t, z)$ for $i = 1, 2$. Taking derivative of(51) on both sides with respect to state variable z, it follows

$$rP_i = \frac{1}{2} \left\{ P_i' \left[a_1 + a_2 + P_1 + P_2 - 2\delta(z) \right] + P_i \left[P_j' - 2\delta'(z) \right] \right\} - 2cz. \tag{52}$$

We notice that, in the second period, the difference between the value function of player 1 and 2 lies solely on the constant term, A_1 and A_2. Thus, we guess similar pattern is true in the first period, thus, $P_1(z) = V_1'(z) = V_2'(z) = P_2 = P_s(z)$. If so, equation (52) can be simplified as the following:

$$\left[\frac{3}{2} P_s(z) + a - \delta(z) \right] P_s'(z) = (r + \delta'(z)) P_s(z) + 2cz. \tag{53}$$

The terminal condition at $\bar{z}$ is determined by the HJB equation (20), which takes the form

$$r\bar{V}_i = \frac{1}{4} \left(a_i + P_s(\bar{z}) \right)^2 + \frac{1}{2} P_s(\bar{z}) \left[a_j + P_s(\bar{z}) - 2\delta(\bar{z}) \right] - c\bar{z}^2.$$

This is a quadratic equation in $P_s(\bar{z})$, in the form[6]

$$3P_s(\bar{z})^2 + 4 \left(a - \delta(\bar{z}) \right) P_s(\bar{z}) + a_i^2 = 4 \left(r\bar{V}_i + c\bar{z}^2 \right). \tag{54}$$

In general, there are two roots for Eq. (54),

$$P_s(\bar{z}) = \frac{1}{3} \left\{ -2 \left(a - \delta(\bar{z}) \right) \pm \sqrt{4 \left(a - \delta(\bar{z}) \right)^2 + 3\Delta_i} \right\} \tag{55}$$

where

$$\Delta_i = 4 \left(rV_i(\bar{z}) + c\bar{z}^2 \right) - a_i^2 \qquad \text{for } i = 1, 2.$$

[6]Note that

$$4r\bar{V}_i - a_i^2 = 4rA_i - a_i^2 + 4r \left(B\bar{z} + \frac{C}{2} \bar{z}^2 \right) = 4aB + 3B^2 + 4r \left(B\bar{z} + \frac{C}{2} \bar{z}^2 \right)$$

is independent of i. So $P_s(\bar{z})$ does not depend on i.

Each gives rise a value function. We choose the one that has the smaller $P_s'(\bar{z})$ to ensure that the value functions $V_i(z)$ are most concave for $z < \bar{z}$ and is near $\bar{z}$. Substituting the right-hand side of (55) into (53) evaluated at $\bar{z}$, it follows that

$$P_s'(\bar{z}) = \frac{r + \delta'(\bar{z})}{3} + \frac{-2(r + \delta'(\bar{z}))(a - \delta(\bar{z})) + 6c\bar{z}}{\pm 3\sqrt{4(a - \delta(\bar{z}))^2 + 3\Delta_i}}.$$

So, the sign is positive if

$$\frac{-2(r + \delta'(\bar{z}))(a - \delta(\bar{z})) + 6c\bar{z}}{3\sqrt{4(a - \delta(\bar{z}))^2 + 3\Delta_i}} \leq \frac{-2(r + \delta'(\bar{z}))(a - \delta(\bar{z})) + 6c\bar{z}}{-3\sqrt{4(a - \delta(\bar{z}))^2 + 3\Delta_i}},$$

which is equivalent to

$$(r + \delta'(\bar{z}))(a - \delta(\bar{z})) - 3c\bar{z} \geq 0, \tag{56}$$

and the sign is negative if the reversed inequality holds.

We can solve (53) in the case where δ is a linear function: $\delta(z) = \alpha - \beta z$ with both α, β positive. Multiplying the both sides by $3/2$, we get

$$\left[\frac{3}{2}P_s(z) + a - \delta(z)\right]\left(\frac{3}{2}P_s(z)\right)' = (r - \beta)\left(\frac{3}{2}P_s(z)\right) + 3cz. \tag{57}$$

Introducing

$$Q_s(z) = \frac{3}{2}P_s(z) + a - \delta(z),$$

the differential equation for $Q_s(z)$ is

$$Q_s(z)\left[Q_s(z) - a + \delta(z)\right]' = (r - \beta)\left[Q_s(z) - a + \delta(z)\right] + 3cz. \tag{58}$$

It leads to

$$Q_s'(z) = r + \frac{a_s + b_s z}{Q_s}$$

where

$$a_s = (\beta - r)(a - \alpha), \quad b_s = \beta(\beta - r) + 3c.$$

We make the substitution $x = z + a_s/b_s$ in the equation, and regards Q_s as a function of

x. It follows that

$$Q_s'(x) = r + \frac{b_s x}{Q_s(x)}.$$

This is a first order equation of homogeneous type. Let $u = Q_s/x$. The equation becomes

$$x\frac{du}{dx} = r + \frac{b_s}{u} - u = -\frac{u^2 - ru - b_s}{u}. \tag{59}$$

In the case where $r^2 + 4b_s > 0$. We let

$$u_s^- = \frac{1}{2}\left[r - \sqrt{r^2 + 4b_s}\right], \quad u_s^+ = \frac{1}{2}\left[r + \sqrt{r^2 + 4b_s}\right].$$

Then

$$\frac{u}{u^2 - ru - b_s} = \frac{p_s}{u - u_s^-} + \frac{q_s}{u - u_s^+}$$

where

$$p_s = -\frac{u_s^-}{u_s^+ - u_s^-}, \quad q_s = \frac{u_s^+}{u_s^+ - u_s^-}.$$

Note that $u_s^- \leq 0 < u_s^+$ (if $\beta \geq r$) and p_s and q_s are both nonnegative. In addition, $p_s + q_s = 1$. So we substitute $1 - p_s$ for q_s. Eq. (59) becomes

$$\left[\frac{p_s}{u - u_s^-} + \frac{1 - p_s}{u - u_s^+}\right] du = -\frac{dx}{x}.$$

By integration, we have

$$p_s \ln\left|u - u_s^-\right| + (1 - p_s)\ln\left|u - u_s^+\right| = -\ln|x| + C.$$

Hence, substituting Q_s for ux, we find

$$\left|Q_s(x) - u_s^- x\right|^{p_s}\left|Q_s(x) - u_s^+ x\right|^{1-p_s} = C.$$

The value of C is determined by the value of Q_s at $\bar{x} \equiv \bar{z} + a_s/b_s$. That is,

$$C = \left|Q_s(\bar{x}) - u_s^- \bar{x}\right|^{p_s}\left|Q_s(\bar{x}) - u_s^+ \bar{x}\right|^{1-p_s}.$$

Hence $Q_s(x)$ satisfies the equation

$$\left|\frac{Q_s(x) - u_s^- x}{Q_s(\bar{x}) - u_s^- \bar{x}}\right|^{p_s} \left|\frac{Q_s(x) - u_s^+ x}{Q_s(\bar{x}) - u_s^+ \bar{x}}\right|^{1-p_s} = 1.$$

Returning the variable z, the equation becomes

$$\left|\frac{Q_s(z) - u_s^- (z + a_s/b_s)}{Q_s(\bar{z}) - u_s^- (\bar{z} + a_s/b_s)}\right|^{p_s} \left|\frac{Q_s(z) - u_s^+ (z + a_s/b_s)}{Q_s(\bar{z}) - u_s^+ (\bar{z} + a_s/b_s)}\right|^{1-p_s} = 1. \tag{60}$$

Finally,

$$P_s(z) = \frac{2}{3}\left[Q_s(z) - a + \delta(z)\right] \tag{61}$$

and the value function $V_i(z)$ can be found by

$$V_i(z) = \bar{V}_i - \int_z^{\overline{z}} P_s(s)ds \text{ for } \overline{z} < z, \quad i = 1, 2. \tag{62}$$

In the case where $r^2 + 4b_s = 0$,

$$\frac{u}{u^2 - ru - b_s} = \frac{u}{(u - r/2)^2} = \frac{1}{u - r/2} + \frac{r/2}{(u - r/2)^2}.$$

Thus

$$\int \left[\frac{1}{u - r/2} + \frac{r/2}{(u - r/2)^2}\right] du = \ln\left|u - \frac{r}{2}\right| - \frac{r}{2u - r} + C.$$

So

$$\ln|2Q_s(x) - rx| - \frac{rx}{2Q_s(x) - x} = C$$

where

$$C = \ln|2Q_s(\bar{x}) - r\bar{x}| - \frac{r\bar{x}}{2Q_s(\bar{x}) - \bar{x}}.$$

This leads to

$$\left|\frac{2Q_s(z) - r(z + a_s/b_s)}{Q_s(\bar{z}) - r(\bar{z} + a_s/b_s)}\right| \exp\left[\frac{r(\bar{z} + a_s/b_s)}{2Q_s(\bar{z}) - r(\bar{z} + a_s/b_s)} - \frac{r(z + a_s/b_s)}{2Q_s(z) - r(z + a_s/b_s)}\right] = 1.$$

Finally, in the case where $r^2 + 4b_s < 0$,

$$\frac{u}{u^2 - ru - b_s} = \frac{u}{\left(u - r/2\right)^2 - \left(r^2/4 + b_s\right)}.$$

Thus,

$$\int \frac{u}{u^2 - ru - b_s} du = \ln \left|(2u - r) - \left(r^2/2 + 2b_s\right)\right| - \frac{r}{2\sqrt{|r^2 + 4b_s|}} \tan \frac{2u - r}{2\sqrt{|r^2 + 4b_s|}} + C.$$

In any case, there is a solution in implicit form

$$S\left(z, Q_s\left(z\right)\right) = S\left(\bar{z}, Q_s\left(\bar{z}\right)\right)$$

for some function S.

The proof of the proposition is complete. □

A.3 Proof of Proposition 4

Part 1. We show that $f_s\left(\bar{z}\right) \equiv P_s\left(\bar{z}\right) + a - \delta\left(\bar{z}\right) \leq 0$. Substituting $f_s\left(\bar{z}\right) - a + \delta\left(\bar{z}\right)$ for $P_s\left(\bar{z}\right)$ in (54), we find

$$3\left(f_s\left(\bar{z}\right) - a + \delta\left(\bar{z}\right)\right)^2 + 4\left(a - \delta\left(\bar{z}\right)\right)\left(f_s\left(\bar{z}\right) - a + \delta\left(\bar{z}^-\right)\right) - \Delta_i = 0.$$

This is a quadratic equation in $f_s\left(\bar{z}\right)$. It can be written as

$$3f_s\left(\bar{z}\right)^2 - 2\left(a - \delta\left(\bar{z}\right)\right)f_s\left(\bar{z}\right) - \Delta_i - \left(a - \delta\left(\bar{z}\right)\right)^2 = 0. \quad (63)$$

By (24), $f_s\left(\bar{z}\right)$ is the smaller root. That is,

$$f_s\left(\bar{z}\right) = \frac{1}{3}\left\{a - \delta\left(\bar{z}\right) - \sqrt{4\left(a - \delta\left(\bar{z}\right)\right)^2 + 3\Delta_i}\right\}. \quad (64)$$

It is clear that $f_s\left(\bar{z}\right) \leq 0$ if $a \leq \delta\left(\bar{z}\right)$. Therefore, the first root of $f_s\left(z\right)$, z_s', is less than or equal to $\bar{z}$. So $\bar{z}$ is never reached, and (25) holds.

Part 2. Suppose (26) holds. We use the result in Corollary 1 (proof is given below) to conclude

$$\frac{1}{3\bar{z}}\left(r+\delta'\left(\bar{z}\right)\right)\left(a-\delta\left(\bar{z}\right)\right)<\frac{ra}{2\bar{z}}<\bar{C}\leq c.$$

Hence (24) holds. Therefore (64) holds. Since (26) holds, it follows that

$$\Delta_i = 4\left(rV_i\left(\bar{z}\right)+c\bar{z}^2\right)-a_i^2 \geq -\left(a-\delta\left(\bar{z}\right)\right)^2.$$

Hence,

$$\sqrt{4\left(a-\delta\left(\bar{z}\right)\right)^2+3\Delta_i} \geq a-\delta\left(\bar{z}\right).$$

This leads to $f_s\left(\bar{z}\right)\leq 0$. As a result, $\bar{z}$ is never reached. If (26) does not hold, then

$$\sqrt{4\left(a-\delta\left(\bar{z}\right)\right)^2+3\Delta_i} < a-\delta\left(\bar{z}\right).$$

Hence, by (64), $f_s\left(\bar{z}\right)>0$. By continuity, $f_s\left(z\right)>0$ for z near $\bar{z}$. Thus if z_0 is within this neighborhood, $f\left(z\right)>0$ if $z_0\leq z\leq\bar{z}$. This means $\bar{z}$ is reachable in finite time from some z_0.

This completes the proof. □

A.4 Proof of Proposition 5

By (27), the sign in (55) is positive. Hence,

$$f_s\left(\bar{z}\right)=\frac{1}{3}\left\{a-\delta\left(\bar{z}\right)+\sqrt{4\left(a-\delta\left(\bar{z}\right)\right)^2+3\Delta_i}\right\}>0$$

if $a>\delta\left(\bar{z}\right)$ or if $a\leq\delta\left(\bar{z}\right)$ and (28) holds. If z_0 is sufficiently close to $\bar{z}$, $f_s\left(z\right)>0$ for any z between z_0 and $\bar{z}$.

To prove the second part, we derive a differential equation for f_s by substituting $P_s\left(z\right)=f_s\left(z\right)-a+\delta\left(z\right)$ in (53), we derive

$$\left[\frac{3}{2}f_s\left(z\right)-\frac{1}{2}\left(a-\delta\left(z\right)\right)\right]f_s'\left(z\right)=\left(r+\frac{\beta}{2}\right)f_s\left(z\right)-\left(r-\frac{\beta}{2}\right)\left(a-\delta\left(z\right)\right)+2cz \qquad \text{for } z<\bar{z}. \tag{65}$$

Let

$$L_1=\frac{1}{3}\left(a-\delta\left(z\right)\right), \qquad L_2\left(z\right)=\frac{2r-\beta}{2r+\beta}\left(a-\delta\left(z\right)\right)-\frac{4cz}{2r+\beta}.$$

Then (65) can be written as

$$\frac{3}{2}\left[f_s\left(z\right)-L_1\left(z\right)\right]f_s'\left(z\right)=\left(r+\frac{\beta}{2}\right)\left[f_s\left(z\right)-L\left(z\right)\right]. \tag{66}$$

We show that $L_1\left(z\right)<L_2\left(z\right)$ for $z_0\leq z\leq\bar{z}$. In the case where $a\geq\alpha$, we observe that

$$L_1\left(0\right)=\frac{1}{3}\left(a-\alpha\right),\qquad L_2\left(0\right)=\frac{2r-\beta}{2r+\beta}\left(a-\alpha\right).$$

Note that since $r>\beta$,

$$\frac{2r-\beta}{2r+\beta}=1-\frac{2\beta}{2r+\beta}>\frac{1}{3}.$$

It follows that $L_1\left(0\right)<L_2\left(0\right)$. Also, at $\bar{z}$, by (27)

$$c\bar{z}<\frac{1}{3}\left(r-\beta\right)\left(a-\delta\left(\bar{z}\right)\right).$$

Thus

$$L_2\left(\bar{z}\right)>\frac{2r-\beta}{2r+\beta}\left(a-\delta\left(\bar{z}\right)\right)-\frac{4\left(r-\beta\right)}{3\left(2r+\beta\right)}\left(a-\delta\left(\bar{z}\right)\right)=\frac{1}{3}\left(a-\delta\left(\bar{z}\right)\right)=L_1\left(\bar{z}\right).$$

Since L_1 and and L_2 are linear functions, it follows that $L_1\left(z\right)<L_2\left(z\right)$ for $z_0\leq z\leq\bar{z}$.

We next show that $f_s\left(z\right)>L_1\left(z\right)$ for $z_0\leq z\leq\bar{z}$. Let $Q\left(z\right)=f_s\left(z\right)-L_1\left(z\right)$. Eq. (66) can be written as

$$\frac{3}{2}Q\left(z\right)Q'\left(z\right)=\left(r+\frac{\beta}{2}\right)\left[Q\left(z\right)+L_1\left(z\right)-L_2\left(z\right)\right]-\frac{\beta}{2}Q\left(z\right). \tag{67}$$

At $\bar{z}$ we have

$$\frac{3}{2}\left[f_s\left(\bar{z}\right)-L_1\left(\bar{z}\right)\right]=\frac{1}{2}\sqrt{4\left(a-\delta\left(\bar{z}\right)\right)^2+3\Delta_i}>0.$$

So $f_s\left(\bar{z}\right)>L_1\left(\bar{z}\right)$. If $f_s\left(\hat{z}\right)=L_1\left(\hat{z}\right)$ for some $\hat{z}$ between z_0 and $\bar{z}$, then $Q\left(z\right)$ has a local minimum at $\hat{z}$. Therefore, the left-hand side of (67), being the same as $3\left[Q\left(z\right)^2\right]'/2$, is zero. Hence the right-hand side is also zero. Since $Q\left(\hat{z}\right)=0$, it follows that $L_1\left(\hat{z}\right)-L_2\left(\hat{z}\right)=0$. This contradicts $L_1\left(z\right)<L_2\left(z\right)$ for $z_0\leq z\leq\bar{z}$.

We now show that $f_s\left(z\right)>0$ for $z_0\leq z\leq\bar{z}$. There are two cases, either $L_2'\left(z\right)>0$ or $L_2'\left(z\right)\leq0$. We first consider the former case. Then, either $f_s\left(\bar{z}\right)\geq L_2\left(\bar{z}\right)$ or $0<f_s\left(\bar{z}\right)<$

$L_2(\bar{z})$. If $f_s(\bar{z}) \geq L_2(\bar{z})$, by (66)

$$\frac{3}{2}\left[f_s(\bar{z}) - L_1(\bar{z})\right] f_s'(\bar{z}) = \left(r + \frac{\beta}{2}\right)\left[f_s(\bar{z}) - L_2(\bar{z})\right] \geq 0.$$

Since $f_s(\bar{z}) > L_1(\bar{z})$, it follows that $f_s'(\bar{z}) \geq 0$. Note that $f_s(z) > L_2(z)$ for $z < \bar{z}$ and is near $\bar{z}$. This follows from continuity of $f_s(z)$ and $L_2(z)$ if $f_s(\bar{z}) > L_2(\bar{z})$. If $f_s(\bar{z}) = L_2(\bar{z})$, the above equation shows that $f_s'(\bar{z}) = 0$. Since $f_s(\bar{z}) = L_2(\bar{z})$ and $L_2'(\bar{z}) > f_s'(\bar{z})$, we again find that $f_s(z) > L_2(z)$ for $z < \bar{z}$ and is near $\bar{z}$. If there is $z_1 < \bar{z}$ such that $f_s(z_1) = L_2(z_1)$, then $f_s'(z_1) \geq L_2'(z_1) > 0$. However, by (66), $f_s'(z_1) = 0$. This is a contradiction. Therefore $f_s(z) \geq L_2(z)$. As a result, $f_s(z) \geq L_2(z) > L_2(0) > 0$.

If $0 < f_s(\bar{z}) < L_2(\bar{z})$. By (66), $f_s'(\bar{z}) < 0$. Hence $f_s(z)$ is decreasing. The derivative is negative for all z such that $f_s(z) < L_2(z)$. Since L_2 is increasing and f_s is decreasing, there is a $\tilde{z}$ such that $f_s(\tilde{z}) = L_2(\tilde{z})$. For $z < \tilde{z}$, by a reasoning similar to the previous paragraph, we find that $f_s(z) > L_2(z)$. Hence, again $f_s(z) > 0$ for $z_0 \leq z \leq \bar{z}$. This completes the proof in the case where $L_2'(z) > 0$.

Suppose $L_2'(z) \leq 0$. If $f_s(\bar{z}) > L_2(\bar{z})$, then $f_s'(z) > 0$ for all $z \leq \bar{z}$ such that $f_s(z) > L_2(z)$. Hence f_s is increasing. Since L_2 is nonincreasing, there is $\hat{z}$ such that $f_s(\hat{z}) = L_2(\hat{z})$. At this point, by (66), $f_s'(\hat{z}) = 0$. Since $L_2' \leq 0$, $f_s(z) \leq L_2(z)$. For any point z at which $f_s(z) < L_2(z)$, $f_s'(z) < 0$. Hence f_s is decreasing. This shows that $f_s(z) \geq f_s(\hat{z})$ for $z \leq \hat{z}$. Therefore $f_s(z_0) \geq f_s(\hat{z}) = L_2(\hat{z}) > 0$.

If $f_s(\bar{z}) \leq L_2(\bar{z})$, the above proof shows that $f_s(z)$ is decreasing for all z such that $f_s(z) < L_2(z)$. Hence, again, $f_s(z_0) \geq f_s(\hat{z}) > 0$.

This completes the proof of the proposition. $\square$

A.5 Proof of Corollary 1

The proof is complete in two steps: step 1 shows the existences of threshold $\overline{c}$ and step 2 proves the statement of the corollary.

Step 1. The existence of $\overline{c}$.

Rewrite inequality condition (26) as

$$c\bar{z}^2 + rV_i(\bar{z}; c) \geq \frac{a_i^2 - (a - \delta(\bar{z}))^2}{4}.$$

Define the left hand side as $F(c)$ and right hand side as $g(c)$, that is,

$$F(c) = c\bar{z}^2 + rV_i(\bar{z}; c), \quad g(c) = \frac{a_i^2 - (a - \delta(\bar{z}))^2}{4}.$$

Obviously $g(c)$ is a constant in term of c.

Denote $\overline{C}$ is given by the root of equation (26) when it is equality, i.e.,

$$F(\overline{C}) = g(\overline{C}) = constant.$$

In the following, we shall prove that the existence of root $\overline{C} \in (0, \infty)$ of the above equation, such that, $\forall c \geq \overline{C}$,

$$F(c) \geq g(c). \tag{68}$$

If so, Corollary 1 is proved, that is, provided the other conditions hold in Corollary 1, if $c > \overline{c}$, $\overline{z}$ is unreachable while if $c < \overline{c}$, $\overline{z}$ is reachable, where $\overline{c}$ is defined in Corollary 1:

$$\overline{c} = \max\left\{\frac{1}{3\bar{z}}\left(r + \delta'(\bar{z})\right)\left(a - \delta(\bar{z})\right), \overline{C}\right\}.$$

To finish the proof, recall

$$V_i(\bar{z}; c) = A_i^m + B^m\bar{z} + \frac{C^m\,\bar{z}^2}{2}$$

with

$$C^m = \frac{r - \sqrt{r^2 + 12c}}{3}(< 0), \quad B^m = \frac{(a_1 + a_2)C^m}{2r - 3C^m}(< 0)$$

and

$$A_1^m = \frac{(a_1 + B^m)^2 + 2B^m(a_2 + B^m)}{4r}, \quad A_2^m = \frac{(a_2 + B^m)^2 + 2B^m(a_1 + B^m)}{4r}.$$

It is easy to see when $c = 0$,

$$C^m(c=0) = 0, \quad B^m(c=0) = 0, \quad A_i^m(c=0) = \frac{a_i^2}{4r}.$$

So

$$F(0) = 0 + V_i(z;0) = A_i^m(c=0) = \frac{a_i^2}{4r} > \frac{a_i^2 - (a - \delta(\bar{z}))^2}{4} = g(0).$$

By continuity, in the small neighborhood of $c = 0$, the above inequality (68) holds always. Furthermore, when c is sufficiently large, we have

$$\lim_{c\to\infty} C^m = -\infty,$$

$$\lim_{c\to\infty} B^m = \lim_{c\to\infty} \frac{(a_1+a_2)C^m}{2r - 3C^m} = \lim_{c\to\infty} \frac{(a_1+a_2)\frac{\partial C^m}{\partial c}}{-3\frac{\partial C^m}{\partial c}} = -\frac{a_1+a_2}{3}$$

by l'Hospital's rule, and

$$\lim_{c\to\infty} A_i^m = \frac{1}{4r}\left[a_i^2 - \frac{(a_1+a_2)^2}{3}\right].$$

So

$$\lim_{c\to\infty} F(c) = \lim_{c\to\infty} \bar{z}^2\left(c + \frac{rC^m}{2}\right) + \lim_{c\to\infty} r(B^m\bar{z} + A_i^m)$$

where the last term is finite as shown above, while the first term is

$$\lim_{c\to\infty}\left[c - \frac{r}{2}\frac{r - \sqrt{r^2+12c}}{3}\right] = \lim_{c\to\infty}\sqrt{c}\left[\sqrt{c} - \frac{r\sqrt{r^2/c+12}}{6}\right] + \frac{r^2}{6} = +\infty.$$

Thus,

$$\lim_{c\to\infty} F(c) = +\infty. \tag{69}$$

In other words, when c sufficiently large, inequality (68) holds as well. The remain is to check $c \in (0,\infty)$.

It is easy to check

$$\frac{\partial C^m}{\partial c} = -\frac{2}{\sqrt{r^2+12c}} < 0, \quad \frac{\partial B^m}{\partial c} = \frac{(a_1+a_2)2r}{(2r-3C^m)^2}\frac{\partial C^m}{\partial c} < 0$$

and
$$\frac{\partial A_i^m}{\partial c} = \frac{1}{2r}(a_1 + a_2 + 3B^m)\frac{\partial B^m}{\partial c} = \frac{a_1 + a_2}{2r - 3C^m}\ \frac{\partial B^m}{\partial c} < 0.$$

Thus,
$$\frac{\partial V_i}{\partial c} = \frac{\partial A_i^m}{\partial c} + \overline{z}\ \frac{\partial B^m}{\partial c} + \frac{\overline{z}^2}{2}\ \frac{\partial C^m}{\partial c} < 0.$$

Furthermore,

$$\frac{dF(c)}{dc} = z^2 + r\frac{\partial V_i}{\partial c} = r\left[\frac{\partial A_i^m}{\partial c} + \overline{z}\ \frac{\partial B^m}{\partial c}\right] + \overline{z}^2\left[1 - \frac{r}{\sqrt{r^2 + 12c}}\right]$$

where the first term is negative and the second term is positive $\forall c > 0$.

Additionally, evaluate the above differential at $c = 0$, it follows

$$\frac{dF(0)}{dc} = r\left[\frac{\partial A_i^m}{\partial c} + \overline{z}\ \frac{\partial B^m}{\partial c}\right]|_{c=0} < 0.$$

Given (69), there must exist $c_{min} \in (0, \infty)$ such that,

$$\frac{dF(c_{min})}{dc} = 0$$

and
$$\frac{dF(c)}{dc}\begin{cases} < 0 \text{ if } c < c_{min}, \\ > 0 \text{ if } c > c_{min}. \end{cases}$$

In other words, $F(c)$ is strictly convex in term of c with minimum value at c_{min}.

Two cases appear: (1) if at c_{min}, $F(c_{min}) > g(c_{min})$, then for any $c \geq 0$, we have $F(c) > g(c)$. In this case, we define $\overline{C} = 0$.

(2) if at c_{min}, $F(c_{min}) < g(c_{min})$, then there must exist c_l and c_h such that,

$$c_l < c_{min} < c_h, \qquad F(c_i) = g(c_i), \quad i = l, h,$$

and
$$F(c) > g(c) \text{ if } c \in [0, c_l) \cup (c_h, \infty) \text{ and } F(c) < g(c) \text{ if } c \in (c_l, c_h).$$

In this case, we define
$$\overline{C} = c_h.$$

Then for any $c > \overline{C}$, it follows

$$F(c) > g(c).$$

Step 2. Proof of the statement of Corollary 1. We first prove (30). Let

$$G\left(c\right) = a_i^2 - 4rV_i\left(\bar{z}\right) - \left(a - \delta\left(\bar{z}\right)\right)^2 - 4c\bar{z}^2.$$

By (20) and (17),

$$\begin{aligned} a_i^2 - 4rV_i\left(\bar{z}\right) &= a_i^2 - 4r\left(A_i^m + B^m\bar{z} + \frac{C^m}{2}\bar{z}^2\right) \\ &= -3\left(B^m\right)^2 - 4\left[B^m\left(a + r\bar{z}\right) + \frac{C^m}{2}r\bar{z}^2\right]. \end{aligned}$$

Hence,

$$\begin{aligned} G\left(c\right) &= -3\left(B^m\right)^2 - 4\left[B^m\left(a + r\bar{z}\right) + \frac{C^m}{2}r\bar{z}^2\right] - 4c\bar{z}^2 - \left(a - \delta\left(\bar{z}\right)\right)^2 \\ &= -3\left(B^m\right)^2 - 4B^m\left(a + r\bar{z}\right) - 2\bar{z}^2\left(C^m r + 2c\right) - \left(a - \delta\left(\bar{z}\right)\right)^2. \end{aligned}$$

Observe from (16) that

$$\begin{aligned} B^m &= \frac{2aC^m}{2r - 3C^m} = \frac{2a}{3}\frac{r - \sqrt{r^2 + 12c}}{r + \sqrt{r + 12c}} = -\frac{8ac}{\left(r + \sqrt{r^2 + 12c}\right)^2}, \\ C^m r + 2c &= \frac{r}{3}\left(r - \sqrt{r^2 + 12c}\right) + 2c = \frac{-4rc}{r + \sqrt{r^2 + 12c}} + 2c \\ &= \frac{2c\left(-r + \sqrt{r^2 + 12c}\right)}{r + \sqrt{r^2 + 12c}} = \frac{24c^2}{\left(r + \sqrt{r^2 + 12c}\right)^2} = -\frac{3c}{a}B^m. \end{aligned}$$

Therefore,

$$G\left(c\right) = -B^m\left[3B^m + 4\left(a + r\bar{z}\right) - \frac{6c}{a}\bar{z}^2\right] - \left(a - \delta\left(\bar{z}\right)\right)^2. \tag{70}$$

As shown above, $G\left(c\right) < 0$ for $c > \bar{C}$. Thus, (30) follows if we can show

$$G\left(\frac{ra}{2\bar{z}}\right) > 0. \tag{71}$$

Substitute $\hat{c} = ra/(2\bar{z})$ for c in (70), we obtain

$$\begin{aligned} G\left(\frac{ra}{2\bar{z}}\right) &= -B^m\left(\frac{ra}{2\bar{z}}\right)\left[3B^m\left(\frac{ra}{2\bar{z}}\right) + 4a + r\bar{z}\right] - (a - \delta(\bar{z}))^2 \\ &= \frac{4a^2r}{\bar{z}\left(r + \sqrt{r^2 + 12\hat{c}}\right)^2}\left[4a + r\bar{z} - \frac{12a^2r}{\bar{z}\left(r + \sqrt{r^2 + 12\hat{c}}\right)^2}\right] - (a - \delta(\bar{z}))^2. \end{aligned}$$

Let

$$U = \frac{4ar}{\bar{z}\left(r + \sqrt{r^2 + 12\hat{c}}\right)^2}, \qquad W = 4 + \frac{r\bar{z}}{a}. \tag{72}$$

Then

$$G\left(\frac{ra}{2\bar{z}}\right) = a^2U\left[W - 3U\right] - (a - \delta(\bar{z}))^2. \tag{73}$$

We show that

$$U\left[W - 3U\right] > 1.$$

This is equivalent to showing

$$3U^2 - UW + 1 < 0.$$

The above inequality holds if and only if

$$W - \sqrt{W^2 - 12} < 6U < W + \sqrt{W^2 - 12}. \tag{74}$$

By (72) we find

$$6U = \frac{12ar}{\bar{z}\left[r^2 + r\sqrt{r^2 + 12\hat{c}} + 6\hat{c}\right]} = \frac{12a}{\bar{z}r + \bar{z}\sqrt{r^2 + 12\hat{c}} + 3a} < 4. \tag{75}$$

On the other hand,

$$W + \sqrt{W^2 - 12} = 4 + \frac{r\bar{z}}{a} + \sqrt{4 + 8\frac{r\bar{z}}{a} + \left(\frac{r\bar{z}}{a}\right)^2} > 4.$$

The second inequality in (74) follows. Furthermore, by calculation,

$$\begin{aligned} W - \sqrt{W^2 - 12} &= \frac{12}{W + \sqrt{W^2 - 12}} = \frac{12}{4 + \bar{z}r/a + \sqrt{4 + 8r\bar{z}/a + (r\bar{z}/a)^2}} \\ &= \frac{12a}{4a + \bar{z}r + \bar{z}\sqrt{4a^2/\bar{z}^2 + 8ar/\bar{z} + r^2}}. \end{aligned}$$

Since $12\hat{c} = 6ar/\bar{z}$, it follows that

$$\sqrt{4a^2/\bar{z}^2 + 8ar/\bar{z} + r^2} > \sqrt{12\hat{c} + r^2}.$$

Hence

$$W - \sqrt{W^2 - 12} < \frac{12a}{4a + \bar{z}r + \bar{z}\sqrt{r^2 + 12\hat{c}}} < 6U$$

This proves the first inequality in (74).

From (73) and (74) we find

$$G\left(\frac{ra}{2\bar{z}}\right) > a^2 - \left(a - \delta\left(\bar{z}\right)\right)^2 \geq 0.$$

Hence

$$\frac{ra}{2\bar{z}} < \bar{C}.$$

Suppose $c < \bar{C}$. Then, either

$$c < \frac{1}{3\bar{z}}\left(r + \delta'\left(\bar{z}\right)\right)\left(a - \delta\left(\bar{z}\right)\right),$$

or

$$\frac{1}{3\bar{z}}\left(r + \delta'\left(\bar{z}\right)\right)\left(a - \delta\left(\bar{z}\right)\right) \leq c < \frac{1}{4\bar{z}^2}\left[a_i^2 - 4rV_i\left(\bar{z}\right) - \left(a - \delta\left(\bar{z}\right)\right)^2\right].$$

In the former case, (27) holds. So $\bar{z}$ is reached from some z_0 by Proposition 5. In the latter case, (24) and the reversed inequality in (26) both hold. The same conclusion follows from Part 2 of Proposition 4.

Conversely, if $c \geq \bar{C}$, then (24) and (26) both hold. By Part 2 of Proposition 4, $\bar{z}$ is never reached.

That completes the proof of Corollary 1.

A.6 Proof of Proposition 6

Part 1. Suppose $a \leq \delta(z)$. By (37), $\sigma_1 = \sigma_2 = -1$ and $\mu \leq \Delta_i$. Hence, the right-hand side of (39) is increasing in μ and is no more than its value at $\mu = \min\{\Delta_1, \Delta_2\}$. Hence,

$$f(\bar{z}) \leq -(a - \delta(\bar{z})) - \frac{1}{2}\left\{2|a - \delta(z)| + \sqrt{4(a - \delta(\bar{z}))^2 + |\Delta_1 - \Delta_2|}\right\} \leq 0.$$

This proves that the first root of $f(z)$ is less than or equal to $\bar{z}$. Hence $\bar{z}$ is never reached.

Part 2. Suppose $a > \delta(\bar{z})$ and (41) holds. We show that $f(\bar{z}) \leq 0$. If $\sigma_i < 0$ for either $i = 1$ or 2, then

$$\begin{aligned} f(\bar{z}) &= -(a - \delta(\bar{z})) + \frac{\sigma_j}{2}\sqrt{4(a - \delta(\bar{z}))^2 + \Delta_j - \mu} - \frac{1}{2}\sqrt{4(a - \delta(\bar{z}))^2 + \Delta_i - \mu} \\ &= \frac{1}{2}P_j(\bar{z}) - \frac{1}{2}\sqrt{4(a - \delta(\bar{z}))^2 + \Delta_i - \mu} \leq 0. \end{aligned}$$

Hence, $f(\bar{z}) > 0$ can occur only if $\sigma_1 = \sigma_2 = 1$. Suppose by contradiction that $f(\bar{z}) > 0$. Then

$$\sqrt{4(a - \delta(\bar{z}))^2 + \Delta_1 - \mu} + \sqrt{4(a - \delta(\bar{z}))^2 + \Delta_2 - \mu} > 2(a - \delta(\bar{z})).$$

By the concavity of the square root function,

$$\sqrt{4(a - \delta(\bar{z}))^2 + \Delta_1 - \mu} + \sqrt{4(a - \delta(\bar{z}))^2 + \Delta_2 - \mu} < 2\sqrt{4(a - \delta(\bar{z}))^2 + \frac{\Delta_1 + \Delta_2}{2} - \mu}.$$

Hence

$$\sqrt{4(a - \delta(\bar{z}))^2 + \frac{\Delta_1 + \Delta_2}{2} - \mu} > a - \delta(\bar{z}).$$

This leads to

$$\mu < 3(a - \delta(\bar{z}))^2 + \frac{\Delta_1 + \Delta_2}{2}.$$

On the other hand, since $P_i \leq 0$, it follows that

$$\sqrt{4(a - \delta(\bar{z}))^2 + \Delta_i - \mu} \leq 2(a - \delta(\bar{z})).$$

Hence $\mu \geq \Delta_i$ for $i = 1, 2$. Therefore,

$$\mu \geq \max\{\Delta_1, \Delta_2\}.$$

Hence, if (41) holds, since

$$\frac{\Delta_1+\Delta_2}{2}=\frac{1}{2}\left[\max\left\{\Delta_1,\Delta_2\right\}+\min\left\{\Delta_1,\Delta_2\right\}\right],$$

it follows that

$$\begin{aligned}\max\left\{\Delta_1,\Delta_2\right\} &\geq 6\left(a-\delta\left(\bar{z}\right)\right)^2+\min\left\{\Delta_1,\Delta_2\right\}\\ &= 6\left(a-\delta\left(\bar{z}\right)\right)^2+\Delta_1+\Delta_2-\max\left\{\Delta_1,\Delta_2\right\}.\end{aligned}$$

This leads to

$$\max\left\{\Delta_1,\Delta_2\right\}\geq 3\left(a-\delta\left(\bar{z}\right)\right)^2+\frac{\Delta_1+\Delta_2}{2}.$$

Hence, no such μ exists. Therefore, $f\left(\bar{z}\right)\leq 0$.

Part 3. We show that $f\left(\bar{z}\right)\leq 0$ if (43) holds. Suppose by contradiction that $f\left(\bar{z}\right)>0$. As proven in Part 2, it is necessary that

$$\max\left\{\Delta_1,\Delta_2\right\}\leq\mu\leq 3\left(a-\delta\left(\bar{z}\right)\right)^2+\frac{\Delta_1+\Delta_2}{2}.$$

Consider the both sides of (38). At $\mu=\max\left\{\Delta_1,\Delta_2\right\}$, the right-hand side is zero, and by (43), the left-hand side is

$$\max\left\{\Delta_1,\Delta_2\right\}>0.$$

At

$$\mu=3\left(a-\delta\left(\bar{z}\right)\right)^2+\frac{\Delta_1+\Delta_2}{2}$$

the right-hand side is

$$\begin{aligned}&2\left\{2\left(a-\delta\left(\bar{z}\right)\right)-\sqrt{\left(a-\delta\left(\bar{z}\right)\right)^2+\frac{\Delta_1-\Delta_2}{2}}\right\}\left\{2\left(a-\delta\left(\bar{z}\right)\right)-\sqrt{\left(a-\delta\left(\bar{z}\right)\right)^2+\frac{\Delta_2-\Delta_1}{2}}\right\}\\ \leq\;& 2\left(a-\delta\left(\bar{z}\right)\right)\left[2\left(a-\delta\left(\bar{z}\right)\right)-\sqrt{\left(a-\delta\left(\bar{z}\right)\right)^2-\frac{\left|\Delta_1-\Delta_2\right|}{2}}\right]<2\left(2-\sqrt{4-2\sqrt{3}}\right)\left(a-\delta\left(\bar{z}\right)\right)^2\\ =\;& 2\left(3-\sqrt{3}\right)\left(a-\delta\left(\bar{z}\right)\right)^2.\end{aligned}$$

On the other hand, the left-hand side is

$$
\begin{aligned}
3\left(a-\delta\left(\bar{z}\right)\right)^{2}+\frac{\Delta_{1}+\Delta_{2}}{2} &\geq 3\left(a-\delta\left(\bar{z}\right)\right)^{2}+\max\left\{\Delta_{1},\Delta_{2}\right\}-\frac{\left|\Delta_{1}-\Delta_{2}\right|}{2} \\
&\geq 2\left(3-\sqrt{3}\right)\left(a-\delta\left(\bar{z}\right)\right)^{2}
\end{aligned}
$$

Furthermore, the right-hand side has the derivative with respect to μ as

$$
\frac{2\mu-8\left(a-\delta\left(\bar{z}\right)\right)^{2}-\Delta_{1}-\Delta_{2}}{\prod_{i=1}^{2}\sqrt{4\left(a-\delta\left(\bar{z}\right)\right)^{2}+\Delta_{i}-\mu}}+\sum_{i=1}^{2}\frac{2\left(a-\delta\left(\bar{z}\right)\right)}{\sqrt{4\left(a-\delta\left(\bar{z}\right)\right)^{2}+\Delta_{i}-\mu}}
$$

which is increasing in μ. Hence, the left-hand side of (38) is strictly greater than the right-hand side. Therefore, (38) has no solution. This proves that $f\left(\bar{z}\right)\leq 0$. So $\bar{z}$ is never reached.

In the case where $\max\left\{\Delta_{1},\Delta_{2}\right\}\leq 0$, then the right-hand side of (38) at $\mu=0$ is

$$
2\prod_{i=1}^{2}\left\{2\left(a-\delta\left(\bar{z}\right)\right)-\sqrt{4\left(a-\delta\left(\bar{z}\right)\right)^{2}+\Delta_{i}}\right\}\geq 0.
$$

and the left-hand side is zero. As shown above, the rigth-hand side of (38) is less than its left-hand side. Therefore, there is a solution to (38). At this point $f\left(\bar{z}\right)>0$. Hence, $\bar{z}$ is reached from some z_{0}.

This completes the proof. □

A.7 Proof of Proposition 7

We first solve the differential game.

Period II, $z>\bar{z}$. Since $0\leq\lambda\leq\min\left\{c_{1},c_{2}\right\}=0$, by (32)

$$
C_{1}=0,\qquad C_{2}=r-\sqrt{r^{2}+8c}
$$

As a result,

$$B_1 = 0, \qquad A_1 = \frac{a_1^2}{4r}, \qquad B_2 = \frac{2aC_2}{2r - C_2}, \qquad A_2 = \frac{a_2^2 + B_2^2}{4r} + \frac{aB_2}{r}.$$

Also, the dynamics is governed by

$$\dot{z} = \frac{a_1}{2} + \frac{1}{2}\left[a_2 + B_2 + C_2 z\right] = a + \frac{1}{2}\left(B_2 + C_2 z\right).$$

The equilibrium is

$$z_e^* = -\frac{2a + B_2}{C_2} = -\frac{4ar}{C_2\left(2r - C_2\right)} = \frac{ar}{2c}. \tag{76}$$

Period I, $z < \bar{z}$**.** Let us still assume that V_1 is the constant $a_1^2/\left(4r\right)$. Then $P_1 = 0$. By (52),

$$\left[\frac{P_2\left(z\right)}{2} + a - \delta\left(z\right)\right] P_2'\left(z\right) = \left(r - \beta\right) P_2\left(z\right) + 4cz. \tag{77}$$

In addition, $P_2\left(\bar{z}\right)$ satisfies (36) with $i = 2$, $c_2 = c/2$, and $P_1 = 0$. That is,

$$r\bar{V}_2 = \frac{1}{4}\left(a_2 + P_2\left(\bar{z}\right)\right)^2 + \frac{1}{2}P_2\left(\bar{z}\right)\left[a_1 - 2\delta\left(\bar{z}\right)\right] - 2c\bar{z}^2.$$

There are two roots,

$$P_2\left(\bar{z}\right) = -2\left(a - \delta\left(\bar{z}\right)\right) \pm \sqrt{4\left(a - \delta\left(\bar{z}\right)\right)^2 + \Delta_2^e}$$

where

$$\Delta_2^e = 4\left(rV_2\left(\bar{z}\right)\right) + 2c\bar{z}^2 - a_2^2.$$

We choose the one so that $P_2'\left(\bar{z}\right)$ is minimum. Substituting the above into (77), we get

$$\begin{aligned} & \pm\frac{1}{2}\sqrt{4\left(a - \delta\left(\bar{z}\right)\right)^2 + \Delta_2^e} P_2'\left(\bar{z}\right) \\ = \quad & \left(r + \delta'\left(\bar{z}\right)\right)\left\{-2\left(a - \delta\left(\bar{z}\right)\right) \pm \sqrt{4\left(a - \delta\left(\bar{z}\right)\right)^2 + \Delta_2^e}\right\} + 4c\bar{z}. \end{aligned}$$

Hence,

$$\frac{1}{2}P_2'\left(\bar{z}\right) = r + \delta'\left(\bar{z}\right) + \frac{-2\left(a - \delta\left(\bar{z}\right)\right)\left(r + \delta'\left(\bar{z}\right)\right) + 4c\bar{z}}{\pm\sqrt{4\left(a - \delta\left(\bar{z}\right)\right)^2 + \Delta_2^e}}.$$

The sign is positive if

$$\frac{-2\left(a-\delta\left(\bar{z}\right)\right)\left(r+\delta'\left(\bar{z}\right)\right)+4c\bar{z}}{\sqrt{4\left(a-\delta\left(\bar{z}\right)\right)^{2}+\Delta_{2}^{e}}}<\frac{-2\left(a-\delta\left(\bar{z}\right)\right)\left(r+\delta'\left(\bar{z}\right)\right)+4c\bar{z}}{-\sqrt{4\left(a-\delta\left(\bar{z}\right)\right)^{2}+\Delta_{2}^{e}}},$$

which is equivalent to

$$\left(r+\delta'\left(\bar{z}\right)\right)\left(a-\delta\left(\bar{z}\right)\right)>2c\bar{z},$$

which is condition 44 in Proposition 7.

We can finally solve the equation in the case where δ is linear. The equation is similar to (57) below except the coefficient of z is $2c$. Let $Q_e\left(z\right)=P_2\left(z\right)/2+a-\delta\left(z\right)$. Then

$$\left|\frac{Q_e\left(z\right)-u_e^-\left(z+a_e/b_e\right)}{Q_e\left(\bar{z}\right)-u_e^-\left(\bar{z}+a_e/b_e\right)}\right|^{p_e}\left|\frac{Q_e\left(z\right)-u_e^+\left(z+a_e/b_e\right)}{Q_e\left(\bar{z}\right)-u_e^+\left(\bar{z}+a_e/b_e\right)}\right|^{1-p_e}=1,$$

where

$$\begin{aligned}
a_e &= \left(\beta-r\right)\left(a-\alpha\right), \qquad b_e=\beta\left(\beta-r\right)+2c,\\
u_e^- &= r-\sqrt{r^2+4b_e}, \qquad u_e^+=r+\sqrt{r^2+4b_e}, \qquad p_e=-\frac{u_e^-}{u_e^+-u_e^-},\\
Q_e\left(\bar{z}\right) &= P_2\left(\bar{z}\right)/2+a-\delta\left(\bar{z}\right).
\end{aligned}$$

Note that $Q_e\left(z\right)=f_e\left(z\right)$. Hence z_e' satisfies the equation

$$\left|\frac{u_e^-\left(z_e'+a_e/b_e\right)}{Q_e\left(\bar{z}\right)-u_e^-\left(\bar{z}+a_e/b_e\right)}\right|^{p_e}\left|\frac{u_e^+\left(z'+a_e/b_e\right)}{Q_c\left(\bar{z}\right)-u_e^+\left(\bar{z}+a_e/b_e\right)}\right|^{1-p_e}=1.$$

It follows that

$$z_e'=-\frac{a_e}{b_e}+\left|\frac{u_e^-}{Q_e\left(\bar{z}\right)-u_e^-\left(\bar{z}+a_e/b_e\right)}\right|^{-p_e}\left|\frac{u_e^+}{Q_c\left(\bar{z}\right)-u_e^+\left(\bar{z}+a_e/b_e\right)}\right|^{p_e-1}.$$

We now prove the reachability results stated in the proposition. In view of $P_1=0$ and $c_2=2c$, Eq. (36) with $i=2$ has the form

$$rV_2\left(\bar{z}\right)=\frac{1}{4}\left(a_2+P_2\left(\bar{z}\right)\right)^2+\frac{1}{2}P_2\left(\bar{z}\right)\left[a_1-2\delta\left(\bar{z}\right)\right]-2c\bar{z}^2.$$

Substituting $f_e(\bar{z}) = P_2(\bar{z})/2 + a - \delta(\bar{z}^-)$ into the above equation, we find

$$4\left[rV_2(\bar{z}) + 2c\bar{z}^2\right] = \left(f_e(\bar{z}) - a + \delta + \frac{a_2}{2}\right)^2 + \left(f_e(\bar{z}) - a + \delta(\bar{z})\right)\left(a_1 - 2\delta(\bar{z})\right).$$

It is simplified to

$$f_e(\bar{z})^2 = 4\left[rV_2(\bar{z}) + 2c\bar{z}^2\right] + \left(a - \delta(\bar{z})\right)^2 - \frac{a_2^2}{4}.$$

So

$$f_e(\bar{z}) = -\sqrt{4\left[rV_2(\bar{z}) + 2c\bar{z}^2\right] + \left(a - \delta(\bar{z})\right)^2 - \frac{a_2^2}{4}}$$

if (44) does not hold, and

$$f_c(\bar{z}) = \sqrt{4\left[rV_2(\bar{z}) + 2c\bar{z}^2\right] + \left(a - \delta(\bar{z})\right)^2 - \frac{a_2^2}{4}}$$

if (44) holds. In the former case, $f_e(\bar{z}) \leq 0$. Hence, the first root of $f_e(z)$ is less than or equal to $\bar{z}$. Therefore, $\bar{z}$ is never reached and $z(t)$ converges to z_e' as $t \to \infty$. In the latter case, $f_e(\bar{z}) > 0$. It remains positive for some $z < \bar{z}$ and near $\bar{z}$ by continuity. Therefore $f_e(z) > 0$ for $z_0 \leq z \leq \bar{z}$ if z_0 is sufficiently close to $\bar{z}$.

We show that $f_e(z) > 0$ for $z \leq \bar{z}$ if δ is linear and (44) and (45) both hold. Subsitituting $P_2(z) = 2\left(f_e(z) - a + \delta(z)\right)$ in (77), we find

$$2f_e(z)\left[f_e'(z) - \beta\right] = 2(r-\beta)\left[f_e(z) - a + \delta(z)\right] + 4cz \qquad \text{for } z < \bar{z}.$$

The equation can be written as

$$\frac{1}{2}\frac{d}{dz}\left[f_e(z)\right]^2 = rf_e(z) - (r-\beta)\left(a - \delta(z)\right) + 2cz.$$

If there is $\hat{z} < \bar{z}$ such that $f_e(\hat{z}) = 0$. Then, $\left[f_e(z)\right]^2$ has a local minimum at $\hat{z}$, Therefore, the left-hand side of the above equation is zero. This leads to

$$(r-\beta)\left(a - \delta(\hat{z})\right) = 2c\hat{z}.$$

However, since both $(r-\beta)\left(a - \delta(z)\right)$ and $2cz$ are linear functions of z, and by (44) and

(45), the former is larger than the latter for $z_0 \leq z \leq \bar{z}$. Therefore

$$(r-\beta)(a-\delta(\hat{z})) > 2c\hat{z}.$$

This is a contradiction. Hence $f_e(z) > 0$ for any z between z_0 and $\bar{z}$.

This completes the proof. □

A.8 Proof of Proposition 8

Proof. It is clear from (4) and (76) that $z_c^* < z_e^*$. To show that $z_e^* \leq z_a^*$, we observe that since $\lambda > 0$, it follows that

$$r\left[r(4c-\lambda)+(4c-3\lambda)\sqrt{r^2+4(2c-\lambda)}\right] < 4c\left[\lambda+r^2+r\sqrt{r^2+4(2c-\lambda)}\right].$$

Hence

$$z_e^* = \frac{ar}{2c} < \frac{2a\left[\lambda+r^2+r\sqrt{r^2+4(2c-\lambda)}\right]}{r(4c-\lambda)+(4c-3\lambda)\sqrt{r^2+4(2c-\lambda)}} = z^*.$$

It remains to show that $z_a^* \leq z_s^*$.

We first show that z_a^* is increasing in λ for $0 \leq \lambda \leq c$. By differentiation, we find

$$\frac{dz_a^*}{d\lambda} = \frac{4a\left[2r\left(r^2+7c-3\lambda\right)\sqrt{r^2+4(2c-\lambda)}+2r^4+2(9c-5\lambda)r^2+16c^2-4\lambda c-3\lambda^2\right]}{\sqrt{r^2+4(2c-\lambda)}\left[r(4c-\lambda)+(4c-3\lambda)\sqrt{r^2+4(2c-\lambda)}\right]^2}.$$

Since $\lambda \leq \min\{c_1, c_2\} \leq c$, it follows that

$$2r\left(r^2+7c-3\lambda\right)\sqrt{r^2+4(2c-\lambda)} \geq 2r^4+2r^2c.$$

Thus, the quantity between brackets in the numerator is no less than

$$4r^4+2(11c-5\lambda)r^2+16c^2-4\lambda c-3\lambda^2 \geq 13c^2 > 0.$$

Hence, z_a^* is increasing in λ for $0 \leq \lambda \leq c$.

We next show the λ is decreasing in $|c_1 - c_2|$. Let $\mu = |c_1 - c_2|/2$. Suppose $c_i \leq c_j$. Then

$c_i = c - \mu$ and $c_j = c + \mu$. By (33),

$$2\lambda = \left(r - \sqrt{r^2 + 4\left(c - \mu - \lambda\right)}\right)\left(r - \sqrt{r^2 + 4\left(c + \mu - \lambda\right)}\right).$$

Regarding λ as a function of μ and differentiating the two sides with respect to μ, we obtain

$$\lambda'\left(\mu\right) = \left(1 + \lambda'\left(u\right)\right)\frac{r - \sqrt{r^2 + 4\left(c + \mu - \lambda\right)}}{\sqrt{r^2 + 4\left(c - \mu - \lambda\right)}} - \left(1 - \lambda'\left(u\right)\right)\frac{r - \sqrt{r^2 + 4\left(c - \mu - \lambda\right)}}{\sqrt{r^2 + 4\left(c + \mu - \lambda\right)}}.$$

The equation can be written as

$$\xi\lambda'\left(\mu\right) = \eta$$

where

$$\begin{aligned} \xi &= 1 - \frac{r - \sqrt{r^2 + 4\left(c + \mu - \lambda\right)}}{\sqrt{r^2 + 4\left(c - \mu - \lambda\right)}} - \frac{r - \sqrt{r^2 + 4\left(c - \mu - \lambda\right)}}{\sqrt{r^2 + 4\left(c + \mu - \lambda\right)}}, \\ \eta &= \frac{r - \sqrt{r^2 + 4\left(c + \mu - \lambda\right)}}{\sqrt{r^2 + 4\left(c - \mu - \lambda\right)}} - \frac{r - \sqrt{r^2 + 4\left(c - \mu - \lambda\right)}}{\sqrt{r^2 + 4\left(c + \mu - \lambda\right)}}. \end{aligned}$$

Note that

$$\frac{r - \sqrt{r^2 + 4\left(c \pm \mu - \lambda\right)}}{\sqrt{r^2 + 4\left(c \mp \mu - \lambda\right)}} = \frac{-4\left(c \pm \mu - \lambda\right)}{\left(r + \sqrt{r^2 + 4\left(c \pm \mu - \lambda\right)}\right)\sqrt{r^2 + 4\left(c \mp \mu - \lambda\right)}} \leq 0,$$

for $\lambda \leq \min\left\{c_1, c_2\right\}$. Hence $\xi > 0$. On the other hand, since $\mu \geq 0$, it follows that

$$\sqrt{r^2 + 4\left(c + \mu - \lambda\right)} \geq \sqrt{r^2 + 4\left(c - \mu - \lambda\right)}.$$

Therefore

$$\frac{r - \sqrt{r^2 + 4\left(c + \mu - \lambda\right)}}{\sqrt{r^2 + 4\left(c - \mu - \lambda\right)}} \leq \frac{r - \sqrt{r^2 + 4\left(c - \mu - \lambda\right)}}{\sqrt{r^2 + 4\left(c + \mu - \lambda\right)}}.$$

This implies that $\eta \leq 0$. Hence,

$$\lambda'\left(\mu\right) = \eta/\xi \leq 0.$$

So, λ is decreasing in μ. As a result,

$$z_a^* = z_a^*\left(\lambda\left(\mu\right)\right) \leq z_a^*\left(\lambda\left(0\right)\right) = z_s^*.$$

This completes the proof. $\square$

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