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Optimal Monetary Policy Rules in the Fiscal Theory of the Price Level *

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Abstract

In the fiscal theory of the price level, inflation and debt dynamics are determined jointly. We derive optimal monetary policy rules that can approximate the Ramsey outcome in this environment. When the government issues a portfolio of bonds of different maturities and buys it back every period the optimal interest rate response to inflation is a simple, transparent function of the average debt maturity. This policy exploits the maturity structure to minimize the intertemporal variability of inflation in response to fiscal shocks. We then turn to the more realistic scenario of a government that does not repurchase and reissue debt in every period. In the case where debt is only long term, the optimal policy equilibrium features oscillations in inflation and simple interest rate rules may lead to explosive inflation dynamics. Issuing both short and long bonds rules out oscillations and implies that simple inflation targeting rules can approximate the Ramsey outcome. Under no repurchases a flat maturity structure of debt is optimal to reduce inflation variability.

Keywords: Fiscal Theory, Optimal Interest Rates, Government Debt Maturity, Ramsey policy.

JEL: E31, E52, E58, E62, C11

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1 Introduction

A considerable literature has analyzed optimal monetary policy in environments where inflation is used to stabilize government debt dynamics. Chari and Kehoe (1999) first considered this in the context of Ramsey optimal policy assuming an economy with flexible prices where debt is issued in a short term bond. Their approach has been subsequently extended to sticky price models (Schmitt-Grohé and Uribe, 2004 and Siu, 2004) and to models in which debt can be both short and long term (e.g. Lustig et al., 2008; Faraglia et al., 2013; Sims, 2013; Leeper and Zhou, 2021 among others).

In these models optimal policy is the solution to the first order conditions of the Ramsey program, which involves the Lagrange multipliers of current and past consolidated budget constraints, the objects that define the dynamics of debt. These multipliers are state variables; they summarize the impact of shocks that have hit the economy and thus also impacted debt. In the optimal policy equilibrium, macroeconomic variables (such as inflation and output) as well as the nominal interest rate become functions of the multipliers.

This approach to optimal policy has thus the limitation that it yields a path of the nominal rate, the main instrument of monetary policy, defined on the basis of variables that are not observed in practice. Even though Lagrange multipliers in these models can be expressed as functions of the histories of economic shocks, a policy instrument that depends explicitly on shocks is also not practical, in that it involves the non trivial task of identifying current and past shocks.

Practically relevant policy rules specify the path of the nominal rate as a function of macroeconomic variables (inflation, output etc). These are the types of rules that we estimate in medium scale DSGE models and have also been considered by a large number of papers studying optimal policy in the context of the baseline New Keynesian model (when inflation does not respond to the debt aggregate).[1] We would like to know whether in the context of a model where inflation and debt dynamics are determined jointly (in the fiscal theory of the price level framework) such rules can approximate the optimal policy and if so, what is the appropriate coefficient on inflation (or on other relevant macroeconomic variables) in the interest rate rule.

We use a simplistic framework which enables us to characterize optimal policy analytically and also allows us to experiment with various modelling assumptions regarding the structure of debt. Our model is a Fischerian New Keynesian economy, augmented with the consolidated budget constraint. To isolate our focus on the role of inflation in stabilizing debt, we assume that taxes are constant, assuming also that the government's surplus fluctuates according to an exogenous shock to spending. Our model is thus broadly similar to that of Cochrane (2001).

In Section 2 we begin by laying out our theoretical framework. Following the optimal policy literature cited above, we model long term bonds with full buybacks, that is assuming that debt is repurchased one period after issuance. We revisit Ramsey optimal policy in this framework since this is the benchmark by which the optimality of the interest rate rules that we will consider will be measured, and explaining the key forces that determine the path of inflation under Ramsey is important. The simplicity of our framework enables us to characterize the Ramsey solution analytically, under any maturity structure of debt.

Section 3 then turns to the analysis of optimal interest rate rules. Our first substantive finding is that commitment to a rule that sets the nominal interest rate only as a function of inflation is sufficient to approximate the Ramsey outcome very well and, under specific debt maturity structures, it even delivers effectively the same outcome as Ramsey policy. One does not need to include many lags of inflation (or other variables) in the policy rule, a property that seems surprising given the dependency of Ramsey policy on the history of Lagrange multipliers and shocks.

The second finding is that, over the broad range of alternative maturity structures of debt that we consider, the optimal inflation coefficient is given by a simple formula of the average debt maturity. In

[1] See for example Giannoni and Woodford (2003a,b); Giannoni (2014), among numerous others.

particular, the coefficient is $1-\frac{1}{\text{Maturity}}$. A higher maturity leads to a stronger reaction of the nominal interest rate to inflation.

To understand this, note first that, as in any other fiscal theory model, the inflation coefficient must be between 0 and 1 in order to have a unique stable equilibrium. This is obviously the case here. As is well known, in this type of environment, inflation becomes a backward looking process and raising the nominal rate will not accomplish a lower inflation rate, rather it will make inflation increase persistently. This is desirable when the average maturity of debt is long, since it enables to spread inflation over more periods and stabilize debt. In contrast, in the case where debt is short, making inflation respond over the longer term to fiscal shocks is wasteful, because it is only short term price growth that can contribute towards debt stability. Under the optimal policy, therefore, the inflation coefficient is zero when debt is only short term and it is strictly positive when both short and long bonds are issued. In the case of a flat maturity structure (equivalently debt is issued in a consol that pays fixed coupons) the coefficient becomes equal to one.

These findings turn out to hold independently of the types of shocks that hit the economy and can also be generalized to interest rate rules that respond to the output gap or to lagged values of the interest rate. The key driver of inflation dynamics in our model is the debt maturity structure and in Section 4 of the paper we consider an extension that departs significantly from the canonical modelling of long term bonds found in the literature.

All papers mentioned previously assume that the entire stock of long bonds is repurchased one period after issuance. This assumption is made for tractability (keeping track of many lags of debt in the state vector is not easy) however it is not in line with observed practices in the US and elsewhere. For example, Faraglia et al. (2019) provide extensive evidence that the US Treasury does not buy back their debt prior to maturity. The Quantitative easing program run by the Federal reserve since the 2008-9 recession can be seen as a partial buyback of long term government bonds.

When we consider *no buy back* as the modelling assumption for long term debt we find strikingly different implications for optimal policy. Under no debt repurchases and when debt is a zero coupon bond of maturity N, optimal inflation features oscillations of periodicity N which persist forever. Simple policy rules that specify the nominal rate as a function of current inflation will not work; these types of rules do not lead to stable equilibria, even when the inflation coefficients are between 0 and 1.

To flex out the intuition behind the first result (that oscillations occur in this model) we provide a simple analytical example assuming $N=2$. A negative shock to the surplus in period t will need to be compensated with higher inflation to reduce the real payout of debt that matures in t, that is debt that has been issued in $t-2$. However, higher period t inflation will also impact the real value of debt that has not matured, debt issued in $t-1$. This impact will destabilize the intertemporal debt constraint in period $t+1$ (when this debt has to be redeemed) and for the constraint to hold it must be that inflation drops in $t+1$. These effects persist indefinitely.

A simple policy rule cannot mitigate the N cycle as it does not pin down a unique stable equilibrium path. Instability is a worse outcome; arbitrary inflation oscillations may occur and their magnitude can grow over time. In the no buyback model a unique stable equilibrium can be reached when the policy rule pins down the growth of the price level between $t-N$ and t, in other words when it determines the sum of inflation rates $\hat{\pi}_t+\hat{\pi}_{t-1}+...+\hat{\pi}_{t-N+1}$ (in terms of the notation in our log linear model). The interest rate rules that can deliver this are highly impractical, featuring $N-1$ lags and leads of inflation to determine the sum.

Our final experiment in Section 4 explores whether under no buyback there are maturity structures that would restore the optimality of simple inflation targeting rules. We find that the answer is yes, the key condition is that the government not only issues long term debt but also positive amounts of short term bonds are being issued. In this case we once again obtain the formula $1-\frac{1}{\text{Maturity}}$. We also establish an equivalence result under buyback and no buyback in the case where long bonds are perpetuities that pay decaying or even constant coupons.

Section 5 shows the robustness of these findings to variants of our baseline model. Most notably we depart from the baseline framework where we have assumed that the policy objective is to minimize the volatility of inflation to consider cases where output and interest rate stabilization become part of the objective of the policy authority. Also in these cases we can obtain analytically policy rules with inflation coefficients that are simple functions of debt maturity and approximately deliver the Ramsey outcome. A final section concludes the paper.

Our paper is related to a vast literature of optimal policy models. First, numerous papers have studied optimal policy under commitment to an interest rate rule in the context of the baseline New Keynesian model, to identify simple rules that can approximate Ramsey outcomes. See Giannoni and Woodford (2003a,b); Giannoni (2014) among others. We apply the arguments of these papers to optimal policy in the fiscal theory of the price level framework.

Second, many papers have studied policy assuming that an optimizing government chooses taxes to finance debt in the context of real models (e.g. Lucas and Stokey, 1983; Aiyagari et al., 2002; Marcet and Scott, 2009; Faraglia et al., 2016 and others). Lucas and Stokey (1983) assume that the government can issue debt in state contingent instruments, whereas Aiyagari et al. (2002); Marcet and Scott (2009); Faraglia et al. (2016) assume 'incomplete markets' letting debt be issued only in a single bond, long or short term. Our optimal Ramsey policy framework also assumes incomplete markets, and thus our approach is methodologically similar to Aiyagari et al. (2002); Marcet and Scott (2009); Faraglia et al. (2016); however, we allow debt to be issued in multiple assets of different maturity. Importantly, Faraglia et al. (2016) consider a distinction between buyback and no buyback in their model showing that under no buyback the Ramsey solution features tax oscillations. Our result in Section 4 that inflation oscillations are unavoidable when repurchases of long term nominal bonds are ruled out, is rooted into their analysis.

Relatedly, a literature on public debt management using real models with distortionary taxes has explored how debt portfolios can be designed to ensure the intertemporal solvency of debt, absorbing shocks that hit the government budget. Angeletos (2002) and Buera and Nicolini (2004) both assuming that debt is repurchased, reach the conclusion that when governments focus on issuing long term debt, then taxes will not need to adjust to shocks to government spending.[2] Faraglia et al. (2019) show that this will not hold in a model where debt repurchases are ruled out. Under no buyback it becomes optimal to issue a mix of both short term and long term bonds. Our result that issuing positive amounts of short bonds removes inflation oscillations and restores the optimality of simple interest rate rules is inspired by their finding.

Finally, a related literature has studied optimal policy in the fiscal theory of the price level using a linear quadratic framework, as we do in this paper. Cochrane (2001) explores the optimal debt management policy under various modelling assumptions for long bonds, to investigate how alternative bond issuance strategies affect the path inflation in a Fischerian model with flexible prices. Benigno and Woodford (2007) study the optimal paths of inflation under various fiscal policy regimes (including the case where taxes are held constant). Most of their results are built on the assumption that debt is short term, though the authors also consider the case where debt maturity becomes a shock absorber, as in Angeletos (2002); Buera and Nicolini (2004). Leeper and Zhou (2021) solve a Ramsey problem assuming that debt is issued in a perpetuity and make progress with deriving analytical results. We use a simpler (Fischerian) setup which allows us to experiment with alternative assumptions regarding the debt structure, including the no buyback assumption whose implications we explore in Section 4. Our analytical formulae complement those of Leeper and Zhou (2021) and Benigno and Woodford (2007). The optimal interest rate rules that we derive are novel and also complement these papers.

[2] In the context of models of optimal inflation (e.g. Lustig et al., 2008; Sims, 2013; Leeper and Zhou, 2021) a similar prediction obtains, long term enables to reduce inflation's reaction to shocks.

2 Theoretical Framework and Optimal Ramsey Policy

Our baseline model is a standard Fischerian New Keynesian model augmented with a fiscal block, the consolidated budget constraint and taxes which can either be lump sum or distortionary (levied on labour income). For simplicity, we will assume that taxes are constant through time.[3] We will further assume that the surplus of the government fluctuates over time due to exogenous shocks to the spending level. Since taxes are constant, spending shocks can only be financed through changes in the inflation rate that adjust the real market value of government debt. The model is thus a standard laboratory of the fiscal theory, as in e.g. Cochrane (2001).

Since this is a well known setup, for brevity, we will define here the competitive equilibrium equations in log-linear form. In the appendix we describe the background non-linear model and derive the equations from the optimality conditions of the households' and firms' optimization problems.

We let $\hat{x}$ denote the log deviation of variable x from its steady state value, $\overline{x}$. The system of the competitive equilibrium equations is the following:

$$\hat{\pi}_t = \kappa_1 \hat{Y}_t + \beta E_t \hat{\pi}_{t+1}, \tag{1}$$

where $\kappa_1 \equiv -\frac{(1+\eta)\overline{Y}}{\theta}\gamma_h > 0$.

$$\sum_{k=1}^{\infty} \overline{p}_k \overline{b}_k (\hat{b}_{t,k} + \hat{p}_{t,k}) = -\overline{S}\hat{S}_t + \overline{b}_1(\hat{b}_{t-1,1} - \hat{\pi}_t) + \sum_{k=2}^{\infty} \overline{p}_k \overline{b}_k (\hat{b}_{t-1,k} + \hat{p}_{t,k-1} - \hat{\pi}_t) \tag{2}$$

where

$$\overline{S}\hat{S}_t = -\overline{G}\hat{G}_t + \overline{R}(1+\gamma_h)\hat{Y}_t$$

$$-\hat{p}_{t,1} = \hat{i}_t = E_t \hat{\pi}_{t+1} \tag{3}$$

$$\overline{p}_k \hat{p}_{t,k} = -\beta^k \sum_{l=1}^{k} E_t \hat{\pi}_{t+l} \tag{4}$$

(1) is the Phillips curve at the heart of our model. $\hat{\pi}_t$ represents inflation and $\hat{Y}_t$ is the output gap. Parameters $\eta < 0$ and $\theta > 0$ govern the elasticity of substitution across the differentiated (monopolistically competitive) goods produced in the economy and the degree of price stickiness respectively.[4] Parameter γ_h is the inverse of the Frisch elasticity of labor supply.

(2) is the consolidated budget constraint. The LHS of this equation represents the value of debt issued in period t. We assume that debt can be issued in bonds of maturities k where $k = 1, 2, ...$ and which pay zero coupons. $\overline{b}_k$ denotes the (steady state) quantity of the k bond. The price of the bond is denoted $\hat{p}_{t,k}$ ($\overline{p}_k$ in steady state).

The first term on the RHS of (2) is the government's surplus ($\overline{S}\hat{S}_t$). $\hat{G}_t$ is the spending of the government and $\overline{R}$ denotes the (steady state) revenue due to distortionary taxation. When all revenue derives from distortionary taxation we have $\overline{S} = \overline{R} - \overline{G}$. In contrast, if taxes are 100 percent lump sum then $\overline{R} = 0$. Notice also that since taxes are assumed constant, changes in revenue can only derive from fluctuations in output $\hat{Y}_t$, as long as $\overline{R} > 0$.

[3] This assumption is not restrictive. We could derive our key results below assuming (for example) that taxes (mildly) adjust to the deviation of debt from a target value or other macroeconomic variables.

[4] θ is the parameter that governs the magnitude of price adjustment costs in the standard quadratic cost function of Rotemberg (1982). When θ equals zero prices are fully flexible.

The second term on the RHS of (2) is the real value of debt that was issued in $t-1$ and repurchased in t. $\overline{p}_k\overline{b}_k(\hat{b}_{t-1,k}+\hat{p}_{t,k-1}-\hat{\pi}_t)$ denotes the value of debt of maturity k issued in $t-1$. This debt is priced in t as $k-1$ maturity debt and the corresponding price is $\hat{p}_{t,k-1}$.

Equations (3) and (4) define the prices of k bonds. (3) is the log-linear IS-Euler equation. Since our setup is Fischerian, the real rate is exogenous and constant. The (log of the) short term nominal interest rate, $\hat{i}_t$, thus equals $-\hat{p}_{1,t}$. (4) defines the formula that determines the price of debt of any k in period t. A long term bond issued in t promises one unit of income in $t+k$. The price is the real value of this claim of income, adjusted according to expected inflation between periods $t+1$ to $t+k$. The steady state price satisfies $\overline{p}_k = \beta^k$ where $\beta < 1$ denotes the standard household discount factor.

2.1 Ramsey Optimal Policies

We first consider the Ramsey policy equilibrium. We assume that a benevolent planner chooses sequences $\left\{\hat{\pi}_t, \hat{Y}_t, \hat{i}_t, \hat{b}_{t,k}, \hat{p}_{t,k}\right\}_{t\geq 0}$ subject to the competitive equilibrium equations to maximize the following objective:

$$-\frac{1}{2}E_0\sum_{t\geq 0}\beta^t\hat{\pi}_t^2 \tag{5}$$

[5] To simplify this problem, we take the standard approach of dispensing with prices and equations. Substituting (4) into (2) we obtain the following expression for the consolidated budget constraint

$$\sum_{k=1}^{\infty}\beta^k\overline{b}_k(\hat{b}_{t,k}-\sum_{l=1}^{k}E_t\hat{\pi}_{t+l}) = -\overline{S}\hat{S}_t + \overline{b}_1(\hat{b}_{t-1,1}-\hat{\pi}_t) + \sum_{k=2}^{\infty}\beta^{k-1}\overline{b}_k(\hat{b}_{t-1,k}-\sum_{l=0}^{k-1}E_t\hat{\pi}_{t+l}) \tag{6}$$

which is independent of bond prices. Moreover, noting that given the optimal path of inflation, $\hat{i}_t$ can be set to satisfy (3) and $\hat{p}_{t,k}$ can be set to satisfy (4), we can drop these equations from the constraint set.

Finally, we can further simplify, noticing that the portfolio $\hat{b}_{t,k}$ will not be uniquely defined in this program. Letting $\overline{d}\hat{d}_t \equiv= \sum_{k=1}^{\infty}\beta^{k-1}\overline{b}_k\hat{b}_{t,k}$ be the value of debt issued in t and bought back in $t+1$ evaluated at steady state prices, we let the planner choose $\hat{d}_t$ along with inflation and output to maximize (5).

The optimal policy solves:

$$\max_{\left\{\hat{\pi}_t,\hat{Y}_t,\hat{d}_t\right\}_{t\geq 0}} -E_0\frac{1}{2}\sum_{t\geq 0}\beta^t\hat{\pi}_t^2$$

subject to (1) and

$$\beta\overline{d}\hat{d}_t - \sum_{k=1}^{\infty}\beta^k\overline{b}_k\sum_{l=1}^{k}\hat{\pi}_{t+l} + \overline{R}\Big(\gamma_h+1\Big)\hat{Y}_t - \overline{G}\hat{G}_t = \overline{d}\hat{d}_{t-1} - \sum_{k=1}^{\infty}\beta^{k-1}\overline{b}_k\sum_{l=0}^{k-1}\hat{\pi}_{t+l} \tag{7}$$

[5] As in Cochrane (2001) we assume that the planner focuses on minimizing the variability of inflation. Though (5) has not been derived as an approximation of the household's preferences in the background non-linear model, for certain parameterizations, it is the preferences-consistent objective (see appendix). Moreover, in Section 5 and in the appendix of the paper we experiment with an alternative policy setup where stabilizing the output gap becomes part of the objective.

2.1.1 Optimality

In the appendix we setup a Lagrangian to derive the optimal paths of output, inflation and $\hat{d}_t$. Attach a multiplier $\psi_{\pi,t}$ to the Phillips curve and $\psi_{gov,t}$ to the consolidated budget. The first order conditions of the Ramsey program are:

$$-\hat{\pi}_t + \Delta\psi_{\pi,t} + \sum_{k=1}^{\infty} \bar{b}_k \sum_{l=1}^{k} \beta^{k-l} \Delta\psi_{gov,t-l+1} = 0 \tag{8}$$

$$-\psi_{\pi,t}\kappa_1 + \overline{R}\Big(1+\gamma_h\Big)\psi_{gov,t} = 0 \tag{9}$$

$$\bar{d}\beta\Big(\psi_{gov,t} - E_t\psi_{gov,t+1}\Big) = 0 \tag{10}$$

where (8), (9) and (10) are the FONC with respect to $\hat{\pi}_t$, $\hat{Y}_t$ and $\hat{d}_t$ respectively.[6]

To inspect these optimality conditions, combine (8) and (9) to substitute out ψ_π and obtain the following expression for $\hat{\pi}_t$

$$\hat{\pi}_t = \overline{R}\frac{(1+\gamma_h)}{\kappa_1}\Delta\psi_{gov,t} + \sum_{k=1}^{\infty} \bar{b}_k \sum_{l=1}^{k} \beta^{k-l}\Delta\psi_{gov,t-l+1} \tag{11}$$

According to (11), under optimal policy, inflation becomes a weighted average of current and lagged values of the growth of the multiplier ψ_{gov}. From (10), the latter object evolves according to a random walk.

These features are standard outcomes of optimal Ramsey policy (e.g. Aiyagari et al. (2002); Schmitt-Grohé and Uribe (2004); Lustig et al. (2008); Faraglia et al. (2013, 2016), among others). Debt and deficit fluctuations in our model can only be financed through distortionary inflation and the multiplier ψ_{gov}, which measures the burden of the distortions, follows a random walk because the planner desires to spread the burden evenly across periods.

Moreover, to clarify the dependence of inflation on the current and lagged values of ψ_{gov} let us iterate forward on constraint (7) to obtain the *intertemporal consolidated budget constraint* as:

$$E_t\sum_{j=0}^{\infty}\beta^j\overline{S}\hat{S}_{t+j} = \bar{d}\hat{d}_{t-1} - \sum_{k=1}^{\infty}\beta^{k-1}\bar{b}_k\sum_{l=0}^{k-1}E_t\hat{\pi}_{t+l} \tag{12}$$

(12) links the present discounted value of the fiscal surplus (LHS) to the real value of debt outstanding in t (RHS). It is equivalent to (7) in terms of the Ramsey policy.[7] Consider a spending shock which lowers the LHS of (12) relative to the RHS. In response to such a shock the constraint tightens, and the value of the multiplier ψ_{gov} increases. To satisfy the constraint, the planner needs to engineer a drop in the real payout of debt, the last term of the RHS of (12). This requires to increase current inflation and also promise to increase future inflation, the latter in the case where long term debt has been issued. The lagged terms $\Delta\psi_{gov,t-l+1}$ in (11) capture the promises made by the planner to adjust inflation in response to past shocks.

Finally, using (3) and (11) we can obtain the following expression for the nominal rate under Ramsey policy:

$$\hat{i}_t = E_t\hat{\pi}_{t+1} = \sum_{k=2}^{\infty}\bar{b}_k\sum_{l=2}^{k}\beta^{k-l}\Delta\psi_{gov,t+2-l}$$

which reveals that the nominal rate also is a function of the state variables $\Delta\psi_{gov,t-l+1}$.

[6] Note that our optimal policy program assumes a 'timeless perspective'. We do not (for instance) let the planner inflate away debt at the beginning of the horizon.

[7] See for example Aiyagari et al. (2002).

2.2 The Optimal Inflation

With the optimality conditions we can go very far towards characterizing analytically key features of optimal policy. We now study the properties of inflation under alternative maturity structures $\bar{b}_k$. To simplify the algebra, we will assume that spending $\hat{G}_t$ follows an i.i.d process.

2.2.1 One Maturity

Consider first the case where the government issues debt in one maturity only. Specifically, let us assume that all debt is issued in maturity N zero coupon bonds, or $\bar{b}_N > 0$ and $\bar{b}_k = 0$ for $k \neq N$. This modelling assumption is made by Schmitt-Grohé and Uribe (2004) and Faraglia et al. (2013) among others. Moreover, isolating our focus on a single maturity enables to transparently characterize the forces that drive inflation under optimal policy.

In the appendix we derive the following expression for equilibrium inflation in this model:

Proposition 1. *Assume that the government issues debt in a single N period bond. Optimal inflation under Ramsey is given by:*

$$\hat{\pi}_t = \sum_{j=0}^{N-1} \eta_{-j} \hat{G}_{t-j} \tag{13}$$

where

$$\eta_{-j} = \left\{ \begin{array}{ll} \dfrac{\tilde{f}\,\overline{G}}{\left[\tilde{f}^2 + (\beta^{N-1}\bar{b}_N)^2\left(\frac{1-\frac{1}{\beta^N}}{1-\frac{1}{\beta}} - 1\right)\right]} & j = 0 \\ \dfrac{\beta^{N-j-1}\bar{b}_N\overline{G}}{\left[\tilde{f}^2 + (\beta^{N-1}\bar{b}_N)^2\left(\frac{1-\frac{1}{\beta^N}}{1-\frac{1}{\beta}} - 1\right)\right]} & j = 1, 2, ..., N-1 \end{array} \right\} \tag{14}$$

$$\tilde{f} = \left(\frac{\overline{R}}{\kappa_1}(1+\gamma_h) + \beta^{N-1}\bar{b}_N\right) > 0$$

Proof: See appendix.

According to Proposition 1, inflation is a weighted average of the current and past ($N-1$ lags) shocks to spending. Since coefficients η_{-j} are positive, following a positive spending shock in t inflation will rise on impact and will remain above zero until period $t+N-1$. The optimal path of inflation may be frontloaded in the sense that if $\overline{R} > 0$ (taxes are distortionary) then $\tilde{f}$ exceeds $\beta^{N-j-1}\bar{b}_N$ and the impact of the shock on inflation in t is larger than in other periods. Otherwise, since β is plausibly close to 1, the rate of inflation will be roughly constant through time.

Turning to the impact of maturity, N, note that the term $\beta^{N-1}\bar{b}_N$ is such that in steady state the value of debt equals the present value of the surplus. Therefore, $\beta^{N-1}\bar{b}_N = \frac{\overline{S}}{1-\beta}$ is independent of N. Debt maturity N affects the coefficients η_{-j} through the term $\left(\frac{1-\frac{1}{\beta^N}}{1-\frac{1}{\beta}} - 1\right)$ in the denominator. Longer maturity increases this term, thus lowering the coefficients η_{-j}, however, the response of inflation to the shock is spread over more periods. When $N = 1$ we have $\eta_0 = \frac{\overline{G}}{\tilde{f}}$ and all of the response of inflation is concentrated in t. As N grows towards infinity, we obtain $\eta_{-j} \approx 0$ and inflation will permanently increase in response to the shock.

To understand the above properties notice first that when debt is of maturity N, setting a positive inflation rate after $t+N-1$ will not contribute towards satisfying the intertemporal constraint (12), it would be wasteful from the point of view of fiscal solvency. This explains why only the first $N-1$ lags of spending shocks matter for inflation as Proposition 1 stated.

Second, coefficient η_0 is larger, and it is optimal to frontload inflation the reason being that inflation in t has a direct impact on output and on the LHS of the intertemporal constraint (12). To see this, notice that the LHS of the constraint can be written as:

$$E_t \sum_{j=0}^{\infty} \beta^j \overline{S} \hat{S}_{t+j} = E_t \sum_{j=0}^{\infty} \beta^j \left(\overline{R}(1+\gamma_h)\hat{Y}_{t+j} - \overline{G}\hat{G}_{t+j} \right)$$

Thus, the surplus will increase when output is higher. Using the Phillips curve $\hat{Y}_{t+j} = \frac{1}{\kappa_1}\left(\hat{\pi}_{t+j} - \beta E_{t+j}\hat{\pi}_{t+j+1}\right)$ we can write

$$E_t \sum_{j\geq 0} \beta^j \overline{R}(\gamma_h+1)\hat{Y}_{t+j} = \frac{\overline{R}}{\kappa_1}(\gamma_h+1)\hat{\pi}_t$$

which reveals that higher inflation in t will increase the present value of revenues from distortionary taxation. Through making inflation higher in t, the planner enables a smaller drop in the intertemporal surplus, following a positive spending shock, and this in turn reduces the increase in inflation in periods $t+1, t+2, t+3, ..., t+N-1$ required to satisfy (12).

Note that besides $\overline{R} > 0$ a further condition that needs to be satisfied for this channel to be important, is that prices are sticky, the slope of the Phillips curve coefficient, κ_1, should not be large. If prices are quite flexible, then frontloading inflation will not impact output and the government's revenue. In contrast, under sticky prices, a change in inflation can impinge a significant effect on output.[8]

Impulse responses. Figure 1 plots the response of inflation to a spending shock under different values N. Table 1 reports the numerical values of the model's parameters assumed to construct the figure. Parameter θ which governs the degree of price stickiness in the model, is calibrated at 17.5 following Schmitt-Grohé and Uribe (2004). The slope of the Phillips curve is thus around one third, a plausible value.

The top panel of the Figure assumes that taxes are distortionary. Notice that even though prices are quite sticky in the model, the incentive to frontload inflation is not particularly strong. Moreover, the longer is the maturity, the less felt is the initial 'blip' in inflation. When $N = 10$ the resulting path of inflation is basically the same as the analogous object in the bottom panel of Figure 1, in which taxes are lump sum and the incentive to concentrate inflation in the initial period is absent.

2.2.2 Multiple Maturities

Let us now consider the more general case where instead of issuing debt in only one maturity, all maturities $\overline{b}_k$ can be issued. In the appendix we derive the following formula for optimal inflation in this model:

[8]Note that this is the only channel through which the degree of price stickiness can exert an influence on the path of inflation. In contrast to the Ramsey literature (e.g. Schmitt-Grohé and Uribe (2004); Faraglia et al. (2013) and others) where it is typically assumed that the planner can finance debt either through inflation or through taxes and a small slope coefficient κ_1 shifts the optimal policy mix towards more taxation, here the planner only has one instrument. Stickier prices will not reduce inflation volatility, rather they will increase the volatility when $\overline{R} > 0$ and frontloading inflation becomes optimal.

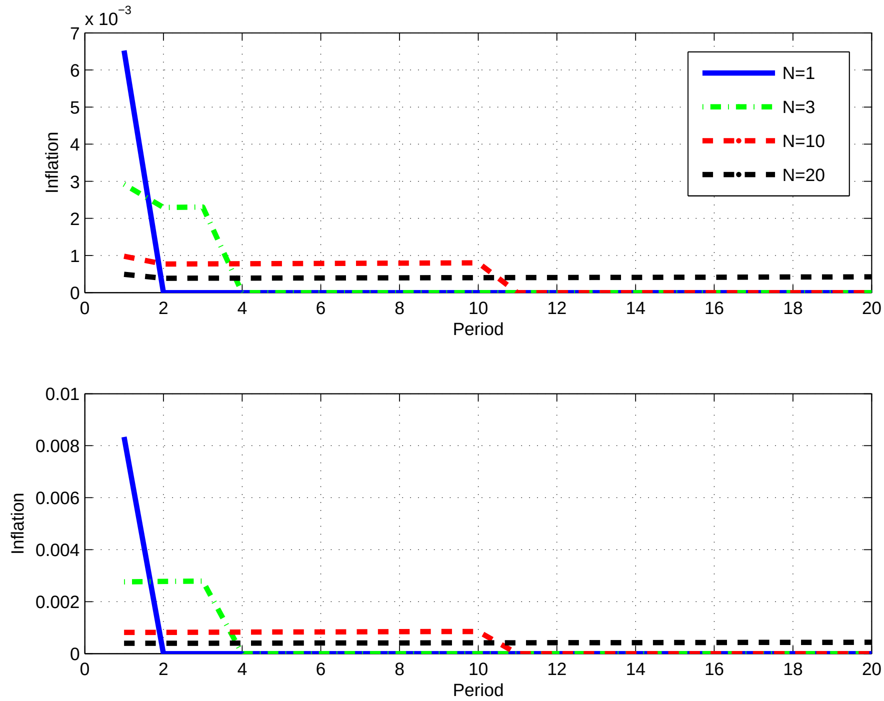
Finally, note that this effect is also present in a non-Fischerian models. When real long bond prices increase in current output, the incentive to distort intertemporally inflation remains, however it should be weaker.

Table 1: **Calibration**

Parameter	Value	Label
β	0.995	Discount factor
θ	17.5	Price Stickiness
η	-6.88	Elasticity of Demand
γ_h	1	Inverse of Frisch Elasticity
$\overline{Y}$	1	Steady State Output
$\overline{G}$	0.1	Steady State Spending

Notes: The table reports the values of model parameters. β notes the discount factor chosen to target a steady state annual real interest rate of 2 percent. Parameter η is calibrated to target markups of 17 percent in steady state. θ determines the cost of adjusting prices and is calibrated as in Schmitt-Grohé and Uribe (2004). Coefficient γ_h, the inverse of the Frisch elasticity of labour supply, equals 1. Finally, the steady state level of debt is assumed equal to 60 percent of GDP (at annual horizon), and the level of public spending is 10 percent of aggregate output, which is normalized to unity in steady state.

Figure 1: **Responses to the spending shock: Optimal Ramsey policy.**



Notes: The figure plots the path of optimal inflation in response to a shock that increases spending by 20% (from 10% of GDP to 12% of GDP). The top panel shows the case of distortionary taxes and the bottom panel assumes lump sum taxation. Each response corresponds to a different debt maturity. See legend of the Figure.

Proposition 2. *Assume that the government issues maturities* $\left\{\overline{b}_k\right\}_{k\geq 1}$. *Optimal inflation under Ramsey is given by:*

$$\hat{\pi}_t = \sum_{j=0}^{t} \eta_{-j}\hat{G}_{t-j}$$

where

$$\eta_{-j} = \begin{cases} \dfrac{\widetilde{f}\,\overline{G}}{\left[\widetilde{f}^2+\sum_{j=2}^{\infty}\beta^{j-1}\lambda_j^2\right]} & j=0 \\ \dfrac{\lambda_j\overline{G}}{\left[\widetilde{f}^2+\sum_{j=2}^{\infty}\beta^{j-1}\lambda_j^2\right]} & j=1,2,...,N-1 \end{cases} \tag{15}$$

$$\lambda_j = \frac{1}{\beta}(\lambda_{j-1} - \overline{b}_{j-1}) \qquad \text{and} \qquad \lambda_1 = \sum_{k=1}^{\infty}\beta^{k-1}\overline{b}_k \tag{16}$$

$$\widetilde{f} = \left(\frac{\overline{R}}{\kappa_1}(1+\gamma_h) + \sum_{k=1}^{\infty}\beta^{k-1}\overline{b}_k\right)$$

Proof: See appendix.

The recursive formula in Proposition 2 enables to easily calculate the optimal inflation path for any maturity structure of debt. To inspect the formula, consider first coefficient η_0. The numerator term $\widetilde{f}$ is determined by two forces. First, $\frac{\overline{R}}{\kappa_1}(1+\gamma_h)$ again measures the impact of inflation on the present value of the surplus, and second, the term $\sum_{k=1}^{\infty}\beta^{k-1}\overline{b}_k$ measures the effect of period t inflation on the real payout of total government debt, long and short bonds outstanding. Coefficients η_{-j} are then determined by objects λ_j which follow the recursive formula (16). According to this formula λ_j will be positive insofar as not all debt is of maturity less than or equal to j and be 0 otherwise.

A standard modelling assumption found in the literature (e.g Angeletos (2002); Buera and Nicolini (2004) and Faraglia et al., 2019) is to assume that the government issues debt in two assets, one short bond of 1 period maturity and one long term asset of maturity N. We then have $\overline{b}_1, \overline{b}_N \neq 0$ and $\overline{b}_k = 0$ for $k \neq 1, N$ and the path of the λs is given by:

$$\lambda_j = \frac{1}{\beta^{j-1}}\beta^{N-1}\overline{b}_N, \quad j=2,...,N-1$$

$$\lambda_N = \frac{1}{\beta}(\lambda_{N-1} - \overline{b}_N) = 0$$

and $\lambda_{N+1} = \lambda_{N+2} = ... = 0$.

We can then compute the second term in the denominator of η_{-j} as:

$$\sum_{j=2}^{\infty}\beta^{j-1}\lambda_j^2 = \sum_{j=2}^{N-1}\frac{1}{\beta^{j-1}}(\beta^{(N-1)}\overline{b}_N)^2 = (\frac{\overline{S}}{1-\beta} - \overline{b}_1)^2\frac{1}{1-\beta}(\frac{1}{\beta^{N-1}} - 1)$$

where the last equality derives from the steady state intertemporal constraint, $\frac{\overline{S}}{1-\beta} = \overline{b}_1 + \beta^{N-1}\overline{b}_N$.

The above condition suggests that $\sum_{j=2}^{\infty}\beta^{j-1}\lambda_j^2$ will be higher the more tilted is the portfolio towards long term debt (the smaller $\overline{b}_1$ is, the larger $\frac{\overline{S}}{1-\beta} - \overline{b}_1$ will be). Then the coefficients η_{-j} become smaller in magnitude and the inflation response to a spending shock weakens. In fact, when the government can issue a very large amount of the long term asset, financing its position through negative debt (savings) in the short term bond, we can have that $\eta_{-j} \approx 0$.

This result resembles the finding of Angeletos (2002); Buera and Nicolini (2004) that issuing long term bonds enables to absorb fiscal shocks by exploiting the variability of long bond prices. Whereas in Angeletos (2002) and Buera and Nicolini (2004) this happens because real long bond prices drop following a spending shock, here the planner can leverage on the persistent increase in inflation which yields a drop in the nominal long bond price after the shock.

Finally, another interesting case, and one to which we will later turn when we study optimized interest rate rules, is when the government issues a portfolio in which the shares of $\bar{b}_k$ bonds decay at constant rate δ, i.e. $\bar{b}_k = \delta^{k-1}\bar{b}$. Equivalently, debt is a perpetuity that pays decaying coupons (e.g. Cochrane, 2001; Leeper and Zhou, 2021). The formula in Proposition 2 then gives us:

$$\eta_0 = \frac{\widetilde{f}\,\overline{G}}{\left[\widetilde{f}^2 + \frac{\bar{b}_\delta^2 \beta\delta^2}{(1-\beta\delta^2)(1-\beta\delta)^2}\right]}$$

$$\eta_{-j} = \frac{\frac{\bar{b}}{1-\beta\delta}\delta^j\overline{G}}{\left[\widetilde{f}^2 + \frac{\bar{b}_\delta^2 \beta\delta^2}{(1-\beta\delta^2)(1-\beta\delta)^2}\right]} \quad , j = 1, 2, ..$$

which shows that the coefficients η_{-j}, $j \geq 1$, that capture the response of inflation to past spending shocks, decay monotonically at rate δ. We can easily show that a higher δ reduces the magnitude of the coefficients.

3 Rules rather than Ramsey

Ramsey policies give rise to the best outcome given the competitive equilibrium conditions of the model, however, from a practical standpoint, a Ramsey policy maybe difficult to implement, when it leads to path of interest rates that is unrelated to macroeconomic variables, e.g. inflation, output or lagged values of interest rates. In practice, actual policies implemented by central banks are informed by macroeconomic conditions summarized by these variables, and a large literature has been devoted to studying how to design interest rate rules that are simple functions of inflation, output, etc, in the baseline New Keynesian model.

We now consider a model where monetary policy sets the nominal interest rate as a function of macroeconomic variables only. We begin assuming a simple inflation targeting rule

$$\hat{i}_t = \phi_\pi \hat{\pi}_t. \tag{17}$$

where coefficient ϕ_π will be set optimally to maximize objective (5). Such exercises (assuming commitment to a rule and optimizing over parameters) have been considered many times in the context of the standard New Keynesian model. We carry out this exercise in the context of the fiscal theory.

Our key finding is this section is that the optimal policy sets

$$\phi_\pi^* \approx 1 - \frac{1}{\text{maturity}}$$

where ϕ_π^* denotes the optimal inflation coefficient. This simple and transparent relation between ϕ_π^* and the maturity of debt will hold across all maturity structures considered. This policy will be shown to approximate the Ramsey outcome very closely.

3.1 Optimal policy with a simple policy rule

To solve for the optimal policy rule we proceed in two steps. We first characterize the equilibrium under a generic value for parameter ϕ_π, then we compute the optimal coefficient ϕ_π as a function of the debt maturity.

3.1.1 One Maturity

Consider first the case where debt is issued in a single maturity N. Under (17), the equilibrium is a solution to the following system of equations:

$$\beta^N \overline{b}_N(\hat{b}_{t,N} - \sum_{l=1}^{N} E_t\hat{\pi}_{t+l}) = -\overline{S}\hat{S}_t + \sum_{k=2}^{\infty} \beta^{N-1}\overline{b}_N(\hat{b}_{t-1,N} - \sum_{l=0}^{N-1} E_t\hat{\pi}_{t+l})$$
$$\phi_\pi\hat{\pi}_t = E_t\hat{\pi}_{t+1}$$
$$\hat{\pi}_t = \kappa_1\hat{Y}_t + \beta E_t\hat{\pi}_{t+1}$$

Consider the second equation which is obtained by substituting out the nominal rate from the Euler equation using (17). When $\phi_\pi > 1$ this equation has an unstable root and can be solved forward to give us a unique solution $\hat{\pi}_t = 0$, for all t. From the Phillips curve it also holds that $\hat{Y}_t = 0$. Then, inflation will not satisfy the consolidated budget constraint and intertemporal solvency will fail.

In the case where $0 \leq \phi_\pi \leq 1$ the model has a unique equilibrium where inflation is not zero. In this case the intertemporal budget constraint will pin down inflation.

These are standard results of course. Monetary policy needs to be 'passive' (e.g. Leeper, 1991) for the equilibrium to be unique in a model where taxes do not adjust to debt to ensure the solvency of the government's budget. Allowing for a richer maturity structure will not change this prediction.

Solving the above system of equations, it is easy to show that equilibrium inflation is given by:

$$\hat{\pi}_t = \sum_{j=0}^{t} \frac{\phi_\pi^{j-1}\overline{G}}{\xi}\hat{G}_{t-j} \tag{18}$$

where $\xi = \overline{R}\frac{1+\gamma_h}{\kappa_1} + \beta^{N-1}\overline{b}_N\frac{1-\phi_\pi^N}{1-\phi_\pi}$.

Note that there are two ways in which ϕ_π influences this solution. First, (18) states that inflation will display persistence ϕ_π, a higher inflation coefficient will translate into a more persistent process of inflation. This follows easily from $\phi_\pi\hat{\pi}_t = E_t\hat{\pi}_{t+1}$, which defines a backward looking process. Second, ϕ_π also influences the denominator of (18). When $N > 1$ a higher ϕ_π implies higher ξ. When $N = 1$, ξ does not depend on ϕ_π.

Why is this so? With long term debt, inflation can contribute towards stabilizing debt up to period $t + N - 1$. A higher ϕ_π will make inflation more persistent in response to the spending shock, and a smaller increase in inflation is needed (in each period) to satisfy the intertemporal solvency condition. This explains why the denominator of (18) increases in ϕ_π. When debt is short term, however, then only inflation in t can absorb the shock, and the persistence of inflation will not matter for debt solvency.

This finding hints at how the optimal ϕ_π will be influenced by debt maturity. In the case where $N = 1$ optimal policy should set a constant interest rate, as letting $\phi_\pi > 0$ will lead inflation to persistently deviate from target, without contributing anything towards debt sustainability. Conversely, if $N > 1$, then persistence of inflation would be desirable, as it would spread the burden of inflation over more periods reducing overall losses.[9]

[9] See Leeper and Leith (2016) for a numerical experiment with ad hoc rules of the form (17) where this result emerges.

Formally, the optimal policy solves:[10]

$$\max_{\phi_\pi} -\frac{1}{2}E_0 \sum_{t\geq 0} \beta^t \hat{\pi}_t^2 = \max_{\phi_\pi} -\frac{1}{2}\frac{\overline{G}^2 \sigma_G^2}{\xi^2}\frac{1}{1-\beta\phi_\pi^2}$$

The first order condition is:

$$-\frac{1}{2}\frac{\overline{G}^2\sigma_G^2}{\xi^2}\frac{1}{1-\beta\phi_\pi^2}\left[\frac{\beta\phi_\pi}{1-\beta\phi_\pi^2} - \beta^{N-1}\bar{b}_N \frac{1}{\xi(1-\phi_\pi)^2}\Big(1+(N-1)\phi_\pi^N - N\phi_\pi^{N-1}\Big)\right] = 0 \tag{19}$$

Let us for simplicity focus on the case where $\beta \approx 1$ and $\overline{R} = 0$. Then, the optimal coefficient ϕ_π^* solves:

$$\frac{\phi_\pi(1-\phi_\pi^N)}{1+\phi_\pi} = \Big(1+(N-1)\phi_\pi^N - N\phi_\pi^{N-1}\Big) \tag{20}$$

In the case where $N = 1$ the optimum is $\phi_\pi^* = 0$. When $N = 2$ we have $\phi_\pi^* = \frac{1}{2}$. For higher N the LHS of (20) defines a concave function which equals 0 when ϕ_π is either 0 or 1. The RHS of (20) defines a strictly downward sloping function, which is equal to 1 at $\phi_\pi = 0$ and 0 at $\phi_\pi = 1$. The LHS is equal to the RHS at a unique $\phi_\pi \in (0, 1)$ which defines the optimum.[11]

The solution cannot be characterized analytically for general N. It holds however that $\phi_\pi^* \approx 1 - \frac{1}{N}$. One way to show this is by comparing the implied paths of inflation in response to a shock to spending. This is done in the top panels of Figure 2. We plot the responses of inflation for different N, when the inflation coefficient is equal to ϕ_π^* (dashed / red lines) and when it is equal to $1 - \frac{1}{N}$ (solid / blue lines). The left panel in the figure sets $N = 3$. On the right we assume $N = 10$. As it is evident, the responses are very similar. Effectively, $\phi_\pi = 1 - \frac{1}{N}$ is a very good approximation of optimal policy.

The bottom panel of the Figure, repeats this exercise assuming that taxes are distortionary (hence $\overline{R} > 0$). We then recover ϕ_π^* as a solution to (19). As can be seen from the plot, once again, the blue lines almost overlap with the red/green/black lines. Even in this case $1 - \frac{1}{N}$ is very close to ϕ_π^* .

3.1.2 A comparison with Ramsey and Ramsey implied rules.

For this model, the Ramsey policy first order conditions can be rearranged to yield an interest rate rule that targets current inflation. Using the Euler equation and (11), assuming one bond of maturity N, we get:

$$\hat{i}_t = E_t\hat{\pi}_{t+1} = \frac{1}{\beta}\bar{b}_N \sum_{l=1}^{N-1} \beta^{N-l}\Delta\psi_{gov,t-l+1}$$

and then using (11) to replace the weighted sum of the multipliers, we obtain:

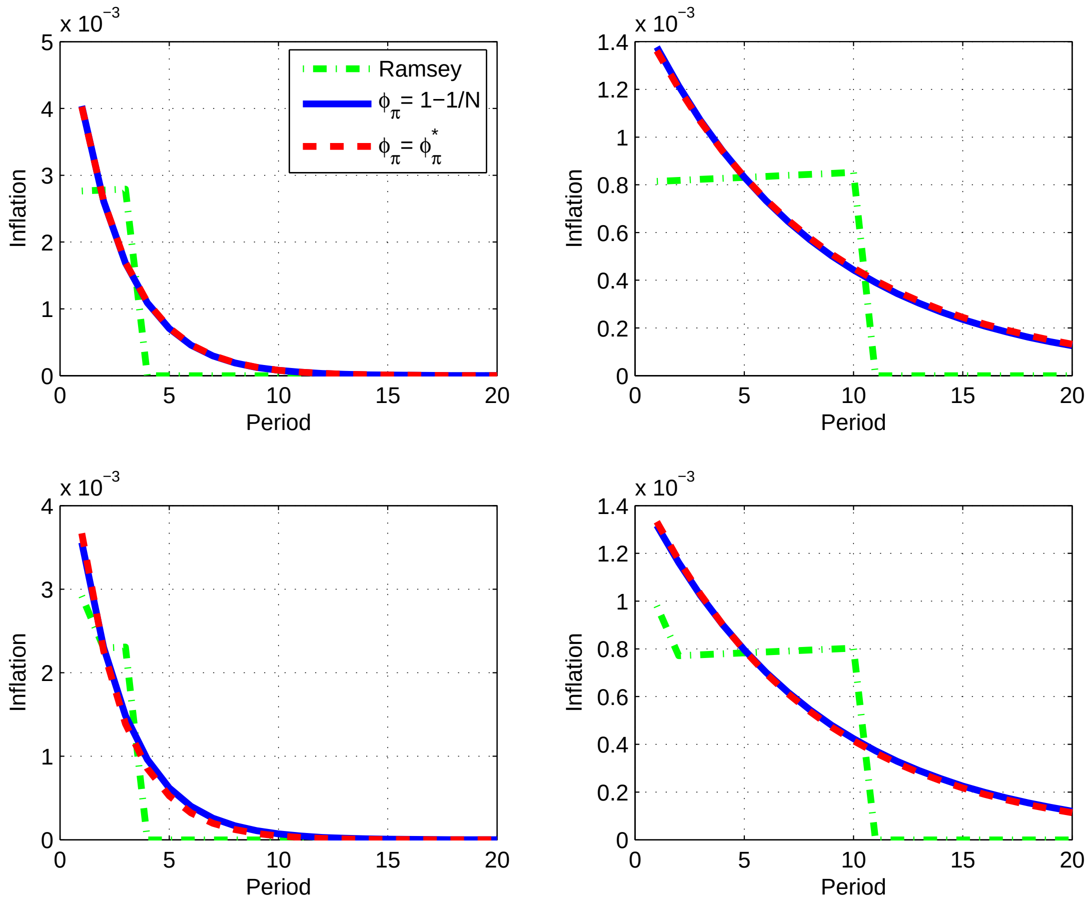
$$\hat{i}_t = \frac{1}{\beta}\Big(\hat{\pi}_t - \widetilde{f}\Delta\psi_{gov,t} - \bar{b}_N\Delta\psi_{gov,t-N+1}\Big) \tag{21}$$

According (21), the nominal interest is a function of inflation (with coefficient $\frac{1}{\beta}$) and of the stochastic intercept terms $\Delta\psi_{gov,t}$ and $\Delta\psi_{gov,t-N+1}$. Moreover, since making interest rate policy contingent on the

[10] We denoted σ_G^2 the variance of $\hat{G}_t$.

[11] It can also be easily seen that in the case where $N = \infty$ (20) gives a unique solution $\phi_\pi^* = 1$.

Figure 2: **Responses to the spending shock under optimal monetary policy rules.**



Notes: The figure plots the path of optimal inflation in response to a shock. Debt is a zero coupon bond of maturity N. The dashed-dotted green line is the Ramsey solution. The top panel assumes $\overline{R} = 0$ (lump sum taxation), the bottom assumes that $\overline{R} > 0$ (distortionary taxation). The solid lines show the response of inflation when coefficient ϕ_π solves (19). The dashed lines set $\phi_\pi = 1 - \frac{1}{N}$.

multipliers $\psi_{gov,t}$ does not seem practically relevant, we could express the multipliers as a function of $\hat{G}$. Under Ramsey policy it holds that :

$$\Delta\psi_{gov,t} = \frac{\overline{G}}{\left[\widetilde{f}^2 + (\beta^{N-1}\overline{b}_N)^2\left(\frac{1-\frac{1}{\beta^N}}{1-\frac{1}{\beta}} - 1\right)\right]}\hat{G}_t$$

and so it follows that (21) expresses the nominal rate as a function of inflation and of spending in t and $t - N + 1$.

It thus seems that a system of equations comprising of the Phillips curve, the Euler equation, the consolidated budget and assuming that monetary policy follows (21) (when $\Delta\psi_{gov}$ is substituted out of the system) will reproduce the Ramsey policy outcome. However, this is not so. The problem is that policy rule (21) will lead to an explosive solution. Since the inflation coefficient exceeds unity, (21) defines an 'active' monetary policy (Leeper, 1991).[12] In contrast, a simple rule of the form (17) leads to a unique stable solution.

[12]Intuitively, when we substitute out from the model $\Delta\psi_{gov,t}$ we also lose the random walk condition $E_t\Delta\psi_{gov,t+1} = 0$.

Figure 2 plots that the responses of inflation to a spending shock, under the rule based policy (17) and setting $\phi_\pi = 1 - \frac{1}{N}$, and under Ramsey. To dashed dotted/green lines which show the Ramsey outcome are essentially the responses shown in Figure 1 but we repeat them in order to facilitate the comparison. Clearly, the inflation paths do not coincide for $N = 3, 10$ considered in the Figure. The rule based policy prescribes a monotonic path for inflation whereas under Ramsey inflation is roughly flat for N periods and then abruptly becomes 0. A rule that would coincide with Ramsey policy only when $N = 1$ or when $N \to \infty$.

Given this result, it may seem that an alternative inflation targeting rule, one that could set inflation to respond to the shock for $N - 1$ periods, and subsequently change drastically the nominal rate so that inflation returns to target (or close to target), will provide a better approximation of the Ramsey policy than the simple rule that sets $\phi_\pi = 1 - \frac{1}{N}$. In this sense, the Ramsey implied policy (21) can serve as a useful benchmark. For example, since $\beta \approx 1$ we can perhaps set policy according to

$$(22) \qquad \hat{i}_t = \hat{\pi}_t - \frac{1}{\beta}\widetilde{f}\frac{\overline{G}}{\left[\widetilde{f}^2 + (\beta^{N-1}\overline{b}_N)^2\left(\frac{1-\frac{1}{\beta^N}}{1-\frac{1}{\beta}} - 1\right)\right]}\hat{G}_t - \frac{1}{\beta}\overline{b}_N\frac{\overline{G}}{\left[\widetilde{f}^2 + (\beta^{N-1}\overline{b}_N)^2\left(\frac{1-\frac{1}{\beta^N}}{1-\frac{1}{\beta}} - 1\right)\right]}\hat{G}_{t-N+1}$$

i.e. set the inflation coefficient to 1 instead of $\frac{1}{\beta}$ to obtain a stable equilibrium.

Ultimately, whether or not it is worthwhile devising a more elaborate interest rate policy based on the Ramsey outcome requires to evaluate the loss function (5) under the Ramsey policy equilibrium and under the rule based policy (17). If the differences are small then there is little margin to improve on the outcome of the rule based policy.

In Figure 3 we plot the loss function under the policy rule (17) for $\phi_\pi \in [0, 1]$. The Ramsey outcome is represented with the dashed-dotted /green line. Note that when $\phi_\pi = 1 - \frac{1}{N}$ the differences are minuscule. Thus even though the inflation paths of the two models differ, this does not translate to a significant loss under rule based policy. We obtain a similar finding for many other calibrations of N.

Finally, we have computed the loss function under the interest rate rule (22). The losses were several orders of magnitude larger than Ramsey.

3.1.3 Multiple maturities: One decaying coupon bond.

Let us now turn to the case where more than one maturity can be issued. The optimal coefficient ϕ_π solves:

$$(23) \qquad \left[\frac{\beta\phi_\pi}{1-\beta\phi_\pi^2}\left(\overline{R}\frac{1+\gamma_h}{\kappa_1} + \sum_{k=1}^{\infty}\beta^{k-1}\overline{b}_k\frac{1-\phi_\pi^k}{1-\phi_\pi}\right) - \sum_{k=1}^{\infty}\beta^{k-1}\overline{b}_k\frac{1}{(1-\phi_\pi)^2}\left(1 + (k-1)\phi_\pi^k - k\phi_\pi^{k-1}\right)\right] = 0$$

To derive an analytical solution let us first assume that debt is a perpetuity that pays decaying coupons, $\overline{b}_k = \delta^{k-1}\overline{b}$. The following Proposition gives the optimal policy rule:

Proposition 3: *Assume that* $\overline{b}_k = \delta^{k-1}\overline{b}$ *and* $\overline{R} = 0$. *The optimal interest rate rule is*

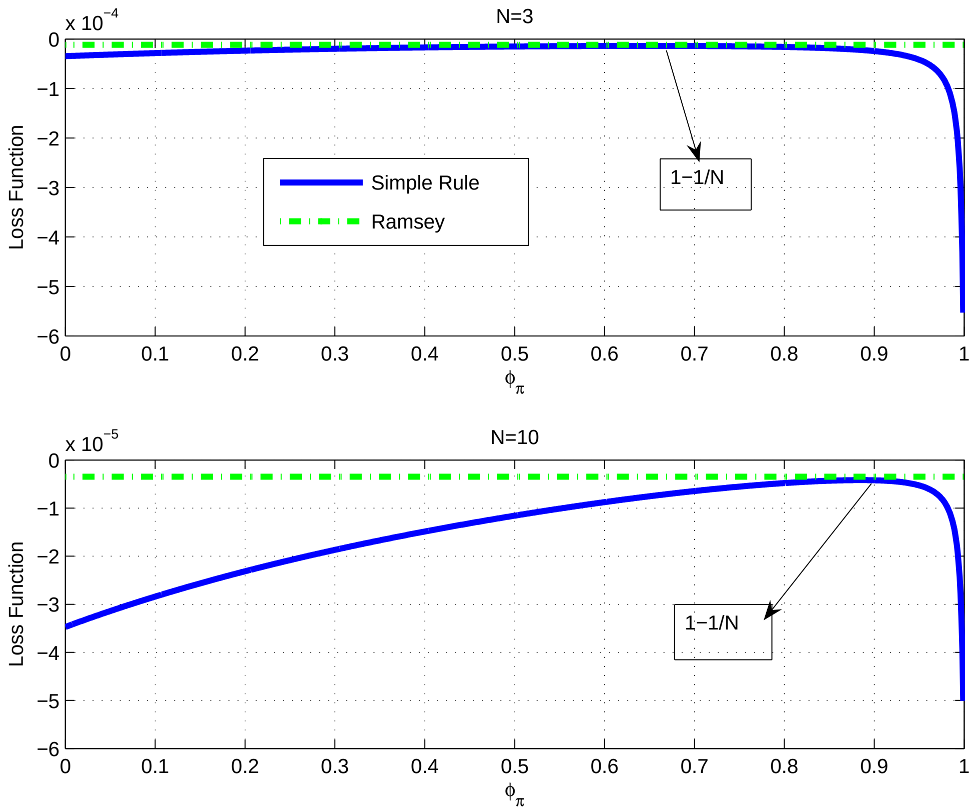
$$(24) \qquad \hat{i}_t = \delta\hat{\pi}_t$$

Proof: See appendix.

Then, the dynamic system has too many unstable roots. Otherwise, it would possible to solve for a unique equilibrium using (21) without substituting out the multiplier and without dropping the martingale condition.

Note also that the claim here is not that (21) is a unique interest rate policy that can implement the Ramsey outcome. Given the model structure it is plausible that there are policy rules that make the nominal rate a function of (many) lags of inflation and possibly deliver a unique equilibrium. Non-uniqueness of *robustly optimal rules* is a standard feature of the New-Keynesian model (see Giannoni and Woodford, 2003a).

Figure 3: **Loss Function under Ramsey and Optimal Interest Rate Rules.**



Notes: The figure plots the loss function under an interest rate rule, as a function of ϕ, and under the Ramsey outcome.

Under the assumed maturity structure, the average maturity (of the face value of debt) is $\frac{1}{1-\delta}$. Thus Proposition 3 confirms the principle that the optimal policy rule sets the inflation coefficient equal to $1-\frac{1}{\text{Maturity}}$. To understand why this is optimal here, note again that when $\phi_\pi = \delta$ then inflation displays first order autocorrelation equal to δ. Thus, following a positive spending shock inflation will rise and gradually revert back towards 0 at this rate.

Assume that the planner had set $\phi_\pi > \delta$ thus making inflation a more persistent process. Then, inflation would be high even when the coupon payments on debt outstanding in t have become low, which implies a higher cost of inflation without bringing any benefit in terms of stabilizing debt. Conversely, in the case where $\phi_\pi < \delta$ inflation becomes too frontloaded. A higher persistence would then enable to spread the costs more efficiently. Making inflation decay at the same rate as the coupons is the optimal policy.[13]

How does this rule based policy fare against the Ramsey outcome? In the appendix we show that the optimal Ramsey plan admits the following solution for the equilibrium interest rate:

$$\hat{i}_t = \delta\hat{\pi}_t - \delta\overline{R}\frac{(1+\gamma_h)}{\kappa_1}\Delta\psi_{gov,t} \tag{25}$$

Clearly, in the case $\overline{R} = 0$ as we assumed in Proposition 3, the nominal interest rate under Ramsey

[13]This also applies in the case where long bonds are consols ($\delta = 1$). Then $\phi_\pi^* = 1$ and inflation becomes a random walk. The formula $\phi_\pi^* = 1 - \frac{1}{\text{Maturity}}$ continues to hold since in this case the maturity of debt is infinite.

policy is simply equal to $\delta\hat{\pi}_t$. In other words, this is the *robustly optimal rule* that we can recover from solving the Ramsey first order conditions (e.g. Giannoni and Woodford, 2003a).

When $\overline{R}\frac{(1+\gamma_h)}{\kappa_1}$ is not zero (i.e. when revenues derive from distortionary taxes), (25) defines a response of the nominal rate to a positive spending shock which has a lower intercept in t. As discussed previously, this accomplishes to make inflation slightly higher in t and increase output to raise the surplus. Afterwards, inflation will decay monotonically at rate δ. It turns out, however, that even in this case $\phi^*_\pi \approx \delta$ under the calibration assumed in Table 1.

We thus conclude that under the maturity structure assumed in this paragraph, a simple inflation targeting rule approximately delivers the Ramsey outcome.

3.1.4 Multiple maturities and decaying coupon bonds.

Assuming a debt structure of the form $\bar{b}_k = \delta^{k-1}\bar{b}$ provides a good approximation of outstanding payments on US government debt. However, Barrett et al. (2021) argue that modelling debt payment profiles as a mixture of multiple decaying coupon bonds, of different maturity, can explain most of the time variation of payments we observe in the data. In particular assuming a structure of the form

$$\bar{b}_k = \bar{b}\sum_{i=1}^{M}\omega_i\delta_i^{k-1}$$

where ω_i represents the weight attached to the bond with repayment profile δ_i can fit the data very well even when a small number ($M = 2$ or $M = 3$) of decaying coupon bonds is used.

We now consider this type of debt structure, assuming $M = 2$. Unfortunately, even with this simple structure deriving analytical formulae is not easy and so we resort to a numerical solution to find the optimal interest rate policy.

Figure 4 sets $\delta_1 = 0.75$, an average maturity of 4 quarters (periods) for the first bond, and $\delta_2 = 0.9917$ which gives an average maturity of 30 years for the long term asset. [14] On the horizontal axis we have the weight ω, the share of the long bond in the portfolio.[15] The solid line represents $1 - \frac{1}{\text{Maturity}}$, the dashed line is the solution to (23). The top panel shows the case of distortionary taxes and the bottom one the case of lump sum taxes.

As is evident from the Figure, the graphs almost completely overlap. Thus the formula $1 - \frac{1}{\text{Maturity}}$ continues to approximate very closely the optimal inflation coefficient.

3.2 Alternative Rules and Shocks

Our findings can be extended to interest rate rules that track the output gap or feature interest rate inertia, i.e. lagged values of $\hat{i}_t$. Consider first the case where the policy rule is of the form:

$$\hat{i}_t = \phi_\pi\hat{\pi}_t + \phi_Y\hat{Y}_t \tag{26}$$

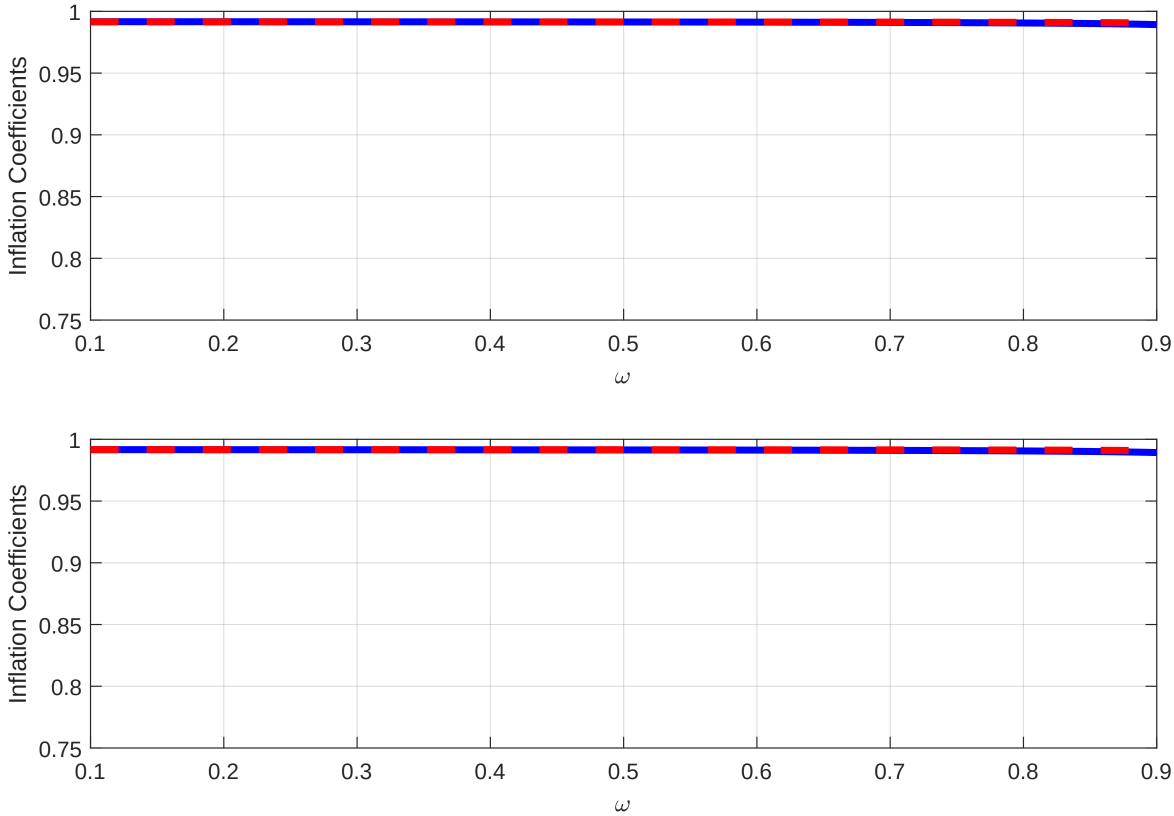
Combining this rule with the Euler equation and the Phillips curve, we can show that inflation dynamics evolve according to:

$$E_t\hat{\pi}_{t+1} = \phi_\pi\hat{\pi}_t + \frac{\phi_Y}{\kappa_1}\Big(\hat{\pi}_t - \beta E_t\hat{\pi}_{t+1}\Big)$$

[14] This provides a good approximation of the data since a large share of the outstanding debt in the US is concentrated at maturities between 1 quarter and 2 years, but also very long bonds of maturity equal to 30 years are being issued.

[15] Obviously, when ω is either 0 or 1 we are back to the case of the previous paragraph. The graph then considers the range $[\delta_1, 1]$ on the vertical axis to cover these plausible outcomes.

Figure 4: **Optimal Inflation Coefficients with two coupon bonds.**



Notes: The figure plots the optimal inflation coefficients (the solution to equation (23)) as a function of the weight ω (solid line). The dashed line is $1-\frac{1}{\text{Maturity}}$. The average maturity for the two bond model has been computed using the formula $\frac{(1-\omega)\frac{1}{(1-\delta_1)^2}+\omega\frac{1}{(1-\delta_2)^2}}{(1-\omega)\frac{1}{1-\delta_1}+\omega\frac{1}{1-\delta_2}}$.

or

$$E_t\hat{\pi}_{t+1} = \frac{\phi_\pi + \frac{\phi_Y}{\kappa_1}}{1+\beta\frac{\phi_Y}{\kappa_1}}\hat{\pi}_t$$

It is obvious that the sufficient condition to have an equilibrium in this model is $\phi_\pi + \frac{\phi_Y}{\kappa_1}(1-\beta) \leq 1$, the standard configuration of the parameters for which monetary policy is passive. The interest rate rule now has to satisfy

$$\frac{\phi_\pi + \frac{\phi_Y}{\kappa_1}}{1+\beta\frac{\phi_Y}{\kappa_1}} = 1 - \frac{1}{\text{Maturity}}$$

to implement the optimal policy outcome.

Next, consider a rule with interest rate inertia

$$\hat{i}_t = \rho\hat{i}_{t-1} + (1-\rho)\phi_\pi\hat{\pi}_t$$

Following the above logic it is simple to show that now it is optimal to set

$$(\rho + (1-\rho)\phi_\pi) = 1 - \frac{1}{\text{Maturity}}$$

(which defines a positive inflation coefficient in the case where $\rho < 1 - \frac{1}{\text{Maturity}}$).

Finally, we have for simplicity considered a model where government spending shocks drive all fluctuations in the surplus. It is simple to extend our analysis, by adding more shocks. In the appendix we show that the formula $\phi_\pi = 1 - \frac{1}{\text{Maturity}}$ continues to hold when the economy can be hit by demand shocks that introduce real interest rate fluctuations.[16] Analogously, we could easily extend our findings to the case where fluctuations in the surplus are driven by shocks to government transfers, another commonly made assumption in the context of the fiscal theory. None of the formulae we derived previously will change since, in our Fischerian model, shocks to spending and transfers deliver effectively the same effects.

4 No buy back

Thus far we modelled long bonds following the bulk of the literature, assuming repurchases of long debt one period after it has been issued. Though this assumption is commonly made in the literature as a simplification it is obvious that it is at odds with the observed practices of both governments and central banks. The stock of long term debt is not repurchased continuously, only to be replaced by new long debt.

Abandoning the assumption of full buybacks, requires to identify an alternative setup for the modelling of long term debt. This might entail allowing for partial buybacks of long bonds prior to maturity (e.g. Quantitative Easing) or even ruling out repurchases altogether and assuming that debt is redeemed at maturity.

[16]For simplicity we consider only the case where debt is a decaying coupon bond. The optimal interest rule is of the form:

$$\hat{i}_t = \hat{r}_t + \delta\hat{\pi}_t$$

where $\hat{r}_t$ defines the exogenous shock to the natural rate of interest.

Our results will also hold in the case where shock processes are not i.i.d. Intuitively, the responses of inflation to shocks were derived to satisfy the intertemporal budget constraint, when a shock hits (see appendix). More persistence will simply translate into a bigger effect of the shock on the intertemporal surplus, the LHS of the constraint. This will only affect the magnitude of the response of inflation, not the time-path of inflation.

Both of these assumptions fit US policy in the post WWII era. Faraglia et al. (2019) provide evidence that buybacks by the US Treasury have been very rare during this period, the Treasury typically redeemed long term debt at maturity.[17] On the other hand, the Federal reserve bought considerable amounts of long term bonds as part of the Quantitative easing program launched in the decade following the 2008-9 crisis and also intervened in secondary bond markets to buy long term assets during the so called Operation Twist in the early 1960s. Since ours is a monetary model featuring the consolidated budget constraint, it is appropriate to think of these episodes as partial buybacks of long bonds.

Our analysis in this section adopts the first alternative and thus we rule out buybacks from the outset and assume that the long bonds are always redeemed at maturity. We do so for simplicity (modeling partial buybacks involves much more notation), but also because having covered the cases of full buybacks in sections 2 and 3 and no buyback in this section, it becomes possible to think of the intermediate scenario of partial buybacks.

Our results in this section show that no buyback can become an important friction for monetary policy. When we assume that debt is only long term (a zero coupon long term bond) optimal Ramsey policy gives rise to oscillations in inflation that persist forever. Simple interest rate rules that target inflation lead to explosive inflation dynamics. Yet, we find that these features are important when the government focuses on issuing only long debt and generally do not carry over to the case where both short and long bonds are issued. We obtain an equivalence result (in terms of the optimal policies) with the buyback model in the case where debt is issued in coupon bearing bonds and in particular when it is a perpetuity that pays constant or decaying coupons.

4.1 Optimal Ramsey policy without repurchases

The only equation of the model that needs to be modified to rule out repurchases from the model is the consolidated budget constraint. We now have

$$\sum_{k=1}^{\infty}\beta^k\overline{b}_k(\hat{b}_{t,k}-\sum_{l=1}^{k}E_t\hat{\pi}_{t+l})+\overline{S}\hat{S}_t=\overline{b}_1(\hat{b}_{t-1,1}-\hat{\pi}_t)+\sum_{k=2}^{\infty}\overline{b}_k(\hat{b}_{t-k,k}-\sum_{l=0}^{k-1}\hat{\pi}_{t-l}) \tag{27}$$

Note that (27) differs from the constraint under buy back (equation (6)) only with respect to the last term on the RHS. This term now measures the real value of debt that has reached maturity and is redeemed in t. Thus, $\hat{b}_{t-k,k}$ denotes debt that was of maturity k in period $t-k$. The real payout of this debt in t depends on the realized inflation between periods $t-k+1$ and t.

Notice also that RHS of (27) does not represent the entire market value of debt issued by the government. Since debt is not redeemed prior to maturity, there are non-maturing bonds in the government's portfolio. These objects will show up in future consolidated constraints and also show up in the intertemporal budget constraint that equates the value of debt to the present value of surpluses. The latter can be written as:

$$E_t\sum_{j=1}^{\infty}\beta^j\overline{S}\hat{S}_{t+j}=\overline{b}_1(\hat{b}_{t-1,1}-\hat{\pi}_t)+\sum_{k=2}^{\infty}\overline{b}_k\left(\sum_{i=1}^{k}\beta^{k-i}(\hat{b}_{t-i,k}-E_t\sum_{l=1}^{k}\hat{\pi}_{t-i+l})\right) \tag{28}$$

[17] A noteworthy exception is the 2001 buyback program. Moreover, until the 1980s a sizable fraction of long term debt outstanding was in callable bonds. This type of debt can be bought back prior to maturity, within a certain call window which however starts long after the bond has been issued. For example, a callable 15 year bond can be bought back 2 years before it matures. Callable bonds are thus approximately redeemed at maturity, their payment profiles are similar to those of non-callable bonds (see Faraglia et al. (2019)).

4.1.1 Optimal policy with one N bond.

For simplicity let us consider first the Ramsey program under the assumption that the government issues only one N bond. The optimal policy solves:

$$\max_{\left\{\hat{\pi}_t,\hat{Y}_t,\hat{b}_{t,N}\right\}_{t\geq 0}} \quad -E_0\frac{1}{2}\sum_{t\geq 0}\beta^t\hat{\pi}_t^2$$

subject to (1) and

$$\beta^N\left(\hat{b}_{t,N}-E_t\sum_{l=1}^{N}\hat{\pi}_{t+l}\right)+\overline{R}\left(\gamma_h+1\right)\hat{Y}_t-\overline{G}\hat{G}_t=\bar{b}_N\left(\hat{b}_{t-N,N}-\sum_{l=0}^{k-1}\hat{\pi}_{t-l}\right) \tag{29}$$

The first order conditions are:

$$-\hat{\pi}_t+\Delta\psi_{\pi,t}+\bar{b}_N\sum_{l=1}^{N}\beta^{N-l}\left(\psi_{gov,t+N-l}-\psi_{gov,t-l}\right)=0 \tag{30}$$

$$-\psi_{\pi,t}\kappa_1+\overline{R}\left(1+\gamma_h\right)\psi_{gov,t}=0 \tag{31}$$

$$\bar{b}_N\beta^N\left(\psi_{gov,t}-E_t\psi_{gov,t+N}\right)=0 \tag{32}$$

and with appropriate substitutions we get the following expression for optimal inflation:

$$\hat{\pi}_t=\overline{R}\frac{\left(1+\gamma_h\right)}{\kappa_1}\Delta\psi_{gov,t}+\bar{b}_N\sum_{l=1}^{N}\beta^{N-l}\Delta^N E_t\psi_{gov,t+N-l} \tag{33}$$

where $\Delta^N\psi_{gov,t+N}=\psi_{gov,t+N}-\psi_{gov,t}$

There are several noteworthy features. First, note that the multiplier $\psi_{gov,t}$ no longer follows a random walk. As equation (32) shows, $\psi_{gov,t}$ is equated to the expected value of the period $t+N$ multiplier, which gives the standard martingale only when $N=1$ (debt is short term). Second, according to (33) inflation is no longer only a function of the current and lagged values of the multiplier; also future multipliers exert an influence in the path of inflation.

To explain these properties we assume for simplicity $N=2$. Iterating forward equation (29) gives:

$$E_t\sum_{j=0}^{\infty}\beta^{2j}\overline{S}\hat{S}_{t+2j}=\bar{b}_2\left(\hat{b}_{t-2,2}-\hat{\pi}_t-\hat{\pi}_{t-1}\right) \tag{34}$$

In (34) the market value of debt outstanding in t compensates for the surplus in $t,t+2,t+4,\ldots$. Consider an i.i.d shock $\hat{G}_t$ that lowers the LHS of (34). Then, since $\hat{\pi}_{t-1}$ is predetermined, only $\hat{\pi}_t$ can adjust to reduce the real value of maturing debt and ensure satisfaction of the constraint.

Future inflation (in particular inflation in $t+1$ when we assume a two period asset) does not help with making the debt solvent in t. The analogue of (34) in $t+1$ is:

$$E_{t+1}\sum_{j=0}^{\infty}\beta^{2j}\overline{S}\hat{S}_{t+1+2j}=\bar{b}_2\left(\hat{b}_{t-1,2}-\hat{\pi}_{t+1}-\hat{\pi}_t\right) \tag{35}$$

Evidently, the shock $\hat{G}_t$ has no impact on this intertemporal constraint. $\hat{\pi}_{t+1}$ will respond to the (expected) surplus sequence $\hat{S}_{t+1}, \hat{S}_{t+3}, \hat{S}_{t+5}, ...$ and not to $\hat{S}_t, \hat{S}_{t+2}, \hat{S}_{t+4}, ...$

Generically, since following a positive shock to spending (34) will tighten, but not necessarily (35), these constraints will affect the solution differently and so the associated Lagrange multipliers will differ. The fact that ψ_{gov} follows a cycle of 2 periods in the model can be understood in terms of this property. To show how it affects the optimal path of inflation, we consider the impulse response with respect to a shock occurring in period t and assuming no shock will hit the economy thereafter. Conditional expectations can then be dropped and we can derive analytically the path of inflation. We have:

Proposition 4: *Assume $N = 2$ and consider a spending shock in period t assuming no further shock thereafter. Optimal inflation in the Ramsey program under no buyback is:*

$$\hat{\pi}_{t+\bar{t}} \begin{cases} \frac{\overline{R}}{\kappa_1}(1+\gamma_h)\overline{\psi} + \overline{b}_2\left(\overline{\psi} + \beta\underline{\psi}\right) & \bar{t} = 0 \\ \frac{\overline{R}}{\kappa_1}(1+\gamma_h)\underline{\psi} + \overline{b}_2\underline{\psi} & \bar{t} = 1 \\ \frac{\overline{R}}{\kappa_1}(1+\gamma_h)(\overline{\psi} - \underline{\psi})I_{\bar{t}=\text{even}} + \frac{\overline{R}}{\kappa_1}(1+\gamma_h)(\underline{\psi} - \overline{\psi})I_{\bar{t}=\text{odd}} & \bar{t} > 1 \end{cases} \tag{36}$$

where $\underline{\psi} \neq \overline{\psi}$ denote the values of the Lagrange multipliers, $\psi_{gov,t+j} = \overline{\psi}$ for $j = 0, 2, 4, ...$ and $\psi_{gov,t+j} = \underline{\psi}$ for $j = 1, 3, 5, ...$

Proof: See appendix.

The appendix provides an analytical formula for $\underline{\psi}, \overline{\psi}$.

There are two key messages in Proposition 4 that are worth highlighting: First, inflation in $t+1$ will generally not equal zero. Second, inflation will persist from period $t+2$ onwards and follow a 2 period cycle. Generically, $\underline{\psi} < \overline{\psi}$ and so inflation will be positive when $\bar{t}$ is even (i.e. in periods $t+2, t+4, ..$ etc) and negative when $\bar{t}$ is odd (periods $t+3, t+5, ..$ etc). From equations (34) and (35) it is clear why this is so. $\hat{\pi}_t$ will adjust to satisfy (34), however, since $\hat{\pi}_t > 0$ when a positive shock has hit, $\hat{\pi}_{t+1}$ must turn negative to satisfy (35). Then, $\hat{\pi}_{t+2}$ will be positive again to satisfy the intertemporal constraint in $t+2$ and subsequently, $\hat{\pi}_{t+3}$ will have to compensate for this, in order to satisfy the constraint in $t+3$. This process goes on indefinitely.

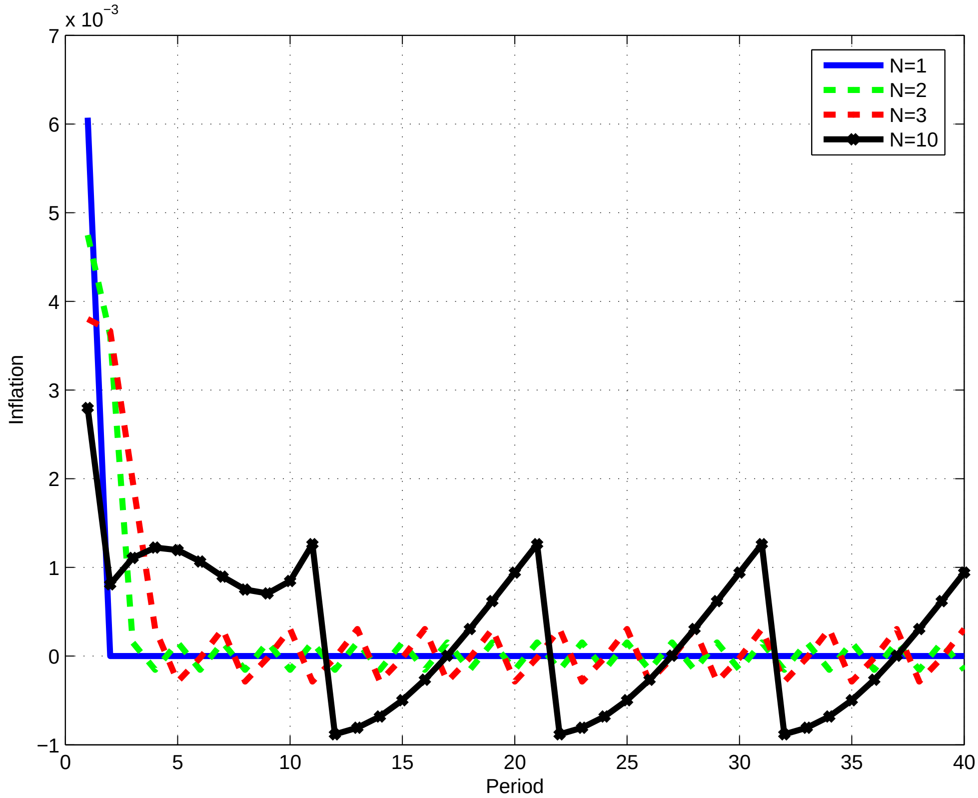
4.1.2 The effect of maturity.

In Figure 5 we use our baseline calibration to solve the optimal policy equilibrium under different values of N.[18] Consider first the cases where debt is long term, i.e. $N \geq 2$. As is evident from the Figure, optimal inflation follows an N period cycle which starts N periods after the shock hits. The pattern highlighted in Proposition 4 thus holds more generally, across all $N \geq 2$ considered in the Figure. Moreover, the Figure shows that the longer the maturity is, the larger is the volatility displayed by inflation. Thus issuing long term debt under no buyback does not enable to spread the burden of inflation over time. In the buyback model of Section 2, the opposite property held.

Finally, consider the case where $N = 1$ (solid blue line). Debt is short term and so repurchasing debt is coincident to redeeming debt at maturity; the dynamics of inflation are effectively the same as the analogous dynamics in the model of Section 2.

[18] Period t is period 1 in the graph.

Figure 5: **Responses to the spending shock under no buyback.**



Notes: The figure plots the path of optimal inflation in response to a shock that increases spending by 20% (from 10% of GDP to 12% of GDP) under various maturity structures and assuming no debt repurchases.

4.2 Interest Rate Rules under No Buy Back

We have seen that when long term bonds are redeemed at maturity, the optimal policy gives rise to oscillations in inflation that persist indefinitely. The longer is the maturity of debt the larger are the fluctuations in inflation after period N.

These features emerge from the Ramsey solution and so clearly they represent the best possible competitive equilibrium outcome given the set of constraints that define the equilibrium and the assumptions that we made in the model. Nonetheless, an equilibrium outcome in which inflation fluctuates periodically may not be desirable from a practical standpoint. First, because the interest rate rule that can implement this outcome will likely be a complicated function of macroeconomic conditions and debt dynamics, and thus do not conform with the principle that a policy rule should be a simple transparent function of macroeconomic variables. Second, foreseeable inflation oscillations may be disruptive in various contexts, including in financial markets or in terms of firms' pricing decisions. Third, oscillations may imply that the nominal interest rate will periodically be at its effective lower bound.

Our purpose here is not to acknowledge these issues explicitly; we want to investigate whether alternative outcomes in which inflation does not feature considerable oscillations are available when a policy rule of the form (17) is implemented by the monetary authority.

We reach a very negative result: In a model where debt repurchases are ruled out and government debt is long term, a policy rule (17) does not yield a unique non-explosive solution. The appendix proves the following Proposition:

Proposition 5. *Consider the no buyback model when monetary policy follows (17), the consolidated budget constraint is given by (29), together with the Phillips curve and the Euler equation. The dynamic system has $N+1$ eigenvalues outside the unit circle for N forward looking variables for all $\phi_\pi \in [0,1]$. Thus, there is no non-explosive solution.*

Proof: See appendix.

This result will also hold for $\phi_\pi \notin [0,1]$. We focus on the usual region where monetary policy is passive, however, assuming (say) $\phi_\pi > 1$ will only add another unstable root to the system. Given that the dynamic system is unstable, it is evident that inflation is an explosive process, a far worse outcome than the stable oscillations we had previously. An equilibrium in which inflation does not feature oscillations and monetary policy follows a simple rule as in (17) is not available.

The result in Proposition 5 suggests that in the presence of long term debt and no buyback, the usual property that a unique stable equilibrium obtains in the fiscal theory when the Taylor principle is violated (i.e. $\phi_\pi < 1$) will not hold. This will also apply if interest rates are set contingently on current output or lagged interest rates. For example, a policy rule (26) will not deliver a non-explosive solution even when $\phi_\pi + \frac{\phi_Y}{\kappa_1}(1-\beta) \leq 1$. This finding should be of interest.

4.2.1 Stability through odd rules

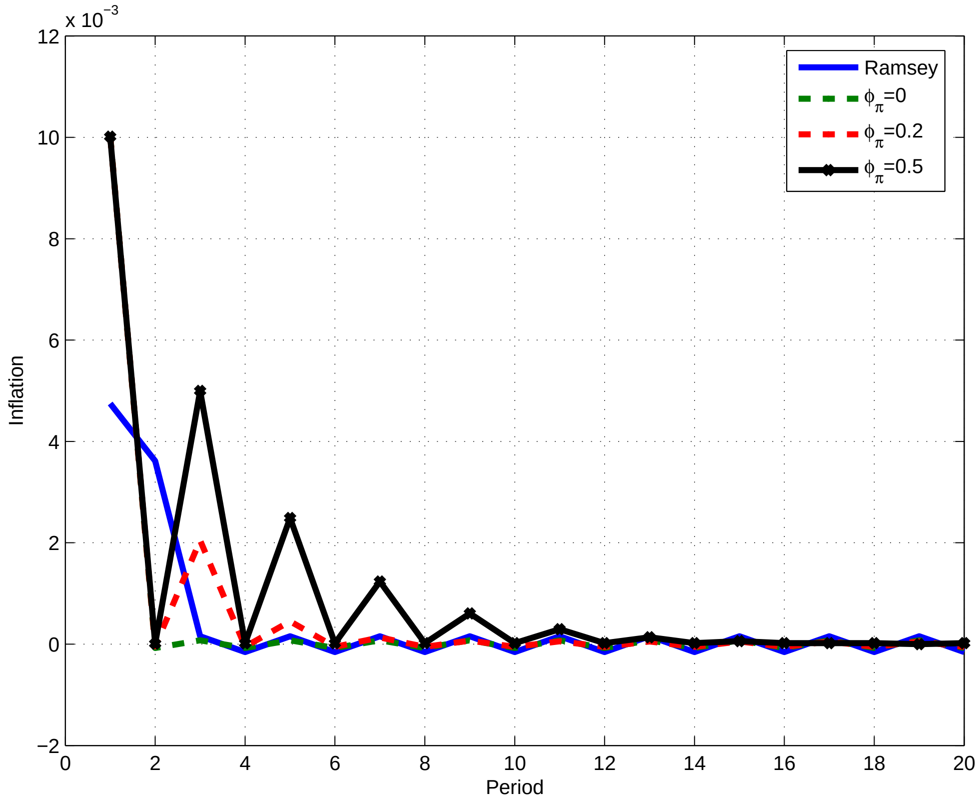
Key to the above result is that policy rule (17) does not pin down the lagged inflation terms on the RHS of (29), $\hat{\pi}_t, \hat{\pi}_{t-1}, ..., \hat{\pi}_{t-N+1}$.

It turns out that to obtain a stable equilibrium in this model, policy has to be specified in such way so that $z^t_{t-N+1} \equiv \hat{\pi}_t + \hat{\pi}_{t-1} + ... + \hat{\pi}_{t-N+1}$ solves a stable difference equation. Thus, a rule that gives

$$z^{t+N}_{t+1} = \epsilon z^t_{t-N+1} + \text{Forcing Terms}$$

will work provided that ϵ lies within the unit circle.

Figure 6: **Responses to the spending shock under no buyback.**



Notes: The figure plots the path of inflation in response to a shock that increases spending by 20% (from 10% of GDP to 12% of GDP). We assume $N = 2$ and no debt repurchases. $\phi_\pi = 0, 0.2, 0.5$ are the assumed values of the coefficient in (37). 'Ramsey' stands for the optimal Ramsey policy outcome.

For simplicity, let us focus on the case $N = 2$. A policy rule that can work is of the form:

$$\hat{i}_t = \phi_\pi \left(\hat{\pi}_t + \hat{\pi}_{t-1} \right) - E_t \hat{\pi}_{t+2} \tag{37}$$

which determines z as the non-explosive solution to

$$E_t \left(\hat{\pi}_{t+2} + \hat{\pi}_{t+1} \right) = \phi_\pi \left(\hat{\pi}_t + \hat{\pi}_{t-1} \right)$$

This solution will not avoid oscillations in inflation. The higher is ϕ_π, the larger will be the oscillations in inflation, and the more persistent z will be. This is shown in Figure 6 which plots the responses of inflation to a spending shock under various values of parameter ϕ_π along with the optimal Ramsey policy. Notice that no matter the size of the coefficient ϕ_π, the initial response of inflation is the same (close to 0.1 in the graph). Future expected inflation oscillations do not add anything in terms of making debt more sustainable when the shock hits, but also do not worsen the equilibrium outcome. Debt issuance will adjust to ensure satisfaction of future intertemporal constraints. These properties suggest that in this model it should be optimal to set $\phi_\pi \approx 0$. This would also be (partially) consistent with Proposition 4, which showed that z becomes zero, three periods after the shock in the Ramsey policy.

4.3 Solving the Problem through Issuing Short Bonds

Policies of the form (37) produce a unique stable equilibrium (albeit one featuring oscillations) but we cannot claim that they are obviously practically relevant. Not necessarily because they prescribe targeting future inflation; rather the issue is that at long debt maturity N, a stable equilibrium path is attained when the nominal interest rate tracks $N-1$ leads and N lags of inflation, to pin down z. A high N makes the number of variables that interest rates must respond to very large. Such a policy would be in practice difficult to implement as well as to interpret.

Rather than insisting on finding a policy rule that relies on fewer variables to deliver a non-explosive solution, we now take a different perspective and consider the role of the debt maturity structure in determining the policy rule. In particular, we ask: Are there maturity structures such that a simple rule of the form (17) gives us a stable equilibrium under no buyback? Under such debt structures what is the optimal inflation coefficient?

Notice that while in the previous examples we focused on a single zero coupon N bond, to develop analytical insights into how no buyback affects feasible allocations under optimal policy, in practice governments issue debt in different instruments, both long and short term. Under short term debt no buyback is coincident with buyback and inflation oscillations do not occur. From our previous analysis we know that in this case, a simple rule that sets $\phi_\pi^* = 1 - \frac{1}{\text{Maturity}}$ will deliver the stable optimal policy equilibrium.

Do we get an analogous result in the case where both short and long bonds are issued, and there are no repurchases of long term debt? Let us first go back to Ramsey policy equilibrium and consider that $\overline{b}_1, \overline{b}_2, ..., \overline{b}_N > 0$. To simplify our derivations we fix the portfolio shares assuming that $\hat{b}_{t,1} = \hat{b}_{t,2} = ... = \hat{b}_t$. We also assume the following debt structure: $\overline{b}_1 = \overline{b}_2, ..., \overline{b}_{N-1} = \overline{b} \leq \overline{b}_N = \overline{\widetilde{b}}$.

The last assumption imposes that the bond quantities are $\overline{b}$ for maturities 1 to $N-1$ and then $\overline{\widetilde{b}}$ for maturity N. This nests the case of a flat maturity structure when $\overline{b} = \overline{\widetilde{b}}$, which becomes a consol bond when $N \to \infty$. In the case where $\overline{b} < \overline{\widetilde{b}}$ the government issues debt in a long bond that pays constant coupons equal to $\frac{\overline{b}}{\overline{\widetilde{b}}}$ and the principal can be normalized to 1. In all these cases some short maturity debt is being issued.

Under these assumptions the budget constraint is:

$$\begin{aligned}\hat{b}_t\left(\overline{b}\sum_{j=1}^{N-1}\beta^j + \beta^N\overline{\widetilde{b}}\right) - \overline{b}\sum_{j=1}^{N-1}\beta^j\sum_{l=1}^{j}E_t\hat{\pi}_{t+l} \quad &- \quad \overline{\widetilde{b}}\beta^N\sum_{l=1}^{N}E_t\hat{\pi}_{t+l} + \overline{S}\hat{S}_t \\ &= \quad \overline{b}\sum_{j=1}^{N-1}\left(\hat{b}_{t-j} - \sum_{l=0}^{j-1}\hat{\pi}_{t-l}\right) + \beta^N\overline{\widetilde{b}}\left(\hat{b}_{t-N} - \sum_{l=0}^{N-1}\hat{\pi}_{t-l}\right)\end{aligned}$$

The first order condition of the Ramsey program with respect to $\hat{b}_t$ is given by:

$$\psi_{gov,t}\left(\overline{b}\sum_{j=1}^{N-1}\beta^j + \beta^N\overline{\widetilde{b}}\right) - \left[\overline{b}\left(\sum_{j=1}^{N-1}\beta^j E_t\psi_{gov,t+j}\right) + \overline{\widetilde{b}}\beta^N E_t\psi_{gov,t+N}\right] = 0 \tag{38}$$

which implies non-trivial dynamics linking the current value of the multiplier with the expected future values up to period $t+N$.

To analyze these dynamics let us revisit the previous example where a shock hits the economy in t and no further shock is expected thereafter. We can then derive the following Nth order difference equation that determines $\psi_{gov,t}$:

$$\psi_{gov,t+N} + \frac{\overline{b}}{\overline{\widetilde{b}}}\left(\frac{1}{\beta}\psi_{gov,t+N-1} + \frac{1}{\beta^2}\psi_{gov,t+N-2} + ... + \frac{1}{\beta^{N-1}}\psi_{gov,t+1}\right) - \left(\frac{\overline{b}}{\overline{\widetilde{b}}}\frac{1-\frac{1}{\beta^N}}{1-\frac{1}{\beta}} + 1\right)\psi_{gov,t} = 0 \tag{39}$$

The above equation has one root that is equal to 1. In the case where $N = 2$ there is another real root, equal to $-(1 + \frac{\bar{b}}{\tilde{\tilde{b}}}\frac{1}{\beta}) < -1$. Let us focus on this case for simplicity.[19] We then have that

$$\psi_{gov,t+2}\Big(1 - L\Big)\Big(1 + (1 + \frac{\bar{b}}{\tilde{\tilde{b}}}\frac{1}{\beta})L\Big) = 0$$

where L denotes the lag operator. Notice that when $\bar{b} = 0$ this gives

$$\Delta\psi_{gov,t+2}\Big(1 + L\Big) = 0 \rightarrow \psi_{gov,t+2} = \psi_{gov,t}$$

and as before we obtain a 2 period cycle. However, in the case where $\bar{b} > 0$ we can write

$$-\frac{1}{(1 + \frac{\bar{b}}{\tilde{\tilde{b}}}\frac{1}{\beta})}\Delta\psi_{gov,t+2} = \Delta\psi_{gov,t+1}$$

Assuming a bounded process, or $\lim_{j\to\infty}(-(1 + \frac{\bar{b}}{\tilde{\tilde{b}}}\frac{1}{\beta}))^j\Delta\psi_{gov,t+j} = 0$, delivers the following:

$$\Delta\psi_{gov,t+1} = 0$$

The random walk property of the multiplier is restored.

This is an important property. Recall that in the Ramsey model with a single long bond, inflation oscillations under no buyback resulted from the fact that inflation in period t would compensate for changes in the value of the surplus in periods $t, t+2, t+4, ..$ but not for the surplus in periods $t+1, t+3, ...$ Then, $\hat{\pi}_t$ would absorb the shock $\hat{G}_t$, and this would ensure satisfaction of (34), but it would also perturb (35) so that $\hat{\pi}_{t+1}$ needs to adjust to satisfy the constraint. The effect carried over in other periods. In terms of the Ramsey program, this then meant that (34) and (35) impact the solution differently, or generically $\psi_{gov,t} \neq \psi_{gov,t+1}$. Proposition 4 showed that inflation oscillations can be described in terms of these multipliers.

The fact that $\psi_{gov,t} = \psi_{gov,t+1}$ when short debt is issued implies that inflation oscillations will not occur. We can show that following a shock in $\hat{G}_t$ inflation will evolve according to:

$$\hat{\pi}_{t+\bar{t}} = \left\{\begin{array}{ll} \frac{\overline{R}}{\kappa_1}\Big(1 + \gamma_h\Big)\overline{\psi} + \overline{b\psi} + \widetilde{\overline{b\psi}}\Big(1 + \beta\Big) & \bar{t} = 0 \\ \widetilde{\overline{b\psi}} & \bar{t} = 1 \\ 0 & \bar{t} > 1 \end{array}\right\}$$

where $\overline{\psi}$ now denotes the difference of the Lagrange multiplier between the pre shock value and the value after the shock.

Quite evidently, the response of inflation to the shock resembles the analogous object in the buyback model of Section 2.

4.3.1 Why are short term bonds so important?

Another way of saying that (34) and (35) influence differently the Ramsey solution under no buyback and only long term debt is to say that we cannot add up these constraints in the Ramsey program. If the

[19] Otherwise for $N > 2$ some of the roots of the characteristic equation are complex and moreover it is difficult to factor the characteristic polynomial.

Lagrange multipliers were equal then solving the Ramsey program using the intertemporal constraint

$$E_t \sum_{j=0}^{\infty} \beta^j \overline{S}\hat{S}_{t+j} = \overline{b}_2\left(\hat{b}_{t-2,2} - \hat{\pi}_t - \hat{\pi}_{t-1}\right) + \overline{b}_2\beta\left(\hat{b}_{t-1,2} - E_t\hat{\pi}_{t+1} - \hat{\pi}_t\right) \tag{40}$$

would suffice. Generally, equation (40) is not sufficient for an optimal policy equilibrium under no buyback (see Faraglia et al. (2016)). Instead (34) and (35) are both important implementability conditions, along with the Philips curve.

When short term debt is issued however, the intertemporal constraint (the analogue of equation (40))

$$E_t \sum_{j=0}^{\infty} \beta^j \overline{S}\hat{S}_{t+j} = \overline{b}_2\left(\hat{b}_{t-2,2} - \hat{\pi}_t - \hat{\pi}_{t-1}\right) + \overline{b}_2\beta\left(\hat{b}_{t-1,2} - E_t\hat{\pi}_{t+1} - \hat{\pi}_t\right) + \overline{b}_1\left(\hat{b}_{t-1,1} - \hat{\pi}_t\right) \tag{41}$$

becomes sufficient. The intuition is that the short bond issuance can adjust to satisfy (41) in $t+1$ (and all future periods) given a smooth path of inflation following the shock in spending.

4.3.2 An equivalence result of buyback and no buyback under coupon bearing bonds

Issuing short term debt brings us back to our previous findings, regarding the optimality of simple interest rate rules. We can show that (17) once again leads to a unique non-explosive solution in the model, and the maturity of debt becomes the key variable in determining the optimal inflation coefficient. Instead of working through all previous model versions, let us focus here on the case where the maturity structure is flat and $N \to \infty$. Then (38) becomes:

$$\frac{1}{1-\beta}\psi_{gov,t} = \sum_{j=1}^{\infty} \beta^j E_t \psi_{gov,t+j} = (1-\beta)\beta E_t \frac{1}{1-\beta L^{-1}}\psi_{gov,t+1}$$

which again gives the random walk. The following Proposition defines the optimal interest rate policy in this model:

Proposition 6. *Assume no buyback and long bonds are consols. The optimal path of $\hat{i}_t$ under Ramsey is given by:*

$$\hat{i}_t = \hat{\pi}_t - \frac{\overline{R}}{\kappa_1}(1+\gamma_h)\Delta\psi_{gov,t}$$

which is the same optimal monetary policy as in the buyback model.

Proof: See appendix.

Clearly, in the case where $\overline{R} = 0$ the optimal policy is a rule (17) where the inflation coefficient is $1 - \frac{1}{\text{Maturity}}$ in this case where the average maturity is infinite. The coefficient will be approximately one when $\overline{R} > 0$.

Finally, note that Proposition 6 can be extended to the case of decaying coupon payments. The proof that we provide in the appendix begins from this assumption and shows the more general result.

4.4 Discussion

Issuing short term debt is thus necessary to avoid inflation oscillations as well as to restore the optimality of simple policy rules in the no buyback model.

This result stands out as an interesting point for the Ramsey optimal policy literature. Previous papers studying optimal policy, find that governments should focus on issuing long term bonds as this enables to smooth the path of inflation.[20] This is also the case in our buyback model. To see it, consider the single N bond case in Proposition 1. Coefficients η_{-j} become 0 as $N \to \infty$. Evidently, focusing on the longest maturity available delivers the best possible inflation outcome.

This property does not carry over to the no buyback model. We showed that when one N bond is issued, longer maturity leads to larger inflation oscillations (e.g. Figure 5). A flat maturity (consol) structure will deliver the best policy outcome in this model.

Finally, note that this finding is similar to Faraglia et al. (2019) who show, in a Ramsey model of optimal taxation, that no buyback generates the incentive to issue short term debt. Their non-linear model predicts an optimal debt structure in which short bonds make up for roughly half of the total market value of debt issued. Here we used a simpler (linear) model, to show that a flat maturity structure can enable monetary policy to smooth inflation with a simple inflation targeting rule. Whether a flat maturity would be optimal also in a non-linear model under incomplete markets (when bond issuance may be (realistically) subject to debt constraints as in Faraglia et al. (2019)) remains to be explored.

5 Extensions

We now briefly extend our analysis to other relevant setups of optimal policy. We first show that our results hold in the case where policy optimizes over inflation and distortionary taxes (as in Schmitt-Grohé and Uribe, 2004; Lustig et al., 2008; Faraglia et al., 2013; Sims, 2013; Leeper and Zhou, 2021). We then extend our analysis assuming that the Phillips curve has a backward looking component and to the case where the policy objective includes output gap stabilization. We show that simple rules in which the inflation coefficient is a function of maturity obtain also in these cases.

5.1 Jointly optimal policies

Our results do not hinge on the assumption that taxes are constant through time. We assumed constant taxes in order to simplify the algebra, however, extending the findings to the case where taxes are set optimally by the planner, or follow rules that link the path of taxes to debt, is possible.[21] For brevity we choose to consider only the first scenario, assuming that the planner optimally sets inflation and a distortionary tax to finance debt as in Siu, 2004; Schmitt-Grohé and Uribe, 2004; Faraglia et al., 2013; Sims, 2013; Leeper and Zhou, 2021. We will show that in this policy setup, optimal interest rates continue to follow a rule that targets inflation, the optimal inflation coefficient is equal to δ when debt payment profiles are $\bar{b}_k = \delta^{k-1}\bar{b}$. We focus on decaying coupons since as previously the Ramsey policy first order conditions in this model can be rearranged to derive the simple inflation targeting rule.[22]

[20] Lustig et al. (2008) solve a non-linear model where the planner can choose a portfolio of 7 different maturities. They obtain a corner solution result whereby only the longest maturity available is in positive net supply. In the context of Ramsey optimal taxation models, Angeletos (2002); Buera and Nicolini (2004) obtain similar results.

[21] Obviously tax rules will have to satisfy the condition that fiscal policy is 'active', that is not entail a strong response of fiscal revenue to debt.

[22] The outcome will thus be the same as in the case where the planner is constrained to set the nominal rate according to (17) and sets optimally the inflation coefficient.

Following Sims, 2013 we assume the following objective function for the policy maker:

$$-\frac{1}{2}E_0\sum_{t\geq 0}\beta^t\left(\hat{\pi}_t^2+\lambda_\tau\hat{\tau}_t^2\right) \tag{42}$$

where $\hat{\tau}_t$ is the distortionary tax and $\lambda_\tau > 0$ defines the relative weight attached to tax distortions. In the presence of distortionary taxation the Phillips curve is

$$\hat{\pi}_t = \kappa_1\hat{Y}_t + \kappa_2\hat{\tau}_t + \beta E_t\hat{\pi}_{t+1}$$

where now $\kappa_2 \equiv -\frac{(1+\eta)\overline{Y}}{\theta}\frac{\overline{\tau}}{(1-\overline{\tau})} > 0$. The consolidated budget constraint (assuming for any debt repayment profile) is:

$$\sum_{k=1}^{\infty}\beta^k\overline{b}_k(\hat{b}_{t,k}-\sum_{l=1}^{k}E_t\hat{\pi}_{t+l})+\overline{R}\left((1+\gamma_h)\hat{Y}_t+\frac{\hat{\tau}_t}{1-\overline{\tau}}\right)-\overline{G}\hat{G}_t=\overline{b}_1(\hat{b}_{t-1,1}-\hat{\pi}_t)+\sum_{k=2}^{\infty}\beta^{k-1}\overline{b}_k(\hat{b}_{t-1,k}-\sum_{l=0}^{k-1}E_t\hat{\pi}_{t+l}) \tag{43}$$

where now the government's revenue $\overline{R}\left((1+\gamma_h)\hat{Y}_t+\frac{\hat{\tau}_t}{1-\overline{\tau}}\right)$ can change both due to output fluctuations and due to changes in taxes.

It is easy to show that the first order conditions for inflation, output and debt, when the Ramsey planner maximizes (42) subject to the Phillips curve and the budget constraint, are again given by equations (8), (9) and (10). Using the assumption $\overline{b}_k = \delta^{k-1}\overline{b}$ this then implies that from this model we can derive the same optimal policy as as in Section 3.1.3. The optimal interest rate rule will be $\hat{i}_t = \delta\hat{\pi}_t - \delta\overline{R}\frac{(1+\gamma_h)}{\kappa_1}\Delta\psi_{gov,t}$.

Finally, to characterize the behavior of taxes in this model, we can use the first order condition and rearranging we will get to the following expression linking the tax rate to the multiplier on the consolidated budget:

$$\lambda_\tau\hat{\tau}_t = \psi_{gov,t}\frac{(1+\eta)\overline{\tau}\overline{Y}}{\eta(1-\overline{\tau})}[1-\frac{\overline{\tau}\gamma_h}{1+\gamma_h}] \tag{44}$$

Notice that $[1-\frac{\overline{\tau}\gamma_h}{1+\gamma_h}]$ exceeds 0, otherwise the economy will be on the wrong side of the Laffer curve. (44) then shows that taxes will respond to shocks that hit the consolidated budget. A positive spending shock will tighten the budget constraint and lead to $\Delta\psi_{gov,t} > 0$. Taxes will adjust upwards permanently; according to (44) they will follow a random walk, a standard property of optimal Ramsey taxation models (see e.g. Aiyagari et al., 2002).

The only way in which the availability of taxes as an additional policy instrument will affect our previous results, is through impacting the magnitude of the response of inflation to the shocks. When the welfare costs of tax distortions are less than the costs due to inflation (that is when λ_τ is low and the degree of price stickiness θ is high), the planner will opt for using taxes to adjust the intertemporal surplus (e.g. Siu, 2004; Schmitt-Grohé and Uribe, 2004; Faraglia et al., 2013). When the opposite holds, relying more on inflation to finance debt becomes optimal (e.g. Sims, 2013; Leeper and Zhou, 2021). The time-path of inflation will not change.

5.2 Alternative Specifications

5.2.1 Output and interest rate stabilization

Our model has been purposefully kept simple to enable us to derive analytical insights exploring the interplay between the debt maturity structure and optimal policy. We assumed a Fischerian setup (as in Cochrane (2001) and Sims (2013)) and also assumed that the policy objective is to minimize the volatility of inflation.[23]

A key finding that emerges from our derivations is that in this standard fiscal theory framework, optimal monetary policy sets the inflation coefficient in response to the maturity of debt through a simple formula. In the case where the debt repayment profile is $\overline{b}_k = \delta^{k-1}\overline{b}$ (i.e. with a decaying coupon bond) simple inflation targeting rules are robustly optimal, under both buyback and no buyback. We now exploit this result to extend our analysis to alternative assumptions regarding the objective of policy. In particular, we assume that policy authorities have an objective that targets not only inflation but also output and a smooth path for the nominal rate and extend our derivations to these cases. In keeping with the goal of deriving optimal interest rate rules analytically we continue to assume a Fischerian economy.

As in Giannoni and Woodford (2003b) let us assume that the objective of policy is

$$-\frac{1}{2}E\sum_{t\geq 0}\left(\hat{\pi}_t^2 + \lambda_Y \hat{Y}_t^2 + \lambda_i \hat{i}_t^2\right)$$

Consider first the case where the $\lambda_i > 0$, assuming $\lambda_Y = 0$. We can show that the optimal inflation path under Ramsey is given by:

$$\hat{\pi}_t = \overline{R}\frac{(1+\gamma_h)}{\kappa_1}\Delta\psi_{gov,t} - \frac{\lambda_i}{\beta}\hat{i}_{t-1} + \frac{b}{1-\beta\delta}\sum_{l=0}^{\infty}\delta^k \Delta\psi_{gov,t-l}$$

Using the Euler equation we can thus show that

$$\text{(45)} \qquad \hat{i}_t = \widetilde{\lambda}^{-1}\frac{\delta b}{1-\beta\delta}\sum_{l=0}^{\infty}\delta^k\Delta\psi_{gov,t-l} = \widetilde{\lambda}^{-1}\delta\left(\hat{\pi}_t - \overline{R}\frac{(1+\gamma_h)}{\kappa_1}\Delta\psi_{gov,t}\right) + \delta\widetilde{\lambda}^{-1}\left(\widetilde{\lambda}-1\right)\hat{i}_{t-1}$$

where $\widetilde{\lambda} = \left(1 + \frac{\lambda_i}{\beta}\right)$. Optimal policy is an inertial rule with inflation coefficient $\widetilde{\lambda}^{-1}\delta$. Once again the maturity of debt is the key variable that determines the optimal response to inflation.

Next, consider the case where $\lambda_Y > 0$ and $\lambda_i = 0$. In the appendix we derive an interest rate policy for this model. We obtain the following:

$$\hat{i}_t = (\lambda_1 + \delta)\hat{\pi}_t - \lambda_1\delta\hat{\pi}_{t-1} + \text{Stochastic Intercept}$$

where $0 < \lambda_1(\beta, \kappa_1, \lambda_Y) < 1$ is the stable root of the second order difference equation that governs the dynamics of optimal inflation. The intercept term, as usual, involves the multiplier $\Delta\psi_{gov,t}$. The appendix delivers the complete analytical expressions.

Both variants of the benchmark model we considered deliver an interest rate rule where the inflation coefficient is a simple function of $1 - \frac{1}{\text{Maturity}}$.

5.2.2 Inflation inertia in the Phillips curve

Our findings can also be extended to the case where the Phillips curve has both a backward and forward looking component. As Giannoni and Woodford (2003b) consider the case where the Phillips curve is

$$(\hat{\pi}_t - \gamma\hat{\pi}_{t-1}) = \kappa_1\hat{Y}_t + \beta E_t(\hat{\pi}_{t+1} - \gamma\hat{\pi}_t)$$

[23] This is justified in our Fischerian model, fluctuations in output or other macroeconomic variables will not matter for welfare, or not matter a lot. See appendix for a derivation of the loss function from a second order approximation of the household's utility.

where $0 < \gamma < 1$ and the objective of the planner is given by:

$$-\frac{1}{2}E\sum_{t\geq 0}(\hat{\pi}_t - \gamma\hat{\pi}_{t-1})^2$$

In the appendix we show that the optimal interest rate rule is:

$$\hat{i}_t = (\gamma + \delta)\hat{\pi}_t - \gamma\delta\hat{\pi}_{t-1} - \delta\frac{\overline{R}}{\kappa_1}(1 + \gamma_h)\Delta\psi_{gov,t}$$

Thus, also in this case, debt maturity is the key determinant of the optimal inflation coefficient.

6 Conclusion

We studied optimized interest rate rules in the context of the fiscal theory of the price level. Our key result is that simple inflation targeting rules can approximate closely the Ramsey outcome. The optimal inflation coefficient depends on the average debt maturity. We derive simple formulae showing this dependence. We also investigate how departing from the canonical modelling of long bonds found in the literature, that is by making the empirically grounded assumption of no repurchases of long term debt affects our results. Under no buyback, simple inflation targeting rules work only when the government issues a portfolio with positive amounts of both short and long term bonds. Otherwise, the optimal policy equilibrium features excess volatility of inflation, which takes the form of persistent oscillations. Contrarily to previous studies that ignore the no buy-back constraint, we conclude that short-term debt has an important role to play in the stabilization of inflation.

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Appendices

Appendix A Proofs of propositions

A.1 Proof of proposition 1

Optimal inflation is given by:

$$\hat{\pi}_t = \overline{R}\frac{(1+\gamma_h)}{\kappa_1}\Delta\psi_{gov,t} + \overline{b}_N\Big(\beta^{N-1}\Delta\psi_{gov,t} + \beta^{N-2}\Delta\psi_{gov,t-1} + ... + \Delta\psi_{gov,t-N+1}\Big) \tag{46}$$

We can write the intertemporal consolidated budget constraint ((12) in text) as

$$\frac{\overline{R}}{\kappa_1}(1+\gamma_h)\hat{\pi}_t - \overline{G}\hat{G}_t = \overline{b}_N\hat{b}_{t-1,N} - \beta^{N-1}\overline{b}_N E_t\Big(\hat{\pi}_t + \hat{\pi}_{t+1} + ... + \hat{\pi}_{t+N-1}\Big)$$

Letting $\omega \equiv \frac{\overline{R}}{\kappa_1}(1+\gamma_h)$, replacing (46) into the constraint and using the random walk property of the multiplier we get:

$$\omega\Big[\omega\Delta\psi_{gov,t} + \overline{b}_N\Big(\beta^{N-1}\Delta\psi_{gov,t} + \beta^{N-2}\Delta\psi_{gov,t-1} + ... + \Delta\psi_{gov,t-N+1}\Big)\Big] - \overline{G}\hat{G}_t =$$

$$\overline{b}_N\hat{b}_{t-1,N} - \beta^{N-1}\overline{b}_N\Big[\underbrace{\omega\Delta\psi_{gov,t} + \overline{b}_N\Big(\beta^{N-1}\Delta\psi_{gov,t} + \beta^{N-2}\Delta\psi_{gov,t-1} + ... + \Delta\psi_{gov,t-N+1}\Big)}_{\hat{\pi}_t}$$

$$+\underbrace{\overline{b}_N\Big(\beta^{N-2}\Delta\psi_{gov,t} + ... + \Delta\psi_{gov,t-N+2}\Big)}_{E_t\hat{\pi}_{t+1}} + ... + \underbrace{\overline{b}_N\Delta\psi_{gov,t}}_{E_t\hat{\pi}_{t+N-1}}\Big]$$

Noting that in the absence of any shock in t the lagged terms on the LHS and RHS of the above equation must cancel out for the intertemporal constraint to hold, we have:

$$\omega\Big[\omega\Delta\psi_{gov,t} + \overline{b}_N\beta^{N-1}\Delta\psi_{gov,t}\Big] - \overline{G}\hat{G}_t = -\beta^{N-1}\overline{b}_N\Big[\omega\Delta\psi_{gov,t} + \overline{b}_N\Delta\psi_{gov,t}\Big(1+\beta+...+\beta^{N-1}\Big)\Big]$$

Thus:

$$\Bigg[\Big(\omega + \overline{b}_N\beta^{N-1}\Big)^2 + (\beta^{N-1}\overline{b}_N)^2\Bigg(\frac{\frac{1}{\beta^N}-1}{\frac{1}{\beta}-1}\Bigg)\Bigg]\Delta\psi_{gov,t} = \overline{G}\hat{G}_t$$

Using this result in (46) delivers the expression in Proposition 1. ■

A.2 Proof of proposition 2

The proof is analogous to that of Proposition 1. We can write the intertemporal constraint (12) as

$$\frac{\overline{R}}{\kappa_1}(1+\gamma_h)\hat{\pi}_t - \overline{G}\hat{G}_t = \overline{d}\hat{d}_{t-1} - \sum_{k=1}^{\infty}\beta^{k-1}\overline{b}_k\hat{\pi}_t - \sum_{k=2}^{\infty}\beta^{k-1}\overline{b}_k\sum_{l=1}^{k-1}E_t\hat{\pi}_{t+l} \tag{47}$$

Using the random walk property of the multiplier we have

$$E_t\hat{\pi}_{t+l} = \sum_{k=l}^{\infty}\overline{b}_k\sum_{i=l+1}^{k}\beta^{k-i}\Delta\psi_{gov,t+l-i+1} \tag{48}$$

Noting that all lagged terms $t-1, t-2, ...$ will cancel out in the intertemporal constraint, we get the following expression that pins down $\Delta\psi_{gov,t}$.

$$\Delta\psi_{gov,t}\left(\frac{\overline{R}}{\kappa_1}(1+\gamma_h) + \sum_{k=1}^{\infty}\beta^{k-1}\overline{b}_k\right)^2 + \Delta\psi_{gov,t}\sum_{k=2}^{\infty}\beta^{k-1}\overline{b}_k\sum_{l=1}^{k-1}\sum_{i=l+1}^{\infty}\overline{b}_i\beta^{i-(l+1)} = \overline{G}\hat{G}_t \tag{49}$$

Moreover we have:

$$\begin{aligned}\sum_{k=2}^{\infty}\beta^{k-1}\overline{b}_k\sum_{l=1}^{k-1}\sum_{i=l+1}^{\infty}\overline{b}_i\beta^{i-(l+1)} &= \sum_{k=2}^{\infty}\beta^{k-1}\overline{b}_k\sum_{l=2}^{k}\sum_{i=l}^{\infty}\overline{b}_i\beta^{i-l} = \sum_{k=2}^{\infty}\beta^{k-1}\overline{b}_k\sum_{l=2}^{\infty}\mathcal{I}_{l\le k}\sum_{i=l}^{\infty}\overline{b}_i\beta^{i-l} \\ &= \sum_{l=2}^{\infty}\beta^{l-1}\underbrace{\left(\sum_{k=l}^{\infty}\beta^{k-l}\overline{b}_k\right)}_{\lambda_l}\underbrace{\left(\sum_{i=l}^{\infty}\overline{b}_i\beta^{i-l}\right)}_{\lambda_l} = \sum_{l=2}^{\infty}\beta^{l-1}\lambda_l^2\end{aligned}$$

To see the last equality, that $\sum_{k=l}^{\infty}\beta^{k-l}\overline{b}_k \equiv \lambda_l$ use the definition of λ in text. We stated that:

$$\begin{aligned}\lambda_1 &= \sum_{j=1}\beta^{j-1}\overline{b}_j = \frac{\overline{S}}{1-\beta} \\ \lambda_2 &= \frac{1}{\beta}\left(\lambda_1 - \overline{b}_1\right) = \frac{1}{\beta}\left(\sum_{j\ge 1}\beta^{j-1}\overline{b}_j - \overline{b}_1\right) = \sum_{j\ge 2}\beta^{j-2}\overline{b}_j \\ \lambda_3 &= \frac{1}{\beta}\left(\lambda_2 - \overline{b}_2\right) = \frac{1}{\beta}\left(\sum_{j\ge 2}\beta^{j-2}\overline{b}_j - \overline{b}_2\right) = \sum_{j\ge 3}\beta^{j-3}\overline{b}_j\end{aligned}$$

and so on. With these formulae, (A.2) becomes

$$\Delta\psi_{gov,t}\left(\widetilde{f}^2 + \sum_{l=2}^{\infty}\beta^{l-1}\lambda_l^2\right) = \overline{G}\hat{G}_t$$

The coefficients η_{-j} follow easily from the above. ■

A.3 Proof of Proposition 3

With the assumptions of Proposition 3 the first order condition can be written as:

$$(50)\qquad \left[\frac{\beta\phi_\pi}{1-\beta\phi_\pi^2}\left(\sum_{k=1}^{\infty}\beta^{k-1}\delta^{k-1}\frac{1-\phi_\pi^k}{1-\phi_\pi}\right)-\sum_{k=1}^{\infty}\beta^{k-1}\delta^{k-1}\frac{1}{(1-\phi_\pi)^2}\left(1+(k-1)\phi_\pi^k-k\phi_\pi^{k-1}\right)\right]=0$$

Expanding the sums and using the geometric formula we get

$$\frac{\beta\phi_\pi}{1-\beta\phi_\pi^2}\frac{1}{1-\phi_\pi}\left(\frac{1}{1-\beta\delta}-\frac{\phi_\pi}{1-\phi_\pi\beta\delta}\right)=\frac{1}{(1-\phi_\pi)^2}\left[\frac{1}{1-\beta\delta}+\frac{\phi_\pi^2\beta\delta-1}{(1-\beta\delta\phi_\pi)^2}\right]$$

This reduces to

$$\frac{\phi_\pi}{1-\beta\phi_\pi^2}=\frac{\delta}{(1-\beta\delta\phi_\pi)}$$

It is obvious that $\phi_\pi=\delta$ is the solution. ■

A.4 Proof of proposition 4

Assume that the economy is hit by a shock in t and after there are no more shocks. To simplify, assume initial conditions $\psi_{gov,t-1}=\psi_{gov,t-2}=...=0$ and $\hat{b}_{t-1,2},\hat{b}_{t-2,2}=0$. Assume further that $\hat{\pi}_{t-1}=0$. Optimal Ramsey inflation satisfies:

$$(51)\qquad \hat{\pi}_{t+\bar{t}}=\overline{R}\frac{(1+\gamma_h)}{\kappa_1}\Delta\psi_{gov,t+\bar{t}}+\bar{b}_2\left(\beta(\psi_{gov,t+1+\bar{t}}-\psi_{gov,t-1+\bar{t}})+(\psi_{gov,t+\bar{t}}-\psi_{gov,t-2+\bar{t}})\right)$$

From $\psi_{gov,t+\bar{t}}=\psi_{gov,t+\bar{t}+2}$ we define:

$$\overline{\psi}=\psi_{gov,t}=\psi_{gov,t+2}=\psi_{gov,t+4}=...$$
$$\underline{\psi}=\psi_{gov,t+1}=\psi_{gov,t+3}=\psi_{gov,t+5}=...$$

We then have the following path for inflation:

$$\hat{\pi}_t=\overline{R}\frac{(1+\gamma_h)}{\kappa_1}\overline{\psi}+\bar{b}_2\left(\overline{\psi}+\beta\underline{\psi}\right)$$
$$\hat{\pi}_{t+1}=\overline{R}\frac{(1+\gamma_h)}{\kappa_1}(\underline{\psi}-\overline{\psi})+\bar{b}_2\left(\underline{\psi}+\beta\overline{\psi}\right)$$
$$\hat{\pi}_{t+2}=\hat{\pi}_{t+4}=...=\overline{R}\frac{(1+\gamma_h)}{\kappa_1}(\overline{\psi}-\underline{\psi})+\bar{b}_2(1-\beta)\left(\overline{\psi}-\underline{\psi}\right)$$
$$\hat{\pi}_{t+3}=\hat{\pi}_{t+5}=...=-\overline{R}\frac{(1+\gamma_h)}{\kappa_1}(\overline{\psi}-\underline{\psi})-\bar{b}_2(1-\beta)\left(\overline{\psi}-\underline{\psi}\right)$$

To verify that indeed $\underline{\psi}\neq\overline{\psi}$ use the intertemporal budget constraints. The following two objects are sufficient for an equilibrium:

$$-\overline{b}_2\hat{\pi}_t = \sum_{j\geq 0}\beta^{2j}\overline{R}(1+\gamma_h)\hat{Y}_{t+2j} - \hat{G}_t \tag{52}$$

$$-\overline{b}_2(\hat{\pi}_t + \hat{\pi}_{t+1}) = \sum_{j\geq 0}\beta^{2j}\overline{R}(1+\gamma_h)\hat{Y}_{t+2j+1} \tag{53}$$

Using the Phillips curve we can write the first condition as:

$$-\overline{b}_2\hat{\pi}_t = -\overline{b}_2\left[\frac{\overline{R}}{\kappa_1}(1+\gamma_h)\overline{\psi} + \overline{b}_2\left(\overline{\psi} + \beta\underline{\psi}\right)\right] = \frac{\overline{R}(1+\gamma_h)}{\kappa_1}\left[(\hat{\pi}_t - \beta\hat{\pi}_{t+1}) + \beta^2(\hat{\pi}_{t+2} - \beta\hat{\pi}_{t+3}) + ...\right] - \hat{G}_t$$

Letting $\omega = \frac{\overline{R}(1+\gamma_h)}{\kappa_1}$, we can simplify this equation:

$$-\overline{b}_2\left[\omega\overline{\psi} + \overline{b}_2\left(\overline{\psi} + \beta\underline{\psi}\right)\right] = \omega\ \left[(\omega + \overline{b}_2)\overline{\psi} + \overline{b}_2\beta\underline{\psi} - \beta(\omega + \overline{b}_2)\underline{\psi}\right] + \frac{\beta^2}{1-\beta}\omega^2(\overline{\psi} - \underline{\psi}) - \hat{G}_t$$

Rearranging we get

$$-\underbrace{\left[(\omega + \overline{b}_2)^2 + \frac{\beta^2}{1-\beta}\omega^2\right]}_{\epsilon_1}\overline{\psi} - \beta\underbrace{\left[\overline{b}_2^2 - \frac{\omega^2}{1-\beta}\right]}_{\epsilon_2}\underline{\psi} = -\hat{G}_t$$

Moreover, rather than using the second intertemporal constraint, we use the sum of the two constraints. This gives us:

$$-\overline{b}_2\hat{\pi}_t - \overline{b}_2\beta(\hat{\pi}_t + \hat{\pi}_{t+1}) = \omega\hat{\pi}_t - \hat{G}_t$$

or

$$-\underbrace{(\overline{b}_2(1+\beta) + \omega)\left[(\omega + \overline{b}_2)\right]}_{\epsilon_3}\overline{\psi} - \underbrace{\overline{b}_2\beta\left[2\omega + \overline{b}_2(2+\beta)\right]}_{\epsilon_3}\underline{\psi} = -\hat{G}_t$$

Solving the two equations gives:

$$\overline{\psi} = \frac{\epsilon_4 - \epsilon_2}{\epsilon_4\epsilon_1 - \epsilon_2\epsilon_3}, \qquad \underline{\psi} = \frac{\epsilon_1 - \epsilon_3}{\beta(\epsilon_4\epsilon_1 - \epsilon_2\epsilon_3)}$$

Generically $\overline{\psi} \neq \underline{\psi}$. ■

A.5 Proof of Proposition 5

For simplicity, we prove the Proposition for $N = 2$. The proof is analogous in the case $N > 2$.

Assume that monetary policy sets $\hat{i}_t = \phi_\pi\hat{\pi}_t$. We will show that for all ϕ_π the equilibrium is explosive.

The model equations are the following:

$$\hat{\pi}_t = \kappa_1 \hat{Y}_t + \beta E_t \hat{\pi}_{t+1}$$

$$\overline{b}_2 \beta^2 \Big(\hat{b}_{t,2} - E_t(\hat{\pi}_{t+1} + \hat{\pi}_{t+2}) \Big) + \overline{R} \Big(\gamma_h + 1 \Big) \hat{Y}_t - \overline{G}\hat{G}_t = \overline{b}_2 \Big(\hat{b}_{t-2,2} - \hat{\pi}_t - \hat{\pi}_{t-1} \Big) \tag{54}$$

$$\phi_\pi \hat{\pi}_t = E_t \hat{\pi_{t+1}}$$

Substituting out the Phillips curve and using the last equation to substitute out expectations we have:

$$\overline{b}_2 \beta^2 \Big(\hat{b}_{t,2} - \phi_\pi (1+\phi_\pi) \hat{\pi}_t \Big) + \overline{R} \frac{\Big(\gamma_h + 1\Big)}{\kappa_1} \hat{\pi}_t (1 - \beta\phi_\pi) - \overline{G}\hat{G}_t = \overline{b}_2 \Big(\hat{b}_{t-2,2} - \hat{\pi}_t - \hat{\pi}_{t-1} \Big)$$

$$\phi_\pi \hat{\pi}_t = E_t \hat{\pi_{t+1}}$$

In matrix form this system can be written as:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta^2 \overline{b}_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} E_t \hat{\pi}_{t+1} \\ \hat{\pi}_t \\ \hat{b}_{t,2} \\ \hat{b}_{t-1,2} \end{bmatrix} = \underbrace{\begin{bmatrix} \phi_\pi & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \beta^2 \overline{b}_2 \phi_\pi (1+\phi_\pi) - \overline{R}\frac{\gamma_h+1}{\kappa_1}(1-\beta\phi_\pi) & -\overline{b}_2 & 0 & \overline{b}_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{B} \begin{bmatrix} \hat{\pi}_t \\ \hat{\pi}_{t-1} \\ \hat{b}_{t-1,2} \\ \hat{b}_{t-2,2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \overline{G} \\ 0 \end{bmatrix} \hat{G}_t$$

Stability and determinacy of the equilibrium can be studied by computing the eigenvalues of $A^{-1}B$. We have

$$A^{-1}B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta^2 \overline{b}_2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_\pi & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \beta^2 \overline{b}_2 \phi_\pi (1+\phi_\pi) - \overline{R}\frac{\gamma_h+1}{\kappa_1}(1-\beta\phi_\pi) & -\overline{b}_2 & 0 & \overline{b}_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \phi_\pi & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \phi_\pi (1+\phi_\pi) - \overline{R}\frac{\gamma_h+1}{\kappa_1}\frac{(1-\beta\phi_\pi)}{\beta^2 \overline{b}_2} & -\frac{1}{\beta^2} & 0 & \frac{1}{\beta^2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus

$$\det\Big(A^{-1}B - \lambda I_{4\times 4} \Big) = (\phi_\pi - \lambda)(-\lambda)(\lambda^2 - \frac{1}{\beta^2})$$

There are 4 eigenvalues: $\phi_\pi, 0, \pm\frac{1}{\beta}$. Thus when $0 \leq \phi_\pi \leq 1$, two eigenvalues are in absolute value greater than 1 and we have 1 forward looking variable. There is thus no stable equilibrium in this model.

Finally, note that it is obvious that system (54) which features 2 forward looking variables ($E_t \hat{\pi}_{t+1}$

and $E_t\hat{\pi}_{t+2}$) has 3 unstable roots. ■

A.6 Proof of Proposition 6

Consider a model where the government does not buyback debt and debt is structured so that cash flows are $1, \delta, \delta^2, \ldots$.

We can write the budget constraint as:

$$\bar{b}\beta\sum_{k=1}^{\infty}\beta^{k-1}\delta^{k-1}\left(\hat{b}_t-\sum_{l=1}^{k}E_t\hat{\pi}_{t+l}\right)=-\overline{S}\hat{S}_t+\bar{b}\sum_{k=1}^{\infty}\delta^{k-1}\left(\hat{b}_{t-k}-\sum_{l=0}^{k-1}\hat{\pi}_{t-l}\right)$$

The first order conditions of the Ramsey program are now:

$$-\hat{\pi}_t+\Delta\psi_{\pi,t}+\frac{\bar{b}}{1-\delta}\sum_{k=0}^{\infty}(\beta\delta)^k E_t\psi_{gov,t+k}-\frac{\bar{b}}{1-\beta\delta}\sum_{k=1}^{\infty}\delta^{k-1}\psi_{gov,t-k}=0 \tag{55}$$

$$-\psi_{\pi,t}\kappa_1+\overline{R}\left(1+\gamma_h\right)\psi_{gov,t}=0 \tag{56}$$

$$\bar{b}\beta\sum_{k=1}^{\infty}(\beta\delta)^{k-1}\left(E_t\psi_{gov,t+k}-\psi_{gov,t}\right)=0 \tag{57}$$

Consider the last of equation. This can be written as:

$$\psi_{gov,t}=(1-\beta\delta)E_t\frac{\psi_{gov,t+1}}{1-\beta\delta L^{-1}}\rightarrow\psi_{gov,t}=E_t\psi_{gov,t+1}$$

From this result we can combine (55) and (56) into:

$$-\hat{\pi}_t+\frac{\overline{R}}{\kappa_1}\left(1+\gamma_h\right)\Delta\psi_{gov,t}+\frac{\bar{b}}{1-\delta}\psi_{gov,t}\sum_{k=0}^{\infty}(\beta\delta)^k-\frac{\bar{b}}{1-\beta\delta}\sum_{k=1}^{\infty}\delta^{k-1}\psi_{gov,t-k}=0$$

or

$$-\hat{\pi}_t+\frac{\overline{R}}{\kappa_1}\left(1+\gamma_h\right)\Delta\psi_{gov,t}+\frac{\bar{b}}{1-\beta\delta}\sum_{k=0}^{\infty}\delta^k\psi_{gov,t-k}=0$$

These first order conditions are thus the same as FONC of the buyback model.

Appendix B Nonlinear model and Additional Results

B.1 A model with real interest rate fluctuations

We now consider a model where that real rate fluctuates according to an exogenous stochastic process. We show that the optimal Ramsey policy under decaying coupons admits an interest rate rule with an inflation coefficient equal to δ.

Let $\hat{r}_t$ denote the real interest rate. We assume that fluctuations in $\hat{r}_t$ occur due to demand shocks . Let $\hat{\xi}_t$ denote the demand shock. Standard modelling of the Euler/Fischer equation gives:

$$\hat{i}_t = \underbrace{\hat{r}_t}_{\hat{\xi}_t - E_t\hat{\xi}_{t+1}} + E_t\hat{\pi}_{t+1}$$

The consolidated budget constraint (6 in text) is

$$\begin{aligned}\sum_{k=1}^{\infty}\beta^k\overline{b}_k\left(\hat{b}_{t,k} + E_t(\hat{\xi}_{t+k} - \hat{\xi}_t) - \sum_{l=1}^{k}E_t\hat{\pi}_{t+l}\right) &= -\overline{S}\hat{S}_t + \overline{b}_1(\hat{b}_{t-1,1} - \hat{\pi}_t) \\ &+ \sum_{k=2}^{\infty}\beta^{k-1}\overline{b}_k\left(\hat{b}_{t-1,k} + E_t(\hat{\xi}_{t+k-1} - \hat{\xi}_t) - \sum_{l=0}^{k-1}\hat{\pi}_{t+l}\right)\end{aligned}$$

It is simple to show that the Ramsey policy leads to the same FONC for inflation and output and debt as in the baseline model of Section 2 (equations (8) to (10)). The optimal inflation rate is given again by (11). When the debt structure is $\overline{b}_k = \delta^{k-1}\overline{b}$ we get

$$\hat{\pi}_t = \overline{R}\frac{(1+\gamma_h)}{\kappa_1}\Delta\psi_{gov,t} + \frac{\overline{b}}{1-\beta\delta}\sum_{l=0}^{\infty}\delta^l\Delta\psi_{gov,t-l}$$

Combining the Fischer equation and the random walk property of $\psi_{gov,t}$ we have:

$$\hat{i}_t - \hat{r}_t = E_t\hat{\pi}_{t+1} = \frac{\overline{b}}{1-\beta\delta}\sum_{l=0}^{\infty}\delta^l E_t\Delta\psi_{gov,t-l+1} = \delta\frac{\overline{b}}{1-\beta\delta}\sum_{l=0}^{\infty}\delta^l\Delta\psi_{gov,t-l} = \delta\left(\hat{\pi}_t - \overline{R}\frac{(1+\gamma_h)}{\kappa_1}\Delta\psi_{gov,t}\right)$$

In the case where $\overline{R}\frac{(1+\gamma_h)}{\kappa_1} \approx 0$ the nominal rate is set according to

$$\hat{i}_t = \hat{r}_t + \delta\hat{\pi}_t$$

as was claimed in text.

B.2 A model with inflation intertia

Assume that the Phillips curve is

$$\hat{\pi}_t - \gamma\hat{\pi}_{t-1} = \kappa_1\hat{Y}_t + \kappa_2\hat{\tau}_t - \kappa_3\hat{G}_t + \beta E_t(\hat{\pi}_{t+1} - \gamma\hat{\pi}_t) \tag{58}$$

where $0 < \gamma < 1$. Assume further that the planner seeks to maximize $-\frac{1}{2}E_0\sum_{t\geq 0}\beta^t\Big(\hat{\pi}_t - \gamma\hat{\pi}_{t-1}\Big)^2$.

Consider the debt concerns model. The first order conditions (with respect to $\hat{\pi}, \hat{Y}, \hat{b}_\delta$) now become:

$$-\Big(\hat{\pi}_t - \gamma\hat{\pi}_{t-1}\Big) + \beta\gamma E_t\Big(\hat{\pi}_{t+1} - \gamma\hat{\pi}_t\Big) + \Delta\psi_{\pi,t} + \frac{\bar{b}_\delta}{1-\beta\delta}\sum_{l=0}^{\infty}\delta^l\Delta\psi_{gov,t-l} = 0 \tag{59}$$

$$-\psi_{\pi,t}\kappa_1 + \omega_Y\psi_{gov,t} = 0 \tag{60}$$

$$\frac{\beta\bar{b}_\delta}{1-\beta\delta}\Big(\psi_{gov,t} - E_t\psi_{gov,t+1}\Big) = 0 \tag{61}$$

Rearranging we get:

$$\Big(\hat{\pi}_t - \gamma\hat{\pi}_{t-1}\Big) = \beta\gamma E_t\Big(\hat{\pi}_{t+1} - \gamma\hat{\pi}_t\Big) + \frac{\omega_Y}{\kappa_1}\Delta\psi_{gov,t} + \frac{\bar{b}_\delta}{1-\beta\delta}\sum_{l=0}^{\infty}\delta^l\Delta\psi_{gov,t-l}$$

which can be solved forward to give :

$$\Big(\hat{\pi}_t - \gamma\hat{\pi}_{t-1}\Big) = \sum_{j\geq 0}(\beta\gamma)^j E_t\Big[\frac{\omega_Y}{\kappa_1}\Delta\psi_{gov,t+j} + \frac{\bar{b}_\delta}{1-\beta\delta}\sum_{l=0}^{\infty}\delta^l\Delta\psi_{gov,t+j-l}\Big]$$

and using the random walk property (61) we obtain:

$$\hat{\pi}_t = \gamma\hat{\pi}_{t-1} + \frac{\omega_Y}{\kappa_1}\Delta\psi_{gov,t} + \frac{\bar{b}_\delta}{1-\beta\delta}\sum_{j\geq 0}(\beta\delta\gamma)^j\sum_{l=0}^{\infty}\delta^l\Delta\psi_{gov,t-l} \tag{62}$$

The optimal policy rule can be found by using the Euler equation:

$$\hat{\imath}_t - \hat{\xi}_t = E_t\hat{\pi}_{t+1} = \gamma\hat{\pi}_t + \frac{\bar{b}_\delta\delta}{1-\beta\delta}\sum_{j\geq 0}(\beta\delta\gamma)^j\sum_{l=0}^{\infty}\delta^l\Delta\psi_{gov,t-l} = \gamma\hat{\pi}_t + \delta\Big(\hat{\pi}_t - \gamma\hat{\pi}_{t-1} - \frac{\omega_Y}{\kappa_1}\Delta\psi_{gov,t}\Big)$$

The optimal policy is therefore:

$$\hat{\imath}_t = \hat{\xi}_t + \Big(\gamma + \delta\Big)\hat{\pi}_t - \gamma\delta\hat{\pi}_{t-1} - \delta\frac{\omega_Y}{\kappa_1}\Delta\psi_{gov,t} \tag{63}$$

Clearly, equation (63) defines a passive monetary policy.

B.3 Closing the output gap objective

Assume now that the objective of the planner is

$$-\frac{1}{2}E\sum_{t\geq 0}\beta^t\left(\hat{\pi}_t^2+\lambda_Y\hat{Y}_t^2\right)$$

Also for simplicity assume that debt pays decaying coupons. We can easily show that the FONC from the Ramsey program will yield the following condition for inflation.

$$-\hat{\pi}_t-\frac{\lambda_Y}{\kappa_1}\Delta\hat{Y}_t+\frac{\overline{R}}{\kappa_1}(1+\gamma_h)\Delta\psi_{gv,t}+\frac{\bar{b}}{1-\beta\delta}\sum_{k=0}^{\infty}\delta^k\Delta\psi_{gov,t-l}=0$$

Using the Phillips curve, we obtain the following :

$$-\hat{\pi}_t-\frac{\lambda_Y}{\kappa_1^2}(\hat{\pi}_t-\beta E_t\hat{\pi}_{t+1})+\frac{\lambda_Y}{\kappa_1^2}(\hat{\pi}_{t-1}-\beta E_{t-1}\hat{\pi}_t)+\frac{\overline{R}}{\kappa_1}(1+\gamma_h)\Delta\psi_{gov,t}+\frac{\bar{b}}{1-\beta\delta}\sum_{k=0}^{\infty}\delta^k\Delta\psi_{gov,t-l}=0$$

Define:

$$\chi_t=(\hat{\pi}_t-E_{t-1}\hat{\pi}_t)+\frac{\kappa_1\overline{R}}{\beta\lambda_Y}(1+\gamma_h)\Delta\psi_{gov,t}+\frac{\kappa_1^2}{\beta\lambda_Y}\frac{\bar{b}}{1-\beta\delta}\sum_{k=0}^{\infty}\delta^k\Delta\psi_{gov,t-l}=0$$

where the term $\hat{\pi}_t-E_{t-1}\hat{\pi}_t$ will be a linear function of $\hat{G}_t$. Then

(64) $$E_t\hat{\pi}_{t+1}-(1+\frac{1}{\beta}+\frac{\kappa_1^2}{\lambda_Y\beta})\hat{\pi}_t+\frac{1}{\beta}\hat{\pi}_{t-1}=-\chi_t$$

We will now resolve the above difference equation. Letting $\widetilde{\kappa}=\frac{\kappa_1^2}{\lambda_Y\beta}$) the characteristic polynomial is $\lambda^2-(1+\frac{1}{\beta}+\widetilde{\kappa})\lambda+\frac{1}{\beta}$.

Skipping a few steps, the two roots are:

$$\lambda_{1,2}=\frac{1}{2}\left((1+\frac{1}{\beta}+\widetilde{\kappa})\pm\sqrt{(1-\frac{1}{\beta}-\widetilde{\kappa})^2+4\frac{\widetilde{\kappa}}{\beta}}\right)$$

It is simple to show that one root is stable and one unstable. Let λ_1 denote the stable root. (64) can be written as:

(65) $$\hat{\pi}_t=\frac{1}{\lambda_2}E_t\hat{\pi}_{t+1}+\frac{1}{\lambda_2}\frac{1}{1-\lambda_1L}\chi_t=\frac{1}{\lambda_2}\frac{1}{1-\lambda_1L}\sum_{j\geq 0}\frac{1}{\lambda_2^j}E_t\chi_{t+j}$$

(for the usual boundary condition that inflation does not explode).

Let us compute the term

$$\sum_{j\geq 0}\frac{1}{\lambda_2^j}E_t\chi_{t+j}=\sum_{j\geq 0}\frac{1}{\lambda_2^j}E_t\left[\epsilon\hat{G}_{t+j}+\widetilde{\kappa}\frac{\overline{R}}{\kappa_1}(1+\gamma_h)\Delta\psi_{gov,t+j}+\widetilde{\kappa}\frac{\bar{b}}{1-\beta\delta}\sum_{k=0}^{\infty}\delta^k\Delta\psi_{gov,t+j-l}\right]$$

The final term on the RHS is

$$\widetilde{\kappa}\frac{\bar{b}}{1-\beta\delta}\sum_{j\geq 0}\frac{1}{\lambda_2^j}E_t\left[\sum_{k=0}^{\infty}\delta^k\Delta\psi_{gov,t+j-l}\right]=\widetilde{\kappa}\frac{\bar{b}}{1-\beta\delta}\frac{1}{1-\frac{\delta}{\lambda_2}}\frac{1}{1-\delta L}\Delta\psi_{gov,t}$$

(this follows from the random walk property of the multiplier). Also it is simple to show that

$$\sum_{j\geq 0}\frac{1}{\lambda_2^j}E_t\left[\epsilon\hat{G}_{t+j}+\widetilde{\kappa}\frac{\overline{R}}{\kappa_1}(1+\gamma_h)\Delta\psi_{gov,t+j}\right]=\epsilon\hat{G}_t+\widetilde{\kappa}\frac{\overline{R}}{\kappa_1}(1+\gamma_h)\Delta\psi_{gov,t}$$

Putting everything together we get

$$\hat{\pi}_t=\lambda_1\hat{\pi}_{t-1}+\epsilon\hat{G}_t+\widetilde{\kappa}\frac{\overline{R}}{\kappa_1}(1+\gamma_h)\Delta\psi_{gov,t}+\widetilde{\kappa}\frac{\bar{b}}{1-\beta\delta}\frac{1}{1-\frac{\delta}{\lambda_2}}\frac{1}{1-\delta L}\Delta\psi_{gov,t}$$

To derive the interest rate rule we can compute

$$E_t\hat{\pi}_{t+1}=\lambda_1\hat{\pi}_t+\epsilon\underbrace{E_t\hat{G}_{t+1}}_{=0}+\widetilde{\kappa}\frac{\overline{R}}{\kappa_1}(1+\gamma_h)\underbrace{E_t\Delta\psi_{gov,t+1}}_{=0}+\widetilde{\kappa}\frac{\bar{b}}{1-\beta\delta}\frac{1}{1-\frac{\delta}{\lambda_2}}\underbrace{E_t\frac{1}{1-\delta L}\Delta\psi_{gov,t}}_{=\frac{\delta}{1-\delta L}\Delta\psi_{gov,t}}$$

which gives us:

$$\hat{i}_t=E_t\hat{\pi}_{t+1}=\lambda_1\hat{\pi}_t+\delta\left(\hat{\pi}_t-\lambda_1\hat{\pi}_{t-1}-\epsilon\hat{G}_t-\widetilde{\kappa}\frac{\overline{R}}{\kappa_1}(1+\gamma_h)\Delta\psi_{gov,t}\right)$$

B.4 Non-linear model equations

In this section we derive the log-linear equations describing the model economy presented in Section 2, starting from the non-linear equations representing its competitive equilibrium. As mentioned in the main text, we consider a standard New-Keynesian model with quasi-linear preferences, that is augmented with a fiscal block.

Households Households maximize

$$E_0\sum_{t=0}^{\infty}\beta^t\left(C_t-\chi\frac{h_t^{1+\gamma_h}}{1+\gamma_h}\right)$$

subject to

$$P_tC_t + \sum_{k=1}^{\infty} P_{t,k}B_{t,k} \leqslant (1-\tau)W_th_t - T + P_tD_t + B_{t-1,1} + \sum_{k=2}^{\infty} P_{t,k-1}B_{t-1,k}$$

where C_t denotes consumption and h_t denotes hours worked. D_t represents firms' profits redistributed to households, and P_t denotes the aggregate price level. $B_{t,k}$ denotes government bonds of maturity $k = 1, 2, ...$ issued at time t. These bonds are risk-free and deliver one unit of the (nominal) consumption good in period $t + k$. They are traded at price $P_{t,k}$. τ is the distortionary tax on labour, and T the lump-sum tax imposed on households. As mentioned in the main text, both are assumed to be constant in our fiscally-led economy.

The first order conditions of the household's problem are:

$$P_{t,1} = \beta E_t \frac{1}{\pi_{t+1}} \tag{66}$$

$$P_{t,k} = \beta E_t \frac{1}{\pi_{t+1}} P_{t+1,k-1} \tag{67}$$

$$h_t^{\gamma_h} = (1-\tau_t)\frac{W_t}{P_t} \tag{68}$$

where $\pi_t \equiv \frac{P_t}{P_{t-1}}$ is the gross inflation rate.

Firms Production takes place in monopolistically competitive firms which operate technologies with labour as the sole input. The final good is a CES aggregate of the intermediate goods $Y_t(j)$:

$$Y_t = \Big(\int_0^1 Y_t(j)^{\frac{1+\eta}{\eta}} dj\Big)^{\frac{\eta}{1+\eta}} \tag{69}$$

where η governs the elasticity of substitution between differentiated goods. Firms set prices to maximize profits subject to the demand curve

$$Y_t(j) = \Big(\frac{P_t(j)}{P_t}\Big)^{\eta} Y_t \tag{70}$$

and given price adjustment costs, modelled as in Rotemberg (1982). The dynamic profit maximization program is:

$$\max_{P_t(j)} \quad E_t \sum_{s=0}^{\infty} Q_{t,t+s}\Big(\frac{P_{t+s}(j)}{P_{t+s}}Y_{t+s}(j) - \frac{W_{t+s}(j)}{P_{t+s}}Y_{t+s}(j) - AC_{t+s}(j)\Big)$$

$$\text{s.t.} \quad Y_{t+s}(j) = \Big(\frac{P_{t+s}(j)}{P_{t+s}}\Big)^{\eta_t} Y_{t+s} \tag{71}$$

$$AC_{t+s}(j) = \frac{\theta}{2}\Big(\frac{P_{t+s}(j)}{P_{t+s-1}(j)} - \bar{\pi}\Big)^2 Y_{t+s} \tag{72}$$

where $Q_{t,t+s} \equiv \beta^s$ is the discount factor of households and W_{t+s} is the wage rate, that is equal to the marginal cost of production. (72) is the quadratic adjustment costs incurred by firms when resetting their price.

Focusing on a symmetric equilibrium the first order condition from the firm's dynamic program gives the following non-linear Phillips Curve:

$$\theta(\pi_t - \pi)\pi_t = 1 + \eta(1 - \frac{W_t}{P_t}) + \beta\theta E_t \frac{Y_{t+1}}{Y_t}(\pi_{t+1} - \pi)\pi_{t+1} \tag{73}$$

The firms' technology is linear in labour and thus $Y_t(j) = h_t(j)$ where $j \in [0, 1]$ denotes the generic firm.

Fiscal policy The flow government budget constraint can be written as:

$$\sum_{k=1}^{\infty} P_{t,k} b_{t,k} = \frac{b_t}{\pi_t} + \sum_{k=2}^{\infty} P_{t,k-1} \frac{b_{t-1,k}}{\pi_t} + G_t - \tau h_t w_t - T \tag{74}$$

where $b_{t,k} \equiv \frac{B_{t,k}}{P_t}$ denotes the real value in t of government bonds with matuirty k, τ is the distortionary tax on labour, and T the lump-sum tax imposed on households. As mentioned above, both of these instrument are assumed to stay constant under the assumptions described in the main text. G_t denotes government spending, which is exogenous and is assumed to follow an i.i.d process.

Log-linear model Making use of the labor supply condition $h_t^{\gamma_h} = (1-\tau)\frac{W_t}{P_t}$, as well as the resource constraint $h_t = Y_t = C_t + G_t$ to dispense with W_t, C_t and h_t, we get the following linear New Keynesian Phillips Curve:

$$\hat{\pi}_t = \kappa_1 \hat{Y}_t + \beta E_t \hat{\pi}_{t+1} \tag{75}$$

where κ_1 is defined in text.

Defining $i_t \equiv -\log P_{t,1}$, log-linearizing the Euler equation for short bonds we get the Fisher equation described in text:

$$\hat{i}_t = E_t \hat{\pi}_{t+1} \tag{76}$$

Log-linearizing the Euler equation (67) for bonds with maturity $k > 1$, we get:

$$\hat{p}_{t,k} = E_t(\hat{p}_{t+1,k-1} - \hat{\pi}_{t+1}) \tag{77}$$

Iterating forward we get the equation displayed in text.

Log-linearizing equation (74) and using the primary surplus expression $S_t = \tau w_t h_t + T - G_t$, we get the intertemporal budget constraint (2).

B.5 Optimal policy problem

In the optimal policy problem we solve in Section 2, the Ramsey planner minimizes its loss function subject to the constraints defining the competitive equilibrium of the economy, as derived in the previous section. The Lagrangian associated to this problem is:

$$\begin{aligned}(78)\quad \mathcal{L} \;=\; & E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ -\frac{1}{2}\hat{\pi}_t^2 + \psi_{\pi,t}\Big(\hat{\pi}_t - \kappa_1 \hat{Y}_t - \beta \hat{\pi}_{t+1}\Big) \\ & + \psi_{gov,t}\Big(\beta \bar{d}\hat{d}_t - \sum_{k=1}^{\infty} \beta^k \bar{b}_k \sum_{l=1}^{k} \hat{\pi}_{t+l}) + \overline{R}\Big(\gamma_h + 1\Big)\hat{Y}_t - \overline{G}\hat{G}_t - \bar{d}\hat{d}_{t-1} + \sum_{k=1}^{\infty} \beta^{k-1}\bar{b}_k \sum_{l=0}^{k-1} \hat{\pi}_{t+l}\Big)\Big\}\end{aligned}$$

The first order conditions of this problem are provided in the main text.

B.6 Social loss function for a specific case

Assume that the setady state is efficient. This can be guaranteed by introducing a constant employment subsidy that cancels out distortions from labor taxes and monopolistic competition at the steady state (see eg Leith and Wren-Lewis, 2013). This subsidy would modify some steady state quantities such as $\bar{R}$ and κ_1 but would be without loss of generality regarding our qualitative results. A second-order approximation of the representative households' utility around that efficient steady state gives

$$U(C_t, Y_t) \approx \bar{C}\hat{c}_t + \frac{1}{2}\bar{C}\hat{c}_t^{\,2} - \chi \bar{Y}^{1+\gamma_h}\hat{y}_t - \frac{1}{2}\chi(1+\gamma_h)\bar{Y}^{1+\gamma_h}\hat{y}_t^2$$

At the efficient steady state, we have $\chi = \bar{Y}^{-\gamma_h}$ and thus we can write

$$U(C_t, Y_t) \approx \bar{C}\hat{c}_t + \frac{1}{2}\bar{C}\hat{c}_t^{\,2} - \bar{Y}\hat{y}_t - \frac{1}{2}(1+\gamma_h)\bar{Y}\hat{y}_t^2$$

Next, a second-order approximation of the resource constraint $C_t + G_t + \frac{\theta}{2}(\pi_t - 1)^2 = Y_t$ gives

$$RC(C_t, G_t, \pi_t, Y_t) \approx \bar{C}\hat{c}_t + \frac{1}{2}\bar{C}\hat{c}_t^{\,2} + \bar{G}\hat{G}_t + \frac{1}{2}\bar{G}\hat{G}_t^{\,2} + \frac{1}{2}\theta\bar{\pi}^2\hat{\pi}_t^2 = \bar{Y}\hat{y}_t + \frac{1}{2}\bar{Y}\hat{y}_t^{\,2}$$

Solving for $\bar{C}\hat{c}_t + \frac{1}{2}\bar{C}\hat{c}_t^{\,2}$, substituting in the approximated utility and using $\bar{\pi} = 1$, we find

$$U(C_t, Y_t) \approx -\frac{1}{2}\Big(\hat{\pi}_t^2 + \lambda_Y \hat{y}_t^{\,2}\Big) + tip$$

with $\lambda_Y \equiv \frac{\gamma_h \bar{Y}}{\theta}$. Hence, when the cost of price adjustment, θ, goes to infinity, social welfare is a quadratic measure of inflation only. Otherwise, both inflation and output enter in the objective (a similar result is found in the online appendix of Leeper and Zhou (2021) when considering a distorted steady state).

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