ON COMPETITION FOR SPATIALLY DISTRIBUTED RESOURCES ON NETWORKS

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ON COMPETITION FOR SPATIALLY DISTRIBUTED RESOURCES ON NETWORKS

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Abstract. We study the dynamics of the exploitation of a natural resource, distributed in space and mobile, where spatial diversification is introduced by a network structure. Players are assigned to different nodes by a regulator, after he/she decides at which nodes natural reserves are established. The game solution shows how the dynamics of spatial distribution depends on the productivity of the various sites, on the structure of the connections between the various locations, and on the preferences of the agents. At the same time, the best locations to host a nature reserve are identified in terms of the parameters of the model, and it turns out they correspond to the most central (in the sense of eigenvector centrality) nodes of a suitably redefined network which takes into account the nodes productivities.

Keywords: Harvesting, spatial models, differential games, nature reserve.

JEL Classification: Q20, Q28, R11, C73

1. Introduction

In settings where a network of stocks migration flows connects the various sites where a resource resides, how does the access to the resource of a number of competing agents should be regulated? And in particular, where natural reserves should be placed? To provide a first exploration of this issue, in this paper we develop a simple model where the $n \geq 2$ nodes of a weighted directed network represent the n regions in which a geographical area is partitioned, and the weights upon the edges give the interregional migration flows of the resource stocks. The n regions are heterogeneous not only because they are differently connected, but also because the rate of growth of the resource is not uniform across them. The regulator's task is the assignment of extraction rights to $f < n$ agents in order to maximize a welfare function that is given by the sum of the agents utilities. For reasons we do not indagate - perhaps there are strong congestion externalities, the regulator is also constrained to assign no more than one agent to each region. Following the assignment stage, the f agents compete for the exploitation of the resource as in the classical Levhari and Mirman (1980) dynamic game, with the two differences that the stock of the resources is not homogeneous but distributed among the n regions, and that each agent can only access the resource through the single node he/she is assigned to. The main aim of this paper is to study how the structure of the network affects the regulator's choice.

A small literature has explored aspects of the problem of dynamic strategic interaction with distributed and moving resources, especially in order to evaluate whether management of the resources through a system of Territorial Use Rights (TURF for fisheries) can effectively mitigate the "tragedy of the commons" (see e.g., Kaffine and Costello, 2011, Costello et al., 2015, Herrera et al., 2016, Costello and Kaffine, 2018, Costello et al., 2019, de Frutos and Martin-Herran, 2019, Fabbri et al., 2020). For example, Kaffine and Costello (2011) have shown, using a discrete time model, that Territorial Use Rights coupled with profits sharing can effectively reduce overexploitation of moving resources. Costello et al. (2015) have extended the same model to show how partial enclosure of the commons can improve the welfare of the common property regime. Costello and Kaffine (2018) compared the relative efficiency of centralized versus decentralized management of a moving resource when users have heterogeneous preferences for conservation and the regulator has incomplete information about these preferences. On the other hand, in a two region model in continuous time, Fabbri et al. (2020) have suggested that modulating the access to the different sites through the assignment of Territorial Use Rights can be effective in rising the rate of growth of moving collapsing resources, in a context of high harvesting effort. None of these works, however, have given explicit attention to the network structures that characterize both the access to the sites and the migration flows.

In the network literature, there are works that study the role of networks in the management of natural resources. Currarini et al. (2016) survey various contributions in which network economics has been used in analyzing issues ranging from the pattern and speed of diffusion of new green technology, to the structure and dynamics of international agreements, from the formation of links in building an environmental coalition, to the role of infrastructural networks in the access to natural resources. Among these contributions, İlkılıç (2011) is closest to the question we explore here. He studies a static game in which a given number of users exploit multiple sources of a common pool, and each user faces marginal costs that are increasing in the total extraction from the site, due to the presence of source specific congestion externalities. The main conclusion is that in the unique Nash equilibrium of the game the rate of extraction at each source is proportional to a centrality measure of the links of the source. More recently, Kyriakopoulou and Xepapadeas (2018) studied the interaction between a global congestion externality and local positive externalities, reflecting collaboration links in the exploitation of a single resource by a given number of agents. They show that the equilibrium rate of extraction of agents is, in this case, proportional to their centrality in the local interactions network.

The aim of this paper is to take further the analysis of the exploitation by a given number of agents of a moving spatially distributed resource, highlighting that the equilibrium extraction intensities depend on both the network structure of the migration flows and the access network. In the model we have $n \geq 2$ regions with general (and not necessarily symmetric) linear migration flows. We find a Markovian equilibrium in which each agent's welfare is decreasing in a suitable centrality measure of his/her assigned node. At the same time we show that, in order to maximize the sum of agents' welfares, the social planner has to set natural reserves in the most "central" regions of an appropriately modified migration network.

Since we focus on migration flows from heterogeneous sources, our centrality measure is not the Katz-Bonacich index used in İlkılıç (2011) or Kyriakopoulou and Xepapadeas (2018) to study complementarities, but depends both on the (net) growth rates of the resource at the different nodes and on how much the nodes broadcast to the other nodes. If all nodes are equally productive (i.e., the net growth rates are equal), then our measure coincides with the outdegree eigenvector centrality of the migration network. In the general case, instead, it is the eigenvector centrality of a derived network, obtained by magnifying outgoing links of each node by a factor, increasing with the net productivity at the node. Moreover such factor is less than one for nodes with negative net productivity, equal to one for nodes with zero net productivity, and greater that one otherwise. In the case of an equally weighted and complete network, in which the network is neutral, our centrality measure ranks nodes in increasing order of productivity. The higher the rate of growth (either net or gross) the higher the rank.

For our model, the highest eigenvalue of the derived network coincides with the von Neumann rate of growth of the system and plays the same role as the productivity parameter α in the homogeneous resource dynamic game analyzed in Tornell and Lane (1999). As in the case examined by Tornell and Lane (1999), we also find that the agents react voraciously to a positive shock that increases the dominant eigenvalue when their elasticity of intertemporal substitution is sufficiently higher than 1 (higher that $\frac{f}{f-1}$ when there are f agents). In turn, this disproportional increase of the extraction rate results in a fall of the long run rate of growth of the stocks. Since changing the weigh on a link from one region to another simultaneously implies a change in the net rate of growth of the broadcasting node, the highest eigenvalue depends in a complex way from the weights of the migration network. However, for symmetric networks we are able to establish that the highest eigenvalue is a decreasing function of the elements of the adjacency matrix. Thus it turns out that, when voracity prevails in these family of networks, removing part of the spatial externalities generated by migration actually reduces the rate of growth of the resource.

The paper is organized as follows. In Section 2 the model is described and preliminaries are discussed. Sections 3 contains the main results of the paper and the description of the Nash equilibrium. In Section 4 the role of the network structure is discussed with the aid of a variety of examples. Section 5 contains the final remarks. The proofs of all analytic results are collected in Appendix A.

2. The model

We consider a geographical area, partitioned in subareas/regions/locations, and a standing natural renewable resource, for example fish, mobile through the different regions.

We represent space through a weighted directed network $\mathcal{G}$ with n nodes – as many as the number of regions, in which the area is partitioned. We denote with $N := \{1, .., n\}$ the set of nodes, and with $g_{ij} \geq 0$ the weight upon the edge connecting a source node i and a target node j, g_{ij} representing the intensity of the outflow from i to j, so that when $g_{ij} = 0$ and $g_{ji} = 0$, then there are not direct paths between the two nodes. We assume $\mathcal{G}$ strongly connected, that is, there exists a path connecting any two nodes with corresponding strictly positive coefficients g_{ij}. We also assume that $\mathcal{G}$ has no loops, so that $g_{ii} = 0$ for all $i \in N$. We denote with G the (weighted) $n \times n$ adjacency matrix with elements g_{ij}, $i, j \in N$, with e_i the i-th vector of the canonical basis on $\mathbb{R}^n$, and with $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^n$. We also denote by $\mathbb{R}_+$ the set of nonnegative real values.

For all $i \in N$, the quantity $X_i(t)$ stands for the biomass at location i at time t, and $X(t)$ for the vector with components $X_1(t), .., X_n(t)$. The evolution in time of biomass $X_i(t)$ on region i depends on several factors:

(a) the natural growth $\Gamma_i X_i(t)$ of the resource at time t at node i, embodied by the (constant) natural growth rate Γ_i;

(b) the outflow of the resource from region i to a linked region j at time t, given by $g_{ij} X_i(t)$, so that the net inflow at location i is given by

$$\left(\sum_{j=1}^{n} g_{ji} X_j(t) \right) - \left(\sum_{j=1}^{n} g_{ij} X_i(t) \right) = \langle Ge_i, X(t) \rangle - \left(\sum_{j=1}^{n} g_{ij} \right) X_i(t)$$

(c) the rate of extraction $c_i(t)$ at time t from region i.

As a whole, we then have for all i

$$\dot{X}_i(t) = \left(\Gamma_i - \sum_{j=1}^{n} g_{ij}\right) X_i(t) + \langle Ge_i, X(t)\rangle - c_i(t).$$

If $A = (a_{ij})$ is the diagonal matrix of the net reproduction factors, namely

$$\begin{cases} a_{ij} = 0 & \text{if } i \neq j \\ a_{ii} \equiv a_i = \Gamma_i - \left(\sum_{j=1}^{j=n} g_{ij}\right), \end{cases}$$

$c(t)$ is the vector with components $c_1(t), .., c_n(t)$, and x is the vector of all initial stocks of the resource at the different nodes, then the evolution of the system in vector form is

$$\begin{cases} \dot{X}(t) = (A + G^\top)X(t) - c(t), \quad t > 0 \\ X(0) = x_0, \end{cases} \tag{1}$$

where $x_0 \in \mathbb{R}^n_+$ is the vector of initial biomass concentrations in the various regions. In addition to that, we require the following positivity constraints

$$c_i(t) \geq 0, \quad t \geq 0, i \in N \tag{2}$$

as well as

$$X_i(t) \geq 0, \quad t \geq 0, i \in N \tag{3}$$

Remark 2.1. *As a particular case of the described setting we have the situation where the diffusion process follows Fick's first law, that is, the flow of the resource from region i to a linked region j at time t is proportional to the difference $X_i(t) - X_j(t)$. In this case the matrix G is symmetric as, for connected locations i and j, we have $g_{ij} = g_{ji}$. As a consequence, $A + G^\top = A + G$ and the problem simplifies.*

Remark 2.2. *Note that the matrix $A + G^\top$ is a Metzler matrix, i.e. it has all non-negative elements, except at most those on the principal diagonal. Moreover, since the non-diagonal elements of $A+G^\top$ are the same as those of $G^\top$, which is the adjacency matrix of a strongly connected network, the graph associated to $A+G^\top$ is also strongly* *connected, implying that $A + G^\top$ is irreducible (see Theorem 2.1 page 36 in Latora et al., 2017). Consequently, the Perron-Frobenius theorem (see e.g. Theorem 1.4.4 page 17 of Bapat and Raghavan, 1997) implies that the greatest eigenvalue of $A+G^\top$ is simple and that the associated normalized eigenvector is positive. The same statement holds for the transpose matrix $A + G$.*

Now we assume that some of the regions are exclusively devoted to reproduction of the resource (natural reserves) while, at the same time, each of the remaining is assigned to an agent for exclusive exploitation, enhancing a TURF policy. More precisely, harvesting is prohibited in a subset M of N, while each of the remaining regions $F := N \setminus M$ is exclusively assigned to an agent. We set $f := |F|$, with $|F|$ denoting the cardinality of the set F, so that $n - f = |M|$.

Finally, we assume that agents strategically interact in a differential game where Player i maximizes the payoff

$$J_i(c_i) = \int_0^{+\infty} e^{-\rho t} \frac{(c_i(t))^{1-\sigma}}{1-\sigma} dt, \quad i \in F \tag{4}$$

for $\sigma > 0$ and $\sigma \neq 1$.

We denote the trajectory of (1) at time t, starting at x_0 and driven by $c(t)$ with $X(t; c(\cdot), x_0)$. We use for strategy profiles the notation $c = (c_i, c_{-i})$, meaning that Player i chooses c_i and the other players choose the vector $c_{-i} \in \mathbb{R}_+^{n-1}$ (including the zero components associated to the nature reserve). Admissible strategy profiles, at an initial state $x_0 \in \mathbb{R}_+^n$, are those measurable functions $c : [0, +\infty) \to \mathbb{R}_+^n$ that generate trajectories $X(t; c(\cdot), x_0)$ which are contained in $\mathbb{R}_+^n$ at all times. *Markovian strategy profiles* are a subset of these strategies which are functions only of the current levels of stock variables. The formal definition follows.[1]

Definition 2.3. (Markovian Admissible strategy profiles) *Consider a given initial state $x_0 \in \mathbb{R}_+^n$. We say that the vector of continuous functions $\psi :=$*

[1]Note indeed that some constraint on the state space is needed in order to have existence of meaningful equilibria, otherwise players would choose to extract infinite amounts of resource, even from a negative stock.

$(\psi_1, ..., \psi_n)\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *is an* admissible (stationary) Markovian strategy profile at x_0 *if:*

(*i*) *for all* $i \in M$, $\psi_i \equiv 0$;

(*ii*) *the equation* (1) *with* $c_i(t)$ *replaced by* $\psi_i(X(t))$, *i.e.*

$$\begin{cases} \dot{X}(t) = (A + G^\top)X(t) - \psi(X(t)), & t > 0 \\ X(0) = x_0, & . \end{cases} \tag{5}$$

has a unique solution $X^\psi(\cdot)$;

(*iii*) $X_i^\psi(t) \geq 0$, *for all* $t \geq 0$ *and for all* $i \in N$.

We denote by $\mathcal{M}(x_0)$ *the set of all admissible Markov strategy profiles for the problem at* x_0.

Definition 2.4. (Markovian Nash equilibrium) *Consider a given initial state* $x_0 \in \mathbb{R}^n_+$, *and let* $\psi \in \mathcal{M}(x_0)$. *We say that* ψ *is a* (stationary) Markovian Nash equilibrium at x_0 *if the following fact is true: for all* $i \in F$, *the control* $c_i(t) = \psi_i(X(t))$ *is optimal for the problem of Player* i *given by: the state equation* (1) *where the other players choose* $c_{-i}(t) = \psi_{-i}(X(t))$; *the constraints* (2); *the functional* $J_i(c_i)$ *given by* (4), *to be maximized over the set of admissible controls*

$$\mathcal{C}^{\psi_{-i}}(x_0) = \left\{c_i\colon [0, +\infty) \to \mathbb{R}_+ \;:\; X_j(t; (c_i, \psi_{-i}(X)); x_0) \geq 0, \forall j \in N, \forall t \geq 0\right\}.$$

In formulas

$$J_i(\psi_i(X)) \geq J_i(c_i), \quad \forall c_i \in \mathcal{C}^{\psi_{-i}}(x).$$

3. Existence of Markovian Equilibria

This section contains the main results of the paper, describing a class of Markovian Nash equilibria for the given problem. We advise the reader that all the proofs of the results stated below are contained in Appendix A. Further notation is now introduced. We denote by λ the real positive eigenvalue of $(A + G)$ having greatest real part (see

Remark 2.2), and by $\{\lambda_2, .., \lambda_n\}$ the remaining ones, ordered with decreasing real parts

$$\lambda > \Re(\lambda_2) \geq \Re(\lambda_3) \geq ... \geq \Re(\lambda_n).$$

We denote by $\eta = (\eta_1, \eta_2, .., \eta_n)^\top$ the positive normalized eigenvector associated to λ. We also define as ξ the vector with components $\xi_i = \eta_i^{-1}$ if $i \in F$, and $\xi_i = 0$ otherwise, and $\xi\, \eta^\top$ the $n \times n$ matrix obtained by multiplying the column vector ξ by the row vector $\eta^\top$, in symbols

$$\xi = \sum_{i \in F} \eta_i^{-1} e_i, \qquad \xi\, \eta^\top = (\xi_i \eta_j)_{ij}. \tag{6}$$

Finally we set

$$\theta := \frac{\rho + (\sigma - 1)\lambda}{1 + (\sigma - 1) f}. \tag{7}$$

Remark 3.1. *Observe that, when $f = n$ (no nature reserves), $\xi\, \eta^\top$ has $n - 1$ eigenvectors associated to the eigenvalue 0, all orthogonal to η (and hence generating $\langle \eta \rangle^\perp$), and the eigenvector ξ (described in* (6)*) associated to the eigenvalue λ. Symmetrically, $\eta\, \xi^\top$ has eigenvector η associated to the eigenvalue λ, while all remaining eigenvectors generate $\langle \xi \rangle^\perp$.*

Remark 3.2. *The expansion in rows of the equality $(A + G)\eta = \lambda \eta$ gives*

$$(\lambda - a_i)\eta_i = \sum_{j=1, j \neq i}^{n} g_{ij} \eta_j > 0, \tag{8}$$

since at least one of the g_{ij} is strictly positive (as the network is strongly connected), which implies in particular

$$a_i < \lambda, \quad \forall i \in F. \tag{9}$$

Theorem 3.3. *Assume $\theta > 0$, $x_0 \in \mathbb{R}^n_+$ and define $\psi \colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ as*

$$\psi_i(x) = \frac{\theta}{\eta_i} \langle x, \eta \rangle, \text{ for all } i \in F, \qquad \psi_i(x) = 0, \text{ for all } i \notin F. \tag{10}$$

Assume also that $\psi \in \mathcal{M}(x_0)$. Then:

(i) ψ is a Markovian equilibrium of the game in the sense of Definition 2.4;

(*ii*) *the welfare of agent i along such equilibrium is*

$$V_i(x) = \frac{\theta^{-\sigma}\eta_i^{\sigma-1}}{1-\sigma}\langle x, \eta\rangle^{1-\sigma};$$

(*iii*) *the corresponding evolution of the system is*

$$\begin{cases} \dot{X}(t) = (A + G^\top - \theta\,\xi\eta^\top)X(t), & t > 0 \\ X(0) = x_0. \end{cases} \tag{11}$$

moreover the trajectory at equilibrium, X^*, satisfies

$$\langle X^*(t), \eta\rangle = e^{gt}\langle x_0, \eta\rangle \tag{12}$$

with

$$g = \lambda - \theta f = \frac{\lambda - f\rho}{1 + (\sigma-1)f}. \tag{13}$$

Remark 3.4. *One can study a version of the model with logarithmic utility. In this case the evolution of the system is again described by system (1) but the i-th agent maximizes the functional*

$$J_i(c_i) = \int_0^{+\infty} e^{-\rho t}\ln(c(t))\,dt.$$

In this case a similar result to that described in Theorem 3.3 can be proven. The value of θ simplifies to ρ, the equilibrium is characterized by the following strategies

$$\psi_i(x) = \frac{\rho}{\eta_i}\langle x, \eta\rangle, \text{ for all } i \in F, \qquad \psi_i(x) = 0, \text{ for all } i \notin F,$$

and the welfare of agent i along the equilibrium is

$$V_i(x) = \frac{1}{\rho}\left(\ln(\langle x, \eta\rangle) + \ln\left(\frac{\rho}{\eta_i}\right) + \lambda - f\rho\right).$$

The growth rate is $\lambda - f\rho$.

Remark 3.5. *Note that the positivity of η implies*

$$\min_i \eta_i \sum_i X_i^*(t) \le \langle X^*(t), \eta\rangle \le \max_i \eta_i \sum_i X_i^*(t),$$

so that (iii) in Theorem 3.3 gives

$$\frac{1}{\max_i \eta_i}\, e^{gt}\langle x_0, \eta\rangle \leq \sum_i X_i^*(t) \leq \frac{1}{\min_i \eta_i}\, e^{gt}\langle x_0, \eta\rangle.$$

and g is also the growth rate in the long run of the aggregate stock $\sum_i X_i(t)$.

Proposition 3.6. *Assume $\theta > 0$ and that X^* is the equilibrium trajectory described in Theorem 3.3. Suppose that G is symmetric, i.e. $g_{ij} = g_{ji}$, and regard the growth rate g as a function of g_{ij} $(i, j \in N,\ i \neq j)$. Then:*

(i) (standard case) *if $1-(1-\sigma)f > 0$, then g is a decreasing function of g_{ij}*

(i) (voracity effect) *if $1-(1-\sigma)f < 0$, then g is an increasing function of g_{ij}.*

Remark 3.7. *We here gain some insight at the condition $\theta > 0$, used in Theorem 3.3. We preliminarily observe that a positive sign of the numerator $\rho+(\sigma-1)\lambda > 0$ implies the boundedness of the functional in the case of a single player ($f = 1$), as well as the boundedness of the functional of a control problem for a social planner maximizing the sum of utilities of the players, as it can be easily proven. Moreover, since λ represents the (asymptotic) growth rate of the resource under null extraction, the result is consistent with the parallel condition $\rho+(\sigma-1)A > 0$ in the standard single-player/social-planner AK-models (for extraction or growth). The interpretation of such condition remains the same for a general number f of players, when both the numerator and the denominator in (7) are positive.*

When instead the denominator $1 + (\sigma - 1)f$ is negative, and then necessarily the numerator $\rho + (\sigma - 1)\lambda$ is also negative, the outcome for the game and the social planner problem diverge, in that a Nash equilibrium exists for the game (in presence of the so- called voracity effect*, see Tornell and Lane, 1999) while it can be proven that a solution for the social planner problem does not exist. This follows from the greater consumption of the resource (and therefore from the sub-optimality of the behavior) that one has in the game compared to the planner case: while with this choice of parameters the functionals of the players remain bounded at the equilibrium, infinite-utility controls are possible for the planner.*

In addition, note that when the denominator is negative (in particular $\sigma \in (0,1)$) and the numerator is positive - hence when $\theta < 0$, the reverse situation takes place. Namely, there exists an optimal control for the planner problem but not our equilibrium in the game, as the growth rate perceived by players (i.e. $\lambda-(f-1)\theta$) is "too negative" and it would push them to consume the whole resource at time 0, a strategy which is nonadmissible in our setting.

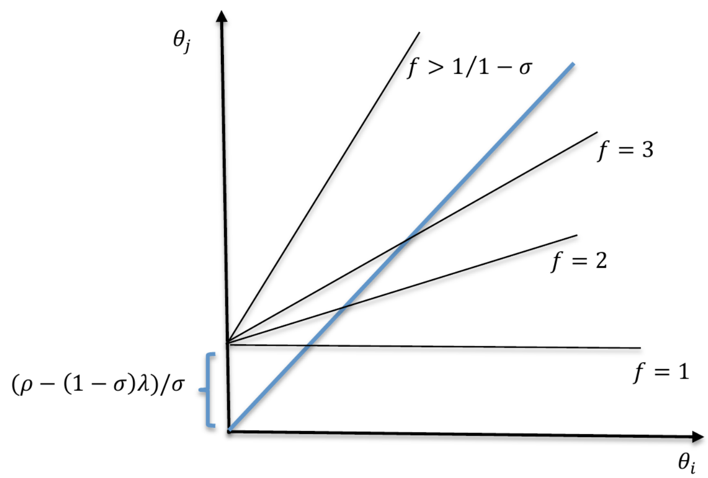


FIGURE 1. Existence and non-existence of the equilibrium varying the number of players. The common value of the equilibrium extraction intensity of the stock ratio solves the linear system: $\theta_i = \frac{\rho-(1-\sigma)\lambda}{\sigma} + (f-1)\frac{1-\sigma}{\sigma}\theta_j$, $\theta_i = \theta_j$. In the case $\rho - (1-\sigma)\lambda > 0$ and $\sigma \in (0,1)$, the line $\theta_i = \frac{\rho-(1-\sigma)\lambda}{\sigma} + \frac{1-\sigma}{\sigma}(f-1)\theta_j$ intersects the bisector of the first orthant if and only if $\frac{1-\sigma}{\sigma}(f-1) < 1$ holds.

To better understand this fact, note that in our Markovian equilibrium the ratio between the i-th agent's extraction intensity and the capital value, normalized with the i-th component of the eigenvector put equal to 1, is uniform across agents (the common value is θ). So one may think that to find the equilibrium one must cross the "reaction functions"

$$\theta_i = \frac{\rho - (1-\sigma)(\lambda - \sum_{j\neq i}\theta_j)}{\sigma},$$

with symmetry implying $\sum_{j \neq i} \theta_j = (f-1)\theta_j = (f-1)\theta_i$. *As shown in Figure 1, if* $\rho - (1-\sigma)\lambda > 0$, *and* $\sigma < 1$, *then there is the solution for the single agent case, but for an increasing number of agents* f *and as soon as* $\frac{1-\sigma}{\sigma}(f-1) > 1$, *the extraction intensity becomes infinite.*

Similar conditions for the aggregate cases with $A = 0$ *are given in Dockner et al. (2000).*

3.1. **Stability.** In order to address convergence of transitional dynamics towards a potential steady state, it is useful to describe the equilibrium trajectory in terms of the eigenvectors/eigenvalues of the matrix of the system in (11), namely $A + G^\top - \theta\, \xi\eta^\top$. Firstly, we observe that η is a left eigenvector of $A + G^\top$ and of $\xi\eta^\top$, with associated eigenvalues λ and f respectively, so η is also a left eigenvalue for $A + G^\top - \theta\, \xi\eta^\top$, associated to the eigenvalue $g = \lambda - \theta f$. Then there exists a (right) eigenvector ζ of $A + G^\top - \theta\, \xi\eta^\top$ associated to the same eigenvalue g.

Remark 3.8. *If in addition* $G = G^\top$, *the remaining eigenvectors of* $A + G - \theta\, \xi\eta^\top$ *are the set* $\{w_2, .., w_n\}$ *of eigenvectors of* $A + G$, *respectively associated to the eigenvalues* $\{\lambda_2, .., \lambda_n\}$ *(now all real). This can be checked by direct proof, making use of Remark 3.1 and of the fact that the vector subspace generated by* $\{w_2, .., w_n\}$ *coincides with* $\langle\eta\rangle^\perp$. *In other words, in the symmetric case, the dynamic of the system* $A + G - \theta\, \xi\eta^\top$ *with (respectively, without) extraction of the resource, namely the case* $\theta > 0$ *(respectively,* $\theta = 0$*), leaves unchanged all eigenvectors and relative eigenvalues except one, that is associated to the dominant root. That eigenvalue changes from* λ *to* $\lambda - \theta f$, *while the associated eigenvector changes from* η *to* ζ. *That means that the trajectory* X^* *is modified only along the direction of such eigenvector.*

This argument implies in particular the next proposition.

Proposition 3.9. *Assume* $\theta > 0$ *and the equilibrium trajectory* X^* *described in Theorem 3.3. Assume in addition* $G = G^\top$. *Then there exist real constants* k_i,

$i = 1, ..n$, such that

$$X^*(t) = \langle x_0, \zeta\rangle e^{gt}\zeta + \sum_{i=2}^{n} \langle x_0, w_i\rangle e^{\lambda_i t} w_i.$$

Moreover, $0 < \theta < \frac{\lambda-\lambda_2}{f}$ (that is, $g > \lambda_2$) implies that the detrended trajectory $X^*(t)e^{-gt}$ satisfies

$$\lim_{t\to+\infty} X^*(t)e^{-gt} = k_1\zeta,$$

that is, it converges asymptotically towards the direction of ζ.

In the general case, i.e. when G is not necessarily symmetric, a similar statement is true. We complete $\{\zeta\}$ into a basis of generalized eigenvectors $\{\zeta \equiv v_1, v_2, .., v_n\}$ of $A + G^\top - \theta\ \xi\eta^\top$. If $\{g \equiv \mu_1, \mu_2, .., \mu_n\}$ are the associated eigenvalues, with Re denoting their real part, then the following results holds.

Proposition 3.10. *Assume $\theta > 0$ and the equilibrium trajectory X^* described in Theorem 3.3. Then there exist continuous coefficients m_i, such that $\lim_{t\to\infty} m_i(t)e^{-\varepsilon t} = 0$ for all $\varepsilon > 0$, and such that*

$$X^*(t) = m_1(t)e^{gt}\zeta + \sum_{i=2}^{n} e^{Re(\mu_i)t} m_i(t) v_i. \tag{14}$$

Moreover, if $0 < \theta < \frac{\lambda - Re(\mu_2)}{f}$ (i.e. $g > Re(\mu_2)$) then

$$\lim_{t\to+\infty} X^*(t)e^{-gt} = \langle x_0, \zeta\rangle \zeta. \tag{15}$$

3.2. **Admissibility.** In the first part of this section, starting with Theorem 3.3, we have always assumed that the strategy profile ψ was in $\mathcal{M}(x_0)$, that is, admissible at $x_0 \in \mathbb{R}^n_+$. Here we want to investigate under which conditions that is true. Some factors are implicated in admissibility of the equilibrium strategy profile ψ:

(*a*) *the magnitude of θ and the positivity of the eigenvector ζ.* This is easily understood from the analysis of the symmetric case. As a consequence of Perron-Frobenius theorem, all eigenvectors w_i for $i \geq 2$ have at least one negative coordinate. If θ is too big, i.e. $\theta > (\lambda - \lambda_2)/f$ (equivalently $g < \lambda_2$), Proposition 3.9 implies that the detrended trajectory $X^*(t)e^{-\lambda_2 t}$ converges along the direction of the second eigenvector

w_2 and leaves definitively the positive orthant, when starting at any initial position x (except for the particular case in which x belongs to the halfline $s = \{t\zeta : t \geq 0\}$). On the other hand, when $\theta < \frac{\lambda - \lambda_2}{f}$ the detrended trajectory $X^*(t)e^{-gt}$ converges along the direction of ζ, so that ζ needs to be a positive vector in order for the trajectory to remain in the positive orthant. Since g and ζ are continuous functions of θ, and for $\theta = 0$ their values are respectively λ and η which are both positive, one may argue that there exists $\theta^* > 0$ such that for all θ such that $0 < \theta < \theta^*$ one has $g(\theta) > \lambda_2$ and $\zeta(\theta)$ is a positive vector.

(b) *the choice of the initial condition.* One simple necessary and sufficient condition of admissibility of ψ at all initial conditions $x_0 \in \mathbb{R}^n_+$ is the following.

Proposition 3.11. *The strategy profile ψ described in Theorem 3.3 is admissible, namely $\psi \in \mathcal{M}(x_0)$, at every $x_0 \in \mathbb{R}^n_+$ if and only if the following condition is true*

$$0 \leq \theta \leq \min_{i \in F, j \in N} \left(g_{ij} \frac{\eta_i}{\eta_j} \right). \tag{16}$$

The condition is meaningful when all nodes are connected to one another, but when at least one of the coefficients $g_{ij},\ i \in F, j \in N$ is null (i.e. there exist two locations which are not directly connected), it implies $\theta = 0$, while for $\theta > 0$ there are always initial positions in the positive orthant at which the equilibrium strategy is not admissible.

This is the case of the following example. Consider a network for $n = 4$ where node i is only connected to nodes $i-1$ and $i+1$ and with weight α, and all natural growth rates are equal to Γ, players are 4 and there is no reserve.

$$G = \begin{pmatrix} 0 & \alpha & 0 & \alpha \\ \alpha & 0 & \alpha & 0 \\ 0 & \alpha & 0 & \alpha \\ \alpha & 0 & \alpha & 0 \end{pmatrix}, \quad A = (\Gamma - 2\alpha)I, \qquad E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

Then (see also Remark 3.8) $\lambda = \Gamma,\ \lambda_2 = \Gamma - 2\alpha,\ \lambda_3 = \Gamma - 4\alpha,\ g = \Gamma - 4\theta,\ \eta = \zeta = \frac{1}{2}(1,1,1,1),\ \theta = (\rho + (\sigma - 1)\Gamma)/(4\sigma - 3)$. Assume in addition that ρ and σ

are such that $\theta > 0$, and $\theta < \alpha/2$ (one such choice is, for instance, $\sigma = 7/8$, and $\Gamma/8 < \rho < \Gamma/8 + \alpha/4$). Then ζ is positive and $g > \lambda_2$, and the detrended equilibrium trajectory tends to the direction of ζ, in view of Proposition 3.9. Now consider the initial condition $x_0 = (0, 0, 1, 0)^\top$ and the associated trajectory X^*. Then $(X_1^*)'(0) = -\theta < 0$ and the trajectory leaves immediately the positive orthant.

Given all previous remarks, we are now ready to deliver a sufficient conditions of admissibility of the equilibrium strategy ψ for a wide class of networks.

Proposition 3.12. *Assume $\theta > 0$ and suppose that the eigenvalue g of $A + G - \theta\, \xi\eta^\top$ satisfies $g > Re(\mu_2)$, and is associated to a positive eigenvector ζ. Then there exists a linear cone C containing ζ, such that the Markovian equilibrium ψ described in Theorem 3.3 is admissible at all initial conditions $x \in C$.*

Remark 3.13. *In Proposition 3.12 we prove the existence of a subset - a cone - of initial states for which the strategy profile ψ of Theorem 3.3 is admissible, and for which ψ is in fact an equilibrium in the sense of Definition 2.4. The reader might be led to think that this means that the described equilibrium is somehow, see e.g. Dockner and Wagener (2014), a* local equilibrium*, and that the state space needs to be restricted to that cone. This is not the case: in our results players maximize their payoff over all admissible strategies, whether they drive the trajectory in or outside the cone. When in addition conditions of Proposition 3.12 are verified, the chosen profile of strategies always maintains the trajectory inside the cone.*

4. The role of network structure

We intend now to interpret the equilibrium strategy in terms of the network structure. From Theorem 3.3 we derive the following property.

Corollary 4.1. *In the hypotheses and with the notation of Theorem 3.3, the overall welfare of players* $\sum_{i\in F} V_i(x)$ *is maximized if the nature reserves are built at the locations* i *where* η_i *are highest.*

The eigenvector η has a straightforward interpretation in network theory. In view of Remark 2.2, if one considers the modified network $\mathcal{G}^*$ associated with the matrix $G+A$, then η represents the so called *eigenvector centrality* of the network $\mathcal{G}^*$. Then Corollary 4.1 establishes that nature reserve maximize social welfare when set at nodes with maximal eigencentrality.

Assume that the Perron-Frobenius eigenvalue for G is $\lambda_\circ$ and that $\eta_\circ$ is the normalized eigenvector associated to it, i.e. $G\eta_\circ = \lambda_\circ \eta_\circ$. To better understand how $\mathcal{G}$ and $\mathcal{G}^*$ are related, one may note that equation $(A+G)\eta = \lambda\eta$ can be rewritten as

$$(I - \lambda^{-1}A)^{-1}G\eta = \lambda\eta, \tag{17}$$

so that the matrix G is magnified by the diagonal matrix $(I-\lambda^{-1}A)^{-1}$ with all positive elements, as

$$\langle e_i, (I-\lambda^{-1}A)^{-1}e_i\rangle = \frac{\lambda}{\lambda - a_i} > 0, \quad \forall i \in N.$$

as a consequence of (9). Nonetheless, not necessarily are the components η_i of η influenced by the initial network eigencentrality $\eta_\circ$ and the natural growth rates Γ_i in a monotonic way, as better explained through the analysis of the following subcases and examples:

(a) *For networks with equal net reproduction rates,* η *coincides with the the eigenvector centrality of* $\mathcal{G}$. Now assume that all net reproduction rates are equal, namely

$$a_i = \Gamma_i - \sum_{j=1}^{n} g_{ij} \equiv a, \text{ for all } i \in N.$$

Then $A+G = aI + G$, eigenvectors of G and $aI+G$ are the same, implying $\eta = \eta_\circ$, and η, η° are associated, respectively, to eigenvalues λ, $\lambda_\circ = \lambda - a$. Components η_i are higher when nodes are better connected to the other nodes and are lower for peripheral nodes. Corollary 4.1 then implies that welfare is higher when reserves are set in more central nodes. An example fitting the description is represented in Figure 2.

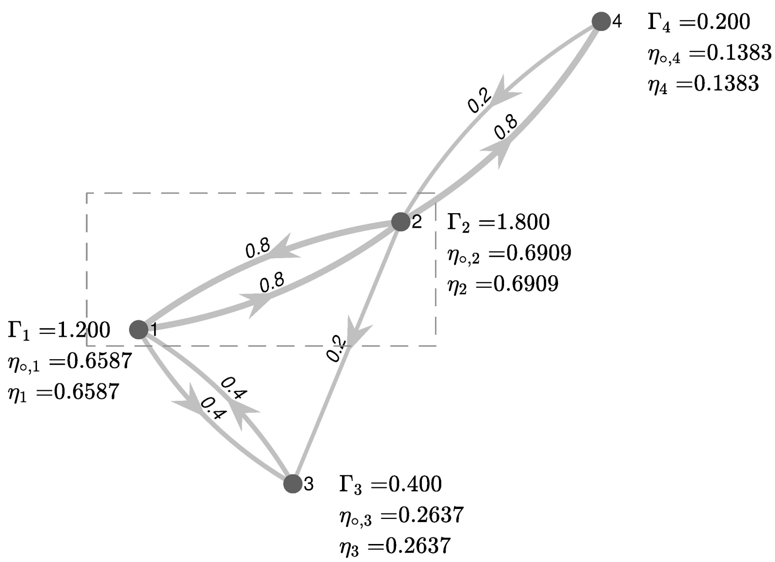


FIGURE 2. A network with equal net reproduction rates, exemplifying those described in Section 4 (a). Here there are 4 nodes and 2 players, so that the planner has to establish a 2-nodes reserve. The values of the the natural growth rates Γ_i, together with the components of the eigenvalue centrality of the networks $\mathcal{G}$ $(\eta_{\circ,i})$ and $\mathcal{G}^*$ (η_i), are reported in each node. Weights are reported over of the edges. The dashed line encloses the two nodes in which it is optimal to establish the reserve.

(b) *In a equally weighted complete network, η_i are ordered like reproduction rates Γ_i.* Assume now a complete network, with $g_{ij} = \alpha$ for some $\alpha > 0$ and all $i \neq j$, and $g_{ii} = 0$.

Combining the i-th and the ℓ-th row of equation (8), one obtains

$$\eta_\ell = \frac{a_i - \lambda - \alpha}{a_\ell - \lambda - \alpha} \eta_i,$$

so that from $a_\ell - \lambda - \alpha < 0$ (see (9)) one derives

$$\eta_\ell \geq \eta_i \quad \Leftrightarrow \quad a_\ell \geq a_i \quad \Leftrightarrow \quad \Gamma_\ell \geq \Gamma_i.$$

Then the overall productivity is highest when reserves are placed in locations with highest reproduction rates. An example fitting the description is represented in Figure 3.

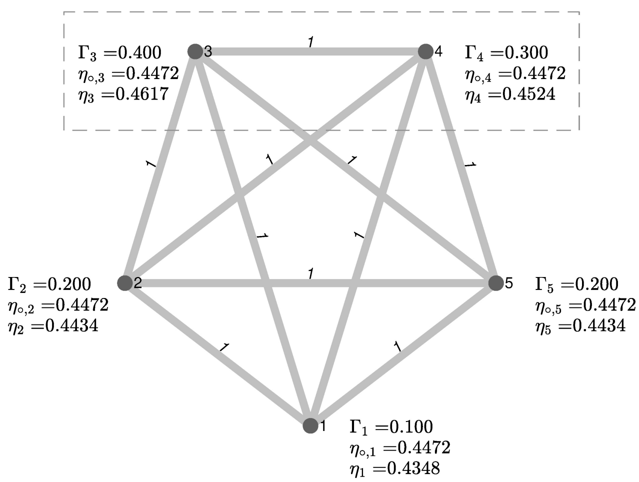


FIGURE 3. An equally weighted complete network, exemplifying those described in Section 4 (b). Here there are 5 nodes and 3 players, and the planner has to establish a 2-nodes reserve. The values of the the natural growth rates Γ_i, together with the components of the eigenvalue centrality of the networks $\mathcal{G}\ (\eta_{\circ,i})$ and $\mathcal{G}^*\ (\eta_i)$, are reported in each node. Weights are reported over of the edges. The dashed line encloses the two nodes in which it is optimal to establish the reserve.

(c) From the analysis of the previous subcases one may wonder if there exists a monotonic relationship between Γ_i, η_i° and η_i. For example, if node i has a greater centrality and reproduction rate than another node j, namely $\eta_i^\circ \geq \eta_i^\circ$ and $\Gamma_i \geq \Gamma_j$, then the reserve is better placed at node i than at node j, namely $\eta_i \geq \eta_j$. The answer is negative, as explained in the following example. Consider

$$G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}, \quad \Gamma_1 = 1, \quad \Gamma_2 = 1 + b, \quad \Gamma_3 = 0,$$

with $b > 0$. By explicit calculation one has $\lambda_\circ = \sqrt[3]{2}$ and $\eta^\circ = \mu^\circ/|\mu^\circ|$ with $\mu^\circ = (2^{-2/3}, 2^{-1/3}, 1)^\top$. Note that $\eta_2^\circ > \eta_1^\circ$ and $\Gamma_2 > \Gamma_1$, that is, node 2 precedes node 1 both in productivity (natural and net) and centrality. Nonetheless, $\eta_1 > \eta_2$ for some choices of a positive b, as we show next. To this extent, if $\eta = \mu/|\mu|$, with $\mu = (1, \mu_2, \mu_3)^\top$, then μ satisfies

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & b & 1 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \mu_2 \\ \mu_3 \end{pmatrix}, \tag{18}$$

whose expansion implies

$$\mu_1 = 1, \quad \mu_2 = \lambda, \quad \mu_3 = \lambda(\lambda - b), \quad b = \lambda - \frac{2}{\lambda(\lambda + 2)}.$$

Note that the last equation implies in particular that b is an increasing function of λ and viceversa. A direct calculation shows that for $b = 0$ one has $\lambda(0) \simeq 0.8$, so that by continuity $\lambda(0) < \lambda(b) < 1$ for a small positive b. Hence $\eta_1 > \eta_2$ and a reserve is better set in node 1 rather than in node 2. The example is represented in Figure 4(C).

(d) The analysis of the previous example with $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = 2 + a$ helps confirming the interpretation of eigencentrality η_i as a measure of productivity and connectiveness not only of the i-th node, but also of the nodes more directly connected to it. In this case

$$\mu_1 = 1, \quad \mu_2 = \lambda, \quad \mu_3 = \lambda^2, \quad a = \lambda - \frac{2}{\lambda^2},$$

with λ is an increasing function of a, moreover for $a = -1$ one has $\mu_1 = \mu_2 = \mu_3 = 1$, and $\lambda = 1$, so that $\lambda > 1$ if and only if $a > -1$. Therefore

$$\mu_1 < \mu_2 < \mu_3, \text{ for } a > -1, \text{ and } \mu_1 > \mu_2 > \mu_3, \text{ for } a < -1.$$

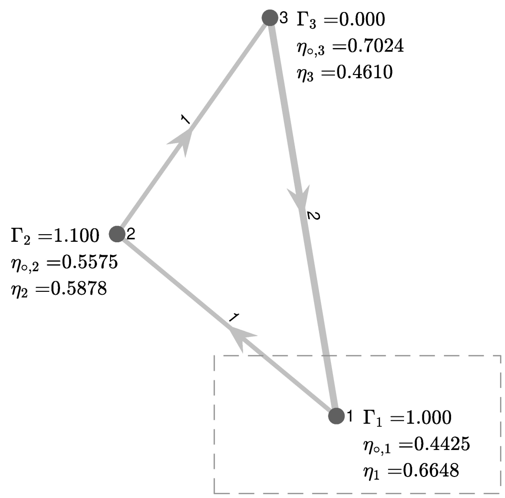
Hence, an increasing reproduction rate Γ_3 not only does increase η_3 making (definitively) Node-3 the most central, but also influences the centrality η_2 of Node-2, which is more directly connected to it than Node-1. The example is represented in Figure 4(D).

5. Concluding remarks

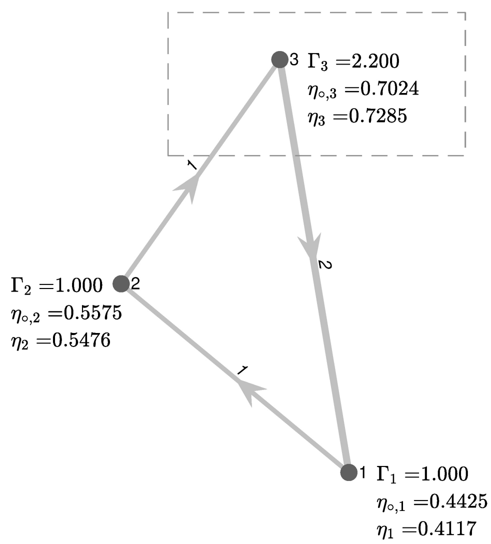
The main aim of this paper has been to explore, in a simple framework with heterogeneous regions and a given number of agents, how the structure of the migration network affects competition for spatially distributed moving resources. We have found that if the regulator's objective is the maximization of the unweighted sum of the utilities of the agents, and he/she is constrained to assign no more than one agent to each region, then the reserves should be localized in the most central regions. Here the relevant centrality measure is given by the eigenvector centrality of a derived network obtained by magnifying the links of each node in the original migration network by a factor that is increasing in the productivity of the node itself.

Although in our analysis both the agents and the regulator care only about consumption of the resource, our model provides a basis for more general analysis where preferences for conservation are considered, introducing for example the resource stocks in the utility functions of the agents and/or in the regulator welfare function. A theme of this analysis will be how the role of the regulator is enhanced under the new hypotheses.

In a different vein, the role of the regulator could be also examined in more general contexts in which a "bad" extreme equilibrium coexists with the interior equilibrium. For example, an extreme equilibrium can be expected to exist in variants of our model if the extracted resource can be stored (e.g., Kremer and Morcom, 2000). In this case, a spatially structured policy could be a useful tool to eliminate the incentives that potentially could lead the agents to coordinate on the "bad" outcome.



(C) The network associated to example (c).



(D) The network associated to example (d).

FIGURE 4. Representation of examples (c) and (d) of Section 4. In each case we have 3 nodes and 2 players (so the planner has to establish a 1-node reserve). The values of the the natural growth rates Γ_i, together with the components of the eigenvalue centrality of the networks $\mathcal{G}$ ($\eta_{\circ,i}$) and $\mathcal{G}^*$ (η_i), are reported in each node. Weights are reported over of the edges. The dashed line encloses the node in which it is optimal to establish the reserve.

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Appendix A. Proofs

Proof of Theorem 3.3. We initially take the standpont of player i, active at node i. For all other players we assume that they play a Markovian strategy, described by

$$c_j(t) = a_j \langle X(t), \eta \rangle, \text{ with } j \in F - \{i\},$$

where a_j are nonnegative real numbers. Then the current value Hamiltonian of i^{th} player is

$$h(x, c_i, p) := \frac{1}{1-\sigma} c_i^{1-\sigma} + \langle x, (A+G)p \rangle - \langle x, \eta \rangle \sum_{j \in F - \{i\}} p_j a_j - c_i p_i \tag{19}$$

so that the maximal Hamiltonian is

$$\begin{aligned} H(x,p) &= \max_{c_i} \left\{ \frac{c_i^{1-\sigma}}{1-\sigma} - c_i p_i \right\} + \langle x, (A+G)p \rangle - \langle x, \eta \rangle \sum_{j \in F - \{i\}} p_j a_j \\ &= \frac{\sigma}{1-\sigma} p_i^{1-\frac{1}{\sigma}} + \langle x, (A+G)p \rangle - \langle x, \eta \rangle \sum_{j \in F - \{i\}} p_j a_j. \end{aligned} \tag{20}$$

with maximum attained at $c_i = p_i^{-\frac{1}{\sigma}}$. As a consequence, the Hamilton-Jacobi-Bellman (briefly, HJB) equation associated to the problem is

$$\rho v(x) = \frac{\sigma}{1-\sigma}\left(\frac{\partial v}{\partial x_i}\right)^{1-\frac{1}{\sigma}} + \langle x, (A+G)\nabla v(x)\rangle - \langle x, \eta\rangle \sum_{j\in F-\{i\}} \left(\frac{\partial v}{\partial x_j}\right) a_j$$

Step 1: we search for a solution of HJB equation of type

$$v(x) = \frac{b_i}{1-\sigma}\left\langle x, \eta\right\rangle^{1-\sigma}, \quad \text{with} \quad \nabla v(x) = b_i \left\langle x, \eta\right\rangle^{-\sigma} \eta \tag{21}$$

where b_i is a suitable positive real number. Substituting v and its partial derivatives into the HJB equation, we obtain that v is a solution if and only if

$$b_i = \frac{1}{\eta_i}\left(\frac{\sigma\eta_i}{\rho - \lambda(1-\sigma) + (1-\sigma)\sum_{j\in F-\{i\}}\eta_j a_j}\right)^{\sigma}.$$

Step 2: Markovian equilibrium. For (20), the candidate optimal strategy for player i satisfies

$$c_i(t) = (b_i\eta_i)^{-\frac{1}{\sigma}}\left\langle X(t), \eta\right\rangle \tag{22}$$

At equilibrium one has $a_i = (b_i\eta_i)^{-\frac{1}{\sigma}}$, implying

$$a_i = \frac{1}{\eta_i}\frac{\rho - \lambda(1-\sigma)}{1-(1-\sigma)f} = \frac{\theta}{\eta_i}, \text{ and } b_i = \eta_i^{\sigma-1}\theta^{-\sigma} \tag{23}$$

from which the formulas (10) and (*ii*) derive.

Step 3: Closed loop equation. Note that, along the equilibrium trajectories,

$$c(t) = \theta\langle X(t), \eta\rangle\xi = \theta\, \xi\eta^{\top}X(t),$$

so that the evolution system in (*iii*) follows. The second part of statement (*iii*) follows from

$$\langle \dot{X}(t), \eta\rangle = \langle X(t), (A+G)\eta\rangle - \langle X(t), \eta\rangle\langle \xi, \eta\rangle = \langle X(t), \eta\rangle(\lambda - \theta f)$$

where $\lambda - \theta f = g = (\lambda - f\rho)(1 + (\sigma-1)f)^{-1}$.

Step 4: Best response. We verify now that the feedback strategy (10) is the best response for Player i, when the other players choose ψ_j, with $j \neq i$, as in (10). Then the problem of Player i is maximizing (4) under the dynamics

$$\begin{cases} \dot{X}(t) = (A + G^\top - \theta\, \xi^i \eta^\top)X(t) - c_i(t)e_i, & t > 0 \\ X(0) = x_0. \end{cases} \tag{24}$$

where the vector ξ^i coincides with ξ except for the i-th component, which is set equal to 0, namely $\xi^i_\ell = \xi_\ell$ for all $\ell \neq i$, and $\xi^i_i = 0$.

Set $c_i^*(t) = \psi(X^*(t))$ and let $c_i(t)$ be any other admissible control, with $X^*(t)$ and $X(t)$, respectively, the associated trajectories. Now we consider the quantity $(c_i^*(t) - c_i(t))\frac{\partial v}{\partial x_i}(X^*(t))$ and use the fact that $c_i^*(t)$ realizes the maximum in (20) with $a_j = \theta/\eta_j$, and $p = \nabla v(X^*(t))$ to derive

$$\frac{1}{1-\sigma}\left(c_i^*(t)^{1-\sigma} - c_i(t)^{1-\sigma}\right) \geq (c_i^*(t) - c_i(t))\, \frac{\partial v}{\partial x_i}(X^*(t)) \tag{25}$$

Next, observe that, adding and subtracting $\big\langle (A + G^\top - \theta \quad \xi^i\eta^\top)(X^*(t) - X(t)), \nabla v(X^*(t))\big\rangle$ and making use of (24), the right and side in (25) equals

(26)

$$\begin{aligned} &\big\langle (A + G^\top - \theta\, \xi^i\eta^\top)(X^*(t) - X(t)), \nabla v(X^*(t))\big\rangle - \big\langle (\dot{X}^*(t) - \dot{X}(t)), \nabla v(X^*(t))\big\rangle \\ &= \big\langle X^*(t) - X(t), (A + G - \theta\eta(\xi^i)^\top)\nabla v(X^*(t))\big\rangle - \big\langle (\dot{X}^*(t) - \dot{X}(t)), \nabla v(X^*(t))\big\rangle. \end{aligned}$$

Recalling (21) and (12) we have

$$\nabla v(X^*(t)) = b_i \langle X^*(t), \eta\rangle^{-\sigma}\eta = b_i e^{-\sigma g t}\langle x_0, \eta\rangle^{-\sigma}\eta.$$

Using this expression and the fact that $(A + G - \theta\eta(\xi^i)^\top)\eta = (\lambda - \theta(f-1))\eta$, the expression in (26) can be written as

$$= b_i\langle x_0, \eta\rangle^{-\sigma} e^{-\sigma g t}\left[\big\langle X^*(t) - X(t), [\lambda - \theta(f-1)]\eta\big\rangle - \big\langle (\dot{X}^*(t) - \dot{X}(t)), \eta\big\rangle\right]$$

Thus, using these estimates, integrating (25) on $[0, T]$ for $T > 0$, we obtain

$$\int_0^T \frac{e^{-\rho t}}{1-\sigma}\left(c_i^*(t)^{1-\sigma} - c_i(t)^{1-\sigma}\right) dt \geq \tag{27}$$

$$b_i\langle x_0, \eta\rangle^{-\sigma}\left[\int_0^T e^{-(\sigma g + \rho)t}\big\langle X^*(t) - X(t), (\lambda - \theta(f-1))\eta\big\rangle dt - \int_0^T e^{-(\sigma g+\rho)t}\big\langle (\dot{X}^*(t) - \dot{X}(t)), \eta\big\rangle dt\right]$$

and, integrating by parts the last term, the right hand side equals

$$
\begin{aligned}
(28) \quad &= b_i\langle x_0,\eta\rangle^{-\sigma}\Big[\int_0^T e^{-(\sigma g+\rho)t}\big\langle X^*(t)-X(t),\big(\lambda-\theta(f-1)\big)\eta\big\rangle dt+ \\
&- e^{-(\rho+\sigma g)T}\left\langle (X^*(T)-X(T)),\eta\right\rangle - \int_0^T e^{-(\sigma g+\rho)t}\big\langle (X^*(t)-X(t)),(\sigma g+\rho)\eta\big\rangle dt\Big] \\
&= b_i\langle x_0,\eta\rangle^{-\sigma}e^{-(\rho+\sigma g)T}\left\langle (X(T)-X^*(T)),\eta\right\rangle \geq -b_i\langle x_0,\eta\rangle^{-\sigma}e^{-(\rho+\sigma g)T}\left\langle X^*(T),\eta\right\rangle
\end{aligned}
$$

where the last equality is a consequence of $\sigma g+\rho=\lambda-\theta(f-1)$, and the last inequality a consequence of $\langle X(T),\eta\rangle\geq 0$, as $X(T)$ is admissible and hence nonnegative. Now $e^{-(\rho+\sigma g)T}\langle X^*(T),\eta\rangle = e^{-(\rho+\sigma g)T}e^{gT}\langle x_0,\eta\rangle$ decreases to 0, as T tends to $+\infty$, as

$$
g(1-\sigma)-\rho=-\theta<0. \tag{29}
$$

Thus, taking the limit as T tends to $+\infty$ of the inequalities (27)(28), implies

$$
\int_0^{+\infty} e^{-\rho t}\frac{c_i^*(t)^{1-\sigma}}{1-\sigma}dt \geq \int_0^{+\infty} e^{-\rho t}\frac{c_i(t)^{1-\sigma}}{1-\sigma}dt,
$$

that is, the optimality of $c_i^*(t)$. (Note that limits exist as integrals are monotonic in T).

□

Proof of Proposition 3.6. We first check the effect of an ϵ-increase of g_{ij}, with $i\neq j$, on the value of λ. To this extent, fix $\epsilon>0$ and define $M_{ij}:=(e_i\,e_j^\top+e_j\,e_i^\top)-(e_i\,e_i^\top+e_j\,e_j^\top)$, and note that the system matrix changes from $A+G$ to $A+G+\epsilon M_{ij}$. Note that this last matrix can be written as the sum of two Metzler matrices

$$
A+G+\epsilon M_{ij}=[A-\epsilon(e_i\,e_i^\top+e_j\,e_j^\top)]+[G+\epsilon(e_i\,e_j^\top+e_j\,e_i^\top)]
$$

so that it is itself a Metzler matrix. On the other hand, M_{ij} is a negative-semidefinite matrix so that $\langle x,M_{ij}x\rangle\leq 0$ for all $x\in\mathbb{R}^n$. We can argue as in Remark 2.2 and denote by η_ϵ its Perron-Frobenius eigenvector of norm 1, and by λ_ϵ the associated Perron-Frobenius eigenvalue. Since the network matrix is symmetric, we can use the variational characterization of eigenvalues (see for instance Corollary III.1.2 of Bhatia, 2013) so that

$$
\begin{aligned}
(30)\qquad \max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\langle x,(A+G+\epsilon M_{ij})x\rangle}{|x|^2} &= \lambda_\epsilon = \frac{\langle \eta_\epsilon,(A+G+\epsilon M_{ij})\eta_\epsilon\rangle}{|\eta_\epsilon|^2} \\
&= \frac{\langle \eta_\epsilon,(A+G)\eta_\epsilon\rangle}{|\eta_\epsilon|^2} + \epsilon\frac{\langle \eta_\epsilon, M_{ij}\eta_\epsilon\rangle}{|\eta_\epsilon|^2} \\
&\le \max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\langle x,(A+G)x\rangle}{|x|^2} \;+\; \epsilon\frac{\langle \eta_\epsilon, M_{ij}\eta_\epsilon\rangle}{|\eta_\epsilon|^2} \le \max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\langle x,(A+G)x\rangle}{|x|^2} = \lambda
\end{aligned}
$$

This means that $\frac{d\lambda}{dg_{i,j}} \le 0$. Using this fact and the expression of the growth rate given in (13) we get the claim. □

Proof of Proposition 3.10 . If J is the real Jordan form of the matrix $A + G^\top - \theta E$, then there exists a real invertible matrix P such that $P^{-1}(A + G^\top - \theta E)P = J$. Consequently there exist real coefficients β_i such that

$$
(31)\qquad X^*(t) = e^{t(A+G^\top-\theta E)}x = Pe^{tJ}\left(\sum_{i=1}^{n}\langle x_0, v_i\rangle P^{-1}v_i\right) = P\sum_{i=1}^{n}\beta_i e^{Jt}P^{-1}v_i.
$$

It follows then from the general theory (see for instance Section 1.3 of Colonius and Kliemann (2014)) that $e^{Jt}P^{-1}v_i = e^{\mathrm{Re}(\lambda_i)t}M_i(t)P^{-1}v_i$ where $M_i(t)$ is a block matrix (which is non-zero only on the Jordan block related to μ_i) whose coefficients are products of sinus and cosinus functions of t and of polynomials of t with maximum degree the dimensions of the generalized eigenspace. Since $Pe^{Jt}P^{-1}v_i$ is again an element of the generalized eigenspace associated to μ_i, it can be written as a linear combination of the eigenvectors related to the same generalized eigenspace, with coefficient having the same described behavior for t, and then the first claim follows. The second statement is a consequence of the same construction, once we observe that $M_i(t)$ for simple eigenvalue is just a real coefficient. □

Proof of Proposition 3.11 . The conditions (16) is equivalent to requiring that the matrix of system (11) (having nondiagonal terms $g_{ij} - \theta\eta_i\xi_j$) is a Metzler matrix, which is equivalent to establishing that the system is positive, that is, it has solutions contained in the positive orthant $\mathbb{R}^n_+$ for all initial conditions $x \in \mathbb{R}^n_+$ (see for example Farina and Rinaldi, 2000, Chapter 2). □

Proof of Proposition 3.12. The generalized eigenvector decomposition described in (14) shows that a trajectory of X^* starting at ζ always remains on the linear subspace generated by ζ. From the same decomposition and from (15) we see that the the origin is asymptotically stable and then (Theorem 1.4.8 page 16 of Colonius and Kliemann (2014)) exponentially stable for the system satisfied by

$$Z(t) = e^{-gt} X^*(t) - \langle x_0, \zeta \rangle \, \zeta.$$

This assures the existence of a positive open linear cone of initial data containing ζ for which trajectories always remain positive. □

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