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Abstract. In this paper, we revisit the theory of spatial externalities. In particular, we depart in several respects from the important literature studying the fundamental pollution free riding problem uncovered in the associated empirical works. First, instead of assuming *ad hoc* pollution diffusion schemes across space, we consider a realistic spatiotemporal law of motion for air and water pollution (diffusion and advection). Second, we tackle spatiotemporal non-cooperative (and cooperative) differential games. Precisely, we consider a circle partitioned into several states where a local authority decides autonomously about its investment, production and depollution strategies over time knowing that investment/production generates pollution, and pollution is transboundary. The time horizon is infinite. Third, we allow for a rich set of geographic heterogeneities across states while the literature assumes identical states. We solve **analytically** the induced non-cooperative differential game under decentralization and fully characterize the resulting long-term spatial distributions. We further provide with full exploration of the free riding problem, reflected in the so-called *border effects*. In particular, net pollution flows diffuse at an increasing rate as we approach the borders, with strong asymmetries under advection, and structural breaks show up at the borders. We also build a formal case in which a larger number of states goes with the exacerbation of pollution externalities. Finally, we explore how geographic discrepancies affect the shape of the border effects.

Key words: Spatial externalities, environmental federalism, transboundary pollution, differential games in continuous time and space, infinite dimensional optimal control problems

JEL classification: Q53, R12, O13, C72, C61, O44.

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1. Introduction

The theory of spatial externalities is an important and quite broad area in economic theory, public economics, and more recently, in the new economic geography. A key aspect in this literature turns out to be the potential free riding problem that might typically arise when actions taken within a jurisdiction have (negative) spillover effects on neighbors, which leads to inefficiency of the decentralized equilibrium (assuming it exists). These are typical considerations for example in the literature of fiscal federalism *à la* Oates (1972). They are also increasingly crucial in another rising literature, labeled *environmental federalism* by Konisky and Woods (2010). In the context of transboundary pollution, the free riding problem has a clear geographic feature: it is associated with a neat border effect. Indeed, if pollution control is decentralized, local governments may be strongly tempted to locate the most polluting facilities near the jurisdictional borders (Monogan III et al., 2017) and/or to enforce less frequently at these borders pro-environmental policies as those promoted by federal or international acts and protocols (see Konisky and Woods, 2010, on the local enforcement of the US federal Clean Air Act).

Earlier empirical assessments suggest indeed that as a consequence of the latter actions, pollution levels are systematically elevated near state borders relative to interior regions (Helland and Whitford, 2003, or Sigman, 2002 and 2005). The Helland and Whitford's 2003 paper is one of the most influential in the area: using toxics release inventory (TRI) US data from 1987 to 1996, they find that facilities' emissions into the air and water are systematically higher in counties that border other states. Focusing on air pollution, Monogan III et al. (2017) have analyzed polluters location in the US using a spatial point pattern model. Again they find that the main air polluters are by far more likely to be located near a state's downwind border than a control group of other industrial facilities. More intriguing: when studying the enforcement of the US federal Clean Air Act from 1990 through 2000, Konisky and Wood (2010) find that while there is a significant negative correlation between enforcement levels and proximity to international borders (Canada, Mexico), proximity to state borders is not associated with fewer inspections or punitive actions. This need not contradict though the general finding outlined above that pollution levels are systematically higher near state borders as this finding is more likely to be caused by the location of the most polluting facilities near the jurisdictional borders (due to fiscal incentives mainly) than by enforcement policy.

This said, the finding of Monogan III et al. (2017) makes very clear that one of the main rationales behind environmental free riding by local governments is maximization of (local and regional) political support. A key subsequent problem to solve at the federal/national level is precisely to eliminate or at least to limit free riding, in order to ultimately reach a socially more efficient environmental federalism. This key issue is treated in several theoretical frames (see below for a brief view). The recent empirical literature reports several relevant natural experiments. In particular, Kahn et al. (2015) have for example studied the impact of a natural experiment set in China in 2005 when the central government changed the local political promotion criteria in order to reduce border pollution. They do find evidence of water pollution reduction at province boundaries.[1]

On the theoretical ground, the free riding problem described above has suggested a quite substantial literature. We refer here for simplicity to the extremely useful works of Hutchinson and Kennedy (2008), and Silva and Caplan (1997).[2] All these papers use statics frameworks, mostly game-theoretic. To give a quite interesting example which uses the same type of geographic space as we will do, Hutchinson and Kennedy (2008) consider a federation of identical states distributed around a latitudinal circle, each of them occupying an arc of length one. A continuum of identical polluting firms is distributed uniformly along the length of each state, and the mass of firms in each state is normalized to one. Pollution is transboundary as wind is blowing (here from west to east). The authors complete their story of pollution diffusion by assuming an *ad hoc* downwind transfer coefficient for emissions per location. In a first stage, the authors show that, under decentralization, states tend to enforce less stringent environmental standards on firms located close to downwind borders, leading to excessive interstate pollution in equilibrium, which is the standard free riding result. Second, they examine how the interplay between the federal policy on standards and the state policies on enforcement may restore efficiency.

In this paper, we depart from the latter abundant environmental federalism literature to dig much deeper in the theoretical foundations and characterization of the fundamental free riding problem highlighted in the associated empirical literature. In particular, instead of assuming *ad hoc* pollution diffusion schemes across space, we start with a realistic spatiotemporal law of motion for pollution, that is a diffusion equation (parabolic partial differential equation) with and without advection. Advection allows to introduce non-homogeneous diffusion across space to account for currents or winds for example.

For simplicity of exposition, on the same line of Hutchinson and Kennedy (2008), we model the space as a circle.[3] The circle is partitioned into several states, which need not be identical, contrarily to Hutchinson and Kennedy (2008). Each state is run by a

[1]In the recent past, water pollution levels have been elevated at political boundaries in China. Using the Hebei province as a case, Duvivier and Xiong (2013) find that dirty firms are more likely to locate in counties borders than in the interior.

[2]Hutchinson and Kennedy's paper includes an excellent survey of related theory papers.

[3]We must clarify here that the findings of the paper still hold if we model the space as a sphere or a portion of it. This would allow to calibrate the model with spatial pollution data and to perform realistic simulations. Here we keep the one dimensional model since this allows to see and explain more clearly the arising stylized phenomena like the border effect.

a local authority which decides about its investment, production and depollution strategies over time knowing that investment/production generates pollution, and pollution is transboundary.[4] The time horizon is infinite. We solve analytically the induced non-cooperative differential game under decentralization, which is itself a far non-trivial task (see the relation to the technical literature below).

With the closed-form solution paths in hands, we are able to illustrate a series of implications of the model. First of all, we are able to generate the border effect, that's a specific equilibrium behavior near the borders of the states. In particular, we show that net pollution emission flow is increasing as we approach the borders, with strong asymmetries under advection, and that structural breaks show up at the borders. Beside being consistent with the basic theory of spatial externalities, analogous phenomena have been recently disclosed by Lipscomb and Mobarak (2017) in their empirical study on water pollution in Brazil. Second, we uncover the predictions of our theory regarding the evolution of the size of spatial externalities (or in other terms, the extent of inefficiency) when the number of states (or jurisdictions) rises. In particular, we pose a formal case in which, as the spatial externalities theory suggests, a larger number of jurisdictions goes with the exacerbation of pollution externalities. Third, we extend the analysis of Hutchinson and Kennedy by departing from the identical states assumption: instead, our analysis allows for a large set of discrepancies across states (starting with the size), which may matter in the shape of the border effects. Last but not least, definitely much easier than the decentralized equilibrium differential game setting, we also characterize the outcomes of the cooperative equilibrium run by, say, a federal government. This is done to permanently outline the distance to efficiency of the equilibrium counterpart.

Our setting applies not only to the large set of interesting questions raised by environmental federalism but also to the currently hot debate around supranational coordination of environmental policies. Our theoretical setting is general enough to accommodate the two global levels (federal and supranational). Even more important, as our approach allows for deep geographic discrepancies, it is perfectly suitable to study some of the fundamental questions in the international agenda, in particular those related to the North/South environmental divide and the associated debate on the compensations to be given to the South to reach a global deal. This is out of the scope of the current paper.

Relation to the existing technical literature. Here come a few technical references for the readers to grasp the scope of this paper from this point of view. Indeed, from the methodological point of view, the differential game problem corresponding to the decentralized equilibrium is rewritten as a (non-zero sum) differential game in an infinite

[4] Note that we allow for heterogeneity also in the discount and risk aversion parameters of the local authorities, heterogeneity across states may also show up in local governance.

dimensional state space. Only very few papers in the mathematical literature (see, e.g., Nisio, 1998 and 1999, Baltas et al., 2019, or Kocan et al., 1997), deal with such problems which, however, arise in a natural way.
In the economic literature, de Frutos and Martin-Herran (2019a, 2019b) and the draft of de Frutos et al. (2019) are to our knowledge the sole papers studying the strategic implications of transboundary pollution in a spatial model. While the dynamics of the pollution stock is given by a diffusion equation like in our setting[5], the underlying economic model is quite different and therefore the economic focus is far from the target of our paper. In particular, the economic objectives of the research above is not rooted in the theorty of spatial externalities and the literature of environmental federalism.
Moreover, the mathematical approach is radically different. In de Frutos and Martin-Herran (2019a and 2019b) the continuous space-time model is not studied: an analogous discrete-space model is solved delivering a feedback Nash equilibrium, which is in turn used to capture the spatial interactions among agents through a truly comprehensive set of carefuly designed numerical exercises. In the draft of de Frutos et al. (2019) the continuous space-time model is studied in a heuristic way as a departure point to perform some interesting numerical simulations. It should be noted here that in our setting, we perform a complete mathematical study of the continuous time-space model, providing the explicit form of the (unique) open loop Nash equilibrium, which indeed is also a Markovian (feedback) Nash equilibrium.[6]

In contrast, the economic literature on spatiotemporal dynamics is rather substantial after the seminal contribution of Brito (2004). In particular, a number of geographic optimal growth models with capital spatiotemporal dynamics have been devised and studied (see Boucekkine et al, 2013, Fabbri, 2016, and Boucekkine et al., 2019). Another contribution in the same vein but with constant (though space dependent) saving rates is due to Xepapadeas and Yannacopoulos (2016). In all these papers, capital flows across space following a parabolic partial differential equations. Just like the transboundary problems and for the very same reason, the induced problem is infinite-dimensional.

Despite such a complexity, we show here that, thanks to the special structure of the problem at hand, we are able to express the unique open loop equilibrium in explicit form. Differently from what is done in other papers (see e.g. Fabbri and Gozzi, 2008, Boucekkine et al, 2013 or Boucekkine et al, 2019), we do not solve the associated HJB equation. Instead we rewrite the objective functional in a suitable way (see Proposition C.8 and Theorem C.9), which allows us to find directly the optimal open-loop strategies

[5]Advection is present only in the draft of de Frutos et al. (2019).

[6]In this very brief review, we abstract away from the huge literature on multi-country dynamic games with a common stock of pollution. See Dockner et al (1993), and Boucekkine et al. (2011) for an earlier contribution.

for the players, and, at the same time, to give a clear economic insight into the properties extracted along the way. The explicit form of the equilibrium also enables us to characterize it comprehensively and to illustrate some key economic findings readily through complementary numerical exercises. This ultimately shows how effective the machinery of infinite dimensional optimal control can be in studying such type of problems. See, for an account of the theory, the books of Li and Yong (1995) and Fabbri et al. (2017).

This paper is organized as follows. Section 2 develops the decentralized equilibrium setting and solves analytically for the Nash equilibrium. Section 3 characterizes the cooperative equilibrium. Section 4 and 5 dig deeper in the concept of border effect alluded to above, combining conceptual and numerical analysis, and finally accounting for a rich variety of inter-state heterogeneities. While the previous sections only consider (pollution) diffusion, Section 6 incorporates advection to clearly highlight the implications of non-homogeneous diffusion across space. Section 7 concludes. All the proofs are reported in the Appendix together with a full explanation of the mathematical setting.

2. The non-cooperative game

We consider a dynamic general equilibrium model for a spatial economy subject to spatial spillovers driven by transboundary pollution dynamics.

Even if generalizations are possible[7] we limit our attention to the case of the circular spatial support S^1:

$$S^1 := \left\{ x \in \mathbb{R}^2 : \ |x|_{\mathbb{R}^2} = 1 \right\}$$

that is the simplest spatial model being compact and without boundary and then having the significant advantage, in terms of modeling characteristics, of preserving the global stock of pollutants during their diffusion processes, without absorbing or reflecting boundaries (as in the case of possible models on sub-domains with Dirichlet or Neumann boundary conditions). By doing so, we are also closer to the related economic literature, in particular to the geographical setting of Hutchinson and Kennedy (2008).

As usual, in the following we will often describe S^1 as the segment $[0, 2\pi]$ with the identification of the two extreme points 0 and 2π. We denote by x the generic spatial point and with $t \geq 0$ the continuous time coordinate. As to demographics, we assume the simplest configuration: one individual per location x at any time t, so that aggregate and per capita variables coincide at any location.

At any time t and location x the production of the final good $y(t, x)$ depends on the quantity of input used in the production $i(t, x)$ according to a linear production function:

$$y(t, x) = A(x)\, i(t, x), \tag{2.1}$$

[7] One can for instance replicate most of the results to the case of the 2-dimensional sphere surface S^2 and indeed many of them can also be generalized to more abstract contexts.

where $A(x)$ is the time-independent exogenous productivity at x.[8] The production activity is polluting and a policy of depollution could be advantageously implemented. At any location, output is produced and used for consumption, input and depollution locally. Denoting by $c(x,t)$ and $b(x,t)$ consumption and resources devoted to depollution respectively, we have the following resource constraint equation at any location x and time t:

$$c(x,t) + i(t,x) + b(t,x) = y(t,x). \tag{2.2}$$

We further assume that using one unit of input produces one unit of emission flow, while a depollution effort $b(t,x)$ can sequestrate a flow $\eta b(t,x)^\theta$ (with $\eta \geq 0$ and $\theta \in (0,1)$) of pollutants. Consequently, net emissions are given by

$$n(t,x) = i(t,x) - \eta(b(t,x))^\theta.$$

The spatio-temporal dynamics of the pollution stock is subject to two natural phenomena: a diffusion process which tends to disperse the pollutants across the locations, and a location-specific decay $\delta(x)$. All in all the evolution of the pollution stock $p(t,x)$ is driven by the following parabolic partial differential equation:

$$\begin{cases} \dfrac{\partial p}{\partial t}(t,x) = \sigma \dfrac{\partial^2 p}{\partial x^2}(t,x) - \delta(x)p(t,x) + i(t,x) - \eta b(t,x)^\theta, \\ \\ p(0,x) = p_0(x), \quad x \in S^1, \end{cases} \tag{2.3}$$

being $\sigma > 0$ the diffusivity coefficient measuring the speed of the spatial diffusion of the pollutants and $p_0(x)$ the initial spatial distribution of the pollution. We shall introduce advection in Section 6.[9]

Let us come back now to geography. The global territory S^1 is partitioned into a finite number of national states or states /regions within the same country. Each of them is governed by a local public authority (for instance a national, state or regional government) which only takes into account the welfare of the people living in its own ground.

More formally we suppose that there are N intervals in the circle $M_j \subset S^1$ with $j = 1, ..., N$ such that

$$M_j \cap M_h = \emptyset \quad \text{for } h \neq j, \quad h, j = 1, ..., N.$$

[8]The production input $i(t,x)$ can be interpreted as a capital good but we do not allow for capital accumulation (or equivalently, we assume full depreciation of capital). This is instrumental to obtaining closed-form solutions while preserving the necessary breadth in the analysis of the environmental free riding problem under scrutiny.

[9]We could have considered a space-dependent diffustion parameter, $\sigma(x)$. Our solution method is unaffected by such a specification just like the many other space-dependent magnitudes incorporated.

These regions can cover the whole space S^1 but we we also admit the possibility that some residual part of the global territory $M_0 := S^1 \setminus \bigcup_{j=1}^N M_j$ is inhabited (for instance oceans).

Each local authority is in charge of production, consumption and depollution decisions in its own territory and so the authority j chooses $i(t,x)$, $b(t,x)$, $c(t,x)$ for all $t \in \mathbb{R}^+$ and $x \in M_j$ subject to production and resource constraints (2.1) and (2.2). In order to emphasize that these decisions only concern the region j we will denote them by $i_j(t,x)$, $b_j(t,x)$ and $c_j(t,x)$ so that their relation with the functions $i(t,x)$ and $b(t,x)$ appearing in (2.3) are indeed

$$\big(i(t,x), b(t,x)\big) := \begin{cases} (i_j(t,x), b_j(t,x)), & \text{if } x \in M_j, \\ 0, & \text{if } x \in M_0. \end{cases} \tag{2.4}$$

We will also denote by $A_j(x)$ the restriction of $A(x)$ to M_j.

The utility of the local public authority j depends, as already mentioned, only on the characteristics of its own territory/population and it takes the following form

$$\int_0^\infty e^{-\rho_j t} \left(\int_{M_j} \left(\frac{\big(c_j(t,x)\big)^{1-\gamma_j}}{1-\gamma_j} - w_j(x) p(t,x) \right) dx \right) dt, \tag{2.5}$$

where $\rho_j > 0$ is the discount factor, $\gamma_j \in (0,1) \cup (1,\infty)$ the inverse of the elasticity of intertemporal substitution and $w_j(x)$ is a measure of the unitary location-specific disutility from pollution. The latter can be roughly interpreted as the measure of environmental awareness at location x in state j. It should be noted that we do not assume that all the inhabitants of territory j share the same environmental awareness, which is somehow more realistic. In contrast, the elasticity of substitution is assumed territory-dependent for simplicity.[10] Finally, each local authority may have a specific view of time discounting, thus the territory-dependent parameter ρ_j.

Observe that, even if the expression above only concerns territory M_j, it also depend on the choices of other authorities through the variable $p(t)$ because, thanks to the diffusion dynamics of pollution, its value at the points $x \in M_j$ depends on all the past production (and thus pollution) decisions of all other players (authorities). We will make explicit this fact in the notation through the index "$-j$", which stands for "all the index but j". Moreover, supposing that A is greater than 1 in all locations[11], we use (2.1) and (2.2) to express $c_j(t,x)$ in terms of $i_j(t,x)$ and $b_j(t,x)$ as $\big((A_j(x)-1)i_j(t,x) - b_j(t,x)\big)$. Finally we can write (2.5) as

$$J_j^{(i_{-j},b_{-j})}\big(p_0; (i_j,b_j)\big) := \int_0^\infty e^{-\rho_j t} \left(\int_{M_j} \frac{\big((A_j(x)-1)i_j(t,x) - b_j(t,x)\big)^{1-\gamma_j}}{1-\gamma_j} - w_j(x) p(t,x) dx \right) dt. \tag{2.6}$$

[10] Allowing for parameter γ to depend on location x does not break down the analytical solution neither.

[11] This assumption is required by the production function specification (2.1) for the ratio investment to production to be lower than 1 everywhere and at any time.

We now get to formalize how decentralization works in our setting. We suppose that the authorities/players engage in a non-cooperative Nash game. By construction, the latter is a differential game where each state authority maximizes the spatiotemporal payoff (2.6) under the state equation (2.3) subject to positivity constraints on i_j, b_j and c_j. See Appendix A for a more detailed formal description of the Nash problem and the corresponding Definitions A.3 and A.4 for open and of Markovian Nash equilibria in the described context.

As repeatedly mentioned above, we are able to solve analytically for the Nash equilibrium involved. This is displayed in the main theorem of our paper here below.

Theorem 2.1. *The unique open-loop Nash Equilibrium strategies for the described game is given, for $j = 1, ..., N$, by*

$$b_j^*(t,x) = [(A_j(x) - 1)\eta\theta]^{\frac{1}{1-\theta}}, \tag{2.7}$$

$$i_j^*(t,x) = \alpha_j(x)^{-\frac{1}{\gamma_j}} (A_j(x) - 1)^{\frac{1-\gamma_j}{\gamma_j}} + (\eta\theta)^{\frac{1}{1-\theta}} (A_j(x) - 1)^{\frac{\theta}{1-\theta}}, \tag{2.8}$$

where α_j is the solution to the following ODE [12]

$$\rho_j\alpha_j(x) - \sigma\alpha_j''(x) + \delta(x)\alpha_j(x) = w_j(x), \quad x \in S^1. \tag{2.9}$$

The welfare of player j at the equilibrium is affine in p_0:

$$v_j(p_0) = \int_{S^1} \alpha_j(x)p_0(x)dx + q_j,$$

for a suitable constant q_j (see Theorem C.9 in the Appendix for its explicit expression). The unique open-loop equilibrium described is also a Markovian Nash equilibrium of the game.

Proof. See Theorem C.9 in the Appendix and its proof. □

In Section 5 we will discuss in more detail the dependence of the strategies chosen by the players on the parameters of the model, looking at the effect of different sorts of spatial heterogeneity. Here we comment briefly on the shape of the players' welfare. One can easily see that the welfare of player j is an increasing function of pollution decay rate (or Nature self-cleaning capacity), δ, and a decreasing function of the pollution disutility parameter, w_j. These properties are rather intuitive when one looks at the objective functionals (for a given set of strategies, it is clearly true that J has this kind of behavior) but they are *a priori* not obvious in the context of a Nash equilibrium (see also Remark C.10 for a related comment). Indeed the variation of δ and/or w in one region also impacts the welfare in the others. More precisely, decreasing w_j pushes player j to produce and

[12] In (2.9) $w_j(x)$ is meant to be extended to 0 outside its initial domain M_j.

pollute more (see Proposition C.6: observe that i is a decreasing function of α) so that the welfare of players different from j decreases when w_j decreases (see again Remark C.10). The effect of δ on the welfare of other players is more complex because it pushes player j to pollute more but it also pushes for a quicker decay of global pollution.

Since the equilibrium values of i_j and b_j (and consequently of c_j, n_j and y_j) are time-independent, we will avoid from now to write the time variable in their expressions. As a consequence of our main theorem, the equilibrium net pollution flow, the production and the consumption in the region j are given respectively by

$$n_j^*(x) = i_j^*(x) - \eta(b_j^*(x))^\theta, \tag{2.10}$$

$$y_j^*(x) = A(x)\, i_j^*(x), \tag{2.11}$$

and

$$c_j^*(t) = y_j^*(x) - i_j^*(x) - b_j^*(x). \tag{2.12}$$

Once we have the set of strategies chosen by the players we can use them in the state equation (2.3) to get the corresponding dynamics of the pollution space distribution and getting the following asymptotic result.

Proposition 2.2. *In the context described by Theorem 2.1 the spatial pollution density $p(t,\cdot)$ converges to the long-run pollution profile p_∞ which is given by the unique solution of the following elliptic equation:*

$$\sigma p''_\infty(x) = \delta(x)p_\infty(x) - n^*(x).$$

Proof. See Corollary C.12 in Appendix C. □

We shall use the expressions given above in our numerical exercises from Section 4 in which we will focus on the spatial distributions of the relevant variable (long-term distribution in the case of pollution). Before, we characterize the cooperative equilibirum, which will serve as a reference for these exercises in order, as we wrote in the introduction, to measure the distance to efficiency of the Nash equilibrium outcomes.

3. The cooperative game

To design the cooperative solution, we suppose that all the players cooperate to maximize a social welfare function defined as the sum of the utility of all the states/territories of S^1:

$$\sum_{j=1}^{N} \int_0^\infty \left(\int_{M_j} e^{-\rho_j t} \left(\frac{\big((A_j(x)-1)i_j(t,x) - b_j(t,x)\big)^{1-\gamma_j}}{1-\gamma_j} - w_j(x)p(t,x) \right) dx \right) dt. \tag{3.1}$$

We limit our attention to the case where the preference parameters are the same for all players: $\rho_j = \rho$ and $\gamma_j = \gamma$ for every $j = 1, ..., N$. In this case the functional (3.1) can be rewritten as

$$\int_0^\infty e^{-\rho t} \left(\int_{S^1} \left(\frac{\big((A(x)-1)i(t,x) - b(t,x)\big)^{1-\gamma}}{1-\gamma} - w(x)p(t,x) \right) dx \right) dt \tag{3.2}$$

where w is defined similarly to (2.4). Indeed, it is the standard Benthamite social functional since we suppose that at each location there is exactly one inhabitant.

The optimal control problem of maximizing (3.2) subject to (2.3) and the positivity constraints on i, b and c can be explicitly solved as described in the following theorem.

Theorem 3.1. *The unique equilibrium for the described cooperation game is given by*

$$\underline{b}^*(t,x) = \big[(A(x)-1)\eta\theta\big]^{\frac{1}{1-\theta}}, \tag{3.3}$$

$$\underline{i}^*(t,x) = \Big[\underline{\alpha}(x)^{-\frac{1}{\gamma}}(A(x)-1)^{\frac{1-\gamma}{\gamma}} + (\eta\theta)^{\frac{1}{1-\theta}}(A(x)-1)^{\frac{\theta}{1-\theta}}\Big] \tag{3.4}$$

where $\underline{\alpha}$ is the solution to

$$\rho\underline{\alpha}(x) - \sigma\underline{\alpha}''(x) + \delta(x)\underline{\alpha}(x) = w(x), \quad x \in S^1. \tag{3.5}$$

The corresponding welfare is

$$\underline{v}(p_0) := \int_{S^1} \underline{\alpha}(x)p_0(x)dx + \underline{q},$$

for a suitable constant $\underline{q}$ (see Theorem E.1 in the Appendix).

Proof. See Theorem E.1 and Corollary E.2 in the Appendix and its proof. □

As for the Nash equilibrium case, since at the equilibrium values $\underline{i}^*$, $\underline{b}^*$, $\underline{n}^*$, $\underline{c}^*$ and $\underline{y}^*$ are time-independent we will avoid to write the time variable in their expressions. A counterpart of the asymptotic result given in Proposition 2.2 is given, in the cooperative context, by the following proposition.

Proposition 3.2. *In the context described by Theorem 3.1 the spatial pollution density $p(t,\cdot)$ converges to the long-run pollution profile $\underline{p}_\infty$ which is given by the unique solution of the following elliptic equation:*

$$\sigma\underline{p}''(x) = \delta(x)\underline{p}(x) - \underline{n}^*(x),$$

where $\underline{n}^*(x) = \underline{i}^*(x) - \eta(\underline{b}^*(x))^\theta$ (and b^* and i^* are defined in Theorem 3.1).

Proof. See Appendix D. □

Not surprisingly, the Nash equilibrium described in Theorem 2.1 is markedly different from the social optimum described above and it is suboptimal in terms of the chosen social target. The suboptimality is of course driven by the spatial pollution externality: the local authority does not completely internalize the damages of the emissions produced in its territory, part of it moves away from its territory and then it does not affect its utility but the utility of other territories, especially the closest. Interestingly enough, our comprehensive modelling of pollution diffusion allows for an accurate appraisal of this externality. A crucial parameter in this respect is parameter σ: when σ drops, the diffusion is slower. When σ is equal to 0, the spatial dynamics vanish. In this case the model is spatially degenerate in the sense that there is no interaction among the economies of the various locations and the dynamics of the model (both in the non-cooperative and in the cooperative case) reduces to the pointwise maximization of the functional

$$\int_0^\infty e^{-\rho t}\left(\frac{\big((A(x)-1)i(t,x)-b(t,x)\big)^{1-\gamma}}{1-\gamma}-w(x)p(t,x)\right)dt$$

for any fixed $x \in S^1$ subject to

$$\frac{\partial p}{\partial t}(t,x) = -\delta(x)p(t,x) + i(t,x) - \eta b(t,x)^\theta.$$

The solution of this maximization problem is given by the same value of b given in (2.7) and

$$i(x) = \left(\frac{w(x)}{\rho+\delta(x)}\right)^{-\frac{1}{\gamma}} (A(x)-1)^{\frac{1-\gamma}{\gamma}} + (\eta\theta)^{\frac{1}{1-\theta}} (A(x)-1)^{\frac{\theta}{1-\theta}}.$$

Since the pollution remains where it is produced, no spatially externality arises in the case $\sigma = 0$, hence, in such case both the cooperative equilibrium and the non-cooperative one coincide. Moreover, the final result of Theorem 3.1 tells us more, i.e. that the no-diffusion non-cooperative case is conceptually close to the cooperative case with any diffusion (Theorem 3.1). Indeed, it can be shown that the two coincide in the particular case where δ and w are constant, in terms of chosen i, b and c. This intriguing property is disclosed in the proposition below.

Proposition 3.3. *Suppose that δ and w are constant in space. Then the strategies $b^*_{j,0}(t,x)$ and $i^*_{j,0}(t,x)$ of player j in the Nash equilibrium described in Theorem 2.1 when $\sigma = 0$ are the same as the equilibrium strategies $\underline{b}^*_j(t,x)$ and $\underline{i}^*_j(t,x)$ of the cooperative case described in Theorem 3.1, or every $\sigma \geq 0$.*

Proof. See Corollary E.2 in the Appendix. □

4. Border Effects

We now perform a series of numerical exercises to uncover the main features of the long-term equilibrium spatial distributions. This will allow us to visualize at glance the shape of the border effects associated with the free riding problem under decentralization. Essentially, we will make clear that the shapes generated are quite consistent with the predictions of the basic (static) theory of spatial externalities, particularly in what concerns the specific equilibrium behaviour near the borders of the states: net pollution flows will be shown to be larger as the borders are approached, and structural breaks emerge at the borders. In this section, we consider that all the states are geographically identical in terms of various parameters of the model, except possibly for the size while the next section will dig deeper in the implications of other spatial heterogeneities.

We shall report at the same time the induced distributions for the cooperative game, which will make the border effects even more striking. We start by showing the situation of two territories while, in a second step, we theoretically address another implication of the basic theory of spatial externalities, that is the evolution of the size of spatial externalities (or in other terms, the extent of inefficiency) when the number of states (or jurisdictions) rises.

4.1. **The shape of border effects.** We concentrate here our attention on cases where all the parameters are constant in space. Accordingly, the unique geographic discrepancies result from the partition in decentralized territories (in the Nash case), that is in the existence of borders and possible differences in territory size.

We will compare the Nash equilibrium described in Theorem 2.1 with the benchmark described in Section 3. In particular, since, as announced above, δ and w are constant in space, we are always in the context of Proposition 3.3 so that the cooperative benchmark with diffusion of Theorem 3.1 and the non-diffusion benchmark are always equivalent. To lighten the notation we will omit the use of the indexes j, this is reasonable because we will mainly look at the aggregate effect of the choices of the agents.

4.1.1. *Symmetric two-region cases.* We start with the case where territories have equal size. In the common economic language, this amounts to studying the symmetric two-region case.

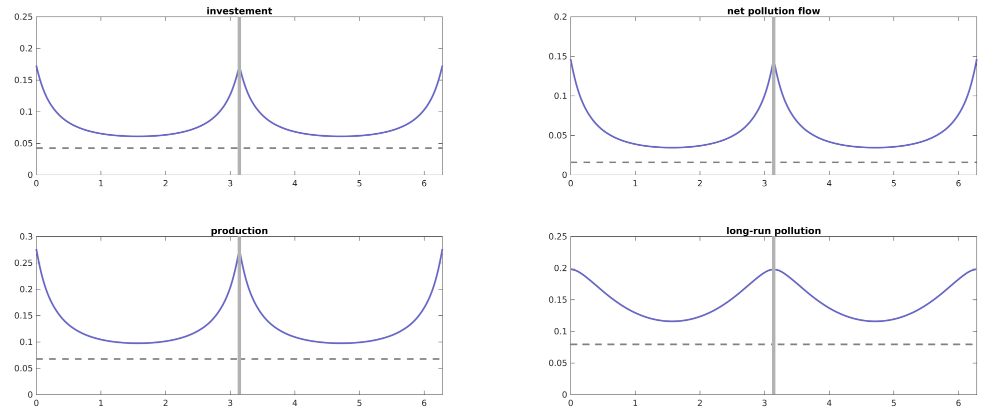

FIGURE 4.1. The case of two players each controlling one half of the circle. All the parameters are space independent: the productivity factor A is 1.6, the discount ρ is 0.03, the diffusivity σ is 0.5, the natural decay of the pollution δ is 0.2, the weight of pollution in the utility parameter w is 1, the intertemporal substitution parameter γ is 0.5, the efficiency of depollution expenditure factor ϕ is 0.2 while the scale returns of depollution activity θ is 0.4. The continuous (and blue) lines represent the Nash equilibrium while the dashed (and gray) lines reproduce the cooperative benchmark or, equivalently, the non-diffusion benchmark.

In Figure 4.1 each local authority controls one half of the circle and then it is interested only in utility of its region (parameters' values are given in the caption of the figure). The continuous (and blu) lines represent the values of the variables in various locations at the Nash equilibrium while the dashed (and gray) lines reproduce the cooperative benchmark or equivalently the non-diffusion benchmark. We represent four variables: the investment i, the net pollution n, the production y and the long-run pollution profile p_∞. The latter is obtained in particular thanks to the representation of the solution in series given in Appendix D. We do not represent in the figure the depollution effort at the equilibrium since, given the particularly simple situation (all the parameters are constant in space), it is constant over space. For this reason the qualitative behavior of net emissions is the same as that of investment. It is also the same as production since the productivity A is space independent.

The first evident element in the distribution of investment (and then in those of production and net pollution) are the big differences among the locations of a same region. In particular we can observe that investment and economic activity are particularly strong near the borders of each region. This *border effect* is due to the spatial structure of the externalities: the negative effects of the emissions on the utility are less and less internalized by the local authority as the location gets closer to the border. That's because a greater part of the pollutants in these locations will flow into another territory and has therefore to be managed by another local authority. In the symmetric two-region case with positive diffusivity that we have here, the emission at the boundary points are immediately equally shared by the two territories while the pollutants coming from a far interior point remains in the short run mostly in their "native" region. The source of the externality inefficiency can be well visualized looking at the long-run distribution of the pollution: indeed the concentration of pollution at the boundary is much less pronounced than the corresponding peak of input because a significant part of the pollutants leaves the locations where they are originally produced.

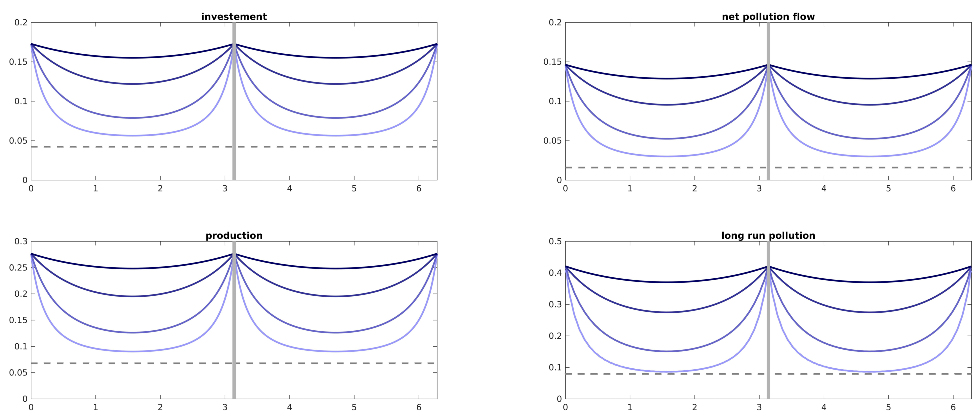

FIGURE 4.2. The case of two players controlling one half of the circle varying the diffusivity coefficient. The values of the parameters (all constant over the whole space) are the same as in Figure 4.1 except the values of diffusivity σ which takes now four values: 0.4 (the lightest line), 0.8, 1.6, and 3.2 (the darkest line). Continuous (and colored) lines represent the Nash equilibrium while the dashed (and gray) lines are the cooperative benchmark or, equivalently, the non-diffusion benchmark.

In Figure 4.2 we dig a little deeper in the mechanisms at work and we see what happens when we vary the diffusivity coefficient σ that is when we change the speed of the diffusion of emissions. We have again the same symmetric two-countries situation as in Figure 4.1 and we represent with a colored continuous (respectively, gray dashed) line the variables at the Nash equilibrium for various value of the parameter σ (respectively, at the benchmark).

We choose four possible values for σ: 0.4 (the lightest line), 0.8, 1.6, and 3.2 (the darkest line), all other parameters are the same as in Figure 4.1 so the values of the variable at the benchmark is the same.

Not surprising, the lower σ, the more the system tends to mimic the behaviour of the 0-diffusion benchmark. Conversely, the higher the value of σ and the faster pollutions disseminates across locations: for σ very big the situation in each location is similar to the situation we have at the boundary since after a short period the produced pollution has an almost equal probability of being in the territory of both authorities. In other words the higher the value of σ, the lesser *future* negative effects on the utility are internalized by the local authority and then the higher the chosen level of input, production and emission in internal points. This mechanism highlights the intertemporal role of the parameter σ that will be emphasized even more in the following.

The two limits of the equilibrium profile of i_j^* when $\sigma \to 0^+$ and $\sigma \to +\infty$ can be computed explicitly and they are (see Proposition C.5 in the Appendix), for the general case specified in Theorem 2.1,

$$i_j^{*,0}(x) = \left(\frac{w_j(x)}{\rho_j + \delta(x)}\right)^{-\frac{1}{\gamma_j}} (A_j(x) - 1)^{\frac{1-\gamma_j}{\gamma_j}} + (\eta\theta)^{\frac{1}{1-\theta}} (A_j(x) - 1)^{\frac{\theta}{1-\theta}}, \tag{4.1}$$

and

$$i_j^{*,\infty}(x) = \left(\frac{\int_{S^1} w_j(x)dx}{\int_{S^1}(\rho_j + \delta(x))dx}\right)^{-\frac{1}{\gamma_j}} (A_j(x) - 1)^{\frac{1-\gamma_j}{\gamma_j}} + (\eta\theta)^{\frac{1}{1-\theta}} (A_j(x) - 1)^{\frac{\theta}{1-\theta}}. \tag{4.2}$$

Notice that the latter expression depends on the space location x only through $A(x)$. In the case of 2 symmetric agents and spatial constants parameters, one gets the two following spatial-independent expressions

$$i^0 = \left(\frac{w}{\rho + \delta}\right)^{-\frac{1}{\gamma}} (A - 1)^{\frac{1-\gamma}{\gamma}} + (\eta\theta)^{\frac{1}{1-\theta}} (A - 1)^{\frac{\theta}{1-\theta}}, \tag{4.3}$$

and

$$i^\infty = \left(\frac{1}{2}\frac{w}{\rho + \delta}\right)^{-\frac{1}{\gamma}} (A - 1)^{\frac{1-\gamma}{\gamma}} + (\eta\theta)^{\frac{1}{1-\theta}} (A - 1)^{\frac{\theta}{1-\theta}}. \tag{4.4}$$

One can readily check that consistently with the intepretations given above, $i^\infty > i^0$.

4.1.2. *Non-symmetric two-region cases.* We now move to asymmetric two-region cases. In Figure 4.3 we represent a situation where parameters are not space dependent and then spatial heterogeneities of agents' behavior is only due to borders and different importance that various agents attribute to the utility of people living in different locations. All the parameters are the same as in Figure 4.1 but here the dimensions of the regions governed by the two authorities are different: the first controls three fourth of the circle while the second only governs on a fourth of the space. The effect of the new division of the territory with respect to the situation of Figure 4.1 is neat: the authority with the larger territory internalizes more the effect of its emissions because it anticipates that it will have them back to its part of the circle in the future. For the very opposite reason, the authority which has a fourth of the circle is less affected by its own emission and then produces and pollutes more than in Figure 4.1 (and a fortiori than the player controlling three fourth of the circle). A major implication of our "circular" and geographically homogenous world is the bigger the country, the cleaner it is. This is a quite interesting result. Of course, by construction, if geographic heterogeneity is added (in technology or ecology for example), the latter result might be reversed. But it is important to visualize the benchmark result with only differences in size and no further geographic discrepancy.

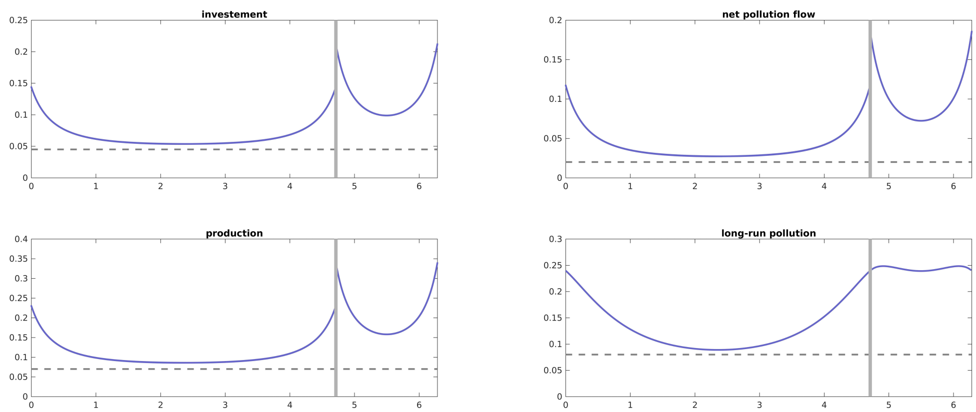

FIGURE 4.3. The case of two players controlling respectively one fourth and three fourth of the circle. The values of the parameters (all constant over the whole space) are the same as in Figure 4.1. Continuous (and colored) lines represent the Nash equilibrium while the dashed (and gray) lines are the cooperative benchmark or, equivalently, the non-diffusion benchmark.

Remark 4.1. *The strength and the structure of the border effects we mentioned also depend on the model of the spatial structure we used. A structure as S^1 allows us to consider the global effects of pollution in the sense that, as already pointed out, the agents (especially those that govern larger areas) are led to consider the fact that a part of the emissions that leave the territory, contribute to increase a spatial stock of pollution that in a certain amount will return to the country in the future. Given the compactness of the support it can also happen that the pollution "crosses" the entire territory of the other* *player before returning. In other types of models the situation is different. For example if one considers the spatial model of the segment with absorbing borders the pollution that leaves the area controlled by the player will no longer return to it, if one considers a model of infinite space the effect of compactness disappears and therefore the difference of behaviors between the player controlling a large territory and the one controlling a small territory is reduced.*

4.2. **Number of players and the size of externalities.** Exactly the same kind of behavior we have seen in the last numerical exercises arises from increasing the number of players: when several players with a per-capita small territory interact, each of them internalizes only a modest portion of the damages of her emission and then she increases input, production and pollution with respect to the two player case. In this way the total amount of pollution tends to increase with the number of players. We devote to this finding a deeper theoretical foundation here.

We consider again the case where the coefficients of the problem are homogeneous in space. Among the sets of territories configurations described in Section 2 (see also Appendix A), we introduce the following partial order relation: given two configurations

$$\Pi^1 = \{M_1^1, ..., M_N^1\}, \qquad \Pi^2 = \{M_1^2, ..., M_K^2\}$$

we say that $\Pi^1 \preceq \Pi^2$ if

$$\forall j = 1, ..., N, \quad \exists i = 1, ...K : \quad M_j^1 \subseteq M_i^2.$$

This means that the territories configuration Π^1 is a fragmentation of the territories configuration Π^2.

Proposition 4.2. *Let Π^1, Π^2 be two territories' configurations such that $\Pi^1 \preceq \Pi^2$ and let $p^{1,*}, p^{2,*}$ the associated optimal pollution paths. Then $p^{1,*}(t,x) \geq p^{2,*}(t,x)$ for all $(t,x) \in \mathbb{R}^+ \times S^1$.*

Proof. See Appendix C. □

The result displayed by the proposition is quite sharp: when decisions in terms of investment, production and emissions of a certain territory move from a central entity to smaller sub-entities, pollution levels systematically increase in line with the different degree of internalization of externalities that we expect from the two players.

Again, this broadens the counterpart property in the basic static theory of spatial externalities. It is also (somehow by construction) consistent with the intuitions one can gain from the numerical exploration above. Note that the result is obtained under the assumption of geographic homogeneity. The next section is designed to highlight some the implications of adding geographic discrepancies into the theory, a feature not considered generally in the standard theory (see for example, Hutchinson and Kennedy, 2008). Our analytical approach allows for these discrepancies as our closed-form solutions do encompass them.

5. Geography and Heterogeneity

We look now at what happens when we introduce natural, technological or preference differences among the regions. We stick to the symmetric two-region case since the relevant mechanisms are already at work there. To be able to disentangle various effects we will look at the effect of one parameter each time. We start with technology.

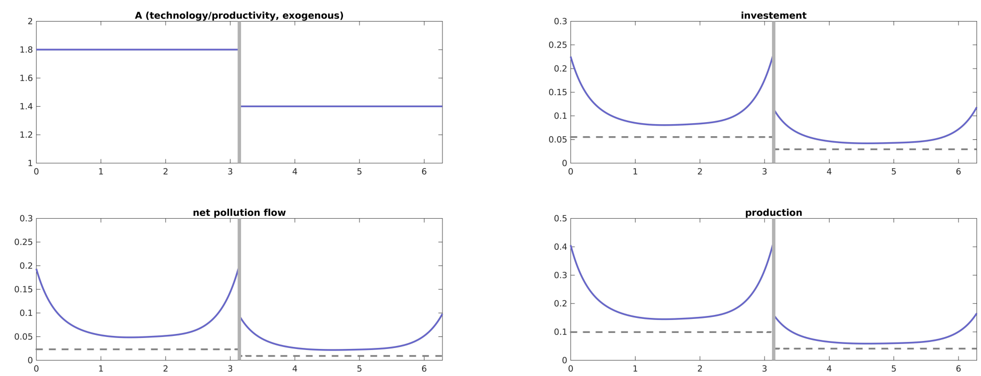

Figure 5.1. The case of two players controlling one half of the circle with different levels of technology A. The productivity of the first player (on the left) is A_1 (constant it her territory) equal to 1.8 while the value of A_2 is 1.4. All other parameters (are constant over the space and) are the same as in Figure 4.1. Continuous (and colored) lines represent the profiles of the variables at the Nash equilibrium while the dashed (and gray) lines are related to the cooperative benchmark or, equivalently, the non-diffusion benchmark.

5.1. **Geographic discrepancy in technology.** In Figure 5.1 we consider the situation when in the economy we have two regions with different technological levels, that's with-ifferent productivities. As we can see from (2.8), the effect on investment of varying A depends on the value of other parameters (in particular on γ, see Remark C.11 in the Appendix for technical details). Indeed a variation of A produces a typical income-substitution trade-off: on the one hand increasing A makes investment more productive leading to a higher increase in investment relative to consumption and depollution effort. On the other hand, a higher level of A can guarantee a higher level of production and thus of consumption together with lower investment and subsequently lower emissions. The predominance of one of the two channels depends on the values of various parameters. In Figure 5.1 the first effect is stronger. Conversely, as one can see from the expression of b_j at the equilibrium given in (2.7), the impact of increasing A on the depollution expenditures is always positive: increasing output always gives more room for pro-environmental actions.

The effects on net pollution depends on the relative strength of the mechanisms described above. For our selected parametrization, the outcomes are represented in Figure 5.1. The long-run spatial distribution of pollution is not explicitly represented in the figure but, similarly to what we have in the figures of Section 4 it is a smoothed version of the distribution of net emission flows (see Remark C.13 for a more technical observation on that).

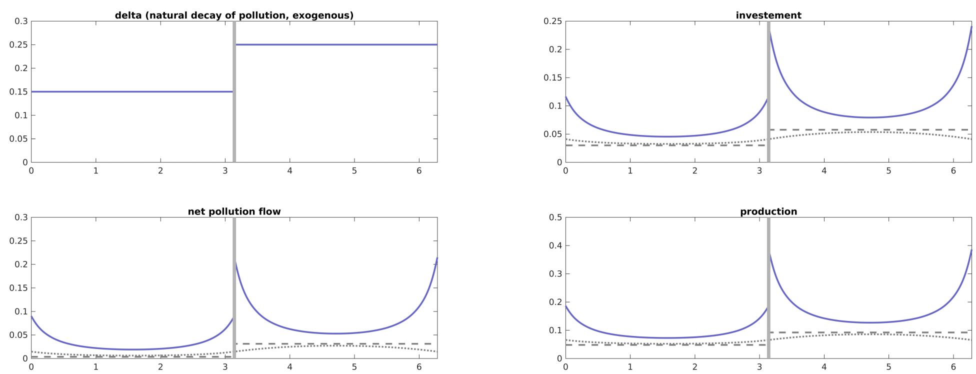

FIGURE 5.2. The case of two players controlling one half of the circle with different levels natural decay of the pollution δ. The value of δ in the territory of the first player (on the left) is δ_1 (constant it her territory) equal to 0.15 while the value of δ_2 is 0.25. All other parameters (are constant over the space and) are the same as in Figure 4.1. Continuous (and colored) lines represent the Nash equilibrium, the dotted (and gray) lines are the cooperative benchmark (with the same σ) while the dashed (gray) lines are the zero-diffusion benchmark.

5.2. **Geographic ecological discrepancy.** In Figure 5.2 we look at the situation where the territories of the two regions have different natural decays of pollution. The effect of increasing δ on investment is positive (see Proposition C.6 in the Appendix for a proof). The intuition is rather straightforward: an higher value of δ reduces, for a given investment strategy, the future stock ofpollution and then it reduces the marginal disutility of polluting with respect to marginal utility of consumption. So it tends to increase input use and thus production and consumption. Conversely, as it transpires from the expression of the optimal level of depollution expenditures (2.7), the latter are not impacted by variations on δ so differences in input use drive mechanically the differences in net emissions flow and production.

We can also note that δ enters in the expression of equilibrium investment (2.8) only through the values of α defined as the solution of (2.9). In this equation δ sounds as a substitute to ρ. This is not so surprising because both parameters have intertemporal implications: they act to discount the future effects of present actions . Nevertheless here substitutability is particularly large (1 to 1 at any time) because, differently from standard growth model, the decay δ directly acts on a variable (pollution) that linearly appears in the utility function. So all the previous remarks on the effects of δ on various endogenous variables, can be replicated exactly for ρ.

Differently from what we had for instance in the examples of Section 4 we can observe that in Figure 5.2 the two benchmarks we use (the cooperative case with the same σ and the no-diffusion situation) give distinct long-term profiles. The same behavior will appear in Figure 5.3. Both benchmarks are associated with less production and emissions than the Nash equilibrium. This is not surprising because in both cases the negative effects of pollution are completely internalized. The reason why the choice of the planner is to pollute more (compared with the zero-diffusion benchmark) in the low-delta zone is that she knows that part of the emission will move to high-delta part and it will decay quickly. This mechanism also explains why the difference between the two is higher, the closer to the border. A symmetric argument is enough to figure out why the investment-choice of the planner tends to be lower relative to the zero-diffusion benchmark in the high-delta zone.

5.3. **Geographic discrepancy in preferences.** In Figure 5.3 we represent again a symmetric two-country example with all the parameters space-independent except the unitary disutility of pollution w which is 0.9 in the region controlled by the first player and 1.1 for the other. As one can infer from Proposition C.6, pollution decreases when w is bigger: a larger marginal disutility from pollution leads the local planner to reduce investment and consumption to be able to reduce emissions. This fact also involves a reduction in production and in the net flow of emissions.

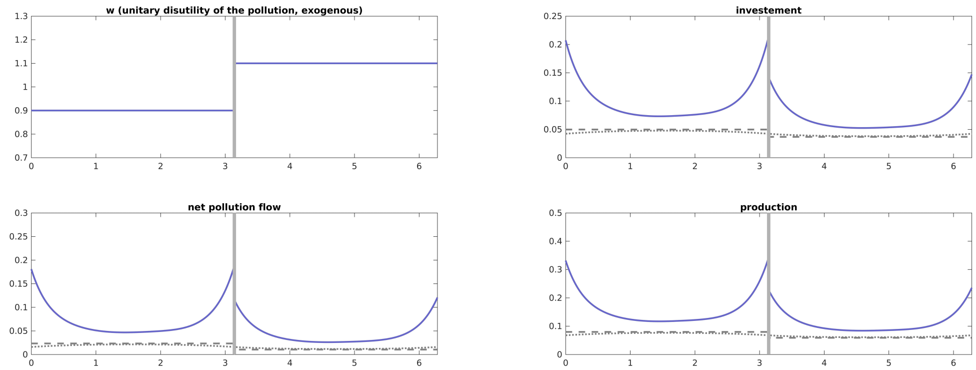

FIGURE 5.3. The case of two players controlling one half of the circle with different unitary disutility of the pollution w. The value of w in the region controlled by the first player (on the left) is w_1 (constant it her territory) equal to 0.9 while the value of w_2 is 1.1. All other parameters (are constant over the space and) are the same as in Figure 4.1. Continuous (and colored) lines represent the Nash equilibrium, the dotted (and gray) lines are the cooperative benchmark (with the same σ) while the dashed (gray) lines are the zero-diffusion benchmark.

6. ADVECTION

So far we have considered, for the natural spatiotemporal dynamics of pollution, a completely homogeneous diffusion/spreading process. Indeed, if we abstract away from the agents' decisions, the dynamics described by (2.3) reduces to

$$\frac{\partial p}{\partial t}(t,x) = \sigma \frac{\partial^2 p}{\partial x^2}(t,x) - \delta(x)p(t,x).$$

In the right side of this expression only the second derivative term describes the spatial dynamics of pollutants while the decay term is, essentially, purely local.

A more general formulation is possible by adding to the right hand side above a term representing exogenous location-specific flow, i.e.

$$v(x)\frac{\partial p}{\partial x}(t,x).$$

This new term is a vector field on S^1 specifying at each location a further movement term (which adds to the already described diffusion term) having speed $v(x)$ at any point x. It is called *advection* term and it allows to take into account the fact that basic dispersion of pollutants need not to be space-homogeneous, for instance due to winds, currents or geographic characteristics. With this new term the evolution equation 2.3 for the pollution stock becomes

(6.1)
$$\begin{cases} \dfrac{\partial p}{\partial t}(t,x) = \sigma \dfrac{\partial^2 p}{\partial x^2} + v(x)\dfrac{\partial p}{\partial x}(t,x) - \delta(x)p(t,x) + i(t,x) - \eta b(t,x)^\theta, \quad (t,x) \in \mathbb{R}^+ \times S^1, \\ \\ p(0,x) = p_0(x), \quad x \in S^1. \end{cases}$$

All the results provided in the previous sections can be generalized to the system including a generic advection term. In particular the counterpart of the Nash equilibrium described in Theorem 2.1 reads now as

$$b_j^{ad,*}(t,x) = [(A_j(x) - 1)\eta\theta]^{\frac{1}{1-\theta}}, \tag{6.2}$$

$$i_j^{ad,*}(t,x) = \alpha_j(x)^{-\frac{1}{\gamma_j}} (A_j(x) - 1)^{\frac{1-\gamma_j}{\gamma_j}} + (\eta\theta)^{\frac{1}{1-\theta}} (A_j(x) - 1)^{\frac{\theta}{1-\theta}}, \tag{6.3}$$

where α_j^{ad} is the unique solution to the ODE

$$\rho_j \alpha_j(x) - \sigma \alpha_j''(x) - v(x)\alpha_j'(x) + \delta(x)\alpha_j(x) = w_j(x), \quad x \in S^1. \tag{6.4}$$

This result is proved in Theorem C.9 in the Appendix.

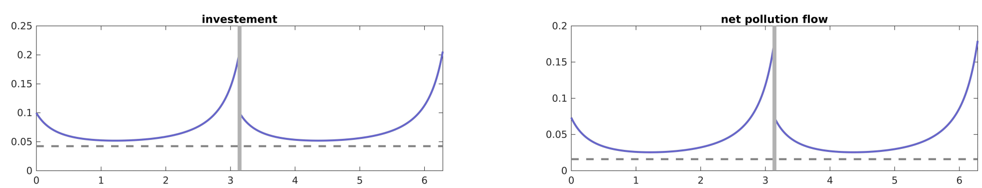

FIGURE 6.1. The case of two players controlling one half of the circle. All the setting and the parameters values are the same as in Figure 4.1 but now a spatially constant advection term $v(x) = 0.08$ is introduced. The continuous (and blue) lines represent the Nash equilibrium while the dashed (and gray) lines are the cooperative benchmark or, equivalently, the non-diffusion benchmark.

To understand how the results can be qualitatively affected by the presence of the advection term we see, in Figure 6.1 how the spatial profiles reported in Figure 4.1 are altered when a constant advection term with $v(x) = v > 0$ is introduced (all other specifications and parameterization remaining the same). This, concretely, means introducing a persistent current which is clockwise when we look at the S^1 as a subset of $\mathbb{R}^2$, that is, roughly speaking a current directed to the "right" in the figure. As a consequence, pollutants tend to move quicker to the other region if they are produced in a location which is close to the right borders (say *est*) and vice versa. For this reason the authority internalizes even less the disutility due to emissions coming from eastern locations and so she has un incentive to produce and pollute more there. A symmetric argument explains why the production and the emission in the western locations are lower than in the no-advection case of Figure 4.1.

7. Conclusion

We have proposed to revisit the foundations of the spatial externalities theory in the case of the free-riding pollution problem, so heavily referred to in the most recent literature on environmental federalism (see for example Hutchinson and Kennedy, 2008) but also in several recent much more applied works on transboundary pollution (see Lipscomb and Mobarak, 2017, as a representative example). In particular, precisely to close the gap between the theory and the latter empirical works, we consider a spatiotemporal framework where, instead of assuming *ad hoc* pollution diffusion schemes across space, we use a realistic spatiotemporal law of motion for air and water pollution (diffusion and advection). This has led us to what we believe to be a strong methodological innovation as we have ultimately to tackle and to solve spatiotemporal non-cooperative (and cooperative) differential games, which is far more complicated than the counterpart static games in the benchmark theory. We also incorporate into the analysis a large set of discrepancies across state and jurisdictions, which broadens even more the scope of our theory and its practical interest.

Also it's worth pointing out that since we solve for both the decentralized non-cooperative and the cooperative equilibrium, our setting can be also used for policy investigations. As outlined in the introduction, a sizeable bunch of interesting questions traditionally raised in the environmental federalism literature can be addressed. Even more interestingly, our setting is general enough to accommodate the two global levels (federal and supranational). Furthermore, since our analytical method allows for deep geographic discrepancies, this in principle enable us to address some of the hot questions in the international agenda, in particular those related to the North/South environmental divide. This is the next step in our project.

Last but not least, because our framework enables us to account for several key spatial heteorgeneities without endangering the availability of closed-form solutions, it can easily accommodate applied work on real multi-regional or multi-country data. This is especially granted because, as we have mentioned already in the introduction, our method works on more realistic spatial sets like spheres.

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Appendices

In the following appendices we describe and we study, in a rigorous mathematical way, the generalized problem with advection presented in Section 6, which includes also the other cases presented in the paper.

Appendix A. Formulation of the problem and main assumptions

Let S^1 be the unitary circle in $\mathbb{R}^2$:

$$S^1 := \{x \in \mathbb{R}^2 : \ |x|_{\mathbb{R}^2} = 1\}.$$

Hereafter, we often identify $S^1 \cong 2\pi\mathbb{R}/Z$ and, according to this identification, we identify functions $\varphi : S^1 \to \mathbb{R}$ with 2π-periodic function $\varphi : \mathbb{R} \to \mathbb{R}$. Moreover, given a function $\psi : \mathbb{R}^+ \times S^1 \to \mathbb{R}$, we denote by ψ_t and ψ_x, respectively, the derivative with respect to the first (time) and the second (space) variable.

We consider the following multiagent problem in S^1. We fix a positive integer number $N \geq 1$ representing the number of players. Each player $j = 1, ..., N$ is endowed with her/his own part of territory $M_j \subset S^1$. Hence,

$$M_j \cap M_h = \emptyset \quad \text{for } h \neq j, \quad h, j = 1, ..., N.$$

Notice that there may be parts of S^1 which do not belong to any player. We assume that M_j is connected and relatively open (hence an open interval in the circle) for every $j = 1, \ldots, N$. We set

$$M_0 := S^1 \setminus \bigcup_{j=1}^{N} M_j.$$

Note that M_0 is closed and contains at least the boundary points of M_j, for $j = 1, \ldots, N$. From now on, when we take an index j without mentioning explicitly where it lies, we mean that $j \in \{1, \ldots, N\}$.

Player j decides the investment policy $i_j(t, x)$ and the depollution policy $b_j(t, x)$ at time $t \in \mathbb{R}^+$ and location $x \in M_j$. Let

$$p_0, \delta, v : S^1 \to \mathbb{R},$$

be given measurable functions and let $\sigma > 0$, $\theta \in (0, 1)$, $\eta \geq 0$ be given constants.

The evolution of the state variable $p(t, x)$, representing pollution, is formally given by the following parabolic PDE([13])

$$\text{(A.1)} \qquad \begin{cases} p_t(t,x) = \sigma p_{xx}(t,x) + v(x)p_x(t,x) - \delta(x)p(t,x) + i(t,x) - \eta b(t,x)^\theta, & (t,x) \in \mathbb{R}^+ \times S^1, \\ \\ p(0,x) = p_0(x), \quad x \in S^1, \end{cases}$$

where

$$\text{(A.2)} \qquad \big(i(t,x), b(t,x)\big) := \begin{cases} \left(i_j(t,x), b_j(t,x)\right), & \text{if } x \in M_j,\ j = 1, \ldots, N \\ 0, & \text{if } x \in M_0 = S^1 \setminus \bigcup_{J=1}^{N} M_j. \end{cases}$$

Given $j = 1, ..., N$, we denote

$$(i_{-j}, b_{-j}) = \left((i_1, b_1), ..., (i_{j-1}, b_{j-1}), (i_{j+1}, b_{j+1}), ..., (i_N, b_N)\right).$$

The payoff functional of player j is

$$\text{(A.3)} \qquad \begin{aligned} & J_j^{(i_{-j}, b_{-j})}(p_0; (i_j, b_j)) \\ & := \int_0^\infty e^{-\rho_j t} \left(\int_{M_j} \left(\frac{\big((A_j(x) - 1) i_j(t,x) - b_j(t,x)\big)^{1-\gamma_j}}{1 - \gamma_j} - w_j(x) p(t,x) \right) dx \right) dt, \end{aligned}$$

where $\rho_j > 0$, $\gamma_j \in (0, 1) \cup (1, \infty)$, and

$$A_j, w_j : M_j \to \mathbb{R}^+,$$

are given measurable functions. The following assumptions will be standing in the remainder of this Appendix.[14].

Assumption A.1.

(1) *$p_0 \in L^2(S^1; \mathbb{R}^+)$;*

[13]In the following, by p_t, p_x, p_{xx} we denote, rspectively, the first partial derivatives of p with respect to t, x, and the second derivative with respect to x.

[14]For the definition of the Lebesgue spaces L^q we refer the reader e.g. to Brezis (2011), Chapter 4. We recall that these spaces are done by equivalence classes of functions according to the equivalence relation which identifies functions which are equal *almost everywhere*.

(2) $\delta \in C(S^1;\mathbb{R}^+)$;
(3) $v \in C^1(S^1;\mathbb{R})$;
(4) $A_j \in L^\infty(M_j;\mathbb{R}^+)$ *and there exists a constant* $l > 0$ *such that* $l \le A_j(x)$ *for all* $j = 1,\dots,N$;
(5) *for each* $j = 1,...,N$, *one has* $w_j \in C(M_j;\mathbb{R}^+)$ *and* w_j *be extended to a function* $\overline{w_j} \in C(\overline{M_j};\mathbb{R})$ *such that* $\overline{w_j}(x) > 0$ *for each* $x \in \overline{M_j}$.

The function p in the objective functional (A.3) above is the solution, in a sense to be specified, to (A.1) corresponding to the initial datum p_0 and to the strategies $((i_j,b_j))_{j=1,\dots,N}$ of the N players.

We consider two classes of admissible strategies: *open loop* and *Markovian (or closed loop)* (see Section 4.1 in Dockner et al. (2000)). Let, for $j = 1,\dots,N$,

(A.4)
$$\mathcal{A}_j := \Big\{ (i_j,b_j) : \mathbb{R}^+ \times M_j \to \mathbb{R}^+ \times \mathbb{R}^+ \ s.t. \\ t \mapsto (i_j(t,\cdot), b_j(t,\cdot)) \in L^1((\mathbb{R}^+, e^{-\rho_j t}dt); L^2(M_j,\mathbb{R}^+)) \times L^1((\mathbb{R}^+, e^{-\rho_j t}dt); L^2(M_j,\mathbb{R}^+)) \\ \text{and } \ (A_j(x)-1)i_j(t,x) - b_j(t,x) \ge 0 \quad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times M_j \Big\}.$$

If we are given, for every $j = 1,\dots,N$, an element $(i_j,b_j) \in \mathcal{A}_j$, we obtain a couple $(i,b) : \mathbb{R}^+ \times S^1 \to \mathbb{R}^2$ defined as in (A.2) and
$$(i,b) \in \mathcal{A} := \mathcal{A}_1 \times ... \times \mathcal{A}_N.$$
Observe that, choosing any $(i,b) \in \mathcal{A}$, the state equation (A.1) has a unique solution in the mild sense for each $p_0 \in L^2(S^1;\mathbb{R}^+)$ (see Remark A.5 and equation (B.7)).

Definition A.2. *The class of* open loop *strategies is the set* $\mathcal{A}$.

Definition A.3. *Let* $p_0 \in L^2(S^1;\mathbb{R}^+)$. *An* open loop Nash equilibrium *for the game starting at* p_0 *is a family of couples*
$$((i_j^*,b_j^*))_{j=1,\dots,N} \in \mathcal{A}$$
such that, for $j = 1,\dots,N$,
$$J_j^{(i_{-j}^*,b_{-j}^*)}\big(p_0;(i_j^*,b_j^*)\big) \ge J_j^{(i_{-j}^*,b_{-j}^*)}\big(p_0;(i_j,b_j)\big), \qquad \forall (i_j,b_j) \in \mathcal{A}_j.$$

We also consider Markovian strategies with perfect state information as follows. Let
$$\mathcal{A}_j^{cl} := \Big\{ (\phi_j,\psi_j) : \mathbb{R}^+ \times M_j \times L^2(S^1;\mathbb{R}) \to \mathbb{R}^2_+ \text{ measurable} \Big\}, \quad j = 1,..,N,$$
and let
$$\mathcal{A}^{cl} := \mathcal{A}_1^{cl} \times ... \times \mathcal{A}_N^{cl}.$$

Definition A.4. *Let* $p_0 \in L^2(S^1;\mathbb{R}^+)$. *An* admissible Markovian strategy *starting at* p_0 *is an* N*-tuple of couples*
$$(\phi,\psi) = \Big((\phi_j,\psi_j)\Big)_{j=1,\dots,N} \in \mathcal{A}^{cl}$$
such that the equation
$$\begin{cases} p_t(t,x) = \sigma p_{xx}(t,x) + v(x)p_x(t,x) - \delta(x)p(t,x) + \phi(t,x,p(t,\cdot)) - \eta\psi(t,x,p(t,\cdot))^\theta, \quad (t,x) \in \mathbb{R}^+ \times S^1, \\ p(0,x) = p_0(x), \quad x \in S^1, \end{cases}$$
where
$$(\phi(t,x,p(t,\cdot)), \psi(t,x,p(t,\cdot))) := 0 \cdot \mathbf{1}_{M_0}(x) + \sum_{j=1}^N \mathbf{1}_{M_j}(x)\left(\phi_j(t,x,p(t,\cdot)), \psi_j(t,x,p(t,\cdot))\right),$$

admits (in the mild sense of equation (B.7)*, see Remark A.5) a unique solution* $p^{(\phi,\psi)}$*, and the functional*

$$J_j(p_0;(\phi,\psi))$$

computed by substituting in (A.3)

$$(i_j(t,x),b_j(t,x)) := \Big(\phi_j(t,x,p^{(\phi,\psi)}(t,\cdot)),\psi_j(t,x,p^{(\phi,\psi)}(t,\cdot)\Big), \quad p(t,x)=p^{(\phi,\psi)}(t,x),$$

is well defined for each $j=1,...,N$*; in particular*

$$(A_j(x)-1)\phi_j(t,x,p^{(\phi,\psi)}(t,\cdot))-\psi_j(t,x,p^{(\phi,\psi)}(t,\cdot))\geq 0 \quad \forall (t,x)\in M_j,\ \forall j=1,...,N.$$

We denote the class of admissible Markovian strategies by $\mathcal{A}^{cl}(p_0)$*.*

Remark A.5. *Notice that the equation in Definition A.4 is a PDE with nonlocal terms in the* x *variable, as* $\phi(t,x,\cdot)$ *and* $\psi(t,x,\cdot)$ *depend, in general, on the structure of* $p(t,\cdot)$ *on* S^1*, not only on its value at* (t,x)*. The precise meaning of this equation and the concept of solution will be provided in the infinite dimensional lifting of the problem that we will perform in the next subsection.*

Given $(\phi,\psi)\in\mathcal{A}^{cl}$ and $j=1,...,N$, we denote

$$(\phi_{-j},\psi_{-j}) := ((\phi_1,\psi_1),...,(\phi_{j-1},\psi_{j-1}),(\phi_{j+1},\psi_{j+1}),...,(\phi_N,\psi_N))\in\mathcal{A}^{cl}_1\times...\times\mathcal{A}^{cl}_{j-1}\times\mathcal{A}^{cl}_{j+1}\times...\times\mathcal{A}^{cl}_N,$$
$$((\phi_{-j},\psi_{-j}),(\phi_j,\psi_j)) := (\phi,\psi).$$

Definition A.6. *Let* $p_0\in L^2(S^1;\mathbb{R}^+)$*. An admissible Markovian strategy* $(\phi^*_j,\psi^*_j)\in\mathcal{A}^{cl}_{ad}$ *is said a* Markovian Nash equilibrium *starting at* p_0 *if, for each* $j=1,..,N$*, it holds*

$$J_j(p_0,(\phi^*,\psi^*))\geq J_j\big(p_0;(\phi_j,\psi_j),(\phi^*_{-j},\psi^*_{-j})\big),$$

for every $(\phi_j,\psi_j)\in\mathcal{A}^{cl}_j$ *such that* $((\phi_j,\psi_j),(\phi^*_{-j},\psi^*_{-j}))\in\mathcal{A}^{cl}(p_0)$*.*

Remark A.7. *The (unique) open loop Nash equilibrium characterized in the next sections is also a Markovian Nash equilibrium in the sense of Definition A.6 (see Section 4.1 in Dockner et al. (2000)). On the other hand, proving that this equilibrium is unique also in the class of Markovian Nash equilibria seems not trivial.*

Remark A.8. *Here the space variable of the model lives in the one dimensional circle* S^1*. Extensions to different space structures are possible but the one dimensional case allows to compute more easily the solution and to perform a more precise analysis of its behavior.*

Appendix B. Reformulation of the problem in infinite dimensional spaces

To solve the problem it is useful to rewrite it in suitable infinite dimensional spaces. We now provide the various ingredients of this reformulation: the spaces, the operators, and the reformulation itself.

B.1. **The spaces.** Consider the spaces

$$H := L^2(S^1) := \left\{ f: S^1\to\mathbb{R} \text{ measurable}: \int_{S^1}|f(x)|^2dx<\infty\right\}$$

and, for $j=0,\dots,N$,

$$H_j := L^2(M_j) := \left\{ f: M_j\to\mathbb{R} \text{ measurable}: \int_{M_j}|f(x)|^2dx<\infty\right\},$$

endowed with the with inner products

$$\langle f,g\rangle_H := \int_{S^1} f(x)g(x)dx, \quad f,g\in H,$$
$$\langle f,g\rangle_{H_j} := \int_{H_j} f(x)g(x)dx, \quad f,g\in H,$$

which render them separable Hilbert spaces. Denote by $|\cdot|_H$, $|\cdot|_{H_j}$ the associated norm, i.e.

$$|f|_H^2 := \int_{S^1} |f(x)|^2 dx, \qquad f \in H,$$

$$|f|_{H_j}^2 := \int_{M_j} |f(x)|^2 dx, \qquad f \in H_j.$$

Finally denote by H^+, H_j^+ the positive cones of H and H_j, respectively. Now, for every $f \in H$ we can write

$$f(x) := \sum_{j=0}^{N} f(x)\mathbf{1}_{M_j}(x), \quad x \in S^1. \tag{B.1}$$

Since the restriction of $f\mathbf{1}_{M_j}$ to H_j is an element of H_j we write, again with a slight abuse of notation

$$H = \bigoplus_{j=0}^{N} H_j, \qquad H^+ = \bigoplus_{j=0}^{N} H_j^+.$$

B.2. **The operators.** Denote by $L(H)$ the space of bounded linear operators on H. Consider the differential operator $(\mathcal{L}, D(\mathcal{L}))$ in H, where

$$D(\mathcal{L}) = W^{2,2}(S^1;\mathbb{R});$$

$$[\mathcal{L}\varphi](x) = \sigma\varphi''(x) + v(x)\varphi'(x) - \delta(x)\varphi(x), \quad \varphi \in D(\mathcal{L}).$$

The latter is a closed, densely defined, unbounded linear operator on the space H (see, e.g. Lunardi, 1995, page 72, Section 3.1.1). A core for it is the space $C^\infty(S^1;\mathbb{R})$ (see, e.g., Engel and Nagel, 1995, pages 69-70). Integration by parts shows that

$$\langle \mathcal{L}\varphi, \psi\rangle_H = \langle \varphi, \mathcal{L}^*\psi\rangle_H, \quad \forall \varphi, \psi \in C^\infty(S^1;\mathbb{R}) \tag{B.2}$$

where $D(\mathcal{L}^*) = D(\mathcal{L}) = W^{2,2}(S^1;\mathbb{R})$ and

$$[\mathcal{L}^*\psi](x) = \sigma\psi''(x) - v(x)\psi'(x) - (v'(x) + \delta(x))\psi(x), \quad \psi \in C^\infty(S^1;\mathbb{R}). \tag{B.3}$$

Since $C^\infty(S^1;\mathbb{R})$ is a core for $\mathcal{L}$ and $\mathcal{L}^*$, (B.3) extends to all couples of functions in $D(\mathcal{L})$. This shows that $\mathcal{L}$ is self-adjoint and dissipative if $v \equiv 0$.

Integration by parts also shows

$$\int_{S^1} v(x)\varphi(x)\varphi'(x)\mathrm{d}x = -\int_{S^1} v(x)\varphi(x)\varphi'(x)\mathrm{d}x - \int_{S^1} v'(x)|\varphi(x)|^2\mathrm{d}x, \tag{B.4}$$

hence,

$$\int_{S^1} v(x)\varphi(x)\varphi'(x)\mathrm{d}x = -\frac{1}{2}\int_{S^1} v'(x)|\varphi(x)|^2\mathrm{d}x \tag{B.5}$$

Next, we compute, again using integration by parts (in particular, (B.4) and (B.5)),

$$\begin{aligned}
\langle \mathcal{L}\varphi, \varphi\rangle_H &= \int_{S^1} \left([\mathcal{L}\varphi](x)\right)\varphi(x)\mathrm{d}x \\
&= -\int_{S^1} \sigma(x)|\varphi'(x)|^2\mathrm{d}x - \int_{S^1} v(x)\varphi(x)\varphi'(x)\mathrm{d}x - \int_{S^1} (v'(x) + \delta(x))|\varphi(x)|^2\mathrm{d}x \\
&= -\int_{S^1} \sigma(x)|\varphi'(x)|^2\mathrm{d}x - \int_{S^1} \left(\frac{1}{2}v'(x) + \delta(x)\right)|\varphi(x)|^2\mathrm{d}x \\
&\leq \left|\left(\frac{1}{2}v' + \delta\right) \wedge 0\right|_\infty |\varphi|_H^2.
\end{aligned}$$

Hence, the operator $\mathcal{L}$ is pseudo-dissipative, and so is $\mathcal{L}^*$. Therefore, by Engel and Nagel (1995) (see in particualr Chapter II), we see that $\mathcal{L}$ generates a strongly continuous semigroup $(e^{t\mathcal{L}})_{t\geq 0} \subset L(H)$. We introduce therefore the following assumption, which will ensure that ρ_j belongs to the resolvent set of the operator $\mathcal{L}$.

Assumption B.1. *For all $j = 1, \dots, N$ we have*

$$\rho_j > \left| \left(\frac{1}{2} v' + \delta \right) \wedge 0 \right|_\infty$$

B.3. **Reformulation of the state equation** (A.1)**.** Setting

$$I_j(t) := i_j(t, \cdot), \quad B_j(t) := b_j(t, \cdot), \quad j = 1, ..., N,$$

we see that $(I_j, B_j) : \mathbb{R}^+ \to H_j^+ \times H_j^+$ and rewrite the set $\mathcal{A}_j^0$, given in (A.4) (and then, consequently, $\mathcal{A}^0$), as follows:

$$\mathcal{A}_j = \Bigg\{ (I_j, B_j) : \mathbb{R}^+ \to H_j^+ \times H_j^+ \text{ measurable} : \int_0^\infty e^{-\rho_j t} (|I_j(t)|^2_{H_j} + |B_j(t)|^2_{H_j}) dt < \infty,$$
$$(A_j(\cdot) - 1) I_j(t)(\cdot) - B_j(t)(\cdot) \geq 0 \quad \text{for a.e. } t \in \mathbb{R}^+ \Bigg\}.$$

Note that the last inequality above means that, for a.e. $t \geq 0$, the function in H_j^+ given by $x \to (A_j(x) - 1) i_j(t, x) - b_j(t, x)$ is nonnegative for a.e. $x \in M_j$. The fact the such function belongs to H^+ follows by Assumption A.1-(1).

Given $\big((I_j, B_j)\big)_{j=1,\dots,N} \in \mathcal{A}$, we set

$$I(t) := 0 \cdot \mathbf{1}_{M_0}(x) + \sum_{j=1}^N \mathbf{1}_{M_j} I_j(t), \quad B(t) := 0 \cdot \mathbf{1}_{M_0}(x) + \sum_{j=1}^N \mathbf{1}_{M_j} B_j(t),$$

and

$$[\eta B(t)^\theta](x) := (\eta B(t)(x))^\theta, \quad x \in S^1.$$

Then, defining

$$[P(t)](x) := p(t, x), \quad x \in S^1,$$

we reformulate (A.1) in H as

$$\text{(B.6)} \qquad \begin{cases} P'(t) = \mathcal{L} P(t) + I(t) - \eta B(t)^\theta, & t \geq 0, \\ P(0) = p_0 \in H. \end{cases}$$

According to Bensoussan et al. (2007) (Part II, Chapter 1. Definition 3.1(v)), we define the *mild solution* to (B.6) as

$$\text{(B.7)} \qquad P(t) = e^{t\mathcal{L}} p_0 + \int_0^t e^{(t-s)\mathcal{L}} \left[I(s) - \eta B(s)^\theta \right] ds, \quad t \geq 0.$$

B.4. **Reformulation of the objective functionals** (A.3)**.** Using (B.7) it is possible to rewrite the objective functionals (A.3) of the players in the following way. First of all, we look at the term

$$\text{(B.8)} \qquad \int_0^\infty e^{-\rho_j t} \left(\int_{M_j} \frac{\big((A_j(x) - 1) i_j(t, x) - b_j(t, x)\big)^{1-\gamma_j}}{1 - \gamma_j} dx \right) dt.$$

Define

$$\left[\frac{\big((A_j - 1) I_j(t) - B_j(t)\big)^{1-\gamma_j}}{1 - \gamma_j} \right](x) := \frac{\big((A_j(x) - 1) i_j(t, x) - b_j(t, x)\big)^{1-\gamma_j}}{1 - \gamma_j}, \quad \forall (t, x) \in \mathbb{R}^+ \times M_j.$$

Then, we can rewrite (B.8) as

$$\int_0^\infty e^{-\rho_j t} \left\langle \frac{\big((A_j-1)I_j(t)-B_j(t)\big)^{1-\gamma_j}}{1-\gamma_j}, \mathbf{1}_{M_j} \right\rangle_{H_j} dt. \tag{B.9}$$

Now we look at the term

$$-\int_0^\infty e^{-\rho_j t} \left(\int_{M_j} w_j(x) p(t,x) dx \right) dt,$$

Setting

$$e^{-(\rho_j-\mathcal{L})t} := e^{-\rho_j t} e^{t\mathcal{L}}, \quad t \geq 0,$$

and defining $\widehat{w_j} : S^1 \to \mathbb{R}$ as

$$\widehat{w_j}(x) := \begin{cases} w_j(x), & \text{if } x \in M_j, \\ 0, & \text{if } x \notin M_j, \end{cases} \tag{B.10}$$

we have

$$\begin{aligned} &\int_0^\infty e^{-\rho_j t} \left(\int_{M_j} w_j(x) p(t,x) dx \right) dt = \int_0^\infty e^{-\rho_j t} \langle \widehat{w_j}, P(t) \rangle_H dt \\ &= \int_0^\infty e^{-\rho_j t} \left\langle \widehat{w_j}, e^{t\mathcal{L}} p_0 + \int_0^t e^{(t-s)\mathcal{L}} \left[I(s) - \eta B(s)^\theta \right] ds \right\rangle_H dt \\ &= \left\langle \widehat{w_j}, \int_0^\infty e^{-(\rho_j-\mathcal{L})t} p_0 \, dt \right\rangle_H + \int_0^\infty e^{-\rho_j t} \left\langle \widehat{w_j}, \int_0^t e^{(t-s)\mathcal{L}} \left[I(s) - \eta B(s)^\theta \right] ds \right\rangle_H dt \end{aligned} \tag{B.11}$$

With the above identifications, the original functional J_j of agent j rewrites as

$$J_j^{(I_{-j},B_{-j})}\big(p_0; (I_j, B_j)\big) := \int_0^\infty e^{-\rho_j t} \left\langle \frac{\big((A_j-1)I_j(t)-B_j(t)\big)^{1-\gamma_j}}{1-\gamma_j}, \mathbf{1}_{M_j} \right\rangle_{H_j} dt.$$

$$\left\langle \widehat{w_j}, \int_0^\infty e^{-(\rho_j-\mathcal{L})t} p_0 \, dt \right\rangle_H + \int_0^\infty e^{-\rho_j t} \left\langle \widehat{w_j}, \int_0^t e^{(t-s)\mathcal{L}} \left[I(s) - \eta B(s)^\theta \right] ds \right\rangle_H \tag{B.12}$$

The reformulated problem for the agent j consists then in maximizing the functional J_j in (B.12), over the set $\mathcal{A}_j$ and under the state equation (B.6). Note that, in this reformulation, the first term of the functional J_j is the only one which depends on the initial datum.

Appendix C. The Nash equilibrium

Here we provide the explicit form of the unique open loop Nash equilibrium to the reformulated problem which give, as a corollary, the solution of the original problem. First of all, we rephrase Definition A.3 in this new framework.

Definition C.1. *An* open loop Nash equilibrium *for the game is a family of couples*

$$\big((I_j^*, B_j^*)\big)_{j=1,\dots,N} \in \mathcal{A}$$

such that, for every $j = 1, ..., N$ and for all $p_0 \in L^2(S^1, \mathbb{R}_+)$,

$$J_j^{(I_{-j}^*,B_{-j}^*)}\big(p_0; (I_j^*, B_j^*)\big) \geq J_j^{(I_{-j}^*,B_{-j}^*)}\big(p_0; (I_j, B_j)\big), \quad \forall (I_j, B_j) \in \mathcal{A}_j.$$

C.1. **The functions α_j and their properties.** By Assumption B.1, we see that ρ_j belongs to the resolvent set of $\mathcal{L}$ for every $j = 1, ..., N$ (see Engel and Nagel, 1995). Hence, the operator

$$\rho_j - \mathcal{L} : \ D(\mathcal{L}) \longrightarrow H$$

is invertible with bounded inverse $(\rho_j - \mathcal{L})^{-1} : H \to H$ and

$$(\rho_j - \mathcal{L})^{-1} h = \int_0^\infty e^{-(\rho_j - \mathcal{L})t} h \, dt \qquad \forall h \in H. \tag{C.1}$$

We define

$$\alpha_j := (\rho_j - \mathcal{L})^{-1} \, \widehat{w_j} \in D(\mathcal{L}) = W^{2,2}(S^1; \mathbb{R}). \tag{C.2}$$

By definition α_j is therefore the unique solution in $D(\mathcal{L}) = W^{2,2}(S^1; \mathbb{R})$ of the abstract ODE

$$(\rho_j - \mathcal{L}) \, \alpha_j = \widehat{w_j}. \tag{C.3}$$

More explicitly, α_j, as defined in (C.2), is the unique solution in the class $W^{2,2}(S^1; \mathbb{R})$ to

$$\rho_j \alpha_j(x) - \sigma \alpha_j''(x) - v(x) \alpha_j'(x) + \delta(x) \alpha_j(x) = \widehat{w_j}(x), \quad x \in S^1, \tag{C.4}$$

meaning that it verifies (C.4) pointwise almost everywhere in S^1 (this will be, from now on, the meaning of solution to such equation).[15] By Sobolev embedding $W^{2,2}(S^1; \mathbb{R}) \subset C^1(S^1; \mathbb{R})$, so $\alpha_j \in C^1(S^1; \mathbb{R})$.

We state an a priori estimate for the solution of the equation $(\lambda - \mathcal{L})g = f$, $\lambda > 0$, when f has a suitable regularity.

Proposition C.2. *Let $M \subset S^1$ be open, nonempty, and connected. Let $f \in L^2(S^1; \mathbb{R})$ be such that $f|_{\overline{M}}$ and $f|_{S^1 \setminus \overline{M}}$ are continuous and $f > 0$ on $\overline{M}$ and $f = 0$ on $S^1 \setminus \overline{M}$. Finally, let $g \in W^{2,2}(S^1; \mathbb{R})$ be such that $(\lambda - \mathcal{L})g = f$. Then there exists $\kappa > 0$ such that*

$$\kappa \le g \le \max_{\overline{M}} f.$$

Proof. First we notice that, since $g \in W^{2,2}(S^1; \mathbb{R}) \subset C^1(S^1; R)$ and S^1 is compact, the function g admits maximum and minimum over S^1.

Estimate from below. We identify M with an open interval (a, b), so $\{a, b\} = \partial M$. The fact that $(\lambda - \mathcal{L})g = f$ and the assumption $\sigma > 0$ yield

$$g''(x) = \frac{1}{\sigma} \left[(\lambda + \delta(x)) g(x) - v(x) g'(x) - f(x) \right], \quad \text{for a.e. } x \in S^1.$$

Since $g \in C^1(S^1; \mathbb{R})$, it follows that $g \in C^2(S^1 \setminus \partial M; \mathbb{R})$ and

$$g''(x) = \frac{1}{\sigma} \left[(\lambda + \delta(x)) g(x) - v(x) g'(x) - f(x) \right], \quad \forall x \in S^1 \setminus \partial M. \tag{C.5}$$

Then, From (C.5) and Assumption A.1, we see that there exist finite $g''(a^+) := \lim_{x \to a^+} g''(x)$ and $g''(b^-) := \lim_{x \to b^-} g''(x)$ and their value is

$$\begin{aligned} g''(a^+) &= \frac{1}{\sigma} \left[(\lambda + \delta(a)) g(a) - v(a) g'(a) - f(a) \right], \\ g''(b^-) &= \frac{1}{\sigma} \left[(\lambda + \delta(b)) g(b) - v(b) g'(b) - f(b) \right], \end{aligned} \tag{C.6}$$

[15] The latter ODE can be also viewed as on ODE on the interval $[0, 2\pi]$ with periodic boundary conditions:

$$\begin{cases} \rho_j \alpha_j(x) - \sigma \alpha_j''(x) - v(x) \alpha_j'(x) + \delta(x) \alpha_j(x) = \widehat{w_j}(x), \quad x \in (0, 2\pi), \\ \\ \alpha_j(0) = \alpha_j(2\pi), \quad \alpha_j'(0) = \alpha_j'(2\pi), \end{cases}$$

falling into the Sturm-Liouville theory with periodic boundary conditions (see Coddington and Levinson, 2013).

Let $x_* \in S^1$ be a minimum point of g over S^1 and set $\kappa := g(x_*)$. Clearly, since $g \in C^1(S^1, \mathbb{R})$ it must be $g'(x_*) = 0$. We distinguish three cases.

Case 1: $x_* \in M$. We have $g'(x_*) = 0$ and $g''(x_*) \geq 0$. Plugging this into (C.5) we get

$$(\lambda + \delta(x_*))\kappa = \sigma g''(x_*) + f(x_*) > 0,$$

hence, we conclude $\kappa > 0$.

Case 2: $x_* \in \{a, b\}$. Assume, without loss of generality, that $x_* = a$. One has $g'(a) = 0$ and $\alpha_j''(a^+) \geq 0$. Plugging this into (C.6) we get

$$0 \leq g''(a^+) = \frac{1}{\sigma}\left[(\lambda + \delta(a))\kappa - f(a)\right], \tag{C.7}$$

Since $f(a) > 0$, we get $\kappa > 0$.

Case 3: $x_* \in S^1 \setminus \overline{M}$. In this case, as $f(x_*) = 0$, arguing as before we get

$$(\lambda + \delta(x_*))\kappa = \sigma g''(x_*) \geq 0,$$

hence $\kappa \geq 0$. If $\kappa > 0$, we have concluded. If $\kappa = 0$, then the fact that $g(x_*) = g_j'(x_*) = 0$ and (C.5) yield $g \equiv 0$ on $S^1 \setminus \overline{M}$. By continuity of g, g', we have also $g(a) = g'(a) = 0$. Moreover, it must be $g \geq \kappa = 0$ on $\overline{M}$, as x_* is a minimum point over S^1. Hence, it must be $g''(a^+) \geq 0$. Hence we agein get (C.7), a contradiction if $k = 0$ as $f(a) > 0$ by assumption.

Estimate from above. This part follows by arguments similar to the ones used for the estimate from below. □

Proposition C.2 immediatly yields the following two corollaries.

Corollary C.3. *Let $\lambda > 0$. The operator $(\lambda - \mathcal{L})^{-1} : H \to D(\mathcal{L}) \subset H$ is poitivity preserving, i.e.*

$$f \in H, \ f \geq 0 \ \ a.e. \quad \Longrightarrow \quad (\lambda - \mathcal{L})^{-1} f \geq 0 \ a.e..$$

Proof. The claim immediately follows by Proposition C.2 due to density of $C(S^1; \mathbb{R})$ in H. □

Corollary C.4. *Let Assumption A.1 hold. Let $\alpha_j \in W^{2,2}(S^1; \mathbb{R}) \subset C^1(S^1; \mathbb{R})$ be the solution to* (C.4). *There exists $\kappa_j > 0$ such that, for every $j = 1, ..., N$, we have*

$$\kappa_j \leq \alpha_j \leq \mathcal{K}_j,$$

where $\mathcal{K}_j := \max_{\overline{M_j}} \frac{1}{\rho_j} \overline{w_j}$.

Proof. Due to Assumption A.1, this is a direct application of Proposition C.2. □

In the next results, we investigate the dependence of α_j on the data. We start with a convergence result on the diffusion coefficient σ.

Proposition C.5. *Let Assumption A.1 hold. Denote by $\alpha_{j,\sigma}$ the unique solution to* (C.4) *when $v(\cdot) \equiv 0$. We have*

$$\lim_{\sigma \to 0^+} \alpha_{j,\sigma}(x) = \frac{w_j(x)}{\rho_j + \delta(x)}, \qquad \lim_{\sigma \to +\infty} \alpha_{j,\sigma}(x) = \frac{\int_{S^1} w_j(x) dx}{\int_{S^1} (\rho_j + \delta(x)) dx}, \quad \forall x \in S^1.$$

Proof. Case $\sigma \to 0^+$. First, notice that under our assumptions, (C.4) reads as

$$\rho_j \alpha_{j,\sigma}(x) - \sigma \alpha_{j,\sigma}''(x) + \delta(x)\alpha_{j,\sigma}(x) = \widehat{w_j}(x), \quad x \in S^1, \tag{C.8}$$

By Proposition C.4 we have

$$(\alpha_j)_*(x) := \liminf_{\overline{\sigma} \to 0^+} \left\{\alpha_{j,\sigma}(z) : \ \sigma \leq \overline{\sigma}, \ z \in S^1, \ |z - x| \leq 1/\overline{\sigma}\right\} \geq 0,$$

$$(\alpha_j)^*(x) := \limsup_{\overline{\sigma} \to 0^+} \left\{\alpha_{j,\sigma}(z) : \ \sigma \leq \overline{\sigma}, \ z \in S^1, \ |z - x| \leq 1/\overline{\sigma}\right\} \leq \mathcal{K}.$$

Clearly $(\alpha_j)_* \le (\alpha_j)^*$. By stability of viscosity solutions (see e.g. Crandal-Ishii-Lions, 1992), the latter functions are, respectively, (viscosity) super- and sub-solution to the limit equation

$$\rho_j \alpha_{j,0}(x) + \delta(x)\alpha_{j,0}(x) = \widehat{w_j}(x)$$

whose unique solution is

$$\alpha_{j,0}(x) = \frac{\widehat{w_j}(x)}{\rho + \delta(x)}.$$

By standard comparison of viscosity solutions one has $(\alpha_j)_* \ge \alpha_{j,0} \ge (\alpha_j)^*$. It follows that

$$\exists \lim_{\sigma \to 0^+} \alpha_{j,\sigma}(x) = (\alpha_j)_*(x) = (\alpha_j)^*(x) = \alpha_{j,0}(x) \quad \forall x \in S^1.$$

Case $\sigma \to +\infty$. First, we rewrite (C.8) as

$$\alpha_j''(x) = \frac{1}{\sigma}\left[\rho_j \alpha_j(x) + \delta(x)\alpha_j(x) - \widehat{w_j}(x)\right], \quad x \in S^1, \tag{C.9}$$

Notice now that $\alpha''_{j,\sigma}$ is equi-bounded and equi-uniformly continuous withe respect to $\sigma \ge 1$. Hence, by Ascoli-Arzelà Theorem we have that, from each sequence $\sigma_n \to +\infty$ we can extract a subsequence σ_{n_k} such that

$$\lim_{k \to +\infty} \alpha_{j,\sigma_{n_k}} = \alpha_{j,\infty} \quad \text{uniformly on } x \in S^1,$$

for some $\alpha_{j,\infty} \in C(S^1;\mathbb{R})$. Again by stability viscosity solutions we see that $\alpha_{j,\infty}$ must solve the limit equation

$$\alpha''_{j,\infty}(x) = 0, \quad x \in S^1,$$

hence, it must be $\alpha_{j,\infty} \equiv c_0$ for some $c_0 \ge 0$. to find the value of c_0 we may integrate (C.8) over S^1 getting

$$\int_{S^1} (\rho_j + \delta(x))\alpha_{j,\sigma}(x)dx = \int_{S^1} \widehat{w_j}(x)dx.$$

Letting $\sigma \to +\infty$ above we get

$$c_0 = \frac{\int_{S^1} \widehat{w_j}(x)dx}{\int_{S^1}(\rho_j + \delta(x))}.$$

As this value does not depend on the sequence σ_n chosen, the claim follows. □

Proposition C.6. *Let Assumption A.1 hold and let $\alpha_j^{\rho_j,\delta(\cdot),\widehat{w}_j(\cdot)}$ be the unique solution to* (C.4) *for given $\rho_j, \delta(\cdot), \widehat{w}_j(\cdot)$. Then α_j is nonincreasing with respect to space homogeneous increments of $\rho_j + \delta(\cdot)$ meaning that*

$$h, k \in \mathbb{R},\ h + k \ge 0 \iff \alpha_j^{\rho_j + h, \delta(\cdot)+k, w_j(\cdot)}(x) \le \alpha_j^{\rho_j,\delta(\cdot) w_j(\cdot)}(x) \quad \forall x \in S^1.$$

Moreover, with the same meaning, α_j is nondecreasing with respect to space homogeneous increments of $\widehat{w}_j(\cdot)$.

Proof. We start proving the first claim. Considering (C.1)–(C.2) we have

$$\alpha_j^{\rho_j,\delta(\cdot),\widehat{w}_j(\cdot)}(x) - \alpha_j^{\rho_j+h,\delta(\cdot)+k,\widehat{w}_j(\cdot)}(x) = \int_0^\infty \left(1 - e^{-t(h+k)}\right) e^{-t\mathcal{L}} \widehat{w_j}\, dt,$$

and the claim follows since $e^{-t\mathcal{L}}\widehat{w_j}$ is a positive operator, i.e. it maps nonnegative functions into nonnegative functions (see, e.g., Section 2 in Chapter II, of Ma and Röckner (1992)).

The second claim follows by Corollary C.3. □

The following proposition establish the dependence on the territory M_j.

Proposition C.7. *Let $M_j \subset \widetilde{M_j} \subset S^1$, let $w_j, \widetilde{w_j}$ coefficients associated to $M_j, \widetilde{M_j}$, respectively, and let α_j, $\widetilde{\alpha_j}$ be the associated solutions to* (C.4). *Assume that $\widetilde{w}|_{M_j} = w_j$. Then $\alpha_j \le \widetilde{\alpha_j}$.*

Proof. It follows from Corollary C.3. □

C.2. **The solution.**

Proposition C.8. *Let Assumption A.1 hold. We have*

$$J_j^{(I_{-j},B_{-j})}\big(p_0;(I_j,B_j)\big) = \int_0^\infty e^{-\rho_j t}\left\langle \frac{\big((A_j-1)I_j(t)-B_j(t)\big)^{1-\gamma_j}}{1-\gamma_j}, \mathbf{1}_{M_j}\right\rangle_{H_j}$$

$$\text{(C.10)}\qquad - \int_0^\infty e^{-\rho_j s}\left\langle \alpha_j|_{M_j}, (I_j(t)-\eta B_j(t)^\theta)\right\rangle_{H_j} ds$$

$$- \langle \alpha_j, p_0\rangle_H - \sum_{k=1,\, k\neq j}^N \int_0^\infty e^{-\rho_j s}\left\langle \alpha_j|_{M_k}, (I_k(t)-\eta B_k(t)^\theta)\right\rangle_{H_k} ds.$$

Proof. We only have to rewrite the second and the third term of J_j in (B.12).

Using (C.1) and (C.2), the second term of J_j in (B.12) can be rewritten as follows (recall the definition of $\widehat{w_j}$ in (B.10))

$$\left\langle \widehat{w_j}, \int_0^\infty e^{-(\rho_j-\mathcal{L})t} p_0\, dt\right\rangle_H = \left\langle \widehat{w_j}, (\rho_j-\mathcal{L})^{-1}p_0\right\rangle_H$$
$$= \left\langle (\rho_j-\mathcal{L})^{-1}\widehat{w_j}, p_0\right\rangle_H = \langle \alpha_j, p_0\rangle_H\,,$$

In the remainder of the proof, for simplicity of notation, we define the net emissions

$$\text{(C.11)}\qquad K(t) := I(t) - \eta B(t)^\theta.$$

Now, using again (C.1) and (C.2), the third term of J_j in (B.12), can be rewritten by exchanging the integrals as follows:

$$\int_0^\infty \left(\int_0^t e^{-\rho_j t}\left\langle \widehat{w_j}, e^{(t-s)\mathcal{L}}K(s)\right\rangle_H ds\right) dt$$
$$= \int_0^\infty \left(\int_0^t e^{-\rho_j s}\left\langle \widehat{w_j}, e^{-(\rho_j-\mathcal{L})(t-s)}K(s)\right\rangle_H ds\right) dt$$
$$\text{(C.12)}\qquad = \int_0^\infty e^{-\rho_j s}\left\langle \widehat{w_j}, \int_s^\infty e^{-(\rho_j-\mathcal{L})(t-s)}K(s)dt\right\rangle_H ds$$
$$= \int_0^\infty e^{-\rho_j s}\left\langle \widehat{w_j}, (\rho_j-\mathcal{L})^{-1}K(s)\right\rangle_H ds$$
$$= \int_0^\infty e^{-\rho_j s}\left\langle (\rho_j-\mathcal{L})^{-1}\widehat{w_j}, K(s)\right\rangle_H ds = \int_0^\infty e^{-\rho_j s}\langle \alpha_j, K(s)\rangle_H ds$$

Now, using (C.11), we get

$$\int_0^\infty e^{-\rho_j s}\langle \alpha_j, K(s)\rangle_H ds = \int_0^\infty e^{-\rho_j s}\left\langle \alpha_j, \sum_{k=1}^N (I_k(s)-\eta B_k(s)^\theta)\right\rangle_H ds$$
$$\text{(C.13)}\qquad = \int_0^\infty e^{-\rho_j s}\left\langle \alpha_j\mathbf{1}_{M_j}, (I_j(t)-\eta B_j(t)^\theta)\mathbf{1}_{M_j}\right\rangle_{H_j} ds$$
$$- \sum_{k=1,\, k\neq j}^N \int_0^\infty e^{-\rho_j s}\left\langle \alpha_j\mathbf{1}_{M_k}, (I_k(t)-\eta B_k(t)^\theta)\mathbf{1}_{M_k}\right\rangle_{H_k} ds.$$

The claim easily follows by rearranging the terms. ☐

The above result is crucial since, from the expression of the functional J_j, we immediately see that player j needs to optimize only the term

$$\int_0^\infty e^{-\rho_j t}\left[\left\langle \frac{\big((A_j-1)I_j(t)-B_j(t)\big)^{1-\gamma_j}}{1-\gamma_j},\mathbf{1}_{M_j}\right\rangle_{H_j} - \left\langle \alpha_j \mathbf{1}_{M_j},(I_j(t)-\eta B_j(t)^\theta)\right\rangle_{H_j}\right]dt \tag{C.14}$$

which does not depend on the choices of the other players. This follows from the separable additive nature of the functional J_j.

Theorem C.9. *Let Assumptions A.1 hold. Then the unique open-loop Nash Equilibrium for our problem is given, for $j=1,...,N$ and $(t,x)\in\mathbb{R}^+\times S^1$, by*

$$b_j^*(t,x)=B_j^*(t)(x)=[(A_j(x)-1)\eta\theta]^{\frac{1}{1-\theta}}, \tag{C.15}$$

$$i_j^*(t,x)=I_j^*(t)(x)=\alpha_j(x)^{-\frac{1}{\gamma_j}}(A_j(x)-1)^{\frac{1-\gamma_j}{\gamma_j}}+(\eta\theta)^{\frac{1}{1-\theta}}(A_j(x)-1)^{\frac{\theta}{1-\theta}}, \tag{C.16}$$

where α_j is the unique solution to the ODE

$$\rho_j\alpha_j(x)-\sigma\alpha_j''(x)-v(x)\alpha_j'(x)+\delta(x)\alpha_j(x)=\widehat{w_j}(x),\quad x\in S^1, \tag{C.17}$$

with $\widehat{w_j}$ defined as

$$\widehat{w_j}(x):=\begin{cases} w_j(x), & \textit{if } x\in M_j,\\ 0, & \textit{if } x\notin M_j.\end{cases} \tag{C.18}$$

Moreover, setting

$$I^*:=0\cdot\mathbf{1}_{M_0}+\sum_{j=1}^N \mathbf{1}_{M_j}I_j^*,\quad B^*:=0\cdot\mathbf{1}_{M_0}+\sum_{j=1}^N\mathbf{1}_{M_j}B_j^*$$

and defining the stationary optimal net emission *as:*

$$N^*:=I^*-(\eta B^*)^\theta,$$

the equilibrium state $p^*(t,\cdot)=P^*(t)$ is

$$P^*(t)=e^{t\mathcal{L}}p_0+\int_0^t e^{(t-s)\mathcal{L}}N^*ds. \tag{C.19}$$

Finally, the welfare of player j is affine in p_0:

$$v_j(p_0):=J_j^{(i^*_{-j},b^*_{-j})}\big(p_0;(i_j^*,b_j^*)\big)=\int_{S^1}\alpha_j(x)p_0(x)dx+q_j,$$

where

$$\begin{aligned} q_j:=&\int_0^\infty e^{-\rho_j t}\left(\int_{M_j}\frac{\big((A_j(x)-1)i_j^*(t,x)-b_j^*(t,x)\big)^{1-\gamma_j}}{1-\gamma_j}dx\right)dt\\ &-\int_0^\infty e^{-\rho_j t}\left(\int_{M_j}\alpha_j(x)(i_j^*(t,x)-\eta b_j^*(t,x)^\theta)dx\right)dt\\ &-\sum_{k=1,\,k\neq j}^N\int_0^\infty e^{-\rho_j t}\left(\int_{M_k}\alpha_j(x)(i_k^*(t,x)-\eta b_k^*(t,x)^\theta)dx\right)dt. \end{aligned}$$

Proof. Observe that (C.14) can be rewritten as

$$\int_0^\infty e^{-\rho_j s}\left(\int_{M_j}\left[\frac{\big((A_j(x)-1)i_j(t,x)-b_j(t,x)\big)^{1-\gamma_j}}{1-\gamma_j}-\alpha_j(x)\left(i_j(t,x)-\eta b_j(t,x)^\theta\right)\right]dx\right)ds. \tag{C.20}$$

Hence, if for every $(t,x)\in\mathbb{R}_+\times M_j$, we maximize the integrand and the maximum point is unique, this will give the unique optimal strategy of player j, independently of the strategies of the other players.

Fix $(t,x)\in\mathbb{R}^+\times M_j$. By strict concavity of the integrand function with respect to $i_j(t,x)$ and $b_j(t,x)$, the unique maximum point can be found just by first order optimality conditions. The resulting system is

$$\begin{cases}\big((A_j(x)-1)i_j(t,x)-b_j(t,x)\big)^{-\gamma_j}(A_j(x)-1)-\alpha_j(x)=0,\\ \\ -\big((A_j(x)-1)i_j(t,x)-b_j(t,x)\big)^{-\gamma_j}+\alpha_j(x)\eta\theta b_j(t,x)^{\theta-1}=0.\end{cases} \tag{C.21}$$

The claims then follows from straightforward computations and by (C.10). □

Remark C.10. *It is clear that, for every $j=1,\dots,N$, the cost functional J_j is decreasing with respect to $\overline{w}_j$; this follows from the fact that a lower cost of pollution makes the welfare bigger, hence also the corresponding welfare function v_j is decreasing in $\overline{w}_j$.*

Moreover, if, for $i\neq j$, $\overline{w}_i$ increases, then J_j, and so v_j also increase. This can be seen looking at the decomposition of Proposition C.8 or simply observing that the increase in $\overline{w}_i$ does not modifies the strategy for the j-th agent, but makes the p to globally decrease since the agent i will pollute less.

Finally, since p is decreasing with respect to δ (this comes from the fact that a higher self-cleaning capacity makes the pollution lower), for every $j=1,\dots,N$, the cost functional J_j is decreasing with respect to δ, hence also the corresponding welfare function v_j is decreasing in δ.

Remark C.11. *We now look at the dependence of the optimal net emissions, i.e. $n(t,x):=i_j^*(t,x)-\eta(b_j^*(t,x))^\theta$, on the data. By a simple computation we see that for every $(t,x)\in\mathbb{R}^+\times S^1$*

$$i_j^*(t,x)-\eta(b_j^*(t,x))^\theta=\alpha_j(x)^{-\frac{1}{\gamma_j}}(A_j(x)-1)^{\frac{1-\gamma_j}{\gamma_j}}-(\eta\theta)^{\frac{1}{1-\theta}}(A_j(x)-1)^{\frac{\theta}{1-\theta}}\left[\theta^{-1}-1\right]. \tag{C.22}$$

From this computation, one can analyze the monotonicity of net emissions with respect to some parameters. For example:

- *when $\gamma_j>1$, the value of $n(t,x)$ is decreasing with respect to $A_j(x)$;*
- *when*
$$\frac{1-\gamma_j}{\gamma_j}\geq\frac{\theta}{1-\theta}\quad\textit{and}\quad\alpha_j(x)^{-\frac{1}{\gamma_j}}\geq\left[\theta^{-1}-1\right](\eta\theta)^{\frac{1}{1-\theta}},$$
the value of $n(t,x)$ is increasing with resect to $A_j(x)$;
- *by Proposition C.6, $n(t,x)$ is increasing with respect to space homogeneous increments of $\rho_j+\delta(\cdot)$ in the sense specified in the same proposition.*

Proof of Proposition 4.2. Just for simplicity of notatation, we prove the claim in a very special case, i.e. when $\Pi^1=\{M_1^1,M_2^1\}$ and $\Pi^2=\{M_1^2\}$. The proof of the general claim is a straightforward generalization. Let $n^{1,*},n^{2,*}$ be the optimal net emissions associated to Π^1,Π^2, respactively. They differ only for the terms containing α. Now, with clear meaning of the symbols, we have

$$\alpha_1^1=(\rho-\mathcal{L})^{-1}\widehat{w}_1^1,\quad \alpha_2^1=(\rho-\mathcal{L})^{-1}\widehat{w}_2^1,\quad \alpha_1^2=(\rho-\mathcal{L})^{-1}\widehat{w}_1^2.$$

Since $\widehat{w}_1^2\geq\widehat{w}_1^1$ and $\widehat{w}_1^2\geq\widehat{w}_1^2$, by Corollary C.3 it follows that

$$\alpha_1^2(x)\geq\alpha_1^1(x)\ \ \forall x\in M_1^1,\quad \alpha_1^2(x)\geq\alpha_2^1(x)\ \ \forall x\in M_2^1.$$

Then the claim follows from (2.10), (2.7), (2.8), (B.7), and since the operator positivity $e^{t\mathcal{L}}$ is positive, i.e. maps nonnegative functions into nonnegative functions (see, e.g., Section 2 in Chapter II, of Ma and Röckner (1992)). □

Corollary C.12. *Let Assumption A.1 hold. Furthermore, assume that there exists $\bar{\delta} > 0$ such that*

$$\frac{1}{2}v'(x) - \delta(x) \geq \bar{\delta} \quad \forall x \in S^1. \tag{C.23}$$

Then

$$\lim_{t\to\infty} P^*(t) = P^*_\infty \quad \text{in } H,$$

where P^*_∞ is the unique solution in H to the ODE $\mathcal{L}P^*_\infty + K^* = 0$, i.e.

$$\sigma \frac{\mathrm{d}^2 P^*_\infty}{\mathrm{d}x^2}(x) + v(x)\frac{\mathrm{d}P^*_\infty}{\mathrm{d}x}(x) - \delta(x)P^*_\infty(x) + K^*(x) = 0, \quad x \in S^1. \tag{C.24}$$

Proof. Let us split

$$\mathcal{L} = \bar{\mathcal{L}} + \bar{\mathcal{D}},$$

where

$$\bar{\mathcal{L}}\varphi := \mathcal{L}\varphi - \bar{\delta}\varphi, \quad \varphi \in D(\mathcal{L}),$$

and

$$\bar{\mathcal{D}}\varphi := -\bar{\delta}\varphi, \quad \varphi \in H.$$

We can rewrite

$$P^*(t) = e^{-\bar{\delta}t}e^{t\bar{\mathcal{L}}}p_0 + \int_0^t e^{-\bar{\delta}(t-s)}e^{(t-s)\bar{\mathcal{L}}}K^* ds,$$

and take the limit above when $t \to \infty$. By (C.23), $\bar{\mathcal{L}}$ is dissipative, hence $e^{s\bar{\mathcal{L}}}$ is a contraction. Therefore, the first term of the right hand side converges to 0 in H, whereas the second one converges to

$$P^*_\infty := \int_0^\infty e^{-\bar{\delta}s}e^{s\bar{\mathcal{L}}}K^* ds$$

in H. Then, the limit state P^*_∞ can be expressed using again Engel and Nagel (1995), Proposition 3.14, page 82 and Chapter II, Theorem 1.10, as

$$P^*_\infty = (\bar{\delta} - \bar{\mathcal{L}})^{-1}K^*,$$

i.e. P^*_∞ is the solution in H to $(\bar{\delta} - \bar{\mathcal{L}})P^*_\infty = K^*$, i.e. to $\mathcal{L}P^*_\infty + K^* = 0$. □

Remark C.13. *The dependence of P_∞ on $A_j(\cdot)$ follows from what observed in Remark C.11. Indeed, since $\mathcal{L}$ does not depend on the $A_j(\cdot)$, P_∞ depends on it only through the stationary optimal net emissions, K^*.*

Appendix D. Series expansion of the α_j's

In this section $A_j \equiv A_j^o > 1$, $w_j \equiv w_j^o > 0$, $\delta \equiv \delta^o \geq 0$, $\sigma \equiv \sigma > 0$, $v \equiv v^o \in \mathbb{R}$. We use the identification $S^1 \cong 2\pi\mathbb{R}/\mathbb{Z}$ and assume, without loss of generality that $M_j = (0, \ell_j)$. Finally, to save notation, we suppress the subscript j. We are going to study the Fourier series expansion of $\alpha = \alpha_j$ in the case without or with advection. We notice that the convergence of the series is uniform on S^1 due to the smoothness of α.

D.1. **The case without advection.** We know that α solves the equation $(\rho - \mathcal{L})\alpha = \widehat{w}$, with $\widehat{w}$ defined as

$$\widehat{w}(x) := \begin{cases} w^o, & \text{if } x \in M, \\ 0, & \text{if } x \notin M. \end{cases}$$

The set of elements of H

$$\left\{ \mathbf{e}_0(x) := \frac{1}{\sqrt{2\pi}} \mathbf{1}_{S^1}(x) \right\} \bigcup \left\{ \mathbf{e}_n^{(1)}(x) := \frac{1}{\sqrt{\pi}} \sin(nx), \ \mathbf{e}_n^{(2)}(x) := \frac{1}{\sqrt{\pi}} \cos(nx), \ n \in \mathbb{N} \setminus \{0\} \right\} \tag{D.1}$$

is an orthonormal basis on H. They are also eigenfunctions of $\mathcal{L}$ with associated eigenvalues

$$\mu_n = -\delta^o - \sigma n^2, \quad n \in \mathbb{N}. \tag{D.2}$$

We can expand in Fourier series

$$\alpha = \langle \alpha, \mathbf{e}_0 \rangle_H \, \mathbf{e}_0 + \sum_{n \in \mathbb{N} \setminus \{0\}, \, i=1,2} \langle \alpha, \mathbf{e}_n^{(i)} \rangle_H \, \mathbf{e}_n^{(i)}. \tag{D.3}$$

Let us compute the coefficients of the series. As for $n = 0$, we notice that

$$\langle \alpha, (\rho - \mathcal{L})\mathbf{e}_0 \rangle_H = \langle (\rho - \mathcal{L})\alpha, \mathbf{e}_0 \rangle_H = \langle \widehat{w}, \mathbf{e}_0 \rangle_H,$$

hence,

$$\langle \alpha, \mathbf{e}_0 \rangle_H = (\rho + \delta^o)^{-1} \langle \widehat{w}, \mathbf{e}_0 \rangle_H = (\rho + \delta^o)^{-1} \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \widehat{w}(x) dx = \frac{1}{\sqrt{2\pi}} \frac{\ell w^o}{\rho + \delta^o}. \tag{D.4}$$

Similarly,

$$\langle \alpha, \mathbf{e}_n^{(i)} \rangle_H = \left(\rho + \delta^o + \sigma n^2 \right)^{-1} \langle \widehat{w}, \mathbf{e}_n^{(i)} \rangle_H, \qquad \forall i = 1, 2, \ \forall n \in \mathbb{N} \setminus \{0\}.$$

We may compute

$$\langle \widehat{w}, \mathbf{e}_n^{(i)} \rangle_H = \begin{cases} \frac{1}{\sqrt{\pi}} \int_0^{2\pi} \widehat{w}(x) \sin(nx) \, dx = \frac{w^o}{\sqrt{\pi}} \int_0^{\ell} \sin(nx) \, dx = \frac{w^o}{\sqrt{\pi}} \frac{1}{n} \left[1 - \cos(n\ell) \right], & \text{if } i = 1, \\ \frac{1}{\sqrt{\pi}} \int_0^{2\pi} \widehat{w}(x) \cos(nx) \, dx = \frac{w^o}{\sqrt{\pi}} \int_0^{\ell} \cos(nx) \, dx = \frac{w^o}{\sqrt{\pi}} \frac{1}{n} \sin(n\ell), & \text{if } i = 2. \end{cases} \tag{D.5}$$

Plugging these results into (D.3) yields

$$\alpha(x) = \frac{1}{2\pi} \frac{\ell w^o}{\rho + \delta^o} + \frac{w^o}{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)\left(1 - \cos(n\ell)\right) + \sin(n\ell)\cos(nx)}{n\left(\rho + \delta^o + \sigma n^2\right)},$$

D.2. **The case with advection.** In this subsection we remove the assumption $v \equiv 0$. Recalling the expression of $\mathcal{L}^*$ provided in (B.3), we have in the present case

$$[\mathcal{L}^* \psi](x) = \sigma \psi''(x) - v^o \psi'(x) + \delta^o \psi(x), \quad \psi \in D(\mathcal{L}^*).$$

Consider again the family (D.1). In this case $\mathbf{e}_0$ is still an eigenfunction of $\mathcal{L}^*$, but $\mathbf{e}_n^{(i)}$ are not eigenfunction of $\mathcal{L}^*$ anymore for $n \in \mathbb{N} \setminus \{0\}$. However, still $H_n := \operatorname{Span}\{\mathbf{e}_n^{(1)}, \mathbf{e}_n^{(2)}\}$ are invariant subspaces for $\mathcal{L}^*$ for each $n \in \mathbb{N} \setminus \{0\}$. Indeed

$$\mathcal{L}^* \mathbf{e}_n^{(1)} = -\sigma n^2 \mathbf{e}_n^{(1)} - v^o n \mathbf{e}_n^{(2)} + \delta^o \mathbf{e}_n^{(1)}; \quad \mathcal{L}^* \mathbf{e}_n^{(2)} = -\sigma n^2 \mathbf{e}_n^{(2)} + v^o n \mathbf{e}_n^{(1)} + \delta^o \mathbf{e}_n^{(2)};$$

Arguing in a similar way as in the previous subsection we get, for each $n \in \mathbb{N} \setminus \{0\}$, the couple of equations

$$(\rho + \sigma n^2 + \delta^o)\langle \alpha, \mathbf{e}_n^{(1)} \rangle_H + v^o n \langle \alpha, \mathbf{e}_n^{(2)} \rangle_H = \langle \widehat{w}, \mathbf{e}_n^{(1)} \rangle_H, \quad (\rho + \sigma n^2 + \delta^o)\langle \alpha, \mathbf{e}_n^{(2)} \rangle_H - v^o n \langle \alpha, \mathbf{e}_n^{(1)} \rangle_H = \langle \widehat{w}, \mathbf{e}_n^{(2)} \rangle_H.$$

yielding, for each $n \in \mathbb{N} \setminus \{0\}$,

$$\langle \alpha, \mathbf{e}_n^{(1)} \rangle_H = \frac{(\rho + \sigma n^2 + \delta^o)\langle \widehat{w}, \mathbf{e}_n^{(1)} \rangle_H - v^o n \langle \widehat{w}, \mathbf{e}_n^{(2)} \rangle_H}{(\rho + \sigma n^2 + \delta^o)^2 + (v^o n)^2},$$

$$\langle \alpha, \mathbf{e}_n^{(2)} \rangle_H = \frac{(\rho + \sigma n^2 + \delta^o)\langle \widehat{w}, \mathbf{e}_n^{(2)} \rangle_H + v^o n \langle \widehat{w}, \mathbf{e}_n^{(1)} \rangle_H}{(\rho + \sigma n^2 + \delta^o)^2 + (v^o n)^2}.$$

Using (D.3)–(D.5) and the expressions above, we have for $x \in S^1$

$$\alpha(x) = \frac{1}{2\pi} \frac{\ell w^o}{\rho + \delta^o} + \frac{w^o}{\pi} \sum_{n=1}^{\infty} \frac{(\rho + \sigma n^2 + \delta^o)\sin(n\ell) + v^o n(1 - \cos(n\ell))}{n\,(\rho + \delta^o + \sigma n^2)} \cos(nx)$$
$$+ \frac{w^o}{\pi} \sum_{n=1}^{\infty} \frac{(\rho + \sigma n^2 + \delta^o)(1 - \cos(n\ell)) - v^o n \sin(n\ell)}{n\,(\rho + \delta^o + \sigma n^2)} \sin(nx).$$

Appendix E. The cooperative game case

In this case we consider the problem when the N players are cooperative and have the same discount factor $\rho_j = \rho$ and preference parameter $\gamma_j = \gamma$ for every $j = 1, ..., N$. We suppose they coordinate to maximize the sum of their utilities:

$$\sum_{j=1}^{N} \int_0^\infty e^{-\rho t} \left(\int_{M_j} \left(\frac{\big((A_j(x) - 1)i_j(t,x) - b_j(t,x)\big)^{1-\gamma}}{1-\gamma} - w_j(x)p(t,x) \right) dx \right) dt, \tag{E.1}$$

over $((i_j, b_j))_{j=1,...,N} \in \mathcal{A}$ and under the state equation (A.1). As we see in Theorem E.1 below, this problem is equivalent to the problem of a unique player acting on $\bigcup_{j=1}^N M_j$. We know the solution of the latter problem, as we only need to implement the results of Theorem C.9 in the case of a unique player. Defining

$$\underline{A}(x) := 0 \cdot \mathbf{1}_{M_0}(x) + \sum_{j=1}^{N} A_j(x)\mathbf{1}_{M_j}(x), \qquad \underline{w}(x) := 0 \cdot \mathbf{1}_{M_0}(x) + \sum_{j=1}^{N} w_j(x)\mathbf{1}_{M_j}(x)$$

and letting $\underline{\alpha}$ be the solution to

$$\rho\underline{\alpha}(x) - \sigma\underline{\alpha}'(x) - v(x)\underline{\alpha}'(x) + \delta(x)\underline{\alpha}(x) = \underline{w}(x), \quad x \in S^1, \tag{E.2}$$

Theorem C.9 provides the solution for the unique player:

$$\underline{b}^*(t,x) := \begin{cases} \left[(\underline{A}(x) - 1)\eta\theta\right]^{\frac{1}{1-\theta}}, & \text{if } x \in S^1 \setminus M_0, \\ 0 & \text{if } x \in M_0, \end{cases} \tag{E.3}$$

and

$$\underline{i}^*(t,x) := \begin{cases} \underline{\alpha}(x)^{-\frac{1}{\gamma_1}} (\underline{A}(x) - 1)^{\frac{1-\gamma_1}{\gamma_1}} + (\eta\theta)^{\frac{1}{1-\theta}} (\underline{a}(x) - 1)^{\frac{\theta}{1-\theta}}, & \text{if } x \in S^1 \setminus M_0, \\ 0, & \text{if } x \in M_0. \end{cases} \tag{E.4}$$

Theorem E.1. *Let Assumptions A.1 hold. Then the unique* $((\underline{i}_j^*, \underline{b}_j^*))_{j=1,..,N} \in \mathcal{A}$ *maximizing* (E.1)*, i.e. the unique optimal choice of the cooperative game when the N players have the same discount factor $\rho_j = \rho$ and preference parameter $\gamma_j = \gamma$ for every $j = 1, ..., N$, is given by*

$$\underline{b}_j^*(t,x) = \left[(\underline{A}(x) - 1)\eta\theta\right]^{\frac{1}{1-\theta}} \mathbf{1}_{M_j}(x), \quad j = 1, ..., N, \tag{E.5}$$

$$\underline{i}_j^*(t,x) = \left[\underline{\alpha}(x)^{-\frac{1}{\gamma}} (\underline{a}(x) - 1)^{\frac{1-\gamma}{\gamma}} + (\eta\theta)^{\frac{1}{1-\theta}} (\underline{a}(x) - 1)^{\frac{\theta}{1-\theta}}\right] \mathbf{1}_{M_j}(x), \quad j = 1, ..., N, \tag{E.6}$$

where $\underline{\alpha}$ is the solution to (E.2)*.The corresponding welfare is*

$$\underline{v}(p_0) := \int_{S^1} \underline{\alpha}(x)p_0(x)dx + \underline{q},$$

where

$$\underline{q} := \int_0^\infty e^{-\rho t} \left(\int_{S^1} \left[\frac{\left((\underline{A}(x)-1)i^*(t,x) - \underline{b}^*(t,x)\right)^{1-\gamma}}{1-\gamma} - \underline{\alpha}(x)(\underline{i}^*(t,x) - \eta \underline{b}^*(t,x)^\theta) \right] dx \right) dt,$$

with $\underline{b}^*(t,x)$ *and* $\underline{i}^*(t,x)$ *given by* (E.3) *and* (E.4).

Sketch of proof. We observe that the sum of utilities given in (E.1) is exactly equal to the utility of the unique player. Since the constraints and the state equation are the same, the equivalence follows. □

Corollary E.2. *Let Assumptions A.1 hold and assume that* $\delta(\cdot) \equiv \delta^o > 0$, $w(\cdot) \equiv w^o > 0$, *and* $\rho_j = \rho$ *for all* $j = 1, ..., N$. *Then*

$$\underline{\alpha} \equiv \frac{w^o}{\rho - \delta^o},$$

which is (constant and) independent of σ*; moreover, if* $\sigma = 0$*([16]), then*

$$\alpha_j \equiv \frac{w^o}{\rho - \delta^o}, \quad \forall j = 1, ..., N,$$

i.e. the same solution of the cooperative game case obtained for each diffusion coefficient $\sigma \geq 0$.

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[16]To be precise, the case $\sigma = 0$ should be treated separately, as it is out of our assumptions. Nonetheless, this case can be easily treated pointwiseless on x and gives raise, in the case under consideration here, to the solutions we illustrate below.

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