Assessing the Parfit's *Repugnant Conclusion* within a canonical endogenous growth set-up

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Assessing the Parfit's *Repugnant Conclusion* within a canonical endogenous growth set-up*

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Abstract

Parfit's *Repugnant Conclusion* stipulates that under total utilitarianism, it might be optimal to choose increasing population size while consumption per capita goes to zero. We evaluate this claim within a canonical AK model with endogenous fertility and a reduced form relationship between demographic growth and economic growth. While in the traditional linear dilution model, the Parfit *Repugnant Conclusion* can never occur for realistic values of intertemporal substitution, we show that it occurs when population growth is linked to economic growth via an inverted U-shaped relationship. Finally, we find moving from the Benthamite to the Millian social welfare function may not only cause optimal population size to go up and consumption to go down, it may also favor the realization of the *Repugnant Conclusion*.

Key words: Parfit's Repugnant Conclusion, AK models, endogenous fertility, intertemporal altruism

JEL classification: O41, I20, J10

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1 Introduction

Population ethics have become increasingly invoked in normative economics. The recent debate on sustainability (see Arrow et al., 2004 and 2010) is an example of the economic mainstream topics in which population ethics do matter. Indeed, there is an increasing interest among economists in questions such that "is it possible to make the world a better place by creating additional happy individuals?" and "is there a moral obligation to have children?"...etc... A major contribution to population ethics is Parfit's 1984 formulation of the so-called the *Repugnant Conclusion*. This theory is a clear criticism of total utilitarianism as it stands in the original statements: "For any possible population of at least ten billion people, all with a very high quality of life, there must be some much larger imaginable population whose existence, if other things are equal, would be better even though its members have lives that are barely worth living". In other words, under total utilitarianism, that is for a Benthamite social welfare function, it might be optimal to choose increasing population size while consumption per capita goes to zero.

Parfit's conclusion has been evaluated in several previous studies, including economic growth studies, one of the most recent being Arrow's et al. (2010). In the latter, growth is exogenous and the planner maximizes a Benthamite social welfare function by choosing an optimal sequence of health expenditures controlling for individuals'life time (and therefore for population size), in addition to consumption. In our paper, we consider the framework of endogenous growth. To minimize technicalities, we build on a canonical AK-like model where population growth affects the accumulation of capital taken in a very wide sense (that's including physical, human or natural capital). In a first step, no assumptions are made on the shape of the relationship between demographic growth on one side and productivity and/or depreciation on the other side. Within this quite flexible structure, we propose to investigate to which extent Parfit's conclusion is inherent to AK-like models. To make the analysis even more appealing, we shall introduce a flexible parameterized social welfare function encompassing the Benthamite and the Millian case (or average utilitarianism) as well.

Our work can be connected with an important stream of economic literature, namely the literature on optimal population size. This literature traces back to Edgeworth (1925) who was the first to claim that the Benthamite welfare functions are those which lead to the largest population size and the lowest living standards. Recent contributions by Nerlove and co-authors went into the same direction (see for example, Nerlove et al., 1985). Our framework is much more in line with Palivos and Yip (1993) who examined the issue of optimal population size within an AK-like setting. A major conclusion of the latter authors is that the Benthamite social welfare function may dominate the Millian under certain realistic parameterizations of the model. In particular, Palivos and Yip (1993) show that if the intertemporal elasticity of substitution is realistically chosen (that's much lower than unity), the Benthamite criterion yields a smaller population size and higher economic growth, in sharp contrast to the wisdom established by Edgeworth decades ago.

Our model generalizes in a way the model of Palivos and Yip and addresses more specifically the Parfit's conclusions. While these authors model the impact of population growth on capital accumulation in the traditional way, namely through a linear dilution effect in the law of motion of capital, we take the more realistic view that the relationship between capital accumulation and population growth is much more complex, going much beyond the simplistic linear dilution mechanism. Several studies have already pointed out the limits of the latter traditional assumption. Among them, Blanchet (1988) showed that a simple accounting of the age structure of physical capital yields depreciation rates that are no longer linear in population growth. This nonlinearity is therefore valid even when one takes a strict concept of capital. Of course, this property is likely to be reinforced when one takes a wider concept of capital, as we do in this paper: it is highly reasonable when natural capital is concerned, and it is even a well-known stylized fact when dealing with human capital (see the large literature on the nonlinear relationship between human capital accumulation and population growth, as surveyed by Kelley, 1988). A further difference with respect to Palivos and Yip is that we remove intratemporal altruism (that's population growth as an argument of the instantaneous utility function) to recover the standard form of social welfare functions in growth theory. Instead, we keep the ingredient of intertemporal altruism through a standard parameter allowing to cover the Millian and Benthamite cases as polar parameterizations.

The resulting trade-offs driving the optimal choice of population size are therefore different. In Palivos and Yip, population growth has a direct positive impact on welfare through intratemporal altruism, an indirect positive one through intertemporal altruism and a negative one through the linear dilution effect. In this paper, the picture is simpler: only the two last effects are present but the negative one is more involved since we do no longer assume the unrealistic linear dilution effect. This said, the objective pursued in this paper is clearly different: we don't aim to compare the outcomes of the Benthamite vs the Millian social welfare but to shed light on the occurrence of Parfit's conclusions in our canonical model, the degree of intertemporal altruism being only one of the relevant parameters of the problem.

We find in particular then while in the traditional linear dilution model, the Parfit *Repugnant Conclusion* can never occur for realistic values of intertemporal substitution, it does occur when the relationship between demographic growth and economic growth is nonlinear. In particular, we get this occurrence when population growth is linked to economic growth via an inverted U-shaped relationship, consistently with the literature (e.g. Kelley and Schmidt, 1995, and Boucekkine et al., 2002). We also find that decreasing technical progress or increasing the time discount rate facilitates the realization of the *Repugnant Conclusion*. Interestingly enough, we find that for realistic values of intertemporal substitution, moving from the Benthamite to the Millian social welfare function may not only cause optimal population size to go up and consumption to go down as in Palivos and Yip (1993), it may also favor the realization of the *Repugnant Conclusion*.

The paper is organized as follows. Section 2 mainly displays the model and characterizes the underlying optimal control problem. Section 3 studies the general model, that's the model with a general reduced form relationship between population growth and economic growth, and states formally Parfit's *Repugnant Conclusion*. Some general results on the extent to which this conclusion holds are also presented. Section 4 solves some particular cases, including the case of nonlinear and possibly non-monotonic dilution mechanisms. Section 5 concludes.

2 The model

Consider the system described by the evolution of the following differential system:

$$\begin{cases} \dot{z}(t) = Az(t) - c(t) - \mu(n(t))z(t), \qquad z(0) = z_0 > 0 \\ \dot{N}(t) = n(t)N(t), \qquad N(0) = N_0 > 0. \end{cases} \tag{1}$$

$n(\cdot)$ stands for population growth rate and $c(\cdot)$ for consumption per capita. Both variables will be the controls of the system while $z(\cdot)$ and $N(\cdot)$ represent the states variables: the former is the stock of composite capital per capita available at t and the latter is the population size at that time. As mentioned repeatedly in the introduction, $z(\cdot)$ is taken in a wide sense including physical, human and natural capital. Population growth rate is given by $n(t)$. Population growth affects capital accumulation via function $\mu(\cdot)$. Traditionally, the latter function is assumed linear as authors typically suppose a linear dilution effect mechanism. Here, the shape of $\mu(.)$ can be any nonlinear function consistently with the arguments outlined in the introduction. We keep the word "dilution" to fix the exposition but is should be clear from now that our state equations are reduced forms that go much beyond the traditional frameworks. The production function is AK, with A the productivity parameter. We could have assumed that A is also influenced by population growth but since the production function is linear, the relevant term in the law of motion of z is $(A - \mu(n(t)))\, z(t)$: without loss of generality, we choose to fix A to a constant while "playing" with the shape of the dilution function. Precisely, we postulate that $\mu(\cdot)$ is a function defined by:

$$\mu \colon [m, M] \to \mathbb{R},$$

where $m < 0 < M$. The justification of this set of choice is the following. We assume that the planner has the full control on the birth rate but not on the death rate. Eventually, one can assume that there is an exogenous death rate, say $d > 0$. If $\bar{M} > 0$ is the maximal (biological) feasible birth rate with $\bar{M} > d$, then $M = \bar{M} - d > 0$ and $m = -d < 0$. Importantly enough, our framework is so general that we have no *a priori* assumption on the sign of $\mu(n)$ though we call it, by analogy, dilution. This is not only due to the fact that by construction, n can be negative. In our model, growth is entirely driven by net productivity of capital, that is $A - \mu(n)$. As mentioned in the introduction, the relationship between population growth and economic growth is known to be nonlinear, possibly non-monotonic. For example, population growth might be good for economic growth for small values of n, then drives economic growth down from a certain threshold value as in Boucekkine et al. (2002). Accordingly, one may consider a function $\mu(n)$ that is decreasing for small positive values of n and then increasing after. We shall consider such a function in Section 4.2.

Consider now the problem of maximizing

$$J_{N_0, z_0}(c(\cdot), n(\cdot)) := \int_0^{+\infty} e^{-\rho t} \frac{c^{1-\sigma}(t)}{1-\sigma} N^{\gamma}(t)\, \mathrm{d}t, \tag{2}$$

where ρ is the time discounting rate, and $\gamma \in [0\ , 1]$ is the intertemporal altruism parameter: $\gamma = 0$ (Resp. $\gamma = 1$) corresponds to the Millian (Resp. Benthamite) case. The utility function is standard, it is isoelastic, with the elasticity parameter $\sigma > 0$. We require that

$$z(t), N(t), c(t) \geq 0 \;\text{ and }\; n(t) \in [m, M] \qquad \text{for all } t \geq 0 \tag{3}$$

so that the set of admissible controls of the system, depending on initial condition (N_0, z_0), is

$$\mathcal{U}_{N_0, z_0} := \Big\{ (c(\cdot), n(\cdot)) \in \big(L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+)\big)^2 \;:\; \text{eqs. (3) hold for all } t \geq 0 \Big\}$$

We aim to maximize functional (2) subject to the state equations (1) and the admissibility constraints (3). We first show that the problem makes sense under some reasonable assumptions. Call $V\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ the value function of the problem, defined as

$$V(N_0, z_0) := \sup_{(c(\cdot),n(\cdot)) \in \mathcal{U}_{N_0,z_0}} J_{N_0,z_0}(c(\cdot), n(\cdot))$$

Let us set the following assumption:

Hypothesis 2.1 *Let* $\arg\max_{n\in[m,M]} \left(\frac{\gamma}{1-\sigma} n + (A - \mu(n))\right)$ *be non-void. Let us choose*

$$\bar{n} \in \arg \max_{n\in[m,M]} \left(\frac{\gamma}{1-\sigma} n + (A - \mu(n))\right) \tag{4}$$

and call

$$\eta := \max_{n\in[m,M]} \left(\frac{\gamma}{1-\sigma} n + (A - \mu(n))\right). \tag{5}$$

Note that the previous hypothesis is very mild, it is always satisfied e.g. when $\mu(\cdot)$ is continuous. Under this assumption and a further traditional non-explosivity condition, our optimization problem is well-posed in the sense that the value function is finite, as stated in the following proposition.

Proposition 2.1 *Assume that Hypothesis 2.1 is satisfied. Then the following condition is sufficient to guarantee the finiteness of the value function at every* $(N_0, z_0) \in \mathbb{R}^2_+$*:*

$$\rho > (1 - \sigma)\eta. \tag{6}$$

If $\sigma \in (0, 1)$ *the condition (6) is also necessary. If it is not satisfied then* $V(N_0, z_0) = +\infty$ *for all* $(N_0, z_0) \in \mathbb{R}^2_+$*.*

Condition (6) can be interpreted as a traditional non-explosivity condition. As it will be clear in the next section, $(1 - \sigma)\eta$ is the "optimal" growth rate of the undiscounted integrand in the objective function. Therefore, condition (6) ensures the boundedness of the integral along the optimal paths as usual in optimal growth models.

3 The general solution

We now provide the general solution to the optimal control problem presented in the previous section, that's the solution for any function $\mu(n)$. In contrast to the typical treatment in the literature of optimal population size which relies on steady state trajectories (see Nerlove et al., 1985, or Palivos and Yip, 1993), we will directly extract the optimal dynamic trajectories using dynamic programming. The next theorem shows that the considered AK production function allows to extract a closed-form solution to the value-function whatever the dilution function and despite population growth is a control.[1]

Theorem 3.1 *Assume that Hypothesis 2.1 and (6) are satisfied. Then the explicit form of the value function is*

$$V(N_0, z_0) = \theta z_0^{1-\sigma} N_0^{\gamma} \tag{7}$$

where

$$\theta = \frac{1}{1-\sigma}\left(\frac{\rho - (1-\sigma)\eta}{\sigma}\right)^{-\sigma}. \tag{8}$$

Moreover V satisfies, on $\mathbb{R}_+ \times \mathbb{R}_+$, the following Hamilton-Jacobi-Bellman equation:

$$\rho v(N, z) - \sup_{\substack{c \geq 0 \\ n \in [m, M]}} \left(\frac{\partial v}{\partial N} nN + \frac{\partial v}{\partial z}[(A - \mu(n))z - c] + \frac{c^{1-\sigma} N^{\gamma}}{1-\sigma} \right) = 0. \tag{9}$$

It should be noted that population size enters the value function as along as the intertemporal altruism parameter γ is nonzero, which is an obvious property of our model. An important feature of the value function is its dependence on the position of σ with respect to unity, which is not inconsistent with the findings of Palivos and Yip (1993) within their particular mathematical setting. Indeed, one can observe that parameter θ has the same sign as $1 - \sigma$: therefore, it is straightforward, to observe that the value function is positive if and only if $\theta > 0$, that is if $\sigma < 1$. It should be also noted that whatever σ, the value function is always increasing in the stock of capital per capita. In contrast, the value function increases with population size at elasticity γ if and only if $\sigma < 1$. This observation allows to fix the finding of Palivos and Yip at first glance: if $\sigma > 1$, the Millian case ($\gamma = 0$) is dominated by the Benthamite case ($\gamma = 1$). Things are even clearer if we visualize the optimal control trajectories. This is done below after the following assumption.

Hypothesis 3.1 *The function $\omega(n)$*

$$\begin{cases} [m, M] \to \mathbb{R} \\ n \mapsto \frac{\gamma}{1-\sigma} n + A - \mu(n) \end{cases}$$

has a unique maximum point $\bar{n}$: $\eta = \omega(\bar{n})$.

[1]The theorem does not include the case $\sigma = 1$. To simplify the exposition, we exclude this special case, which can be also treated with the dynamic programming approach used in this paper but requires a different proof.

This assumption allows to assure that we have a unique optimal control. It rules out the case of complicated dilution functions yielding multiple equilibria. Under this assumption, it is easy to describe the optimal controls.

Theorem 3.2 *Assume that Hypothesis 2.1 and (6) are satisfied. Then there exists an optimal strategy for the optimal control problem described by (1), (2) and (3). It is given by*

$$\begin{cases} n^*(t) \equiv \bar{n}, & t \geq 0 \\ c^*(t) = c_0 e^{\left(\frac{-\rho+A-\mu(\bar{n})}{\sigma}+\frac{\gamma}{\sigma}\bar{n}\right)t} \end{cases} \tag{10}$$

where

$$c_0 = \frac{\rho - (1-\sigma)\eta}{\sigma} z_0 > 0 \tag{11}$$

(note that c_0 is positive thanks to (6)). The related optimal trajectory is

$$\begin{cases} N^*(t) = N_0 e^{\bar{n}t} \\ z^*(t) = z_0 e^{\left(\frac{-\rho+A-\mu(\bar{n})}{\sigma}+\frac{\gamma}{\sigma}\bar{n}\right)t}. \end{cases} \tag{12}$$

Moreover, if Hypothesis 3.1 is satisfied, then the described optimal strategy is unique.

The optimal trajectories feature the absence of transitional dynamics whatever function $\mu(n)$: the optimal demographic growth is constant over time, equal to the maximizer of function $\omega(n)$, and all the other variables are immediately on their balanced growth paths. In order to understand better the optimal control produced, some comparative statics are in order. Call

$$g_c := \left(\frac{-\rho + A - \mu(\bar{n})}{\sigma} + \frac{\gamma}{\sigma}\bar{n}\right),$$

the growth rate of the per-capita consumption. We have the following properties.

Proposition 3.1 *Under the assumptions of Theorem 3.2, we have:*

1. *$\bar{n}$ does not depend on ρ, g_c is decreasing in ρ, c_0 is increasing in ρ.*
2. *if $\sigma \in (0,1)$ then $\bar{n}$ is increasing in γ, if $\sigma > 1$ then $\bar{n}$ is decreasing in γ*

The proposition states some properties which are independent of the dilution function specification. The second property is highly interesting if one has in mind the literature on optimal population size. Clearly, this property is consistent with the main finding of Palivos and Yip: The Benthamite case yields a smaller population whenever $\sigma > 1$. That's to say our model, with a general dilution function and without intertemporal altruism yields the same results as those extracted by Palivos and Yip at least for the size of population. Our analysis re-emphasizes the crucial role of the intertemporal rate of substitution in the optimal population dynamic problems. As to living standards as captured by consumption in our model, and since growth is endogenous, one has to look primarily at the growth rates. We need to explicit the dilution functions to this end. This is done in the next section. Before, some definitions are needed. The first one describes an economy meeting the Parfit's *Repugnant Conclusion*.

Definition 3.1 *The economy meets the Parfit's* Repugnant Conclusion *if condition (6) is satisfied and* $\bar{n} > 0$ *with* $g_c < 0$.

The next definition will simplify the exposition.

Definition 3.2 *Consider some parameter of the economy, say* ψ *(for example* $\psi = m, M, \sigma, \gamma, ...$ *or equal to any parameter of the dilution function). We say that increasing (resp. decreasing)* ψ *facilitates the Parfit's* Repugnant Conclusion *if, everything else equal,* $\psi_1 < \psi_2$ *(resp.* $\psi_2 < \psi_1$*), and the economy meets the Parfit's* Repugnant Conclusion *for* ψ_1 *imply that it meets the conclusion for* ψ_2.

An immediate application of the previous definition is the following property.

Corollary 3.1 *Increasing* ρ *facilitates the Parfit's* Repugnant Conclusion.

This property is an immediate implication of Proposition 3.1: the larger the discount rate, the lower the growth rate of consumption while population size is unaffected. We now come to the analysis with explicit dilution functions.

4 Some particular cases

For illustration purposes, we start with the traditional linear dilution case. After, a non-monotonic case will be considered.

4.1 The linear case

Let us consider the linear case:

$$\mu(n) = \alpha n$$

for some positive constant α.[2]In that case we have two possibilities

$$\bar{n} = \arg \max_{n \in [m,M]} \left(\frac{\gamma}{1-\sigma} n + A - \mu(n) \right) = \begin{cases} M & \text{if } \alpha < \frac{\gamma}{1-\sigma} \\ m & \text{if } \alpha > \frac{\gamma}{1-\sigma}. \end{cases}$$

In both cases we have a corner solution. Notice that if $\sigma > 1$, we have always $\bar{n} = m < 0$, and the *Repugnant Conclusion* cannot hold. Instead suppose $\sigma < 1$ and $\alpha < \frac{\gamma}{1-\sigma}$. Then the optimal population (we avoid here the *) is:

$$N(t) = N_0 e^{Mt},$$

while the optimal per-capita consumption is

$$c(t) = c_0 e^{\frac{A-\alpha M-\rho+\gamma M}{\sigma} t}$$

where $c_0 = \left(\frac{\rho-(1-\sigma)\left(\frac{\gamma}{1-\sigma}M+(A-\alpha M)\right)}{\sigma} z_0 \right) > 0$. The positivity of c_0 comes from assumption (6) with $\eta = \omega(M)$. It follows that we have the Parfit's *Repugnant Conclusion* in this case, $\alpha < \frac{\gamma}{1-\sigma}$, if and only if $A+(\gamma-\alpha)M-\rho < 0$: in this case, we observe an optimally exponentially growing population growth together with an exponentially decreasing per-capita consumption. The following proposition clarifies the conditions under which the Parfit's *Repugnant Conclusion* does occur when $\alpha < \frac{\gamma}{1-\sigma}$.

Proposition 4.1 *Under the assumptions of Theorem 3.2, the Parfit's* Repugnant Conclusion *occurs when $\alpha < \frac{\gamma}{1-\sigma}$ and $A < \alpha M$. This conclusion can never happen if $\sigma > 1$.*

The proof is trivial. It is readily shown that (6) is equivalent to

$$\rho > A + (\gamma - \alpha)M - \sigma(A - \alpha M),$$

which implies

$$A + (\gamma - \alpha)M - \rho < 0,$$

under the sufficient condition $A < \alpha M$. First, notice that the *Repugnant Conclusion* only occurs for non-realistic values of the intertemporal elasticity of substition, that is when $\frac{1}{\sigma}$ is larger than unity. Therefore, one conclusion to draw is that the *Repugnant Conclusion* is a non-realistic event under linear dilution effects. We shall see if this finding is robust to nonlinearities in the dilution function in the next sub-section. Here, one can note that the

[2]Considering the more general parameterization, $\mu(n) = \alpha n + \beta$, with $\beta > 0$ measuring the non-demographic depreciation rate of capital, will not alter the main results listed below.

Repugnant Conclusion occurs when $\alpha < \frac{\gamma}{1-\sigma}$, so under $\sigma < 1$, and it is clearly facilitated when technological progress, A, goes down. Also, in the normalized case $\alpha = 1$ corresponding to the traditional dilution effects modelling, increasing M (for example through an exogenous drop in the mortality rate, d) facilitates Parfit's *Repugnant Conclusion*. The same property holds for γ: if $\sigma < 1$ and as long as $\gamma < \alpha(1-\sigma)$ holds , a larger parameter of intertemporal altruism drives down the growth rate of consumption per capita without affecting population size in our linear dilution case. How could nonlinearity in function $\mu(.)$ alter these properties? This is investigated just below.

4.2 The case $\mu(n) = n^2 + an + b$

Now suppose that

$$\mu(n) = n^2 + an + b$$

with $a \in \mathbb{R}$ and $b > 0$. Indeed, $b = \mu(0)$ measures non-demographic depreciation of capital. Notice also that if $a < 0$, $\mu(n)$ is decreasing for $n < \frac{a}{2}$, then increasing: as mentioned before, this features the realistic case where population growth is good for economic growth up to a certain threshold value of demographic growth. We shall show that in this case, one may have the Parfit's *Repugnant Conclusion* with realistic values of σ, that's for $\sigma > 1$. In this situation we have

$$\bar{n} = \frac{1}{2}\left(\frac{\gamma}{1-\sigma} - a\right) \vee m \wedge M. \tag{13}$$

So $\bar{n}$ is greater than zero if and only if

$$a < \frac{\gamma}{1-\sigma}. \tag{14}$$

Notice that if $\sigma > 1$, we have $a < 0$ and the shape between economic growth and demographic growth is an inverted U-shaped cuerve as in Boucekkine et al. (2002). Now note that as long as $a \in \left(\frac{\gamma}{1-\sigma} - 2M, \frac{\gamma}{1-\sigma} - 2m\right)$ the maximum is in the interior of (m, M) and $\bar{n} = \frac{1}{2}\left(\frac{\gamma}{1-\sigma} - a\right)$. The related per-capita consumption growth rate is (after some algebra)

$$g_c = \frac{1}{\sigma}(A - \rho - b) + \left(\frac{1}{1-\sigma}\right)^2 \frac{1}{4\sigma}\left[\gamma^2(1-2\sigma) + a^2(1-\sigma)^2 - 2\gamma a(1-\sigma)^2\right],$$

and, when $\bar{n} = M$,

$$g_c = \frac{1}{\sigma}(A - \rho - b) + \frac{1}{\sigma}(-M^2 + (\gamma - a)M).$$

This allows us to state the following proposition:

Proposition 4.2 *Assume $\sigma > 1$, and suppose assumption (14) is checked. Then the Parfit's* Repugnant Conclusion *holds for ρ and/or b large enough. Moreover we have the following properties: $\frac{\partial g_c}{\partial b} < 0$, $\frac{\partial g_c}{\partial A} > 0$, $\frac{\partial g_c}{\partial \rho} < 0$. So, since $\bar{n}$ does note depend on the choice of these three parameters increasing b or decreasing A or increasing ρ facilitates the Parfit's Repugnant Conclusion. Finally whenever $a \in \left(\frac{\gamma}{1-\sigma} - 2M, \frac{\gamma}{1-\sigma} - 2m\right)$, we have that, if $\sigma > 1$, in the region $a < \frac{2\gamma}{1-\sigma}$, $\frac{\partial g_c}{\partial \gamma} > 0$ so decreasing γ facilitates the Parfit's* Repugnant Conclusion.

In contrast to the traditional linear dilution model, we have the Parfit *Repugnant Conclusion* realized even for realistic values of $\sigma > 1$. In particular, if we get this occurrence when population growth is linked to economic growth via an inverted U-shaped relationship, which is also a quite realistic fact. Also notice that just like in the previous sub-section, decreasing technical progress through A facilitates the *Repugnant Conclusion.* Interestingly enough, we find that when $\sigma > 1$, decreasing the intertemporal altruism parameter, γ, does favor the *Conclusion*, which pushes the argument of Palivos and Yip (1993), further: moving from the Benthamite to the Millian social welfare function may not cause optimal population size to go up and consumption to go down, it will also favor the realization of the *Repugnant Conclusion.*

5 Conclusion

In this paper, we have evaluated the so-called Parfit's Repugnant Conclusion in a canonical AK model with endogenous fertility and a reduced form relationship between demographic growth and economic growth. While in the traditional linear dilution model, the Parfit *Repugnant Conclusion* can never occur for realistic values of intertemporal substitution, it does occur when the relationship between demographic growth and economic growth is nonlinear. In particular, we get this occurrence when population growth is linked to economic growth via an inverted U-shaped relationship. Finally, we find that when $\sigma > 1$, decreasing the intertemporal altruism parameter, γ, does favor the *Conclusion*, which pushes the argument of Palivos and Yip (1993), further: moving from the Benthamite to the Millian social welfare function may not only cause optimal population size to go up and consumption to go down, it will also favor the realization of the *Repugnant Conclusion.*

References

K. Arrow, P. Dasgupta, L. Goulder, G. Daily et al. , "Are we consuming too much?", *Journal of Economic Perspectives* 18, 147-172, 2004.

K. Arrow, A. Bensoussan, Q. Feng and S. Sethi, "The genuine savings criterion and the value of population in an economy with endogenous fertility rate" in R. Boucekkine, N. Hritonenko and Y. Yatsenko (eds.), *Optimal Control of Age-structured Populations in Economy, Demography, and the Environment*, Taylor and Francis, 2010.

M. Bardi, and I. Capuzzo Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Springer, Berlin, 1997.

D. Blanchet, "Age structure and capital dilution effects in neo-classical growth models", *Journal of Population Economics* 1, 183-194, 1988.

R. Boucekkine, D. de la Croix and O. Licandro, "Human vintage capital, demographic trends and endogenous growth", *Journal of Economic Theory* 104, 340-375, 2002.

F. Edgeworth. *Papers relating to political economy.* Vol. III, Macmillan, London, 1925.

A. Kelley and R. M. Schmidt, "Aggregate Population and Economic Growth Correlations: The Role of the Components of Demographic Changes", *Demography* 32, 543-555, 1995.

A. Kelley, "Economic consequences of population change in the third world", *Journal of Economic Literature* 26, 1685-1728, 1988

D. Parfit. *Reasons and persons.* Oxford University Press, Oxford, 1984.

M. Nerlove, A. Razin, and E. Sadka, "Population size: individual choice and social optima", *Quarterly Journal of Economics* 100, 321-334, 1985.

T. Palivos and Ch. K. Yip, "Optimal population size and endogenous growth", *Economics Letters* 41, 107-110, 1993.

A Proofs

Proof of Proposition 2.1. Here we prove only the sufficiency part. The necessity is a corollary of Theorem 3.1 and it will be proved in Proposition A.1.

Given $(N_0, z_0) \in \mathbb{R}^2_+$ and $(c(\cdot), n(\cdot)) \in \mathcal{U}_{N_0,z_0}$ we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(z(t)N^{\frac{\gamma}{1-\sigma}}(t)\right) &= [(A-\mu(n(t)))z(t)-c(t)]\,N^{\frac{\gamma}{1-\sigma}}(t) + \frac{\gamma}{1-\sigma}n(t)z(t)N^{\frac{\gamma}{1-\sigma}}(t)\\ &= \left((A-\mu(n(t))) + \frac{\gamma}{1-\sigma}n(t)\right)z(t)N^{\frac{\gamma}{1-\sigma}}(t) - c(t)N^{\frac{\gamma}{1-\sigma}}(t) \quad (15)\end{aligned}$$

thanks to the definition of η in (5) the expression above is

$$\leq \eta z(t)N^{\frac{\gamma}{1-\sigma}}(t) - c(t)N^{\frac{\gamma}{1-\sigma}}(t).$$

So

$$z(t)N^{\frac{\gamma}{1-\sigma}}(t) \leq z_0 N_0^{\frac{\gamma}{1-\sigma}} e^{t\eta} - \int_0^t e^{\eta(t-s)} c(s) N^{\frac{\gamma}{1-\sigma}}(s)\,\mathrm{d}s.$$

Thus

$$\int_0^t e^{-s\eta} c(s) N^{\frac{\gamma}{1-\sigma}}(s)\,\mathrm{d}s \leq z_0 N_0^{\frac{\gamma}{1-\sigma}} - z(t)N(t)^{\frac{\gamma}{1-\sigma}} e^{-t\eta} \leq z_0 N_0^{\frac{\gamma}{1-\sigma}}$$

and passing to the limit in $t \to \infty$:

$$\int_0^{+\infty} e^{-s\eta} c(s) N^{\frac{\gamma}{1-\sigma}}(s)\,\mathrm{d}s \leq z_0 N_0^{\frac{\gamma}{1-\sigma}}. \quad (16)$$

Now, thank to (6) we have that $\rho > (1-\sigma)\eta$ so we deduce that

$$\begin{aligned}&\int_0^{+\infty} e^{-\rho t}\left|\frac{c^{1-\sigma}(t)}{1-\sigma}\right| N^{\gamma}(t)\,\mathrm{d}t\\ &\quad= \left|\frac{1}{1-\sigma}\right| \int_0^{+\infty} e^{-(\rho-(1-\sigma)\eta)t}\left(e^{-\eta t}c(t)N^{\frac{\gamma}{1-\sigma}}(t)\right)^{1-\sigma}\mathrm{d}t\\ &\quad\leq \left|\frac{1}{1-\sigma}\right| \int_0^{+\infty} e^{-(\rho-(1-\sigma)\eta)t}\left(1+e^{-\eta t}c(t)N^{\frac{\gamma}{1-\sigma}}(t)\right)^{1-\sigma}\mathrm{d}t\\ &\quad\leq \left|\frac{1}{1-\sigma}\right| \int_0^{+\infty} e^{-(\rho-(1-\sigma)\eta)t}\left(1+e^{-\eta t}c(t)N^{\frac{\gamma}{1-\sigma}}(t)\right)\mathrm{d}t\\ &\quad\leq \left|\frac{1}{1-\sigma}\right|\left(\frac{1}{\rho-(1-\sigma)\eta} + z_0 N_0^{\frac{\gamma}{1-\sigma}}\right) \quad (17)\end{aligned}$$

where in the last step we used (16). Since the estimate does not depend on the chosen control $(c(\cdot), n(\cdot)) \in \mathcal{U}_{N_0,z_0}$ the same bound holds for the value function:

$$V(N_0, z_0) \leq \left|\frac{1}{1-\sigma}\right|\left(\frac{1}{\rho-(1-\sigma)\eta} + z_0 N_0^{\frac{\gamma}{1-\sigma}}\right) < +\infty$$

and this conclude the proof. □

Proof of Theorem 3.1. To prove that the function V defined in (7) is a solution of (9) we need only to verify directly: computing expression in the left side of (9) for V defined in (7) we have:

$$\rho\theta z^{1-\sigma}N^{\gamma} - \sup_{\substack{c\geq 0\\ n\in[m,M]}} \Bigg(\theta\gamma n N^{\gamma} z^{1-\sigma} + (1-\sigma)\theta[(A-\mu(n))N^{\gamma}z^{1-\sigma} - cN^{\gamma}z^{-\sigma}] + \frac{c^{1-\sigma}N^{\gamma}}{1-\sigma}\Bigg). \tag{18}$$

Thanks to Hypothesis 2.1 the supremum in the equation above is a maximum and it is attained at [note that θ has the same sign of $(1-\sigma)$] $n \in \arg\max_{n\in[m,M]} \left(\frac{\gamma}{1-\sigma}n + (A-\mu(n))\right)$ and $c = ((1-\sigma)\theta)^{-1/\sigma}z$, so the expression above is equal to

$$\rho\theta z^{1-\sigma}N^{\gamma} - (1-\sigma)\eta\theta z^{1-\sigma}N^{\gamma} + \left(1-\frac{1}{1-\sigma}\right)((1-\sigma)\theta)^{1-1/\sigma}z^{1-\sigma}N^{\gamma}$$
$$= \theta z^{1-\sigma}N^{\gamma}\left(\rho - (1-\sigma)\eta - \sigma((1-\sigma)\theta)^{-1/\sigma}\right)$$

that is zero thanks to the definition of θ given in (8).

From the general theory we know (see for example Bardi and Capuzzo Dolcetta (1997)) that the value function is the unique solution of the Hamilton-Jacobi-Belmann equation related to the optimal control problem. Since we have explicitly found a solution it has to be the value function.[3] □

Lemma A.1 *Assume that (6) is satisfied and consider* $(N_0, z_0) \in \mathbb{R}^2_+$. *Givenn* $(\hat{c}(\cdot), \hat{n}(\cdot)) \in \mathcal{U}_{N_0,z_0}$ *and the related trajectory* $(\hat{N}(\cdot), \hat{z}(\cdot))$ *we have that*

$$\limsup_{t\to\infty} e^{-\rho t}V(\hat{N}(t), \hat{Z}(t)) \leq 0. \tag{19}$$

Proof. Let us compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\theta\hat{z}^{1-\sigma}(t)\hat{N}^{\gamma}(t)\right) = (1-\sigma)\theta(A-\mu(\hat{n}(t)))\hat{z}^{1-\sigma}(t)\hat{N}^{\gamma}(t) - (1-\sigma)\theta\hat{z}^{-\sigma}(t)\hat{N}^{\gamma}(t)\hat{c}(t) + \gamma\theta\hat{n}(t)\hat{z}^{1-\sigma}(t)\hat{N}^{\gamma}(t)$$

The expression above is

$$\leq (1-\sigma)\theta\eta\hat{z}^{1-\sigma}(t)\hat{N}^{\gamma}(t)$$

and then $\theta\hat{z}^{1-\sigma}(t)\hat{N}^{\gamma}(t) \leq \theta z_0^{1-\sigma}N_0^{\gamma}e^{(1-\sigma)\eta t}$. Eventually

$$e^{-\rho t}V(\hat{N}(t), \hat{z}(t)) = \theta e^{-\rho t}\hat{z}^{1-\sigma}(t)\hat{N}^{\gamma}(t) \leq \theta e^{-\rho t}z_0^{1-\sigma}N_0^{\gamma}e^{(1-\sigma)\eta t}$$

and we conclude, passing to the limit in $t\to\infty$, thanks to (6). □

[3]Actually in our case we will not need to use the general theory but we will see directly that the solution we have found is the value function, see on this Remark A.1 at page 18.

Proof of Theorem 3.2. As usual in the dynamic programming approach we use the solution of the Hamilton-Jacobi-Belmann equation (9) found in Theorem 3.1 to give a candidate-optimal solution in feedback form (i.e. to have the control (n, c) as function of the state (N, z)):

$$\begin{cases} F\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \times \mathbb{R}_+ \\ F\colon (N, z) \mapsto \arg\max_{\substack{n\in[m,M] \\ c\geq 0}} \Big(\frac{\partial V}{\partial N} nN \\ \qquad\qquad + \frac{\partial V}{\partial z}[(A-\mu(n))z - c] + \frac{c^{1-\sigma}N^{\gamma}}{1-\sigma}\Big) \end{cases} \tag{20}$$

The point of maximum for c is unique and it is $((1-\sigma)\theta)^{-1/\sigma}z$ so the explicit form for F when we use V as defined in (7) is

$$(n^*, c^*) = F(N, z) := (\bar{n}, ((1-\sigma)\theta)^{-1/\sigma} z). \tag{21}$$

The candidate-optimal trajectory is then found using the feedback control F control in the state equation:

$$\begin{cases} \dot{N}^*(t) = \bar{n}N^*(t), \quad N^*(0) = N_0 \\ \dot{z}^*(t) = (A-\mu(\bar{n}))z^*(t) - \left[((1-\sigma)\theta)^{-1/\sigma} z^*(t)\right], \quad z^*(0) = z_0 \end{cases}$$

that gives $\begin{cases} N^*(t) = N_0 e^{\bar{n}t} \\ z^*(t) = z_0 e^{\left(A-\mu(\bar{n}) - ((1-\sigma)\theta)^{-1/\sigma}\right)t} = z_0 e^{\left(\frac{-\rho+A-\mu(\bar{n})}{\sigma} + \frac{\gamma}{\sigma}\bar{n}\right)t} \end{cases}$, that are exactly the expressions appearing in (12), so using again the function F defined in (21) we can eventually find the explicit solution of the candidate-optimal control:

$$\begin{aligned} (u^*(t), c^*(t)) &= F(N^*(t), z^*(t)) \\ &= \left(\bar{n}, ((1-\sigma)\theta)^{-1/\sigma} z_0 e^{\left(\frac{-\rho+A-\mu(\bar{n})}{\sigma} + \frac{\gamma}{\sigma}\bar{n}\right)t}\right) \\ &= \left(\bar{n}, \frac{\rho-(1-\sigma)\eta}{\sigma} z_0 e^{\left(\frac{-\rho+A-\mu(\bar{n})}{\sigma} + \frac{\gamma}{\sigma}\bar{n}\right)t}\right) \end{aligned} \tag{22}$$

that are exactly the expressions of the control given in (10)-(11).

Now we have to prove that the candidate-optimal controls are in fact optimal controls. Let us introduce the function

$$\begin{cases} W\colon \mathbb{R}^3_+ \to \mathbb{R}_+ \\ W(t, N, z) := e^{-\rho t} V(N, z). \end{cases}$$

As we look for optimal trajectories we can restrict our analysis to the trajectories for which the limsup appearing in the claim of Lemma A.1 is proper limit and is equal to 0. We have the following: considered $(N_0, z_0) \in \mathbb{R}^2_+$ and given $(\hat{c}(\cdot), \hat{n}(\cdot)) \in$

$\mathcal{U}_{N_0,z_0}$ with the related trajectory $(\hat{N}(\cdot), \hat{z}(\cdot))$, we have[4]

$$V(N_0, z_0) = W(0, N_0, z_0) = -\lim_{T\to\infty} \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} W(t, \hat{N}(t), \hat{z}(t))\, \mathrm{d}t.$$

So, using the expression above and computing explicitly the derivative inside the integral, we have

$$\begin{aligned} T(\hat{c}(\cdot), \hat{n}(\cdot)) &:= V(N_0, z_0) - \int_0^{+\infty} e^{-\rho t} \frac{\hat{c}^{1-\sigma}(t)}{1-\sigma} N^\gamma(t) dt \\ &= \lim_{T\to\infty} \int_0^T \rho e^{-\rho t} V(\hat{N}(t), \hat{z}(t)) - e^{-\rho t} \frac{\partial V}{\partial N}(\hat{N}(t), \hat{z}(t)) \hat{n}(t) \hat{N}(t) \\ &\qquad - e^{-\rho t} \frac{\partial V}{\partial z}(\hat{N}(t), \hat{z}(t)) \Big((A - \mu(\hat{n}(t))) \hat{z}(t) - \hat{c}(t) \Big) dt \\ &\qquad\qquad - \int_0^{+\infty} e^{-\rho t} \frac{\hat{c}^{1-\sigma}(t)}{1-\sigma} \hat{N}^\gamma(t) dt. \end{aligned} \quad (23)$$

Since V is a solution of (9) the expression above is equal to (we write only $\frac{\partial V}{\partial z}$ instead of $\frac{\partial V}{\partial z}(\hat{N}(t), \hat{z}(t))$)

$$\begin{aligned} \limsup_{T\to\infty} \int_0^T e^{-\rho t} \Bigg(\sup_{\substack{c\geq 0 \\ n\in[m,M]}} \left[\frac{\partial V}{\partial N} n \hat{N}(t) + \frac{\partial V}{\partial z} [(A - \mu(n)) \hat{z}(t) - c] + \frac{c^{1-\sigma} \hat{N}^\gamma(t)}{1-\sigma} \right] \\ - \left[\frac{\partial V}{\partial N} \hat{n}(t) \hat{N}(t) + \frac{\partial V}{\partial z} [(A - \mu(\hat{n}(t))) \hat{z}(t) - \hat{c}(t)] + \frac{\hat{c}^{1-\sigma}(t)}{1-\sigma} \hat{N}^\gamma(t) \right] \Bigg) dt. \end{aligned} \quad (24)$$

This last expression allows us to conclude the proof. Indeed first we note that, from the definition of $T(\hat{c}(\cdot), \hat{n}(\cdot))$, finding a minimizer $(\hat{c}(\cdot), \hat{n}(\cdot))$ for T is equivalent to find a maximizer for our optimal control problem. Then we observe (it is clear from last expression) that $T(\hat{c}(\cdot), \hat{n}(\cdot)) \geq 0$. Third we check that along the candidate-optimal solution $(c^*(\cdot), n^*(\cdot))$ we have $T(c^*(\cdot), n^*(\cdot)) = 0$ (it follows immediately from the technique we used to find $(c^*(\cdot), n^*(\cdot))$, indeed they we first introduced them as argmax of the supremum appearing in the equation above, see (20) and (21)). So we can conclude that the candidate-optimal solution is indeed optimal.

Observing the last form of T and using Hypothesis 3.1 we have the uniqueness of the optimal trajectory too: a trajectory $(\hat{c}(\cdot), \hat{n}(\cdot))$ is optimal if and only if $T(\hat{c(\cdot)}, \hat{n}(\cdot)) = 0$, since the integrand is always ≥ 0 we have $T(\hat{c}(\cdot), \hat{n}(\cdot)) = 0$ if and only if the integrand is (almost) everywhere equal to zero i.e. $(\hat{c}(t), \hat{n}(t)) = F(\hat{N}(t), \hat{z}(t))$. Then we need $(\hat{c}(\cdot), \hat{n}(\cdot)) = (c^*(\cdot), n^*(\cdot))$. This concludes the proof. □

[4]Since, given $T > 0$, $W(0, N_0, z_0) - W(T, \hat{N}(T), \hat{z}(T)) = -\int_0^T \frac{\mathrm{d}}{\mathrm{d}t} W(t, \hat{N}(t), \hat{z}(t))\, \mathrm{d}t$ and Lemma A.1, along the considered trajectory, ensures that $\lim_{T\to\infty} W(T, \hat{N}(T), \hat{z}(T)) = 0$.

Remark A.1 Note that we have never really used that V defined in (7) is the valued function in our proofs. Now, as a corollary of the last proof we can see this fact directly, without referring to the general theory. It follows easily from the definition of T: if $(c^*(\cdot), n^*(\cdot))$ is optimal (and we proved it is) and $T(c^*(\cdot), n^*(\cdot)) = 0$ then $V(N_0, z_0) = \int_0^{+\infty} e^{-\rho t} \frac{c^*(t)^{1-\sigma}}{1-\sigma} (N^*(t))^\gamma \, dt$ and then V is the value function.

Proposition A.1 *Assume that Hypothesis 2.1 is satisfied. If (6) is not satisfied and* $\sigma \in (0, 1)$ *then* $V(N_0, z_0) = +\infty$ *for all* $(N_0, z_0) \in \mathbb{R}^2_+$.

Proof. We have only to verify that using $c^*(\cdot)$ and $N^*(\cdot)$ (that are not optimal in that case but are still admissible) we have $J_{N_0,z_0}(c^*(\cdot), n^*(\cdot)) = +\infty$. It is an easy computation. $\square$

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