Search Frictions on Product and Labor markets : Money in the Matching Function

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Search Frictions on Product and Labor markets: Money in the Matching Function[*]

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Abstract

This paper builds a macroeconomic model of equilibrium unemployment in which firms persistently face difficulties in selling their production and this affects their decisions to create jobs. Due to search-frictions on the product market, equilibrium unemployment is a U-shaped function of the ratio of total demand to total supply on this market. When prices are at their Competitive Search Equilibrium values, the unemployment rate is minimized. Yet, the Competitive Search Equilibrium is not efficient. Inflation is detrimental to unemployment.

Keywords: Equilibrium unemployment, Matching, Inflation, Demand Constraints.

JEL Code: E12, E24, E31, J63.

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I Introduction

In the labor-matching model with wage bargaining (Mortensen and Pissarides (1999) and Pissarides (2000), henceforth MP), produced goods are instantaneously sold and consumed. In reality, firms need time to sell their production and consumers spend time looking for the goods they like. In the US, on the basis of the data provided by Kahn and McConnell (2002), it takes nowadays on average about 5 months to sell inventories. This paper builds a macroeconomic model where interactions between frictions on the product and the labor markets can be easily analyzed.

We consider an economy where the act of purchase is separated from the act of sale (no double coincidence of wants). With anonymous agents, money is needed for transactions (see e.g. Kocherlakota 1998). Most research in monetary macroeconomics uses shortcuts to introduce money in the models. The most common ones are the Clower's Cash-In-Advance (CIA) constraint and Sidrauski's Money In the Utility function assumption (MIU) ; see Walsh (1998). The MIU specification is justified by the idea that money makes transaction easier and as such generates utility. Shopping-time models go a step further in this direction by making the technology of trade explicit. We add congestion externalities on the product market. Following the MP research where the introduction of a matching function on the labor market is very convenient, we introduce a similar matching function on the product market[1]. Given the assumptions made about the goods and the preferences, we show that the arguments of this matching function are the real money stock and the aggregate stock of inventories.

The usual explanation for the demand constraints on the product market is the Keynesian assumption of nominal rigidities. However, the New Keynesian micro-foundations for these rigidities are not very convincing in our long run perspective. We build our flexible price model step by step. For pedagogical reason, we start with a fixed price model and then turn to a model with endogenous price on the product market. Throughout the paper, wage bargaining is assumed on the labor market.

The equilibrium unemployment rate is characterized as in the MP model with a single change: The productivity level is multiplied by an inversely U-shaped function of tightness on the product market (i.e. the real money stock divided by the level of inventories). The unemployment rate is then a U-shaped function of tightness on the product market. Two opposite effects are at work. When the product market becomes more tight, firms sell their inventories more rapidly and this stimulates the creation of vacancies. However, entrepreneurs are also consumers. When the product market becomes more tight, the

[1]In Wasmer and Weil (2004) the modeling of frictions in the credit market is inspired by the modeling approach on the labor market.

money made by the sale of their production is less rapidly transformed in a purchase. As only consumption affects the utility level of agents, this effect is detrimental to the opening of vacancies. When tightness is sufficiently low on the product market, we show that the first effect dominates. The opposite is true above a certain threshold. This explains the U-shaped relationship between unemployment and tightness on the product market.

To make prices endogenous, we assume a Competitive Search mechanism à la Moen (1997). Sellers post take-it-or-leave-it prices. This information is common knowledge. Consumers direct their search according to the observed price offers. Once agents have selected a submarket characterized by a specific price, they still have to search for goods they like. At our unique symmetric equilibrium, tightness on the product market maximizes the value of inventories and minimizes the equilibrium unemployment rate.

We then introduce money growth. Money is neutral in this economy but not super neutral. Inflation introduces a depreciation process of money holdings. It affects the creation of vacancies since this depreciation reduces the purchasing power of money held by entrepreneurs in the interim between sale and purchase. We show that the level of equilibrium unemployment is an increasing function of the inflation rate.

From a social welfare viewpoint, what only matters is the pace at which a unit of produced good is consumed by an agent, whoever (s)he is. We show that the equilibrium value of tightness is inefficiently low on the product market. For this reason, the unemployment rate is inefficiently high.

Other papers have already stressed that firms face difficulties in selling their output because of search frictions on the product market. The seminal paper being Diamond (1982) considers a barter economy. Because the matching technology implicitly exhibits increasing returns to scale, there is a multiplicity of equilibria. Howitt (1985) replicates the argument of Diamond in a model that distinguishes labor and product markets. Diamond (1984) extends Diamond (1982) to a monetary economy.

Kiyotaki and Wright (1993) explain how the existence of frictions on the product market gives a convincing micro-foundation for the use of money as a medium of exchanges. Money has an essential role to play because of three things. 1) A double coincidence problem arises in an environment where trade is decentralized and time-consuming; 2) There is a lack of commitment and 3) there is private information concerning trading histories (Kocherlakota 1998). However, in these papers, money is indivisible and agents either own zero or one unit of money.

Monetary policy issues require the introduction of divisible money, which leads to analytical problems. The reason is the key role that is typically played by distribution of money holdings. Therefore, these models can only be solved numerically (see Molico

2006) or some simplifying assumptions are needed. Shi (1997, 1998) considers a model where a large family is composed of different members. Each of them produce, consume and trade in decentralized frictional markets. But at the end of each period, money is redistributed equally across household members. Lagos and Wright (2005) propose an alternative. Agents act individually but in addition to presence of frictional markets, there exists a frictionless market. Moreover, individuals have quasi-linear preferences over the good exchanged in the latter market. Therefore, they leave this market with the same amount of money. These alternatives enable to reduce the distribution of money holding to a mass point.

In Kiyotaki and Wright (1993), Shi (1997), Lagos and Wright (2005) and Molico (2006) among others, the probabilities to find a trading partner are exogenous. In Shi (1998) and Berentsen Rocheteau and Shi (2007), buyers and sellers can increase these probabilities by exerting a costly search effort. In Rocheteau and Wright (2005), these probabilities are functions of the relative numbers of buyers and sellers. An alternative is proposed by Diamond and Yellin (1990) where firms supply all their inventories. In this model, each worker can only search for a single unit of good at a time and only if she owns enough money. Therefore, as in Molico (2006) the distribution of money holdings is key to determine the equilibrium. Conversely, in our model, any consumer can costlessly and simultaneously demand several goods with an upper limit, namely her real money holdings. Therefore, a rise in individual money holdings raises linearly the flow of purchased goods. This linearity makes the distribution of money holdings useless to define the equilibrium.

Our conclusion that inflation increases equilibrium unemployment is finally related to Cooley and Hansen (1989) and Cooley and Quadrini (1999, 2004). Cooley and Hansen provide some empirical support to this conclusion. However, non-employment is driven by labor supply decisions in their theoretical model. Conversely, Cooley and Quadrini (1999, 2004) introduce search-matching frictions on the labor market. However, in these three papers, money is introduced thanks to Cash-In-Advance assumptions. Bertensen, Menzio and Wright (2006) and Lehmann (2006) investigate the superneutrality of money on inflation in models with search unemployment and micro-founded use of money along the search paradigm. They too find that higher inflation increases unemployment.

The paper is organized as follows. We introduce the environment in the next section. The fixed price equilibrium is described in Section III. Assuming exogenous price is not a very relevant assumption for a steady-state equilibrium. We made it essentially for pedagogical reasons in order to build a more general model step by step. Endogenous pricing is introduced in section IV, through price posting. This model is further extended in Section V, where money grows at a constant rate to investigate the superneutrality of money. Section VI challenges the efficiency of the equilibrium, and the last section concludes.

II Environment

Time is discrete. All agents are risk neutral, live forever and have a common discount rate $r > 0$. Only steady states are considered. Money is storable, divisible and is the unique medium of exchange. There is no credit market. Trades in the labor and in the product markets are uncoordinated and time-consuming. We assume two types of agents: entrepreneurs and workers. There is a continuum of both types.

Workers are either employed or unemployed and their total measure is normalized to 1. While employed, their earnings allow them to accumulate money holdings. Whether employed or unemployed, they use their money holdings to express a demand for produced goods.

Entrepreneurs are simultaneously employers, sellers and buyers. Entrepreneurs post an endogenous number of vacancies. Posting a vacancy entails a flow disutility cost. An entrepreneur recruit only unemployed individuals [2]. A job is created when a vacancy matches an unemployed worker and the Nash bargain over the wage leads to an agreement. Produced goods are indivisible, storable and heterogeneous. Each good is unique and cannot be reproduced. Put differently, a given type of good can only be produced once and in quantity one. During a period, a filled job produces a (large) number q of different types of goods. These produced goods are added to the firms' inventories and supplied on the product market.

Individuals, whether they are workers or entrepreneurs, derive utility whenever they find a good they are looking for. In particular, neither an entrepreneur nor an employed worker is interested in consuming the goods she has produced. Therefore, each entrepreneur has to supply her inventories on the product market, waiting for a buyer that will be interested in one of the goods she is supplying. There is a double-coincidence problem: When an entrepreneur finds a good to consume, the producer of this good is not interested in consuming what the former entrepreneur produces. Trade is anonymous and there is no record keeping technology. Therefore, money is used as a medium of exchange (see Kocherlakota 1998). In case of a meeting, the money made by the sale is used for two purposes: to pay the worker and to express a demand on the product market.

[2] We rule out the possibility that an entrepreneur becomes self-employed or recuit another entrepreneur. We also rule out on-the-job search.

III The Fixed Price model

In this section, the prices are fixed on the product market while wages are bargained over. A given period is divided between a labor market (production) sub-period and a product market (consumption) sub-period. During the labor market sub-period, Firms decide whether or not they open vacancies. The aggregate number of vacancies at that moment is[3] υ_{-1}. Job destruction and Job creation take place simultaneously. The wage W is then bargained over. The wage is actually a kind of piece rate in the sense that it is proportional to the (random) quantity of goods sold in step 2c. Since workers and entrepreneurs are risk-neutral, they bargain over the expected (monetary) wage $\mathbb{E}(W)$. In stage 1d each filled vacancy produces q units of goods. This ends the labor market subperiod. The product market sub-period starts with entrepreneurs' supply decisions. Then, workers and entrepreneurs express their demand decisions. Trade takes place on the product market. Finally, entrepreneurs use part of the money made by the sales to pay their employees (see Figure 1).

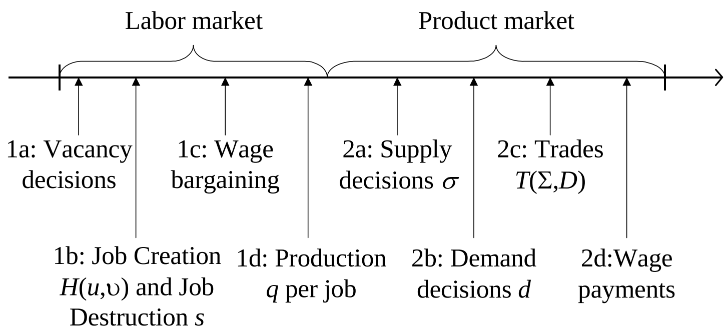

Figure 1: The timing of events in the Fixed Price Model

This section first describes the functioning of the labor (III.1) and of the product (III.2) markets. The emphasis is on the way we model frictions. Then we turn to the behavior of workers (III.3) and the one of entrepreneurs (III.4). We then characterizes the labor market equilibrium (III.5) and finally the fixed price equilibrium (III.6).

III.1 Job Creation and Job destruction

We start by describing the job creation and job destruction processes in the labor market. Firms open υ_{-1} vacancies during stage 1a. At the beginning of stage 1b, there are u_{-1} unemployed workers and $1 - u_{-1}$ employed workers. We define tightness on the labor

[3] In all the paper, values at the past (next) period are indexed -1 $(+1)$.

market as the ratio $\theta_{-1} = v_{-1}/u_{-1}$. To form a job, an unemployed worker and a job have to meet. Following the literature, we assume that this meeting process is imperfect. Let $H(u, v)$ be the number of meetings between u unemployed workers and v vacancies. The following assumption is standard in the labor-matching literature.

Assumption AS 1 *The (labor market) matching function is assumed to be continuously differentiable, increasing and concave in both arguments, to yield constant returns to scale and to verify the following boundary properties for any (u, v):*

$$H(u,0) = H(0,v) = 0 \qquad \lim_{v \mapsto +\infty} H(u,v) = u \qquad \textit{and} \qquad \lim_{u \mapsto +\infty} H(u,v) = v$$

The monotonicity and boundary conditions imply that for any (u, v), $H(u, v) \leq \min(u, v)$. Define the labor market matching elasticity as $\eta(\theta) = u \cdot H'_u/H(u, v)$. Each vacant job meets an unemployed worker with probability

$$h(\theta) \equiv H\left(\frac{1}{\theta}, 1\right) = \frac{H(u,v)}{v}$$

This *job filling* probability $h(.)$ is a decreasing function of θ. It varies from 1 to 0 when θ increases from 0 to $+\infty$. Let $-\eta(\theta) = \theta h'(\theta)/h(\theta)$ be its elasticity. Symmetrically, each unemployed worker finds a job with probability

$$\theta h(\theta) \equiv H(1, \theta) = \frac{H(u,v)}{u} = \frac{v}{u} \cdot \frac{H(u,v)}{v}$$

This *job finding* probability is an increasing function of θ and varies from 0 to 1 when θ increases from 0 to $+\infty$. Its elasticity equals $1 - \eta(\theta)$. Hence, $\eta(\theta) \in (0, 1)$.

We assume that each job ends with an exogenous probability $s \in (0, 1)$. Hence, in each period, $s(1 - u)$ jobs are dissolved and $H(u, v) = u \cdot \theta h(\theta)$ jobs are created. Unemployment therefore evolves according to:

$$u = u_{-1} + s(1 - u_{-1}) - H(u_{-1}, v_{-1}) = s(1 - u_{-1}) + (1 - \theta_{-1} h(\theta_{-1})) \cdot u_{-1}$$

In steady state, the labor market flow equilibrium implies:

$$u = \frac{s}{s + \theta h(\theta)} \tag{1}$$

So, characterizing θ is sufficient to obtain the steady-state value of unemployment. The tighter the labor market, the lower the unemployment rate.

III.2 The working of the product market

Trade in the product market requires a meeting between a consumer and an entrepreneur (a seller). With a certain probability, this meeting leads to trade, that is to the exchange of

p units of money against a single unit of good. For the consumer, this exchange generates a utility level normalized to unity for the consumer at the end of the period.

Consider an entrepreneur who owns ζ units of inventories at Stage 2a of a given period. We assume that an entrepreneur can simultaneously supply many different goods. Let σ be the number of goods she supplies. An entrepreneur is not allowed to supply goods that she has not yet produced. We therefore impose:

$$0 \leq \sigma \leq \zeta \tag{2}$$

We treat consumers in a symmetric way. Consider a consumer with money holdings $m \geq 0$ at Stage 2b of a given period. This consumer can be an entrepreneur, an employed worker or an unemployed one. We assume that any consumer can simultaneously search for different types of goods. Let $d \geq 0$ be the number of goods she is currently searching for or equivalently the number of units of demand she expresses. Expressing a unit of demand means searching for a certain type of good. Under our matching technology on the product market (to be described), there is always a positive, if small, probability that all units of demand will be satisfied. Furthermore, we assume that when a consumer finds the good she is looking for, she cannot default on paying. Hence, by demanding d units of goods, a consumer must be able to spend p times d units of money. In the absence of a credit technology, the demand expressed by a consumer has to satisfy:

$$0 \leq p \cdot d \leq m \tag{3}$$

Modelling the meeting process between demand for and supply of heterogeneous goods is beyond the scope of this paper. Our objective is instead to build an analytically tractable model that incorporates trading frictions in both markets. As each good is assumed to be unique, product market frictions do not qualitatively differ from those observed on the labor market. We therefore model trades in the product market and in the labor market in a similar way. Let Σ and D be respectively the total amounts of supply and demand expressed at the beginning of stage 2c. We assume that the number of trades is given by a product market matching function $T(\Sigma, D)$.

We introduce a matching effectiveness parameter denoted μ_0, with $\mu_0 \in [0, 1]$. This parameter is an implicit argument in function $T(\Sigma, D)$. Basically, one has $T(\Sigma, D) = \mu_0 \cdot \tau(\Sigma, D)$, where function $\tau(.,.)$ is independent to μ_0.

Assumption AS 2 *The (product market) matching function $T(.,.)$ is assumed to be continuously differentiable, increasing and concave in both arguments, to exhibit constant returns to scales, and to verify the following boundary properties for any (Σ, D) :*

$$T(\Sigma, 0) = T(0, D) = 0 \qquad \lim_{\Sigma \to \infty} T(\Sigma, D) = \mu_0 \cdot D \qquad \textit{and} \qquad \lim_{D \to \infty} T(\Sigma, D) = \mu_0 \cdot \Sigma$$

Since $\mu_0 \leq 1$, for any (Σ, D), we have $T(\Sigma, D) \leq \min(\Sigma, D)$. Let $\phi = D/\Sigma$ denote tightness on the product market. The *demand (satisfying) probability* [4] is defined as the probability according to which a unit of demand is satisfied:

$$\mu(\phi) \equiv T\left(\frac{1}{\phi}, 1\right) = \frac{T(\Sigma, D)}{D}$$

As the product market becomes tighter, the congestion-search externality makes the demand probability decreasing from μ_0 to 0. Symmetrically, the *supply (satisfying) probability* is defined as the probability at which a unit of inventory is sold:

$$\phi \cdot \mu(\phi) \equiv T(1, \phi) = \frac{T(\Sigma, D)}{\Sigma} = \frac{D}{\Sigma} \cdot \frac{T(\Sigma, D)}{D}$$

As the product market becomes tighter, supplied goods are sold more quickly. So, the supply probability increases from 0 to μ_0. Hence, for a given tightness ϕ, $\mu(\phi)$ and $\phi\mu(\phi)$ are linear in μ_0. As μ_0 increases, matching in the product market become easier.

We define the elasticity of the product market matching function as $\varepsilon(\phi) \equiv -\phi \cdot \mu'(\phi)/\mu(\phi)$. From above, we get that $\varepsilon(\phi) \in (0,1)$. The following technical assumption will appear very useful:

Assumption AS 3 *$\varepsilon(\phi)$ is nondecreasing.*

An example of matching function that satisfies AS2 is the CES specification: with a low elasticity of substitution, i.e:

$$T(\Sigma, D) = \mu_0 \cdot \left[\Sigma^{1-\frac{1}{\sigma}} + D^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

This specification further verifies Assumption AS3 when $\sigma \in (0,1)$.

Each of the $1-u$ jobs produces q units of goods per period. The flow of produced goods is therefore $(1-u)q$. A flow of $T(\Sigma, D)$ units of supply is purchased. Finally, through a depreciation process, we assume that each unit of inventory is dissolved with probability $\delta \geq 0$. This depreciation probability is exogenous. Aggregate inventories S therefore evolve according to:

$$S_{+1} = (1-\delta)S + (1-u)q - T(\Sigma, D) = (1-\delta) \cdot S + (1-u)q - \phi\mu(\phi) \cdot \Sigma \qquad (4)$$

III.3 The workers' program

The matching process on the product market being stochastic, there is a (small but) positive probability that an entrepreneur sells no good during stage 2c. If furthermore her money holdings are very small, she can be liquidity constrained and therefore unable to

[4] Our *demand satisfying rate* corresponds to the *money velocity* rate in monetary economics.

pay a fixed wage at stage 2d. To avoid the complexities induced by a fixed wage, we notice that risk neutral workers would value equally an appropriately chosen random payment, provided that the expected wage is unchanged. So, we assume that the current total wage paid to an employee, W, is proportional to the quantity of inventories sold at stage 2c. We further assume that the bargaining is over the expected wage $\mathbb{E}(W)$, so the firm commits to adjust the piece rate so that the expected wage of a worker is independent of her employer's demand and supply behaviors.

At the beginning of stage 2b, a worker can be either employed or unemployed. Her maximized lifetime expected utility depends on her money holdings m at that moment. Let respectively $V_e(m)$ and $V_u(m)$ be the value functions of an employed (unemployed) worker at stage 1c.

Consider first the program of an unemployed worker with money holdings m. We ignore unemployment benefits. Because of frictions on the product market, the number of goods bought during the period is a random variable, denoted X. To characterize the distribution of X, we notice that each of the d units of demand expressed can be seen as a random trial with a probability of success of $\mu(\phi)$. At an individual level, these trials are independent. Hence, X follows a binomial distribution with d trials and success probability $\mu(\phi)$. Therefore, the discrete density function of X is given by $\Pr(X = x) = \mathcal{B}(x, d, \mu(\phi))$, where:

$$\mathcal{B}(x, d, \mu(\phi)) = \begin{pmatrix} d \\ x \end{pmatrix} \cdot (\mu(\phi))^{x} \cdot (1 - \mu(\phi))^{d-x}$$

In particular, we notice that $\Pr(X = d) > 0$. So, there is always a positive probability that all the d units of demand expressed are satisfied. The expected purchase, conditional[5] on expressing d units of demand, is:

$$\mathbb{E}(X) = \mu(\phi) \cdot d \tag{5}$$

For any realization x of X, the unemployed worker enjoys x units of utility and will start the next period with $m - p \cdot x$ units of money. Furthermore, at stage 1a of the next period, she will find a job with probability $\theta h(\theta)$ and therefore gets an expected lifetime utility[6] of $V_e(m - p \cdot x)$. Finally, with the remaining probability, she remains unemployed. Her expected lifetime utility is then $V_u(m - p \cdot x)$. Therefore, the value function of an unemployed worker solves the following Bellman equation:

$$\begin{aligned} V_u(m) &= \max_{0 \leq d} \; \mathbb{E}\left\{\frac{X + \theta h(\theta) \cdot V_e(m - p \cdot X) + (1 - \theta h(\theta)) \cdot V_u(m - p \cdot X)}{1 + r}\right\} \\ s.t &: \; p \cdot d \leq m \end{aligned} \tag{6}$$

[5] To ease notations, this conditionality does not appear in the notations.

[6] Recall that we consider only steady states. Hence value functions are time-invariant.

Consider now an employed worker. She will receive a random wage W. For any realization x of X and n of W, the employed worker enjoys x units of utility. At the beginning of the next period, she will own $m+n-p\cdot x$ units of money. With probability $1-s$, she will be employed and will therefore get an expected lifetime utility of $V_e(m+n-p\cdot x)$. With probability s, she will loose her job and will therefore benefit from an expected lifetime utility of $V_u(m+n-p\cdot x)$. Hence, the value function of an employed worker solves the Bellman equation:

$$\begin{aligned} V_e(m) &= \max_{0\leq d} \; \mathbb{E}\left\{\frac{X+(1-s)\cdot V_e(m+W-p\cdot X)+s\cdot V_u(m+W-p\cdot X)}{1+r}\right\} \\ s.t &: \; p\cdot d \leq m \end{aligned} \tag{7}$$

Since individuals' preferences are linear with respect to the number of goods consumed, we guess that value functions are linear in m and of the form (Appendix A.1 formally verifies this claim):

$$V_e(m) = V_e + m\cdot A \qquad \text{and} \qquad V_u(m) = V_u + m\cdot A \tag{8}$$

where V_e, V_u and A are constant values. A stands for the marginal value of money. V_e (V_u) is the value of being employed (unemployed). Under the specification (8), and given (5), the first-order[7] and envelope conditions to programs (6) and (7) are the same. They are respectively:

$$0 = \mu(\phi)\frac{1-p\cdot A}{1+r} - \lambda\cdot p \qquad A = \frac{A}{1+r} + \lambda$$

where $\lambda \geq 0$ is the (non-negative) Kuhn-and-Tucker multiplier associated to the constraint[8] $m \geq p\cdot d$. Substituting λ from the first-order conditions and multiplying the envelope conditions by p gives:

$$p\cdot A = \frac{\mu(\phi) + (1-\mu(\phi))\cdot p\cdot A}{1+r}$$

To interpret this asset equation, we notice that with p additional units of money at the start of stage 2b, the consumer can express one additional unit of demand. With probability $\mu(\phi)$, this additional unit of demand will be satisfied, that is, the consumer will find the good she is searching for, will buy and consume it. This will generate a utility level normalized to $1/(1+r)$. With the remaining probability, the consumer ends stage 2c of the current period with p units of money, that are valued $p\cdot A$. The last equation can be rewritten to express $p\cdot A$ as a function of *the demand satisfying probability* $\mu(\phi)$ and the discount rate r:

$$p\cdot A = \frac{\mu(\phi)}{r+\mu(\phi)} \in (0,1) \tag{9}$$

[7] We henceforth consider that d is a real number and not an integer, which simplifies the problem.

[8] Since $r>0$, the envelope condition implies $\lambda>0$. Hence, the exclusion condition $\lambda(m-p\cdot d)=0$ implies $d=m/p$. Therefore, the constraint $d\geq 0$ does not bind.

We call the right-hand side of (9) the *rationing of demand* term. This term accounts for the fact that it is better to consume a unit of good than to have p units of money. In the absence of discounting, the *rationing of demand* term is equal to 1. When conversely consumers discount the future, they have to wait at least one period before spending their money and the rationing of demand term is lower than 1. As tightness on the product market ϕ increases, the *demand satisfying probability* $\mu(\phi)$ declines and the *rationing of demand* term decreases. The decreasing curve in Figure 2 represents this phenomenon.

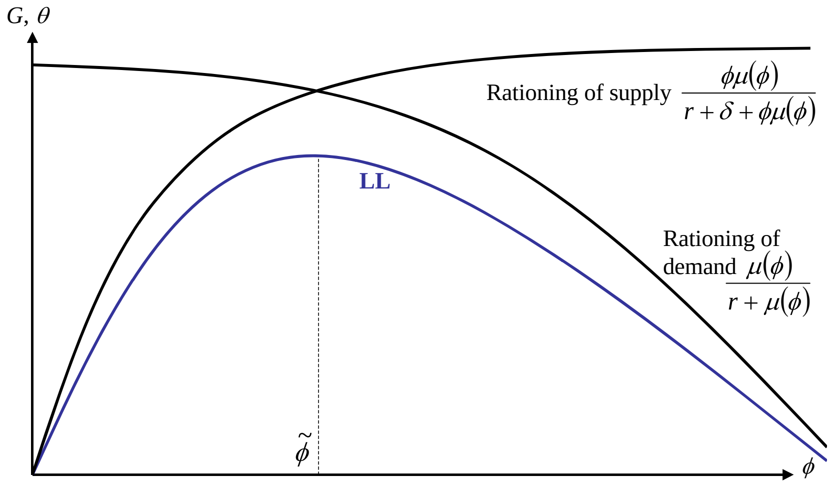

Figure 2: Rationings on the Product market

Furthermore, since $p \cdot A < 1$, a consumer always expects a positive net gain $\mu(\phi)(1 - p \cdot A)$ when she expresses an additional unit of demand. Therefore, she expresses the maximum amount of demand, given her money holdings. So, $d = m/p$. Finally, using (8), equations (6) and (7) implies

$$r \cdot V_u = \theta h(\theta)(V_e - V_u) \qquad \text{and} \qquad r \cdot V_e = \mathbb{E}(W) \cdot A + s(V_u - V_e) \tag{10}$$

which are the usual asset equations for, respectively, the value of being unemployed and of being employed (see e.g. Pissarides 2000 or Cahuc and Zylberberg 2004). The only difference is that wage W being random, only its expectation enters (10).

III.4 The entrepreneurs' program

At the beginning of stage 1b, an entrepreneur owns m units of money, a level of inventory ζ and employs l workers. She must choose how many units of demand d to express, how many units of supply σ to express and how many vacancies ω to open, subject to constraints (2) and (3). Let $V_f(m, \zeta, l)$ be her value function at that moment.

As in the case of a worker, the number X of goods consumed, conditional on the level of demand d is a random variable that follows a binomial distribution. Symmetrically, let the random number of goods sold be denoted by Y. Each unit of good supplied will be sold according to the supply satisfying probability $\phi\mu(\phi)$. At the individual level, these trials are independent and independent of X. Therefore Y follows a binomial distribution with σ trials and a probability of success $\phi\mu(\phi)$. Hence the density function is $\mathcal{B}(y, \sigma, \phi\mu(\phi))$, with:

$$\mathbb{E}(Y) = \phi\mu(\phi) \cdot \sigma \tag{11}$$

The worker and the firm bargain over the expected wage $\mathbb{E}(W)$. For each unit of good sold, the firm pays w units of money per employee, with $W = w \cdot Y$. So:

$$w = \frac{\mathbb{E}(W)}{\mathbb{E}(Y)} = \frac{\mathbb{E}(W)}{\phi\mu(\phi) \cdot \sigma} \tag{12}$$

The number of workers that will be hired during stage 1a of the next period is also a random variable denoted Z. Since the job filling probability is $h(\theta)$, Z follows a binomial distribution of ω independent trials with success probability $h(\theta)$. The number of job dissolved is also a random variable R that follows a binomial distribution with l employment and a success probability s. Finally, the number of inventories depreciated is a random variable Δ that follows a binomial distribution with ζ trials and a success probability δ. We hence get:

$$\mathbb{E}(Z) = h(\theta) \cdot \omega \qquad \mathbb{E}(R) = s \cdot l \qquad \mathbb{E}(\Delta) = \delta \cdot \zeta \tag{13}$$

For any realization x of X, y of Y, z of Z, r of R and ϵ of Δ, the entrepreneur consumes x units of goods, which generates x units of utility. Opening ω vacancies induces a total disutility of $k \cdot \omega$, where $k > 0$ is a parameter. She sells y of units of goods, thereby getting $p \cdot y$ units of money. For each unit of goods sold, she pays w units of money to each of her l employees. Hence, her money holdings at the beginning of stage 1a of the next period will be $m_{+1} = m + p \cdot (y - x) - W \cdot l$, where $W = w \cdot y$. The output of l workers increases inventories by an amount of $q \cdot l$, of which y units are sold and ε units are depreciated. Therefore her inventories at the beginning of the next period will be $\zeta_{+1} = \zeta + q \cdot l - y - \varepsilon$. r jobs are dissolved, but z new workers are recruited. So, her future employment level after stage 1a of the next period will be $l_{+1} = l - r + z$. Finally, her expected lifetime utility at the start of stage 2a of the next period will be given by $V_f\left(m_{+1}, \zeta_{+1}, l_{+1}\right)$. Therefore, the entrepreneur's value function solves the following Bellman equation:

$$\begin{aligned} V_f(m, \zeta, l) &= \max_{w, 0 \leq p \cdot d \leq m, 0 \leq \sigma \leq \zeta, 0 \leq \omega} \mathbb{E}\left\{\frac{X - k \cdot \omega + V_f\left(m_{+1}, \zeta_{+1}, l_{+1}\right)}{1 + r}\right\} \\ s.t: m_{+1} &= m + p \cdot (Y - X) - W \cdot l \qquad \zeta_{+1} = \zeta - \Delta - Y + q \cdot l \\ l_{+1} &= l - R + Z \qquad w = \frac{\mathbb{E}(W)}{\mathbb{E}(Y)} \end{aligned} \tag{14}$$

Since entrepreneurs' preferences are linear with respect to the number of goods consumed and the number of jobs opened, we guess that the value function is linear in m, ζ and l and that it takes the form (Appendix A.2 formally verifies this claim):

$$V_f(m,\zeta,l) = m\cdot\overline{A} + \zeta\cdot G + l\cdot J \tag{15}$$

where $\overline{A}$, G and J are constants that stand for the marginal values of respectively money, inventory and a filled job. Let λ and η be the (non-negative) Kuhn-and-Tucker multipliers associated to constraints $m \geq p\cdot d$ and $\zeta \geq \sigma$. Given (5), (11), (12), (13), and the specification (15), the first-order conditions of program (14) with respect to individual demand d, supply σ and the number of vacancies ω are respectively:

$$0 = \frac{\mu(\phi)\left(1-p\cdot\overline{A}\right)}{1+r} - \lambda\cdot p \qquad 0 = \frac{\phi\mu(\phi)\left(p\cdot\overline{A}-G\right)}{1+r} - \eta \qquad 0 \geq \frac{-k+h(\theta)\cdot J}{1+r}$$

The envelope conditions over respectively money holdings m, inventories ζ and employment l are[9]:

$$\overline{A} = \frac{\overline{A}}{1+r} + \lambda \qquad G = \frac{1-\delta}{1+r}G + \eta \qquad J = \frac{(1-s)\,J + q\cdot G - \mathbb{E}(W)\cdot\overline{A}}{1+r}$$

Substituting λ from the first-order condition with respect to d into the envelope condition with respect to money holdings m gives (9). Therefore, one has $\overline{A} = A$ which means that entrepreneurs and workers value money identically. Furthermore, as for workers, entrepreneurs express the maximum amount of demand permitted by their money holdings and $d = m/p$.

Turning now to supply decisions, substituting η from the first-order condition with respect to supply σ into the envelope condition with respect to inventories ζ gives:

$$G = \frac{\phi\mu(\phi)\cdot p\cdot A + (1-\delta-\phi\mu(\phi))\cdot G}{1+r}$$

The interpretation for this equation parallels the one of equation (9). An additional unit of inventory will be sold during stage 2c with probability $\phi\mu(\phi)$. The entrepreneur then gets p units of money, which are valued $p\cdot A$. With probability $1-\delta-\phi\mu(\phi)$ the good is neither sold nor depreciated. This asset equation can be rewritten as:

$$G = \frac{\phi\mu(\phi)}{r+\delta+\phi\mu(\phi)}\cdot p\cdot A \tag{16}$$

We call the right-hand side of (16) the *rationing of supply* term. This term accounts for the fact that it is better to have p units of money than a unit of inventory. In the absence

[9] From the enveloppe conditions over respectively m and ζ, we see that $\lambda > 0$ and $\eta > 0$ so constraints (2) and (3) are binding and therefore, constraints $0 \leq d$ and $0 \leq \sigma$ are not bidding.

of discounting and depreciation, the rationing of supply term is equal to 1. Otherwise, this term is lower than 1. As tightness on the product market ϕ increases, the *supply satisfying* probability $\phi\mu(\phi)$ rises, and the *rationing of supply* term increases. The increasing curve in Figure 2 represents this phenomenon. Furthermore, since $G < p \cdot A$, an entrepreneur always expects a positive net gain $\phi\mu(\phi)(p \cdot A - G)$ from the supply of one additional unit of good. Therefore, she supplies all her inventory: $\sigma = \zeta$.

The envelope condition over employment l implies:

$$(r+s)\,J = q \cdot G - \mathbb{E}(W) \cdot A \tag{17}$$

During each period, a job produces q additional units of inventories, each of them being valued G. Furthermore, the firm has to pay an expected wage $\mathbb{E}(W)$. Hence, the right hand side of (17) represents the expected current profit on a filled job. Given the probability s of destruction, the lifetime value of a job discounts this current profit at rate $r+s$.

The first-order condition with respect to vacancies ω relates the cost of posting a vacancy k to the lifetime expected profit from searching for a worker. The latter equals the job filling probability $h(\theta)$ times the value of a filled job J. If the cost is larger than the expected profits, firms prefer not to search for workers. Then, the aggregate number υ of vacancies decreases. Tightness on the labor market θ therefore decreases (since the number of unemployed workers is predetermined) and so does the job filling probability $h(\theta)$. If $h(0) \cdot J \leq k$, firms post no vacancies. Otherwise, the preceeding phenomenon continues until $k = h(\theta) \cdot J$. This equality corresponds to the so-called free-entry condition in the standard labor matching model. Given (17) we get the usual free-entry condition (see e.g. Pissarides 2000):

$$\frac{k}{h(\theta)} = J \tag{18}$$

III.5 The labor market equilibrium

At stage 1c, each employed worker negotiate its expected wage $\mathbb{E}(W)$ with her employer. We assume generalized Nash bargaining. Let $\beta \in (0,1)$ denote workers' bargaining power. From (8), the surplus extracted by the worker from employment $V_e(m) - V_u(m)$ is independent of her money holdings m. Symmetrically, from (15), the marginal value of a job J for an entrepreneur is independent of her money holdings m, inventories ζ, or employment l. The expected wage $\mathbb{E}(W)$ maximizes the generalized Nash Product:

$$\max_{\mathbb{E}(W)} \qquad (V_e - V_u)^{\beta} \cdot J^{1-\beta}$$

taking workers' outside option V^u as given. Given (10) and (17), the first-order condition leads to the following sharing rule:

$$(1-\beta)(V_e - V_u) = \beta \cdot J \tag{19}$$

Hence, the total surplus generated by a job $V_e - V_u + J$ is split in shares β and $1-\beta$ between the worker and her employer. From (10), we then have:

$$(r+s)(V_e - V_u) = \mathbb{E}(W) \cdot A - \beta \cdot \theta h(\theta) \cdot (V_e - V_u + J)$$

Together with (17), we get:

$$(r+s+\beta \cdot \theta h(\theta)) \cdot (V_e - V_u + J) = q \cdot G$$

Together with the free-entry condition (18) and the sharing rule (19), the last equation leads to the labor market equilibrium condition:

$$\left(\frac{r+s}{h(\theta)} + \beta \cdot \theta\right) \frac{k}{1-\beta} = q \cdot G \tag{20}$$

Following Assumption AS1, the left-hand side of (20) is an increasing function of θ. When θ increases from 0 to $+\infty$, the left hand side increases from $(r+s)\,k/(1-\beta)$ to $+\infty$. Therefore, if $q \cdot G > (r+s)\,k/(1-\beta)$, there exists a unique equilibrium value of the labor market tightness θ that satisfies (20). Otherwise, a filled job generates too few utility, so entrepreneurs prefer not to open vacancies and $\theta = 0$. This is summarized in the following lemma

Lemma 1 *There exists a single equilibrium value for the labor market tightness. If* $G > (r+s)\,k/(q(1-\beta))$*, this values is the unique solution of (20). Otherwise,* $\theta = 0$.

The functioning of the product market affects the labor market only through the value of a unit of inventory G. This is because the wage payment scheme is adjusted so that the total surplus generated by a job is split between the worker and her employer. Hence, the vacancy supply behavior only depends on the total surplus from a job. This surplus is proportional to the total value of the q units of goods produced by a job within a period. Equilibrium labor market tightness is therefore an increasing function of the value G of a unit of inventory and of the productivity q of a job. We also obtain the usual comparative static properties with respect to the labor market parameters. A rise in the workers' bargaining power β, in the job separation probability s, or in the vacancy cost k decreases equilibrium labor market tightness θ.

Finally, using (10) and (17), the sharing rule (19) implies

$$\mathbb{E}(W) \cdot A = \beta \cdot q \cdot G + (1-\beta) \cdot \theta h(\theta) \cdot (V_e - V_u)$$

Using again (19) with the free-entry condition (18) gives finally the value of the expected wage:

$$\mathbb{E}(W) \cdot A = \beta \{q \cdot G + k \cdot \theta\} \tag{21}$$

III.6 The fixed price Equilibrium

In the two last subsections, we have seen that workers and entrepreneurs choose to express as many units of demand as they are allowed to, given the constraint (3). Let M be the exogenous aggregate money stock. At the macroeconomic level, aggregate demand D is then equal to M/p. Symmetrically, entrepreneurs choose to supply all their inventories. Hence, at the macroeconomic level, aggregate supply of product Σ equals the total level of stock S. Hence, we have:

$$\phi = \frac{M}{p \cdot S} \tag{22}$$

From (4) rewritten in steady state, we get the flow equilibrium condition on the product market:

$$S = \frac{(1-u) \cdot q}{\delta + \phi\mu(\phi)} \tag{23}$$

The aggregate level of inventories S is an increasing function of the productivity level q and of the steady-state level of employment $1-u$. The steady-state level of inventories decreases with the depreciation rate δ and the probability of selling supplied goods $\phi\mu(\phi)$.

From $D = M/p$, we notice that our model amounts to putting "Money in the Matching Function". It is worth stressing that the distribution of the money holdings does not affect the level of demand for produced goods. This is because every agent values money identically. This property comes from the fact that the returns on money holdings are linear at the individual level: holding a twice bigger amount of money permits to simultaneously search for twice more goods and thereby to double the flow of consumed goods.

We are now ready to define the fixed price equilibrium. A fixed price equilibrium is a list $\{A, G, \phi, \theta, S, u, \mathbb{E}(W)\}$ that verifies *i)* The asset equations for the marginal values of money (9) and inventories (16); *ii)* the labor market equilibrium condition (20); *iii)* the demand and supply behaviors, as summarized by (22); *iv)* The flow equilibrium condition on the labor market, which implies (1); *v)* The flow equilibrium condition on the product market (23); *vi)* The wage equation (21).

We now explain how to reduce this equilibrium to the analysis of two curves in the (ϕ, θ) space. Combining (9) and (16), we get:

$$G = \Gamma(\phi) \qquad \text{where} \qquad \Gamma(\phi) \equiv \frac{\phi\mu(\phi)}{r + \delta + \phi\mu(\phi)} \cdot \frac{\mu(\phi)}{r + \mu(\phi)} \tag{24}$$

A unit of good produced is transformed into utility in two steps. First, production is not sold instantaneously. The imperfection of this process is represented by the first term defining $\Gamma(.)$, i.e., the *rationing of supply term.* Second, p units of money cannot instantaneously be used to buy a good. The imperfection of this process is represented by the second term, i.e. the *rationing of demand term.* Each of these two steps are uncertain and take time due to the presence of frictions on the product market. Function $\Gamma(.)$ summarizes the consequence of these two imperfections. We prove in Appendix B the following lemma:

Lemma 2 *Under assumptions AS2 and 3, function* $\Gamma(\phi)$ *is hump shaped, with zero limit for* $\phi \to 0$ *and* $\phi \to \infty$*, It admits a unique maximum value for* $\phi = \tilde{\phi}$ *defined by:*

$$\frac{r}{r+\delta} \cdot \frac{r+\delta+\tilde{\phi}\mu\left(\tilde{\phi}\right)}{r+\mu\left(\tilde{\phi}\right)} = \frac{1-\varepsilon\left(\tilde{\phi}\right)}{\varepsilon\left(\tilde{\phi}\right)} \tag{25}$$

The hump-shaped profile of function $\Gamma(.)$ is a key property of our model. When tightness on the product market is sufficiently low ($\phi < \tilde{\phi}$), a rise in ϕ relaxes more the *rationing of supply* than it reinforces the *rationing of demand.* That is $G/(p \cdot A)$ increases more than $p \cdot A$ decreases with ϕ. Then, increasing ϕ raises the pace at which a unit of production is transformed into utility $\Gamma(\phi)$. The opposite holds when $\phi > \tilde{\phi}$. (See Figure 2).

For a given value of ϕ, when the discount rate r increases, agents are more impatient and both values of inventories G and of money holdings A decrease. As a consequence, the three curves in Figure 2 shift downwards, and $\Gamma(\phi)$ decreases. The opposite hold when the product market matching function parameter μ_0 increases. Finally, a rise in the depreciation rate δ increases the depreciation flow of inventories. The *rationing of supply term* in (16) and therefore Γ shift downwards in Figure 2.

Combining (20) and (24), we can express the equilibrium labor market tightness θ as a function of the product market tightness ϕ through:

$$\left(\frac{r+s}{h(\theta)} + \beta \cdot \theta\right) \frac{k}{1-\beta} = q \cdot \Gamma(\phi) \tag{26}$$

The effects of μ_0, ϕ and δ on tightness θ on the labor market come through changes in the value of Γ. Since the left-hand side of (26) is an increasing function of θ, these effects can immediately be illustrated by rescaling the y-axis of Figure 2 in terms of θ. Equation (26) is represented in Figures 2 and 3 by the curve labelled LL. Following lemma 2, a rise in ϕ increases (decreases) θ if $\phi < \widetilde{\phi}$ (resp. $\phi > \widetilde{\phi}$). For any value of ϕ, a rise in δ or a decline in the scale parameter of the matching function μ_0 lower $\Gamma(.)$ and hence θ. Finally, an increase in the discount rate r decreases θ through two channels. In addition to the negative effect on $\Gamma(\phi)$, there is the standard negative effect on job creation (see Pissarides 2000).

Combining the flow equilibrium conditions on the labor (1) and the product market (23), we get:

$$\frac{\theta h(\theta)}{s+\theta h(\theta)} q = S\left(\delta + \phi \mu(\phi)\right)$$

Using (22), we finally get a second relation between θ and ϕ:

$$\frac{\theta h(\theta)}{s+\theta h(\theta)} q = \frac{M}{p} \cdot \left\{\frac{\delta}{\phi} + \mu(\phi)\right\} \tag{27}$$

A rise in tightness on the labor market θ implies a higher steady-state level of employment $1-u$. Therefore, more output is produced during each period, which raises inventories S. Since the aggregate Money supply M is fixed and the price p is exogenous, the steady-state level of tightness on the product market ϕ decreases, too. This can be seen from the right-hand side of (27), which decreases with ϕ. Therefore, the two flow equilibrium conditions define ϕ as a decreasing function of θ at the steady-state. This second relation is denoted DD in Figure 3.

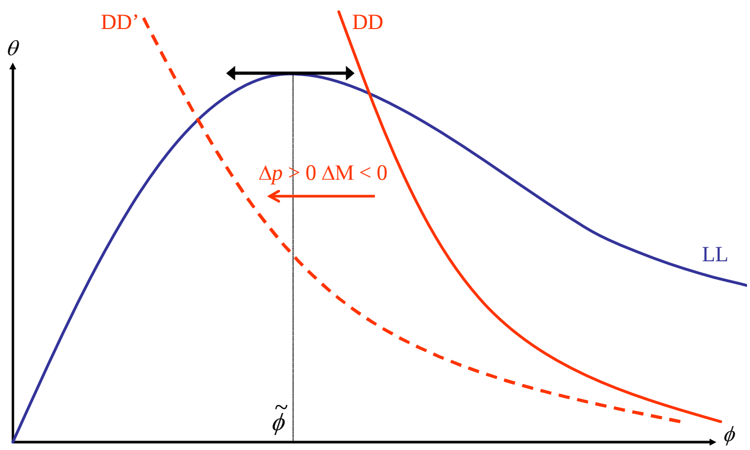

Figure 3: The Fixed Price Equilibrium

Assuming exogenous price is not a very relevant assumption for a steady-state equilibrium. We made it essentially for pedagogical reasons in order to build a more general model step by step. We therefore do not devote too much attention to the study of this equilibrium. Neither existence nor uniqueness are guaranteed. However, a noticeable property is the ambiguous effect of a decline in Aggregate Demand $D = M/p$ on equilibrium, coming from either a rise in price p or a decrease in money supply M.

A decrease in D has no effect on the LL curve (see Equation 26). However, such decline reduces the amount of goods exchanged on the product market $T(S,D)$. For a given level of tightness on the labor market θ, thereby a given flow of produced goods $q(1-u)$, the level of inventories S increases, which offset the reduction in $T(S,D)$. Hence, from (22), the decrease in D and the increase in S lead to a decrease in ϕ (see 27). This is illustrated in Figure 3 by the shift of the DD curve to the left.

Now, such a decline of ϕ has an ambiguous effect on the value of inventories G, thereby on tightness on the labor market θ and on unemployment u. If $\phi < \widetilde{\phi}$, the decline in ϕ decreases $\Gamma(\phi)$ and we get the standard effect: a rise in price p, or a decrease in money supply M decreases labor market tightness θ and increases unemployment u. However, if $\phi > \widetilde{\phi}$, we get the reverse effects: the reduction in ϕ improves more the value of money holdings (by increasing the probability of finding a seller) than it deteriorates the value of inventories relative to the value of money holdings (by decreasing the probability of finding a customer). Therefore, the value of inventories increases, thereby increasing tightness on the labor market and decreasing unemployment.

IV Endogenous price

Various assumptions could be made about price determination in the product market. We assume that sellers post prices, that is they annonce take-it-or-leave-it prices and commit to sell production at this price. To us, this seems to be the most natural assumption for price setting in the product market. However, we are aware that the price of goods are also sometimes bargained over.

We build upon the price posting equilibrium of Moen (1997), also known as the Competitive Search Equilibrium (henceforth CSE). Prices are observable at no cost, search is directed and mobility of agents is perfect. The product market is now made of a continuum of identical submarkets indexed by $i \in I$. Each submarket is characterized by a price p_i. At the beginning of stage 2c of a given period, there are Σ_i units of supply and D_i units of demand in market i. The matching technology is identical across submarkets. Then, a unit of demand (supply) in submarket i is satisfied (is sold) with probability $\mu(\phi_i)$ $(\phi_i \mu(\phi_i))$ in submarket i, where $\phi_i = D_i/\Sigma_i$ is the tightness in the i^{th} submarket.

In a CSE, submarkets with a higher price attract relatively more supply and less demand. Tightness is therefore lower in submarkets with a higher price, which implies a higher (a lower) demand (supply) satisfying probability. Hence, consumers trade off a lower price against a higher probability of finding the desired goods, whereas entrepreneurs trade off a higher price against a lower probability of selling their inventories. We now detail the agents' behaviors. The timing of events is given in Figure 4.

Consider first a consumer with money holdings m. In stage 2b, she has now to decide the number d_i of units of demand to located in each submarket i. The number of goods found in submarket i is therefore a random variable X_i, with a discrete density function

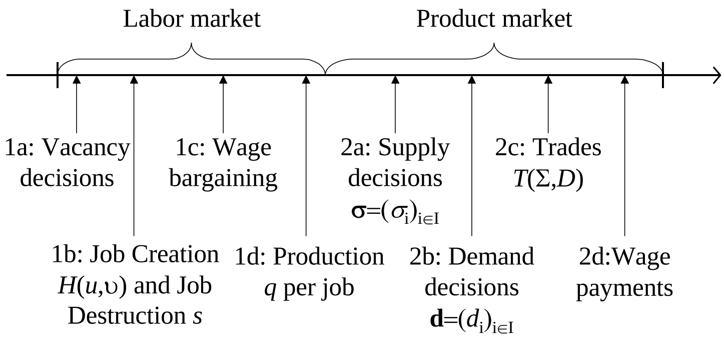

Figure 4: The timing of events along a CSE

$\mathcal{B}(x, d_i, \mu(\phi_i))$ and mean:

$$\mathbb{E}(X_i) = \mu(\phi_i) \cdot d_i \tag{28}$$

The X_i variables are independent. At the end of stage 2c, the consumer enjoys a utility level $\sum_{i \in I} X_i$, whose mean is:

$$\mathbb{E}\left(\sum_{i \in I} X_i\right) = \sum_{i \in I} \mu(\phi_i) \cdot d_i$$

As for the fixed price model, a consumer is not allowed to search for a good she cannot afford. Constraint (3) is here replaced by:

$$0 \leq \sum_{i \in I} p_i \cdot d_i \leq m \tag{29}$$

Consider now the supply behavior of an entrepreneur with total inventory ζ at the beginning of stage 2a. She has now to decide how many units of inventory to supply in each submarket. Let σ_i be its supply on submarket i and let Y_i be the random number of goods sold in submarket i. The (discrete) density function of Y_i is $\mathcal{B}(y, \sigma_i, \phi_i \mu(\phi_i))$, with mean:

$$\mathbb{E}(Y_i) = \phi_i \mu(\phi_i) \cdot \sigma_i \tag{30}$$

The Y_i variables are independent. At the end of stage 2c, the entrepreneur receives a quantity of money $\sum_{i \in I} p_i \cdot Y_i$, whose mean is:

$$\mathbb{E}\left(\sum_{i \in I} p_i \cdot Y_i\right) = \sum_{i \in I} p_i \cdot \phi_i \mu(\phi_i) \cdot \sigma_i$$

As for the fixed price model, an entrepreneur cannot sold a good she has not yet produced. Therefore, constraint (3) should now be replaced by:

$$0 \leq \sum_{i \in I} \sigma_i \leq \zeta \tag{31}$$

Finally, for each unit of good sold in submarket i, the firm has to give $w_i \leq p_i$ units to each of her l employees. The w_i are set so as to guaranty a negotiated level of the expected wage W received by workers. Hence equation (12), should now be replaced by

$$\sum_{i \in I} w_i \cdot \mathbb{E}(Y_i) = \mathbb{E}(W) \cdot l$$

However, since the wage bargain is over the expected wage, entrepreneurs take their demand and supply decisions taking $\mathbb{E}(W)$ as given. We are now ready to write workers and entrepreneurs' programs. For workers, the only difference is on the multidimensionality of the demand decisions. Therefore programs (6) and (7) should now be respectively rewritten as:

$$\begin{aligned} V_u(m) &= \max_{0 \leq d_i} \mathbb{E} \left\{ \frac{\sum_{i \in I} X_i + \theta h(\theta) \cdot V_e(m_{+1}) + (1 - \theta h(\theta)) \cdot V_u(m_{+1})}{1+r} \right\} \\ s.t: m_{+1} &= m - \sum_{i \in I} p_i \cdot X_i \qquad \text{and} \qquad 0 \leq m - \sum_{i \in I} p_i \cdot d_i \end{aligned} \tag{32}$$

and

$$\begin{aligned} V_e(m) &= \max_{0 \leq d_i} \mathbb{E} \left\{ \frac{\sum_{i \in I} X_i + (1-s) \cdot V_e(m_{+1}) + s \cdot V_u(m_{+1})}{1+r} \right\} \\ s.t: m_{+1} &= m + W - \sum_{i \in I} p_i \cdot X_i \qquad \text{and} \qquad 0 \leq m - \sum_{i \in I} p_i \cdot d_i \end{aligned} \tag{33}$$

Rewriting the entrepreneurs' program is slightly more complex since the supply decisions are multidimensional too. Hence (14) becomes:

$$\begin{aligned} V_f(m, \zeta, l) &= \max_{w, 0 \leq d_i, 0 \leq \sigma_i, 0 \leq \omega} \mathbb{E} \left\{ \frac{\sum_{i \in I} X_i - k \cdot \omega + V_f(m_{+1}, \zeta_{+1}, l_{+1})}{1+r} \right\} \\ s.t: m_{+1} &= m + \sum_{i \in I} p_i \cdot (Y_i - X_i) - W \cdot l \qquad \zeta_{+1} = \zeta - \Delta - \sum_{i \in I} Y_i + q \cdot l \\ l_{+1} &= l - R + Z \qquad \sum_{i \in I} w_i \cdot \mathbb{E}(Y_i) = \mathbb{E}(W) \\ 0 &\leq m - \sum_{i \in I} p_i \cdot d_i \qquad \text{and} \qquad 0 \leq \zeta - \sum_{i \in I} \sigma_i \end{aligned} \tag{34}$$

Since preferences are linear in consumption, we again guess that value functions are linear and of the form specified in equations (8) and (15). We start with the analysis of demand decisions at stage 2b[10]. Let λ be the (non negative) Kuhn-and-Tucker multiplier associated to the inequality constraint (29). Given (28) and the specifications (8) and (15), the first-order conditions with respect to d_i of programs (32) (33) and (34) are:

$$0 \geq \frac{\mu(\phi_i)(1 - p_i \cdot A)}{1+r} - \lambda \cdot p_i \qquad \text{with} = \text{ if } d_i > 0$$

while the envelope condition over money holdings m is given by:

$$A = \frac{A}{1+r} + \lambda$$

From the envelope condition, we conclude that $\lambda > 0$. Hence, by the exclusion condition $\lambda\left(m - \sum_{i \in I} p_i \cdot d_i\right) = 0$, we conclude that constraint (29) binds. Therefore, there are at least some submarkets where consumers express a positive demand $d_i > 0$. For these submarkets, the interpretation of the first-order condition is similar to the one given in Section III.3. With the envelope condition, it can be written as

$$p_i \cdot A = \frac{\mu(\phi_i)}{r + \mu(\phi_i)} \in (0,1) \tag{35}$$

This equation defines a decreasing relationship between the price level in submarket i and tightness in this submarket. During stage 2b, the supply decisions had been chosen. Hence Σ_i are predetermined. If in one submarket one has:

$$p_i \cdot A > \frac{\mu(\phi_i)}{r + \mu(\phi_i)} \qquad \Leftrightarrow \qquad r \cdot p_i \cdot A > \mu(\phi_i)(1 - p_i \cdot A)$$

(resp. $<$), the opportunity cost of expressing a unit of demand in submarket i, namely $r \cdot p_i \cdot A$, is higher (lower) than the expected gain from expressing a unit of demand on submarket i $\mu(\phi_i)(1 - p_i \cdot A)$. Consumers therefore exit (enter) this submarket, which shifts total demand D_i downwards (upwards). Therefore, tightness ϕ_i on this submarket decreases (increases) and the demand satisfying probability $\mu(\phi_i)$ increases (decreases). This process continues until (35) is met. Through this free-entry mechanism, consumers accept to express a demand on a submarket with a higher price if and only if the probability $\mu(\phi_i)$ is higher, that is when tightness ϕ_i is lower. Hence (35) holds with equality for all submarkets.

We now turn to supply decisions. Let η be the (non negative) Kuhn-and-Tucker multiplier associated to inequality constraint (31). Given (30), (13) and the specification (15), the first-order conditions with respect to σ_i of program (34) writes:

$$0 \geq \frac{\phi_i \mu(\phi_i)(p_i \cdot A - G)}{1+r} - \eta \qquad \text{with} = \text{ if } \sigma_i > 0$$

[10] That $A = \bar{A}$ isn taken for granted here. See, the arguments in III.4.

while the envelope condition over inventories ζ writes:

$$G = \frac{1-\delta}{1+r} G + \eta$$

From the envelope condition, we conclude that $\eta > 0$. Hence, by the exclusion condition, we know that constraint (31) binds. This induces that there exists at least one submarket e where firms supply some inventories. This leads to the following condition

$$G = \frac{\phi_e \mu(\phi_e)}{r + \delta + \phi_e \mu(\phi_e)} p_e \cdot A$$

that can be interpreted as in III.4. Using (35), we finally get for this (these) submarket(s):

$$G = \Gamma(\phi_e) \tag{36}$$

Conversely, for the other submarkets labelled d, one has

$$(r+\delta) \cdot G > \phi_d \mu(\phi_d)(p_d \cdot A - G)$$

The opportunity cost of supplying one additional good on one of these submarkets $(r+\delta) \cdot G$ is higher than the probability to sell a good $\phi_d \mu(\phi_d)$ times the gain of having p_d units of money instead of a unit of inventory $p_d \cdot A - G$. Using (35), we get for these submarkets:

$$G > \Gamma(\phi_d)$$

Hence, in a CSE, only the submarket(s) for which tightness maximizes function $\Gamma(.)$ are active. From Lemma 2, we know that a unique value $\widetilde{\phi}$ maximizes $\Gamma(\phi)$. Therefore, we have $\phi_e = \widetilde{\phi}$ and $G = \Gamma\left(\widetilde{\phi}\right)$.

It is straightforward to verify that the envelope condition over the entrepreneur's employment level and the first-order condition with respect to vacancies ω are unchanged. Hence equation (17) characterizing the value of filled job and the free-entry condition (18) are still valid. It is also straightforward to verify that Equation (10) remains unchanged. Therefore, the outcome of the wage bargain is the same in the fixed price equilibrium and in the CSE. As a consequence, equation (20) expressing labor market tightness θ as a function of the value of inventory G is unchanged.

We are now ready to characterize the CSE. The price setting mechanism sets tightness on the product market to the unique maximizer of $\Gamma(.)$, so $\phi = \tilde{\phi}$. We then get G thanks to (36). Then, tightness on the labor market is given by (26). From Lemma 1, either $\Gamma\left(\widetilde{\phi}\right) > (r+s)k/(q(1-\beta))$ and there exists a single value of θ that solves (26) and this value is positive; or firms find not profitable to post vacancies and $\theta = 0$.

The flow equilibrium condition on the labor market (1) yields a unique unemployment rate u. Then, the level of inventories S is given by (23). These relationships fully characterize the real part of the economy. The equilibrium price level p is given by (22), as a function of ϕ, money supply M and inventories S. From above, we conclude:

Proposition 1 *There exist a unique CSE. If* $\Gamma\left(\widetilde{\phi}\right) > (r+s)\,k/\left(q\,(1-\beta)\right)$*, then* $\theta > 0$ *and* $u \in (0,1)$*.*

We can now turn to the comparative statics. An improvement in the labor market determinants (i.e. a rise in the productivity level q or a decrease in the bargaining power β, in the vacancy disutility cost k or in the separation rate s) does not influence the equilibrium value of ϕ and has the same effect as before on the labor market: θ increases and unemployment decreases. Inventories increases (see (23)). Finally, aggregate demand $D = M/p$ has to increase to absorb the additional inventories keeping tightness on the product market ϕ unchanged. Hence the price p decreases.

A less efficient matching function on the product market (a decline in μ_0), or an increase or in the depreciation rate δ change ϕ. However, since the equilibrium value of ϕ maximizes $\Gamma(.)$, we can use the envelope theorem and deduce that $\Gamma\left(\widetilde{\phi}\right)$ decreases, thereby G. Tightness on the labor market thus decreases (see (26)) and unemployment increases. An increase in the discount rate r has the same qualitative effects. However, the labor market is affected directly and indirectly through ϕ.

Finally, a rise in the money supply has no effect on tightness ϕ in the product market. As a consequence, G, θ, u and S remain unchanged. Then, price p increases in the same proportion as the money supply to keep aggregate demand D unchanged. This proves the *long-run neutrality* of money in our model. A permanent increase in the *level* of the money supply has no real effect and change only nominal variables proportionally. In the next section, we will investigate the *long-run super-neutrality* of Money, which concerns the effects of a permanent change in the *rate of growth* of money supply.

V Monetary growth

In this section, we introduce monetary growth in the model of Section IV. At the end of each period, after stage 2d, the aggregate money supply increases according to;

$$M_{+1} = \frac{M}{1-\pi}$$

This additional money is distributed to the mass 1 of workers in a lump-sum fashion. Let T be the corresponding transfer, with: $T = M_{+1} - M = \pi \cdot M_{+1}$. The timing of events is hence described by Figure 5.

In steady state, all real variables are constant, whereas nominal variables evolves proportionally to money supply M. It is therefore convenient to rewrite these variables in intensive terms. To do so, we divide nominal variables by the aggregate Money supply. Intensive money holdings $\widehat{m} = m/M$ are the proportion of aggregate money supply hold by

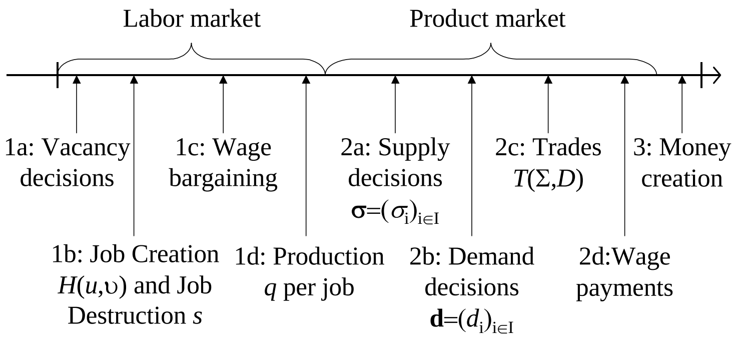

Figure 5: The timing of events along a CSE with monetary growth

an individual whose money holdings is m. We similarly denote $\widehat{w} = w/M$ and $\widehat{p}_i = p_i/M$. It turns out to be very convenient to take *intensive* $\widehat{m}$ rather than *gross* money holdings m as a state variable in the Bellman equations. Consider a worker with gross money holdings $\widetilde{m}$ at the end of stage 2d of the current period. She will receives the lump sum transfer $T = \pi \cdot M_{+1}$ at stage 3. Her next period intensive money holdings is therefore $\widehat{m}_{+1} = (\widetilde{m} + T)/M_{+1} = (1-\pi)(\widetilde{m}/M) + \pi$. Keeping this in mind, Bellman Equations (32) (33) and (34) become:

$$\begin{aligned} V_u(\widehat{m}) &= \max_{0 \leq d_i} \mathbb{E} \left\{ \frac{\sum_{i \in I} X_i + \theta h(\theta) \cdot V_e(\widehat{m}_{+1}) + (1-\theta h(\theta)) \cdot V_u(\widehat{m}_{+1})}{1+r} \right\} \quad (37) \\ s.t: \widehat{m}_{+1} &= (1-\pi)\left(\widehat{m} - \sum_{i \in I} \widehat{p}_i \cdot X_i\right) + \pi \quad \text{and} \quad 0 \leq \widehat{m} - \sum_{i \in I} \widehat{p}_i \cdot d_i \end{aligned}$$

for unemployed workers,

$$\begin{aligned} V_e(\widehat{m}) &= \max_{0 \leq d_i} \mathbb{E} \left\{ \frac{\sum_{i \in I} X_i + (1-s) \cdot V_e(\widehat{m}_{+1}) + s \cdot V_u(\widehat{m}_{+1})}{1+r} \right\} \quad (38) \\ s.t: \widehat{m}_{+1} &= (1-\pi)\left(\widehat{m} + \widehat{W} - \sum_{i \in I} \widehat{p}_i \cdot X_i\right) + \pi \quad \text{and} \quad 0 \leq \widehat{m} - \sum_{i \in I} \widehat{p}_i \cdot d_i \end{aligned}$$

for employed workers, and

$$\begin{aligned}
V_f(\widehat{m},\zeta,l) &= \max_{w,0\le d_i,0\le\sigma_i,0\le\omega} \mathbb{E}\left\{\frac{\sum_{i\in I} X_i - k\cdot\omega + V_f\left(\widehat{m}_{+1},\zeta_{+1},l_{+1}\right)}{1+r}\right\} \qquad (39)\\
s.t: \widehat{m}_{+1} &= (1-\pi)\left(\widehat{m}+\sum_{i\in I}\widehat{p}_i\cdot(Y_i-X_i)-\widehat{W}\cdot l\right)\\
\zeta_{+1} &= \zeta-\Delta-\sum_{i\in I}Y_i+q\cdot l \qquad l_{+1}=l-R+Z \qquad \sum_{i\in I}w_i\cdot\mathbb{E}(Y_i)=\mathbb{E}\left(\widehat{W}\right)\\
0 &\le \widehat{m}-\sum_{i\in I}\widehat{p}_i\cdot d_i \qquad \text{and} \qquad 0\le\zeta-\sum_{i\in I}\sigma_i
\end{aligned}$$

for entrepreneurs. We again guess that value functions are of the form given by (8) and (15), except that we denote $\widehat{A}$, the marginal value of intensive money holdings. Let λ and η be again the non-negative Kuhn-and-Tucker multiplier associated respectively to constraints (29) and (31). The first-order conditions with respect to demand d_i and the envelope condition over intensive money holdings $\widehat{m}$ are now:

$$0\ge\frac{\mu(\phi_i)\left(1-\widehat{p}_i\cdot\widehat{A}(1-\pi)\right)}{1+r}-\lambda\cdot\widehat{p}_i \qquad \text{and} \qquad \widehat{A}=\frac{1-\pi}{1+r}\widehat{A}+\lambda$$

An additional unit of intensive money during stage 2d of the current period implies M additional gross units of money at that time, thereby at stage 1b of the next period. Therefore, the induced increase of intensive money at the next period will only be $1-\pi$. Hence, the growth of money supply generates a depreciation process for intensive money holdings that appears in the envelope condition. Moreover, the expected gain when a unit of demand is satisfied in submarket i is $1-\widehat{p}_i\cdot\widehat{A}(1-\pi)$. Hence, applying the same reasoning as earlier leads to:

$$\widehat{p}_i\cdot\widehat{A}=\frac{\mu(\phi_i)}{r+\pi+(1-\pi)\mu(\phi_i)} \qquad (40)$$

The rationing of demand term now includes a parameter that represents the depreciation effect of monetary growth on money holdings: the rate of inflation π in (40) plays a similar role as the depreciation probability δ in the rationing of supply (see (16) or (41) below).

For the supply decisions, the first-order conditions with respect to supply σ_i and the envelope condition for inventories ζ now write:

$$0\ge\frac{\phi_i\mu(\phi_i)\left(\widehat{p}_i\cdot\widehat{A}(1-\pi)-G\right)}{1+r}-\eta \qquad \text{and} \qquad G=\frac{1-\pi}{1+r}G+\eta$$

A unit of good sold at price p_i (hence at the intensive price $\widehat{p}_i$), is not valued $\widehat{p}_i\cdot\widehat{A}$ but $\widehat{p}_i\cdot\widehat{A}(1-\pi)$ because of inflation. Therefore, for submarkets e where firms choose to supply some inventories, one has $(r+\delta)\,G = \phi_e \mu(\phi_e) \left(\widehat{p}_e \cdot \widehat{A}(1-\pi) - G\right)$, and so:

$$G = \frac{\phi_e \mu(\phi_e)}{r+\delta+\phi_e \mu(\phi_e)} \widehat{p}_e \cdot \widehat{A}(1-\pi) \tag{41}$$

This equation, together with (40), induces $G = \Gamma(\phi_e)$, where function $\Gamma(.)$ is redefined according to:

$$\Gamma(\phi) = \frac{\phi\mu(\phi)}{r+\delta+\phi\mu(\phi)} \cdot \frac{(1-\pi)\cdot\mu(\phi)}{r+\pi+(1-\pi)\cdot\mu(\phi)} \tag{42}$$

Conversely, for submarkets d where firms find non-profitable to supply goods, one has $(r+\delta)\,G > \phi_d \mu(\phi_d) \left(\widehat{p}_d \cdot \widehat{A}(1-\pi) - G\right)$. This, together with (40) induces $G > \Gamma(\phi_d)$. Therefore, at equilibrium, only submarkets where tightness maximizes function $\Gamma(.)$ are active. Appendix 2 show that, under Assumptions AS2 and AS3 the redefined function $\Gamma(.)$ admits a single maximum. Denoting this maximum by $\tilde{\phi}$, it now solves:

$$\frac{r+\pi}{r+\delta} \cdot \frac{r+\delta+\tilde{\phi}\mu\left(\tilde{\phi}\right)}{r+\pi+(1-\pi)\,\mu\left(\tilde{\phi}\right)} = \frac{1-\varepsilon\left(\tilde{\phi}\right)}{\varepsilon\left(\tilde{\phi}\right)} \tag{43}$$

Hence, as before, only a single submarket is active at a time where $\phi_e = \tilde{\phi}$. From the envelope condition over employment l, Equation (17) is replaced by:

$$(r+s)\,J = q \cdot G - \mathbb{E}\left(\widehat{W}\right) \cdot \widehat{A}(1-\pi) \tag{44}$$

From (37) and (38), we get that equation (10) is replaced by

$$r \cdot V_u = \pi \cdot \widehat{A} + \theta h(\theta)(V_e - V_u) \quad r \cdot V_e = \mathbb{E}\left(\widehat{W}\right) \cdot \widehat{A}(1-\pi) + \pi \cdot \widehat{A} + s(V_u - V_e) \tag{45}$$

This is because the wage is only paid at stage 2d of the current period, and so, will only be valued in the next period. Since this process is symmetric for workers and firms, inflation does not affect the sharing rule (19). Finally, from the first-order condition over vacancies in (39), the free-entry condition (18) remains. Hence, equation (20) still determines the equilibrium value of the labor market tightness θ as a function of the value of inventory G. Then the CSE with inflation is still uniquely recursively characterized by (43), (20), (1) and (23). At each period, the price level is given by (27). The only novelty is the redefinition of function $\Gamma(.)$ in (42).

We now consider how a rise in money growth affects the real part of the economy. A rise in π reinforces the *rationing of demand*, since $(1-\pi)\cdot\mu(\phi)\,/\,(r+\pi+(1-\pi)\cdot\mu(\phi))$ decreases with π for any given ϕ. Hence, as for the depreciation rate of inventories δ, the maximized value of Γ decreases (applying the envelope theorem). From equation (26), tightness θ on the labor market decreases, so unemployment u increases (see 1). Finally, according to Appendix B, the equilibrium value $\tilde{\phi}$ decreases.

Proposition 2 *In steady state, a permanent rise in money growth π decreases θ, increases the unemployment rate u and decreases tightness on the product market.*

VI Inefficiency

In this section we investigate whether the CSE equilibrium is socially efficient. We adopt a utilitarian social welfare fucntion. There is a positive flow of utility derived from transactions and a disutility flow derived from vacancy posting. Since $\Sigma = S$, the social objective Ω therefore equals the discounted sum of $T(S, D) - k \cdot v$.

To see whether the equilibrium value of ϕ is socially efficient, we consider a marginal departure from the CSE, taking tightness on the labor market θ (hence the unemployment rate u, and the mass of vacant jobs v) as fixed. The social planner controls tightness on the product market ϕ, while total inventories S evolve according to (4). Taken $S = \Sigma$ into account, (4) becomes:

$$S_{+1} = (1 - u) q + (1 - \delta - \phi\mu(\phi)) S \tag{46}$$

This is equivalent to assuming that the social planner controls aggregate demand $D = M/p$, but that inventories' dynamics remains determined by the matching frictions on the product market. In steady state, (46) together with $T(S, D) = \phi\mu(\phi) \cdot S$ gives:

$$T(S, D) = \frac{\phi\mu(\phi)}{\delta + \phi\mu(\phi)} \cdot q(1 - u) \tag{47}$$

Starting from a steady state, Figure 6 considers the effect of a permanent rise in ϕ when u and v are fixed (see panel a). As a consequence, the probability to sell each good increases. If $\delta > 0$, goods have therefore less chance to depreciate. Then, the flow of consumed goods $T(S, D)$ increases in steady state, thereby increasing social welfare Ω. The higher the depreciation rate δ, the more important is this effect.

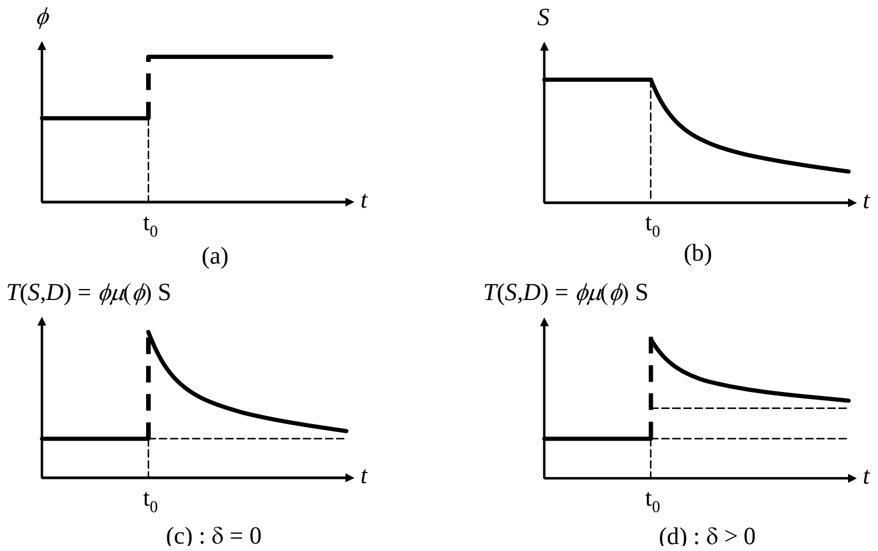

Figure 6: The effect of a permanent rise of ϕ on the social objective.

Moreover, there is an additional effect that occurs along the transition to the new steady state. A permanent rise in ϕ instantaneously rises $T(S,D)$, but has no immediate effect on inventories S (see (46) and panels (b) and (c) or (d) of Figure 6). The amount of supply S decreases progressively to its new steady state value (see panel b). Therefore, along this transition process, the flow of consumed goods $T(S,D) = \phi\mu(\phi) \cdot S$ is higher than its new steady-state value. As we have seen, if $\delta > 0$, the new steady-state value is higher (see panel d), while if $\delta = 0$, the new steady-state value of $T(S,D)$ is unchanged (see panel c). However, in both case, $T(S,D)$ overshoots. As the discount rate increases, this overshooting has a bigger effect on welfare. Actually, applying our reasoning to any value of ϕ, we conclude that welfare always increases with a rise in ϕ, suggesting that the optimal value of ϕ is infinite.

This inefficiency property of a CSE may look surprising given the efficiency result of Moen (1997) in a non-monetary economy. The reason for our inefficiency result is that firms and the social criterion value a transaction on the product market differently. When they sell their inventories, entrepreneurs take into account that they will typically not instantaneously find the goods they want to consume. This *rationing of demand* is due to the monetary feature of the economy. As we have seen in Section IV, the equilibrium value of ϕ trades off the *rationing of demand* and the *rationing of supply*. Conversely, the social criterion values the flow of transactions, no matter who consumes the good exchanged. Therefore, the *rationing of demand* does not matter, only the *rationing of supply* does. This explains why the equilibrium value of tightness on the product market is always inefficiently low.

To conclude this section, we discuss how monetary policy should be conducted to attenuate this inefficiency. From above, monetary policy should be used to increase tightness ϕ on the product market. This requires to lower inflation π. So doing, money depreciates less and the *rationing of demand* term $(1-\pi) \cdot \mu(\phi) / (r + \pi + (1-\pi) \cdot \mu(\phi))$ increases. Furthermore, the equilibrium value of tightness ϕ increases. The latter effect increases the probability $\phi\mu(\phi)$. The *rationing of supply* term $\phi\mu(\phi) / (r + \delta + \phi\mu(\phi))$ increases and so does social welfare.

We can now wonder what is the optimal rate of inflation, that is, what is the minimum *feasible* level of inflation. If inflation was negative $\pi < 0$, the lump sum transfer $T = \pi \cdot M_{+1}$ would become a tax $T < 0$ that an unemployed worker with a very low money holdings cannot pay. Then, the optimal monetary policy would be $\pi = 0$. However, if it was possible, the Friedman rule according to which $\pi = -r$ would be the optimal monetary policy. As explained in Appendix B, when monetary policy tends to the Friedman rule, the *rationing of demand* term tends to 1. Furthermore, since ϕ tends to $+\infty$ infinity, the *rationing of supply* term increases and tends to $1/(1+r+\delta)$. These two effects together maximizes $\Gamma(.)$.

VII Conclusion

In this paper, we have extended the MP labor-matching model by introducing search-frictions on the product market without the introduction of nominal rigidities. We account for the persistent difficulties that firms face in selling their production and we are able to model how these difficulties affect equilibrium unemployment. We show that unemployment is a U-shaped function of tightness on the product market. The parameters characterizing the labor market have the usual effect on unemployment. Price posting and directed search on the product market lead to a Competitive Search Equilibrium on the product market. The equilibrium tightness on the product market is the one that minimizes the unemployment rate, but is inefficiently low. Moreover, a higher level of inflation increases unemployment.

This model can be extended in different directions. First, we have only been concerned with properties in steady state. The transitional dynamics and the cyclical properties of this economy should be investigated. Second, our model abstracts from public finance issues. One can wonder whether fiscal deficits can increase aggregate demand and thereby decrease the equilibrium unemployment rate. To answer this question, we should empirically investigate whether demand expressed by the government matches more rapidly the supply of private goods. Finally, one should consider the introduction of a credit market.

A Bellman equations

In this paper, we solve Bellman equations for workers and entrepreneurs by the "Guess and verify" method (see Ljungvist and Sargent 2000, Page 32). We guess that workers' (resp. entrepreneurs') value functions are of the form given by (8) (resp. 15) and derive the values of A, V_e and V_u (resp. $\overline{A}$, G and J) that satisfy Bellman equations. We have now to verify:

- That each Bellman equation admits a single solution[11]. Hence, if we exhibit a function that solves the Bellman equation, this function necessarily coincides with the value function. The proof here follows very closely Stockey, Lucas and Prescott (1989, henceforth SLP).

[11] Bellman equation are functional equations, that is, equations where the unknown is not (a list of) number(s) but a function.

- That given the obtained values of A, V_e, V_u, $(A,\, G,\, J)$, the obtained value functions effectively solve the Bellman equations.

We proove these two results for both workers and entrepreneurs' Bellman equations. The proofs are written for the most general case with endogenous prices and inflation. The proof without inflation is directly obtained by substituting 0 for π. The proof with fixed prices is again directly obtained with the additional restriction that the set I is reduced to a singleton.

A.1 Workers' program

The important point to notice is that a worker can never hold more (intensive) money $\widehat{m}$ than the aggregate (intensive) money holdings $\widehat{M}$ [12], so $0 \leq \widehat{m} \leq \widehat{M}$. Hence, the state space is bounded. SLP can then be directly applied.

To be more precise, let $\mathcal{F}_w$ be the space of continuous functions mapping $\Omega_w = \left[0, \widehat{M}\right] \times \{e, u\}$ to $\mathbb{R}$. Let $f \in \mathcal{F}_w$. We denote $f_i(m)$ the image of $(m, i) \in \Omega_f$ by f. We use the norm[13] $\|f\|_w = \max\limits_{(m,i)\in\Omega_w} |f_i(m)|$. Hence, $(\Omega_w, \|.\|_w)$ is a complete metric space.

We now define an operator T on $\mathcal{F}$. For any $f \in \mathcal{F}$, $T(f)$ is a function whose image is given by the following equalities. If $i = u$, one has:

$$\begin{aligned} (T(f))_u(\widehat{m}) &= \max_{0\leq d_i}\mathbb{E}\left\{\frac{\sum\limits_{i\in I} X_i + \theta h(\theta)\cdot f_e(\widehat{m}_{+1}) + (1-\theta h(\theta))\cdot f_u(\widehat{m}_{+1})}{1+r}\right\} \\ s.t: \widehat{m}_{+1} &= (1-\pi)\left(\widehat{m} - \sum_{i\in I}\widehat{p}_i\cdot X_i\right) + \pi \qquad \text{and} \qquad 0 \leq \widehat{m} - \sum_{i\in I}\widehat{p}_i\cdot d_i \end{aligned} \tag{48}$$

If $i = e$, one has:

$$\begin{aligned} (T(f))_e(\widehat{m}) &= \max_{0\leq d_i}\mathbb{E}\left\{\frac{\sum\limits_{i\in I} X_i + (1-s)\cdot f_e(\widehat{m}_{+1}) + s\cdot f_u(\widehat{m}_{+1})}{1+r}\right\} \\ s.t: \widehat{m}_{+1} &= (1-\pi)\left(\widehat{m} + \widehat{W} - \sum_{i\in I}\widehat{p}_i\cdot X_i\right) + \pi \qquad \text{and} \qquad 0 \leq \widehat{m} - \sum_{i\in I}\widehat{p}_i\cdot d_i \end{aligned} \tag{49}$$

According to the Bellman equations (37) and (38), the value functions $V_u(.)$ and $V_e(.)$ define a fixed point of operator $T(.)$ on $\mathcal{F}$.

[12] Without inflation, forget the adjective "intensive". One has then $0 \leq m \leq M$. With inflation, one has $0 \leq \hat{m} \leq 1$. Let in both cases $\hat{M}$ be this max.

[13] Since Ω_w is a bounded subset of $\mathbb{R}^2$, any continuous function mapping Ω_w into $\mathbb{R}$ is necessarily bounded. Hence the norm is well defined over $\mathcal{F}_w$.

- The terms in brackets under the max of (48) and (49) are continous functions of $\widehat{m}$ for any given $(d_i)_{i\in I}$. Hence, by the theorem of the Maximum (see. e.g. SLP, section 3.3) the right-hand side of (48) and (49) are continous functions of $\widehat{m}$. Hence $T(f)$ is continuous over $\widehat{m}$, which insures that T maps $\mathcal{F}$ into itself.

- T verifies the Blackwell condition (see. e.g. SLP, theorem 3.3). This is easy to see by fixing d_i. Therefore, T is a contracting mapping function of $\mathcal{F}$ into $\mathcal{F}$.

Therefore, T admits a unique fixed point in $\mathcal{F}$. Since workers' value functions $V_e(.)$ and $V_u(.)$ define a fixed point of T, the fixed point of T coincides with values functions $V_e(.)$ and $V_u(.)$.

We now verify that given the values of V_e, V_u and $\widehat{A}$ given in (45) and (40), the value functions defined by $V_i(m) = V_i + \widehat{A}\cdot\widehat{m}$ verify $(T(V))_i(\widehat{m}) = V_i(\widehat{m})$, for any $\widehat{m}$ and any $i = e, u$. We get from (48):

$$
\begin{aligned}
(T(V))_u(\widehat{m}) &= \max_{0\le d_i}\mathbb{E}\left\{\frac{\sum_{i\in I} X_i + \widehat{A}\cdot\widehat{m}_{+1} + \theta h(\theta)\cdot V_e + (1-\theta h(\theta))\cdot V_u}{1+r}\right\} \\
s.t: \widehat{m}_{+1} &= (1-\pi)\left(\widehat{m} - \sum_{i\in I}\widehat{p}_i\cdot X_i\right) + \pi \qquad \text{and} \qquad 0 \le \widehat{m} - \sum_{i\in I}\widehat{p}_i\cdot d_i
\end{aligned}
$$

Together with (45), this leads to $(T(V))_u(\widehat{m}) = V_u + \mathcal{T}(\widehat{m})$ where,

$$
\mathcal{T}(\widehat{m}) = \max_{0\le d_i, 0\le \widehat{m} - \sum_{i\in I}\widehat{p}_i\cdot d_i} \frac{1}{1+r}\mathbb{E}\left\{\sum_{i\in I} X_i + \widehat{A}\cdot(1-\pi)\left(\widehat{m} - \sum_{i\in I}\widehat{p}_i\cdot X_i\right)\right\}
$$

Doing the same for $(T(V))_e$ from (49), leads to $(T(V))_e(\widehat{m}) = V_e + \mathcal{T}(\widehat{m})$. From (28), we get:

$$
\mathcal{T}(\widehat{m}) = \frac{1-\pi}{1+r}\cdot\widehat{A}\cdot\widehat{m} + \max_{0\le d_i, 0\le \widehat{m} - \sum_{i\in I}\widehat{p}_i\cdot d_i} \frac{\sum_{i\in I}\mu(\phi_i)\cdot d_i\left(1 - \widehat{p}_i\cdot\widehat{A}(1-\pi)\right)}{1+r}
$$

Given (35), we have

$$
\mathcal{T}(\widehat{m}) = \frac{1-\pi}{1+r}\cdot\widehat{A}\cdot\widehat{m} + \frac{r+\pi}{1+r}\max_{0\le d_i, 0\le \widehat{m} - \sum_{i\in I}\widehat{p}_i\cdot d_i}\sum_{i\in I}\frac{\mu(\phi_i)}{r+\pi+(1-\pi)\mu(\phi_i)}\cdot d_i
$$

Using (40):

$$
\mathcal{T}(\widehat{m}) = \frac{1-\pi}{1+r}\cdot\widehat{A}\cdot\widehat{m} + \frac{r+\pi}{1+r}\max_{0\le d_i, 0\le \widehat{m} - \sum_{i\in I}\widehat{p}_i\cdot d_i}\sum_{i\in I}\widehat{A}\cdot\widehat{p}_i\cdot d_i
$$

Therefore, the constraint $0 \leq \widehat{m} - \sum_{i \in I} \widehat{p}_i \cdot d_i$ binds and we obtain:

$$\mathcal{T}(\widehat{m}) = \widehat{A} \cdot \widehat{m} \left\{ \frac{1-\pi}{1+r} + \frac{r+\pi}{1+r} \right\} = \widehat{A} \cdot \widehat{m}$$

which leads to:

$$(T(V))_i(\widehat{m}) = V_i + \widehat{A} \cdot \widehat{m} \qquad \text{for any } i = e, u \text{ and } \widehat{m} \in \left[0, \widehat{M}\right]$$

and therefore ends the proof.

A.2 Entrepreneurs' program

For the entrepreneurs' program, we notice that along a steady state, the three state variables of an entrepreneur, namely $\widehat{m}$, ζ and l, are bonded, respectively by $\widehat{M}$, S and $1-u$. We now consider the space $\mathcal{F}_f$ of continuous functions mapping $\Omega_f = \left[0, \widehat{M}\right] \times [0, S] \times [0, 1-u]$ into $\mathbb{R}$. For any $f \in \mathcal{F}_f$, we define the norm $\|f\| = \max_{(m,\zeta,l) \in \Omega_f} |f_i(m, \zeta, l)|$. Hence, $(\Omega, \|.\|)$ is a complete metric space. We define the operator T_f over $\mathcal{F}_f$ by:

$$\begin{aligned}
(T_f(f))(\widehat{m}, \zeta, l) &= \max_{w, 0 \leq d_i, 0 \leq \sigma_i, 0 \leq \omega} \mathbb{E} \left\{ \frac{\sum_{i \in I} X_i - k \cdot \omega + f\left(\widehat{m}_{+1}, \zeta_{+1}, l_{+1}\right)}{1+r} \right\} \qquad (50) \\
s.t: \widehat{m}_{+1} &= (1-\pi)\left(\widehat{m} + \sum_{i \in I} \widehat{p}_i \cdot (Y_i - X_i) - \widehat{W} \cdot l\right) \\
\zeta_{+1} &= \zeta - \Delta - \sum_{i \in I} Y_i + q \cdot l \qquad l_{+1} = l - R + Z \qquad \sum_{i \in I} w_i \cdot \mathbb{E}(Y_i) = \mathbb{E}(W) \\
0 &\leq \widehat{m} - \sum_{i \in I} \widehat{p}_i \cdot d_i \qquad \text{and} \qquad 0 \leq \zeta - \sum_{i \in I} \sigma_i
\end{aligned}$$

According to the Bellman equation (39), $V_f(.,.,.)$ defines a fixed point of the operator T_f. We can replicate the same arguments as before to show that T_f defines a contraction mapping $\mathcal{F}_f$ into itself. Therefore, a single function solves the Bellman equation (39), which coincides with entrepreneurs' value function V_f. It remains to show that given (40), (41) and (44), one has for any $(\widehat{m}, \zeta, l) \in \Omega_f$, $(T_f(V_f))(\widehat{m}, \zeta, l) = V_f(\widehat{m}, \zeta, l)$. We get from (50):

$$(1+r) \cdot (T_f(V_f))(\widehat{m}, \zeta, l) = \max_{w, 0 \leq d_i, 0 \leq \sigma_i, 0 \leq \omega} \mathbb{E} \left\{ \sum_{i \in I} X_i - k \cdot \omega + f\left(\widehat{m}_{+1}, \zeta_{+1}, l_{+1}\right) \right\}$$

subject to the constraints in (50). Hence, with (28), (30) and (13)

$$\begin{aligned}(1+r)\cdot(T_f(V_f))(\widehat{m},\zeta,l) \;=\;& \widehat{A}(1-\pi)\cdot\widehat{m}+\max_{0\le d_i,}\left\{\sum_{i\in I}\mu(\phi_i)\cdot d_i\left(1-\widehat{p}_i\cdot\widehat{A}(1-\pi)\right)\right\} \\ &+(1-\delta)\zeta+\max_{0\le\sigma_i}\left\{\sum_{i\in I}\phi_i\mu(\phi_i)\cdot\sigma_i\left(\widehat{p}_i\cdot\widehat{A}(1-\pi)-G\right)\right\} \\ &+(1-s)J+q\cdot G-\widehat{A}(1-\pi)\cdot\mathbb{E}(W)+\max_{0\le\omega}\{(-k+h(\theta)\cdot J)\omega\}\end{aligned} \tag{51}$$

From (40), we get

$$1-\widehat{p}_i\cdot\widehat{A}(1-\pi)=\frac{r+\pi}{r+\pi+(1-\pi)\mu(\phi_i)}$$

so,

$$\begin{aligned}\sum_{i\in I}\mu(\phi_i)\cdot d_i\left(1-\widehat{p}_i\cdot\widehat{A}(1-\pi)\right) &= (r+\pi)\sum_{i\in I}\frac{\mu(\phi_i)}{r+\pi+(1-\pi)\mu(\phi_i)}\cdot d_i \\ &= (r+\pi)\cdot\widehat{A}\cdot\sum_{i\in I}\widehat{p}_i\cdot di\end{aligned}$$

Hence, the constraint $0\le\widehat{m}-\sum_{i\in I}\widehat{p}_i\cdot d_i$ binds and we get:

$$\widehat{A}(1-\pi)\cdot\widehat{m}+\max_{0\le d_i,}\sum_{i\in I}\mu(\phi_i)\cdot d_i\left(1-\widehat{p}_i\cdot\widehat{A}(1-\pi)\right)=(1+r)\widehat{A}\cdot\widehat{m} \tag{52}$$

From the first-order condition over σ_i, either $\sigma_i=0$ or the i^{th} market is active. For this (these) latter submarkets, we get from (41):

$$\widehat{p}_i\cdot\widehat{A}(1-\pi)-G=\frac{r+\delta}{r+\delta+\phi_i\mu(\phi_i)}$$

for markets where firms effectively supply their outputs ($i=e$). Therefore,

$$\sum_{i\in I}\phi_i\mu(\phi_i)\cdot\sigma_i\left(\widehat{p}_i\cdot\widehat{A}(1-\pi)-G\right)=(r+\delta)\sum_{i=e}\frac{\phi_i\mu(\phi_i)}{r+\delta+\phi_i\mu(\phi_i)}\cdot\sigma_i=(r+\delta)G\sum_{i=e}\sigma_i$$

Hence, constraint $0\le\zeta-\sum_{i\in I}\sigma_i$ binds and we have:

$$(1-\delta)\zeta+\max_{0\le\sigma_i}\left\{\sum_{i\in I}\phi_i\mu(\phi_i)\cdot\sigma_i\left(\widehat{p}_i\cdot\widehat{A}(1-\pi)-G\right)\right\}=(1+r)G\cdot\zeta \tag{53}$$

Finally, the first order condition over ω implies that either $\omega=0$ or $k=h(\theta)J$. In both cases, we get $(-k+h(\theta)\cdot J)\omega=0$. Therefore,

$$\begin{aligned}&(1-s)J+q\cdot G-\widehat{A}(1-\pi)\cdot\mathbb{E}(W)+\max_{0\le\omega}\{(-k+h(\theta)\cdot J)\omega\} \\ =\;&(1-s)J+q\cdot G-\widehat{A}(1-\pi)\cdot\mathbb{E}(W)=(1+r)J\end{aligned} \tag{54}$$

The last equality following (44). Substituting (52), (53) and (54) in (51), we finally get

$$(T_f(V_f))(\widehat{m},\zeta,l)=\widehat{A}\cdot\widehat{m}+G\cdot\zeta+l\cdot J=V_f(\widehat{m},\zeta,l)$$

which ends the proof.

B Function $\Gamma(\phi)$

In this appendix, we take the definition of $\Gamma(.)$ that includes money growth (see 42). To get the properties without inflation (as in 24), one should only replace π by 0 in the following algebra. Function $\Gamma(.)$ is differentiable and thereby continuous on $\mathbb{R}_*^+$. Asymptotically, the boundary properties of $\mu(\phi)$ and of $\phi\mu(\phi)$ imply:

$$\begin{aligned}\lim_{\phi\mapsto 0}\frac{\phi\mu(\phi)}{r+\delta+\phi\mu(\phi)} &= \lim_{\phi\mapsto+\infty}\frac{(1-\pi)\mu(\phi)}{r+\pi+(1-\pi)\mu(\phi)}=0\\ \lim_{\phi\mapsto 0}\frac{\mu(\phi)}{r+\pi+\mu(\phi)} &= \lim_{\phi\mapsto+\infty}\frac{(1-\pi)\mu(\phi)}{r+\pi+(1-\pi)\mu(\phi)}\leq 1\end{aligned}$$

so:

$$\lim_{\phi\to 0}\Gamma(\phi)=\lim_{\phi\to+\infty}\Gamma(\phi)=0$$

For all $\phi>0$, $\Gamma(\phi)>0$. Since Γ is continuous, there exists at least one value $\widetilde{\phi}$ such that for any ϕ, $\Gamma(\phi)\leq\Gamma\left(\widetilde{\phi}\right)$. In particular, we get $\Gamma'\left(\widetilde{\phi}\right)=0$. To determine uniqueness of $\widetilde{\phi}$, we consider the derivative of $\ln\Gamma(\phi)$:

$$\frac{\phi\Gamma'(\phi)}{\Gamma(\phi)}=\phi\frac{\partial\ln\Gamma(\phi)}{\partial\phi}=\frac{r+\delta}{r+\delta+\phi\mu(\phi)}\left(1+\frac{\phi\mu'(\phi)}{\mu(\phi)}\right)+\frac{r+\pi}{r+\pi+(1-\pi)\cdot\mu(\phi)}\cdot\frac{\phi\mu'(\phi)}{\mu(\phi)}$$

Since $-\frac{\phi\mu'(\phi)}{\mu(\phi)}\equiv\varepsilon(\phi)\in(0,1)$, one has:

$$\frac{\phi\Gamma'(\phi)}{\Gamma(\phi)}=\frac{r+\delta}{r+\delta+\phi\mu(\phi)}(1-\varepsilon(\phi))-\frac{r+\pi}{r+\pi+(1-\pi)\cdot\mu(\phi)}\varepsilon(\phi)$$

From Assumptions AS2 and AS3, $\frac{\phi\Gamma'(\phi)}{\Gamma(\phi)}$ decreases in ϕ. Hence, there exists a single value $\tilde{\phi}$ of tightness on the product market such that $\Gamma'\left(\tilde{\phi}\right)=0$. We further get that $\Gamma'(\phi)\lesseqgtr 0$ when $\phi\gtreqless\tilde{\phi}$. Consequently, function $\Gamma(.)$ is hump-shaped. From above, $\tilde{\phi}$ is implicitly defined by:

$$0=\Phi(\phi,r,\delta,\pi)\equiv\frac{r+\delta}{r+\delta+\phi\mu(\phi)}(1-\varepsilon(\phi))-\frac{r+\pi}{r+\pi+(1-\pi)\cdot\mu(\phi)}\varepsilon(\phi)$$

This last equality gives (43), and (25) if $\pi=0$. Since $\Phi'_\phi<0$ and $\Phi'_\delta>0$ and $\Phi'_\pi<0$, $\tilde{\phi}$ increases with the depreciation rate of inventories δ and decreases with the inflation rate π.

At the Friedman rule (i.e. when $\pi=-r$), for any $\phi\in\mathbb{R}_*^+$, the *rationing of demand* term is constant and equal to 1 and for all ϕ, we have that $\Phi(\phi,....)>0$. Therefore, an equilibrium does not exist at the Friedman rule. However, as the inflation rate tends to the Friedman rule, $\tilde{\phi}$ tends to $+\infty$.

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