Complementarities and Substitutabilities in Matching Models

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Abstract

This paper describes an equilibrium matching model with two types of workers producing two different intermediate goods. Labour markets are perfectly segmented, but productive complementarities between sectors and productive substitutability within sectors arise. This deeply changes the effects of labour market policies. A welfare analysis is also conducted. Under constant returns to scale in the matching technology, the so-called Hosios condition is sufficient to guarantee the efficiency of the decentralized equilibrium.

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1 Introduction

The equilibrium matching model has become the standard framework of analysis for aggregate labour markets. In a standard Pissarides (2000) framework, the matching process between one firm and one worker is not instantaneous, because of some generic frictions present in the labour market. The good market, on the contrary, is perfectly competitive and firms face a completely elastic demand, so that an increase in supply does not affect the optimal price (see for instance Mortensen and Pissarides, 1999).

Starting with the work of Merz (1995) and Andolfatto (1996), search frictions have been also introduced in pure RBC models. In these papers, the assumption of perfect elasticity of the demand for goods is abandoned and a two-tier productive scheme is usually considered. Workers are employed in intermediate goods sectors. Such goods, together with capital, are sold to a final representative firm that produces the unique consumption good. However, these models are usually solved only numerically and scant attention has been paid about their analytical properties.

In this paper I consider a simplified framework where there are only two intermediate sectors and individual have no access to capital markets. I show that an equilibrium exists and is unique under mild assumptions on preferences and technology. Then, I perform some comparative statics and I compare the results with those obtained in a standard matching framework. When the demand for goods is no longer perfectly elastic, new complementarities and substitutabilities arise in addition to that generated by the matching technology. Every additional vacancy created in one sector decreases firms' revenues in that sector and enhances firms' revenues of the other one. These effects come trough the price of the intermediate goods. Since a standard matching framework does not take these effects into account, I show it overestimates the impact of a shock or policy intervention in the sector where such shock has occurred and, at the same time, it ignores the effects that emerge on other sectors of the economy.

This setup is similar to that developed by Acemoglu (2001) and Cahuc and Zylberberg (2004, pages 618-622). The difference with respect to Acemoglu's paper is that he proves the existence of (at least) one equilibrium only for a CES final good production function while I consider any constant returns to scale technology. In Cahuc and Zylberberg's textbook, on the other hand, only policy implications of their model are shown but without explaining the mechanism that is at work in such set-up. Moreover, existence and uniqueness is not proved.

2 The Model

2.1 Production Technology

Assume an economy with one final good (the numeraire), two intermediate goods sectors and two types m and n of infinitely-lived and risk-neutral workers. This analysis can be carried out even with more intermediate sectors. Here, I consider only two for simplicity. The goods markets are perfectly competitive. Each producer of an intermediate good hires only one type of worker. Moreover, every m-skilled employee produces one unit of the intermediate good m and every n-skilled employee produces one unit of the intermediate good n. Let E_m (respectively, E_n) denote both the amount of the m intermediate good (respectively, the n intermediate good) used to produce the final good and the number of workers employed in the m-th (resp. n-th) sector. The final good production function is homogeneous of degree one and written as:

$$Y = F(E_m, E_n), \quad \text{with } \frac{\partial F}{\partial E_i} > 0 \text{ and } \frac{\partial^2 F}{\partial {E_i}^2} < 0, i \in \{m, n\}. \tag{1}$$

Furthermore, the two inputs are p-substitutes ($0 < \frac{\partial^2 F}{\partial E_m \partial E_n} < +\infty$)[1]. Let p_i denote the real price of the intermediate good i. Cost minimization leads to $p_i = \partial F(E_m, E_n)/\partial E_i$, with $i \in \{m, n\}$. The price of each intermediate good depends negatively on the number of workers employed in that sector and positively on the number of workers employed in the other sector.

2.2 Search Technology

The model is markovian and developed in steady state. Time is continuous. Each type of worker can be either unemployed and receive an unemployment benefit b_i or be employed in his sector. The labour market is perfectly segmented. That is, every m-type worker can be hired only by firms in the m sector and the same holds for n-type workers. Due to various imperfections, the matching process is not instantaneous. The matching function is by assumption identical in both intermediate sectors and it is written respectively $M_i = m(U_i, V_i)$, with U_i being the number of unemployed people and V_i the number of job vacancies in sector i. The function $m(.,.)$ is assumed to be increasing, concave and homogeneous of degree 1. Search intensity is exogenous and normalized to 1. Due to the constant return to scale in the matching process, the model can be developed in terms of tightness indicator, namely $\theta_i \equiv \frac{V_i}{U_i}$. The rate at which vacant jobs become filled is $q(\theta_i) = M_i/V_i = m(\frac{1}{\theta_i}, 1), q'(\theta_i) < 0$. Every

[1] I also assume Inada conditions: $\lim_{E_n \to 0} \frac{\partial F}{\partial E_i} = +\infty$ and $\lim_{E_i \to +\infty} \frac{\partial F}{\partial E_n} = 0$.

unemployed worker moves into employment according to a Poisson process with rate $\alpha(\theta_i) \equiv M_i/V_i = \theta_i q(\theta_i)$, with $\alpha'(\theta_i) > 0^2$. The elasticity of the probability q_i of filling a vacancy with respect to tightness θ_i is denoted by $\eta(\theta_i) \equiv -\frac{dq(\theta_i)}{d\theta_i}\frac{\theta_i}{q(\theta_i)}$.

In steady state, the stocks of individuals in each position are constant. With an exogenous size of the labor force, L_i, the employment rate $e_i = E_i/L_i$ in steady state is given by:

$$e_i = \frac{\alpha(\theta_i)}{\phi_i + \alpha_i(\theta_i)}, i \in \{m, n\} . \tag{2}$$

2.3 Preferences and job creation

Individuals are risk-neutral and have no access to capital market. Let r be the discount rate common to all agents. In steady state, the expected lifetime income for an unemployed worker is:

$$rV_{U,i} = b_i + \alpha(\theta_i)\left(V_{E,i} - V_{U,i}\right). \tag{3}$$

This type of equation is standard in search literature. Being unemployed is similar to holding an asset that pays a dividend of b_i, the unemployment benefit, and it has a probability $\alpha(\theta_i)$ of being transformed in employment. In this case, the worker obtains $V_{E,i}$, the asset value of being employed, and he loses $V_{U,i}$. Similarly, the steady state discounted present value of employment can be written as:

$$rV_{E,i} = w_i + \phi_i(V_{U,i} - V_{E,i}), \tag{4}$$

where w_i is the wage bargained in the i-th intermediate sector.

On the other side of the market, let $\Pi_{E,i}$ denote the firm's discounted expected return from an occupied job if the firm produces the ith intermediate good, namely it hires workers endowed with skill i. For simplicity, taxation is not considered here and therefore there is no Government budget condition. The discounted expected return of vacant job is $\Pi_{V,i}$. I denote k_i the cost of posting a vacancy and of selecting applicants. For $i \in \{m, n\}$, the discounted expected returns satisfy the following conditions:

$$r\Pi_{E,i} = p_i - w_i + \phi_i\left(\Pi_{V,i} - \Pi_{E,i}\right), \tag{5}$$

$$r\Pi_{V,i} = -k_i + q(\theta_i)\left(\Pi_{E,i} - \Pi_{V,i}\right). \tag{6}$$

In equilibrium, firms open vacancies as long as they yield a positive expected return. Therefore, the equilibrium condition $\Pi_{V,i} = 0$, combined with (5) and (6), yields the following vacancy-supply curve for each j:

$$w_i = VS_i(\theta_i, \theta_j) \equiv p_i - (r + \phi_i)\frac{k_i}{q(\theta_i)}. \tag{7}$$

[2] Moreover, $\lim_{\theta_i \to 0} q(\theta_i) = +\infty$ and $\lim_{\theta_i \to 0} \alpha(\theta_i) = +\infty$.

We can easily see that the vacancy-supply curve represents a decreasing relationship between the net wage and labor market tightness, θ_i. Note also that (7) depends on tightness indicators of both sectors, θ_i and θ_j, through the price of the intermediate good p_i. So, with a zero-profit condition holding, an increase in θ_i implies a lower wage w_i for two reasons. First, because higher labor market tightness raises the expected cost of filling a vacancy, $k_i/q(\theta_i)$. Second, because higher labor market tightness in sector i enhances employment through the steady state equation (2). More people employed in one intermediate sector lowers the marginal productivity of that intermediate good. So i-firms' revenues are reduced (p_i goes down). Moreover, an increase in labor market tightness in the other sector, θ_j, raises the marginal productivity of the intermediate good i, via Equation (2) and the condition we imposed in (1) about the cross derivative.

2.4 Wage formation

I assume that wages are bargained in both sectors of the economy according to the axiomatic Nash solution. If β denotes the bargaining power of the worker ($0 < \beta < 1$), the solution to the game can be written as $V_{E,i} - V_{U,i} = \gamma(V_{E,i} - V_{U,i} + \Pi_{E,i})$. This property, the Bellman equations (4) and (3) and the free-entry condition ($\Pi_{E,i} = k_i/q(\theta_i)$ lead then to the following "wage-setting curve":

$$w_i = WS_i \equiv \beta_i(p_i + \theta_i k_i) + (1 - \beta_i)b_i. \tag{8}$$

For each skill, the wage-setting curve $w_i = WS_i$ cannot be shown to be always upward sloping in (θ_i, w_i) space. The reason is that, as θ_i increases, $k_i\theta_i$ obviously rises, but p_i decreases. Nevertheless, I can equate the RHS of the vacancy-supply curve (7) and the RHS of the wage-setting curve curve (8) and consider an implicit equation for θ_i , namely $\mathbb{G}_i = 0$:

$$\begin{gathered}\mathbb{G}_i(\theta_i,\, \theta_j) \equiv VS_i - WS_j \; = \\ (1 - \beta_i)p_i - k_i\left(\tfrac{r+\phi_i}{q(\theta_i)} + \beta_i\theta_i\right) - (1 - \beta_i)b_i = 0\end{gathered} \tag{9}$$

Note that $\mathbb{G}_i = 0$, the equilibrium condition in labor market i, depends on θ_j only through the marginal productivity p_i . Differentiating $\mathbb{G}_i$ with respect to θ_i , I obtain:

$$\frac{d\mathbb{G}_i}{d\theta_i} \;=\; \left[(1 - \beta_i)\frac{\partial p_i}{\partial E_i}\frac{\partial E_i}{\partial \theta_i} \;+\; k_i(r + \phi_i)\frac{q'(\theta_i)}{q(\theta_i)^2} \;-\; \beta_i\right], \tag{10}$$

that is negative because all the terms inside the square brackets are negative. Moreover, note that $\lim_{\theta_i\to 0} \mathbb{G}_i(\theta_i, \theta_j) = +\infty$ and $\lim_{\theta_i\to +\infty} \mathbb{G}_i(\theta_i, \theta_j) = -\infty$ because of the Inada conditions imposed both in the final good production function and in the vacancy entry rate. So, since $\mathbb{G}_i$ is continuous and always decreasing in θ_i, it exists a θ_i such that $\mathbb{G}_i = 0$, $\forall \theta_j$.

I also write $d\mathbb{G}_i/d\theta_i = A_i + B_i$ with

$$A_i \equiv \left[k_i(r+\phi_i)\frac{q'(\theta_i)}{q(\theta_i)^2} - \beta_i \right] \text{ and } B_i \equiv \left[(1-\beta_i)\frac{\partial p_i}{\partial E_i}\frac{\partial E_i}{\partial \theta_i} \right],$$

$i \in \{m, n\}$. Both A_i and B_i are negative; A_i is the part of the derivative in (10) that stems from the search technology, whereas B_i is the part that stems from the production technology. Finally, I differentiate $\mathbb{G}_i$ with respect to θ_j:

$$\frac{d\mathbb{G}_i}{d\theta_j} \equiv C_{i,j} = (1-\beta_i)\frac{\partial p_i}{\partial E_j}\frac{\partial E_j}{\partial \theta_j} > 0. \tag{11}$$

The term is positive because the intermediate goods are p-substitutes.

2.5 Existence and uniqueness of the equilibrium

The steady-state equilibrium values of tightness in both sectors are characterized by a system of two equations:

$$\begin{cases} G_n(\theta_n, \theta_m) = 0 \\ G_m(\theta_m, \theta_n) = 0 \end{cases} \tag{12}$$

where every function $\mathbb{G}_i$ represents the equilibrium condition in the i-th labor market. The novelty with respect to the standard matching model hinges on the link between the two intermediate sectors: each labor market depends on tightness of the other one through the price (the marginal productivity) of the intermediate good.

Proposition 1 *There is a unique steady-state equilibrium in tightness levels*

Proof. I divide the proof in three steps:

1. *Intercepts of* $\mathbb{G}_n(\theta_n, \theta_m) = 0$ *and* $\mathbb{G}_m(\theta_m, \theta_n) = 0$.

   Think of $\mathbb{G}_n = 0$ and $\mathbb{G}_m = 0$ as two functions in (θ_n, θ_m) space. Then, consider $\lim_{\theta_j \to 0} \mathbb{G}_i(\theta_i, \theta_j) = 0$. Since $\lim_{\theta_j \to 0} E_j = 0$, I conclude that the solution θ_i of $\lim_{\theta_j \to 0} \mathbb{G}_i(\theta_i, \theta_j) = 0$ is positive. Hence, the intercept of $\mathbb{G}_n = 0$ along the horizontal axis is positive and the intercept of $\mathbb{G}_m = 0$ along the vertical axis is also positive. See Figure 1.

2. $\mathbb{G}_n = 0$ *and* $\mathbb{G}_m = 0$ *define monotonously increasing relationships in a* (θ_n, θ_m) *space.*

   Using (10) and (11) and applying the implicit function theorem, I get:

$$\left.\frac{d\theta_i}{d\theta_j}\right|_{\mathbb{G}_i=0} = -\frac{\partial \mathbb{G}_i/\partial \theta_j}{\partial \mathbb{G}_i/\partial \theta_i} = -\frac{C_{i,j}}{A_i + B_i} > 0. \tag{13}$$

   with $i \in \{m, n\}$. and $i \neq j$.

3. *Existence and uniqueness.*

   See Figure 2. From point 1 and 2, it is straightforward to conclude that, if

$$\left.\frac{d\theta_m}{d\theta_n}\right|_{\mathbb{G}_n=0} > \left.\frac{d\theta_m}{d\theta_n}\right|_{\mathbb{G}_m=0} \forall \theta_n, \tag{14}$$

   then an equilibrium pair (θ_n, θ_m) exists and it is unique. From (13), one derives:

$$\left.\frac{d\theta_m}{d\theta_n}\right|_{\mathbb{G}_n=0} = \frac{B_n + A_n}{C_{n,m}} \tag{15}$$

$$\left.\frac{d\theta_m}{d\theta_n}\right|_{\mathbb{G}_m=0} = \frac{C_{m,n}}{B_m + A_m} \tag{16}$$

   I multiply the numerator of (15) by the denominator of (16) and the numerator of (16) with the denominator of(15). I get four positive terms in the LHS and only one positive term in the RHS. For (14) to hold, the four positive terms on the LHS must be greater than the term on the RHS. One of the term on the LHS is:

$$B_m B_n = (1 - \beta_m)(1 - \beta_n)\frac{\partial p_n}{\partial E_n}\frac{\partial E_n}{\partial \theta_n}\frac{\partial p_m}{\partial E_m}\frac{\partial E_m}{\partial \theta_m} \tag{17}$$

   The positive term on the RHS is:

$$C_{n,m} C_{m,n} = (1 - \beta_m)(1 - \beta_n)\frac{\partial p_n}{\partial E_m}\frac{\partial E_m}{\partial \theta_m}\frac{\partial p_m}{\partial E_n}\frac{\partial E_n}{\partial \theta_n}. \tag{18}$$

   Expressions (17) and (18) are equal because of the Euler's formula for linear homogeneous functions[3]. Then, inequality (14) is verified. I have two increasing functions in (θ_m, θ_n) space. Moreover, the slope of the function with a positive intercept with the horizontal axis ($\mathbb{G}_n = 0$) is always larger than the slope of the function with positive intercept in the vertical axis ($\mathbb{G}_m = 0$). An equilibrium exists and it is unique.

■

[3]Recall that $\frac{\partial^2 F}{\partial^2 Q_n}\frac{\partial^2 F}{\partial^2 Q_m} = \left(\frac{\partial^2 F}{\partial Q_n \partial Q_m}\right)^2$.

3 Complementarities and substitutabilities

The model of this paper is not new in the search-matching literature. However, to the best of my knowledge, this paper is the first one devoted to studying the analytical effects that are at work in such framework. These effects crucially depend on the search and production technologies presented in Section 1. Therefore, I divide my analysis in two parts.

3.1 Production technology

The presence of a demand for goods not perfectly elastic makes productive complementarity *between sectors* and substitutability *within sectors* coexist in the economy. Consider the equilibrium system (12). When firms open new vacancies in sector i, labor market tightness θ_i increases and the productivity p_j in the other intermediate sector, j, increases too. This effect occurs because the two intermediate inputs are p-substitutes. Since in equilibrium the price of the intermediate good is equal to its marginal productivity, firms in sector j will get higher revenues and, to restore the zero profit condition, new vacancies in this sector will be opened too. So a productivity complementarity arise between the sectors. In terms of tightness, such complementarity is equivalent to Inequality (13) and it can be seen in Figure 1, where both curves are upward sloping. However, this is not the only effect present in the economy. When a new vacancy is opened in sector i, labor market tightness θ_i and, in turn, the level of employment E_i increase too. That has a negative impact on the price of the intermediate good p_i. Lower price of the intermediate good i reduces the expected revenues of a filled vacancy in sector i, so less vacancies will be opened in i. In other words, the increase in the activity of a firm in sector i induces other firms to create less vacancies in sector i. Productive substitutability is present in the economy; the term B_i in (10) captures such effect in the model.

3.2 Matching technology

The matching technology also affects the decentralized equilibrium outcome. As Cooper (1999) observed, any search-matching model gives rise to strategic complementarity and substitutability[4]. Strategic substitutability is present because a firm deciding to post a new vacancy lowers the probability that other vacancies can be filled and, therefore, it induces other firms to decrease their activity (that is, less vacancies are created). Such strategic substitutability must be distinguished from that we discussed in the previous

[4]In his book, Cooper analyzes only the Diamond (1982) model of search and unemployment, but his reasoning can be extended to all models with a matching technology.

subsection. There, both complementarity and substitutability are generated through a price mechanism and therefore they cannot be defined as strategic. In the goods market, agents take the price as given and adjust their behaviour according to that. On the contrary, in the labour market, the Walrasian auctioneer is replaced by a matching function and the wage is deprived by any allocative role (see Hosios, 1990). In this context, we can adopt the term strategic substitutability because the action of one agent negatively affects the best response of the other agents not through a price but through an aggregate matching technology. In this model, productive substitutability is represented by term B_i, strategic substitutability by A_i.

In a standard matching model, only the strategic substitutability is present: A marginal increase in the number of vacancies lowers the probability that they can be filled. In models with endogenous productivity, on the contrary, every vacancy created in sector i decreases both the probability that vacancies in i are filled and the productivity of the i-th intermediate good: search substitutability and productivity substitutability arise[5].

3.3 Comparative statics and the multiplier

To better understand the policy implications of such complementarities and substitutabilities, I here analyze what happens when a policy parameter changes. Suppose for instance a reduction in the level of unemployment benefits in sector n, b_n. I consider first the standard setup where the demand for intermediate goods is perfectly elastic so that the price does not change. Then, I study this model and compare the results. A reduction in b_n affects equation $\mathbb{G}_n = 0$: lower unemployment benefits reduce workers' outside option in the Nash bargain and the wage w_n decreases. Hence, firms will be induced to post more vacancies in the n sector and θ_n will go up. This is the standard mechanism at work in simple matching models: when goods demand is perfectly elastic, it does not exist any link between the two intermediate sectors n and m and both the complementarity and the substitutability generated by the production function are equal to zero. Only the strategic substitutability that comes from the matching technology is present. Differentiating $\mathbb{G}_n = 0$ with respect to θ_n and b_n and

[5]If in our matching framework the labor force participation rate was endogenous, the symmetry between search and production technology would be perfect. Another strategic substitutability would arise, because an agent deciding to enter the labour market would negatively affect the maximizing behaviour of other agents outside the labour force. Moreover, also strategic complementarity between the sides of the market would arise: a new vacancy posted would induce agents to enter the labour market and the other way round.

applying the implicit function theorem, I obtain:

$$\left.\frac{d\theta_n}{db_n}\right|_{\mathbb{G}_n^{\bar{p}}=0} = -\frac{1-\beta_n}{A_n} < 0, \tag{19}$$

where the index $\bar{p}$ indicates that we are dealing with the case of perfectly elastic demand where the increase in supply does not affect the price. As expected, such derivative is negative: a decrease in the level of unemployment benefits in n increases labor market tightness θ_n and so E_n.

Consider now the hypothesis of not completely elastic demand. In this case, differentiating system (12) with respect to θ_n, θ_m and b_n and again applying the implicit function theorem leads to:

$$\frac{d\theta_n}{db_n} = -\frac{det\begin{bmatrix} \partial\mathbb{G}_n/\partial b_n & \partial\mathbb{G}_n/\partial\theta_m \\ 0 & \partial\mathbb{G}_m/\partial\theta_m \end{bmatrix}}{det\begin{bmatrix} \partial\mathbb{G}_n/\partial\theta_n & \partial\mathbb{G}_n/\partial\theta_m \\ \partial\mathbb{G}_m/\partial\theta_n & \partial\mathbb{G}_m/\partial\theta_m \end{bmatrix}}. \tag{20}$$

Dividing the denominator and the numerator by $\frac{\partial\mathbb{G}_n}{\partial\theta_n}\frac{\partial\mathbb{G}_m}{\partial\theta_n}$ and using (13) , I get the following expression:

$$\frac{d\theta_n}{db_n} = -\frac{\partial\mathbb{G}_n/\partial b_n}{\partial\mathbb{G}_n/\partial\theta_n} \cdot \frac{1}{1-\frac{d\theta_m}{d\theta_n}_{\mathbb{G}_m=0} \cdot \frac{d\theta_n}{d\theta_n}_{\mathbb{G}_n=0}} < 0. \tag{21}$$

The first term of the product on the RHS of (21) is negative. It represents the effects that arise *within* the intermediate sector n when b_n marginally changes. Note that this first term is *greater* than (19), because $-\partial\mathbb{G}_n/\partial b_n = -\partial\mathbb{G}_n^{\bar{p}}/\partial b_n = 1-\beta_n$, and at the denominator $\partial\mathbb{G}_n/\partial\theta_n$ is lower than $\mathbb{G}_n^{\bar{p}}/\partial\theta_n$[6] . The reason lies on the fact that in $\partial\mathbb{G}_n/\partial\theta_n$ there is also the component B_i that captures the productive substitutability effect: an increase in θ_n not only reduces the probability of vacancies in n to be filled, but also lowers the productivity p_n. Therefore, the first term in (21) is greater (namely, less negative) than (19) because of the productive substitutability effect within sector n. This effect tends to reduce the positive impact on θ_n caused by a decrease in the unemployment benefits. On the other hand, the second term on the RHS of (21) captures the effects that intervene between the two sectors when b_n marginally changes. Both derivatives at the denominator are positive (see equation (13)): They represent

[6]Note that

$$\frac{d\mathbb{G}_i}{d\theta_i} = A_i + B_i < \frac{d\mathbb{G}_i^{\bar{p}}}{d\theta_i} = A_i < 0,$$

with $i \in \{m, n\}$.

the complementarities existing between sector n and sector m. The denominator is also positive and lower than one[7]. Therefore, the second term in (21) is greater than one. It is a multiplier, because it amplifies the effects of a shock on a single sector (in this case a change in b_n) through the induced response of the other sector: lower unemployment benefits in n increase the number of vacancies V_n and so θ_n; this enhances productivity p_m and, consequently, V_m and θ_m. In turn, lower θ_m will increase θ_n even more. Of course, for the existence of a multiplier, it is necessary that complementarities are present in both sectors: the increase in θ_m caused by lower unemployment benefits b_n must in turn affect θ_n in order to have a multiplicative impact in n sector. We can see graphically the effects of a decrease in b_n looking at Figure 1: the curve $\mathbb{G}_n = 0$ shifts to the right and in the new equilibrium point E' both θ_m and θ_n are higher[8].

So the presence of productive substitutability in the first term of (21) tends to mitigate the impact of a shock on tightness and employment, whereas productive complementarity in the second term tends to enhance it. To see which of two effects is stronger, I compute the derivatives in (20) and, after some algebra, I obtain:

$$\frac{d\theta_n}{db_n} = \frac{1-\beta_n}{A_n} . \frac{A_m + B_m}{A_m + B_n \frac{A_m}{A_n} + B_m}. \tag{22}$$

The first ratio on the RHS of (20) is equal to (19). The second ratio is positive and less than one because all the terms at the denominator an at the numerator are negative. Then:

$$\left.\frac{d\theta_n}{db_n}\right|_{\mathbb{G}_n^{\bar{p}}=0} < \frac{d\theta_n}{db_n}$$

Productive substitutability outweighs productive complementarity. So the effect of a reduction in unemployment benefits b_n on θ_n and employment E_n is lower in a model with endogenous than in a standard textbook matching model. There is an intuitive interpretation for that. Complementarities influence E_n only indirectly, via an increase in employment in the other sector, E_m. On the contrary, the productive substitutability effect influences employment E_n directly: when unemployment benefits b_n are reduced, less firms will enter the n-th sector and post vacancies V_n than in a standard matching model, because in this framework an increase in V_n produces not only a reduction in the probability $q(\theta_n)$ of filling a vacancy (strategic substitutability), but also a lower price p_n (productive substitutability). To sum up:

[7] To see this, note that showing that the denominator in the second term of (21) is greater than zero is equivalent to proving inequality (14).

[8] Under the hypothesis of zero complementarities, the equation $\mathbb{G}_n = 0$ would be a vertical line and $\mathbb{G}_n = 0$ a horizontal line in (θ_n, θ_m) space. A decrease in b_n would shift $\mathbb{G}_n = 0$ to the right and the new equilibrium point would present a higher θ_n but the same value of θ_m.

Proposition 2 *Removing the assumption of a perfectly elastic demand for goods in a Pissarides model yields the following results:*

1. *Productive substitutability: it diminishes in absolute value the effects of a sector-specific shock on tightness and employment in that sector.*

2. *Productive complementarity: it amplifies through a multiplier the effects of a sector-specific shock on the level of tightness and employment in the that sector. At the same time, it affects the level of tightness and employment in the other sector.*

3. *Productive substitutability is stronger than complementarity: in the original sector, employment and labour market tightness change less than in a standard matching model*

Proof. The second part of point 2 can be easily checked by applying implicit function theorem and Cramer's rule to compute $d\theta_m/db_n$. Otherwise, one can conclude about the negative sign of $d\theta_m/d\, b_n$ without going through the algebraic passages; since policy in sector n affects employment in m only through p_m and p_m depends positively on E_n, any change in sector n that raises (lowers) E_n will also raise (lower) p_m and so E_m. In figure 1, when the curve $\mathbb{G}_n = 0$ shifts to the right, both θ_n and θ_m increase.

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4 Welfare analysis

Since the work of Hosios (1990), it is well known that a standard matching model with Nash bargaining does not necessarily reach efficiency, that is the steady state value of the social output is not maximized. More precisely, the so-called Hosios condition states that if the bargaining power of workers, β, equals the absolute value of $\eta(\theta)$, the elasticity of the probability q of filling a vacancy with respect to tightness, the search externalities are internalized by the ex post Nash bargain. When these values are different, the decentralized equilibrium does not ensure an efficient allocation of the two inputs (unemployment and vacancies) in the matching technology[9].

[9]The source of such inefficiency depends on the way in which the wage formation is formalized. In the decentralized economy, the extent of substitution between unemployment and vacancies is not governed by the matching technology but by the bargaining solution. Imposing $\beta = \eta(\theta)$ is therefore equivalent to imposing that the substitution between unemployment and vacancies is adjusted according to the matching function. See e.g. Pissarides (2000).

In this section, I analyse the welfare properties of the model presented in the previous sections, where new complementarities and substitutabilities arise. As usual in the literature, I consider the simplest setup: the economy is in steady state and the discount rate r tends to 0. I interpret b_i as the value of leisure enjoyed by job-seekers and no longer as the level of unemployment benefits. In the Appendix it is shown that for a social planner *without distributional concerns* maximizing social output is equivalent to maximizing the production of the consumption good and the value of leisure for the unemployed workers net the cost of posting job vacancies in both sectors. I want to disentangle the effects arising form one sector and those arising in the other; so, since for linear homogeneous functions $Y = \frac{\partial F}{\partial E_n} E_n + \frac{\partial F}{\partial E_m} E_m$, I write the social planner's maximization problem as:

$$\begin{aligned} &\max_{\theta_n,\, \theta_m} \sum_{i \in \{m,n\}} p_i E_i \, + b_i(L_i - E_i) \, - k_i V_i \\ &s.t. \;\; E_i = \frac{\alpha(\theta_i)}{\phi_i + \alpha(\theta_i)} \, L_i \, , \;\; i \in \{m, n\}. \end{aligned} \tag{23}$$

Steady-state equalities in labour market flows are the constraints of social planner's maximization problem. Knowing that $V_i = \theta_i(L_i - E_i)$ with $i \in \{m, n\}$, I obtain the following F.O.C.s:

$$\begin{aligned} &(p_m - b_m + k_m\theta_m)\frac{dE_m}{d\theta_m} \; - \; k_m(L_m - E_m) + \\ &+ \; E_m \frac{\partial p_m}{\partial E_m}\frac{\partial E_m}{\partial \theta_m} \; + \; E_n \frac{\partial p_n}{\partial E_m}\frac{\partial E_m}{\partial \theta_m} = 0, \end{aligned} \tag{24}$$

$$\begin{aligned} &(p_n - b_n + k_n\theta_n)\frac{dE_n}{d\theta_n} \; - \; k_n(L_n - E_n) + \\ &+ \; E_n \frac{\partial p_n}{\partial E_n}\frac{\partial E_n}{\partial \theta_n} \; + \; E_m \frac{\partial p_m}{\partial E_n}\frac{\partial E_n}{\partial \theta_n} \; = \; 0 \end{aligned} \tag{25}$$

The first two terms in (24) represent the net marginal gain of increasing θ_m with p_m fixed. These terms are the usual ones obtained in the social planner optimization problem. The third term in (24) represents the marginal productivity loss suffered by the m sector when θ_m increases marginally. The fourth term is the marginal productivity gain that firms in the n sector obtain if θ_m increases. Equation (25) is symmetric. The last two terms of (24) can be written as:

$$+E_m \frac{\partial p_m}{\partial E_m}\frac{\partial E_m}{\partial \theta_m} + E_n \frac{\partial p_n}{\partial E_m}\frac{\partial E_m}{\partial \theta_m} = \frac{\partial E_m}{\partial \theta_m}\left(E_m \frac{\partial p_m}{\partial E_m} + E_n \frac{\partial p_n}{\partial E_m} \right) = 0. \tag{26}$$

This is true for every constant return to scale production function[10]. Of course, the same holds for the last two terms in (25). The economic intuition of this result is the following. As we have seen in Section 3, when a firm decides to open a new vacancy in the intermediate sector i , productive complementarity and productive substitutability arise at the same time. For CRS production functions these two effects are equal in absolute value. The marginal loss in the aggregate revenues in sector i ($E_i \frac{\partial p_i}{\partial E_i}\frac{\partial E_i}{\partial \theta_i}$) is totally offset by the marginal gain in the aggregate revenues in sector j ($E_j \frac{\partial p_j}{\partial E_i}\frac{\partial E_j}{\partial \theta_j}$).

Proposition 3 *Consider the system (12) with $r \to 0$. Under Hosios conditions, $\beta_i = \eta(\theta_i)$ for $i \in \{m, n\}$, the decentralized equilibrium is efficient.*

Proof. Using (26) and after some algebra, the F.O.C. (24) becomes:

$$\phi_m \alpha'(\theta_m)\left[p_m - b_m + k_m\theta_m\right] - k_m\phi_m[\alpha(\theta_m) + \phi_m] \;=\; 0.$$

Rearranging the terms and knowing that $\alpha'(\theta_m) = q(\theta_m)[1 - \eta(\theta_m)]$, I get:

$$q(\theta_m)[1 - \eta(\theta_m)](p_m - b_m) = k_m[\alpha(\theta_m)\eta(\theta_m) + \phi_m].$$

With $\beta_m = \eta(\theta_m)$ (and $r \to 0$), this equation is equivalent to (9). Obviously, the same passages lead to the same conclusion in sector n.

■

The Hosios condition applied both in sector n and in sector m are sufficient to ensure the efficiency of a two sectors matching model. This makes sense, since in this model the only departure from the Walrasian framework is the presence of search externalities in the labour market. However, the hypothesis about a social planner that does not care about distributional issues must be taken into account. In fact, the productivity effects that are present in this model do not influence the efficient value of labor market tightness that a social planner would choose: if p_n and p_m were exogenous (i.e. if the only link between the two intermediate sector disappeared), then a social planner without distributive concern would select exactly the same values of θ_m and θ_n. What the introduction of productivity externalities changes is the distribution of the resources between the sectors. Put in other terms, the four productivity effects present in (24) and (25) can be viewed as a mechanism that shifts resources from one sector to the other. The net marginal amount of resources that accrues to sector m by means of productivity complementarity and substitutability is:

$$E_m \left(\frac{\partial p_m}{\partial E_m}\frac{\partial E_m}{\partial \theta_m} + \frac{\partial p_m}{\partial E_n}\frac{\partial E_n}{\partial \theta_n} \right), \tag{27}$$

[10] Recall that for every CRS function $\frac{\partial^2 F}{\partial^2 E_m} E_m + \frac{\partial^2 F}{\partial E_n \partial E_n} E_n = 0$.

while the net marginal amount of resources going to sector n is equal to:

$$E_n \left(\frac{\partial p_n}{\partial E_m} \frac{\partial E_m}{\partial \theta_m} + \frac{\partial p_n}{\partial E_n} \frac{\partial E_n}{\partial \theta_n} \right). \tag{28}$$

As I showed above, the sum of (27) and (28) is equal to zero; so expressions in (27) and (28) either have the same absolute value with opposite sign or are both equal to zero. If it is true the latter, we have no transfer of resources between sectors. Otherwise, with (27) positive (negative), resources go from sector n to sector m (from sector m to sector n). If we imagine a social planner that prefers one sector to the other it is no longer true he would select the same values of θ_m and θ_n that he would have chosen in the case of perfectly elastic demand.

5 Conclusions

In this paper I tried to explore the consequences of introducing a more realistic set-up for goods market in a standard matching framework. Models where two or more labour markets with frictions are linked together through a two-tier productive scheme are widespread in the macroeconomic and labour literature. However, to the best of my knowledge, nobody has studied their analytical properties. I have shown that in this kind of set-up, complementarities arise between sectors and substitutabilities arise within sectors. The formers emerge because an increase in the number of vacancies posted in one sector raises also the number of vacancies posted in the other one (given the assumption of p-substitutes inputs), and this in turn enhances employment. Substitutabilities emerge because a new vacancy posted in one sector decreases both the price in that sector and the probability for another vacancy to be filled. The effects on employment are the following: in the sector where the shock occurred, employment changes less than in the standard matching framework. In the other(s) sector(s), employment also varies, and in the same direction than in the "original" sector.

I have also looked at the welfare properties of the model. Under constant returns to scale both in the matching and in the production technology the so-called Hosios condition is sufficient to guarantee the efficiency of the *laissez faire* equilibrium.

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Appendix: Derivation of the steady-state value of the social output

At time t, the social output can be written as

$$\mathbb{W}(t) = \int_t^{+\infty} e^{-r(T-t)} W(T) dT$$

From which, we have $W(t) = r\mathbb{W}(t) - \dot{\mathbb{W}}(t)$. Writing $\sum$ for $\sum_{i\in\{n,m\}}$ and taking all parameters as fixed, I assume that:

$$\mathbb{W}(t) = \mathbb{P}(t) + \sum \left[E_i(t)V_{E,i}(t) + U_i(t)V_{U,i}(t) + \Pi_i(t)\right]$$

in which,

$$\begin{aligned}
\mathbb{P}(t) &= \int_t^{+\infty} e^{-r(T-t)} \left[F(E_n(T), E_m(T)) - p_n(T)E_n(T) - p_m(T)E_m(T)\right] dT \\
\Pi_i(t) &= \int_t^{+\infty} e^{-r(T-t)} \left[E_i(T)\left(p_i(T) - w_i(T)\right) - k_i V_i(T)\right] dT \\
rV_{E,i}(t) &= w_i(t) + \phi_i(V_{U,i}(t) - V_{E,i}(t)) + \dot{V}_{E,i}(t), \\
rV_{U,i}(t) &= b_i + \alpha(\theta_i(t))(V_{E,i}(t) - V_{U,i}(t)) + \dot{V}_{U,i}(t).
\end{aligned}$$

in which V_i denotes the number of vacancies in sector i. From the last expressions,

$$\begin{aligned} W(t) &= r\mathbb{W}(t) - \dot{\mathbb{W}}(t) \\ &= r\mathbb{P}(t) - \dot{\mathbb{P}}(t) + \sum [E_i(t)(rV_{E,\,i}(t) - \dot{V}_{E,\,i}(t))] - \sum \dot{E_i}(t)V_{E,\,i}(t) \\ &\quad + \sum U_i(t)(rV_{U,i}(t) - \dot{V}_{U,\,i}(t)) - \sum \dot{U_i}(t)V_{U,\,i}(t) \\ &\quad + \sum E_i(t)(r\Pi_i(t) - \dot{\Pi_i}(t)) \end{aligned}$$

$$\begin{aligned} W(t) &= r\mathbb{W}(t) - \dot{\mathbb{W}}(t) \\ &= F(E_n(t), E_m(t)) - p_n(t)E_n(t) - p_m(t)E_m(t) \\ &+ \sum [E_i(t)(w_i(t) + \phi_i(V_{U,\,i}(t) - V_{E,\,i}(t)))] \\ &\quad + \sum U_i(t)(\, w_i(t) + \, \alpha(\theta_i(t))(V_{E,i}(t) - V_{U,i}(t))) \\ &\quad - \sum \dot{E_i}(t)V_{E,\,i}(t) - \sum \dot{U_i}(t)V_{U,\,i}(t) \\ &\quad + \sum E_i(t)(p_i(t) - w_i(t)) - k_iV_i(t). \end{aligned}$$

Simplifying,

$$\begin{aligned} W(t) &= r\mathbb{W}(t) - \dot{\mathbb{W}}(t) \\ &= F(E_n(t), E_m(t)) \\ &\quad \sum [E_i(t)(\phi_i(V_{U,\,i}(t) - V_{E,\,i}(t)))] \\ &\quad + \sum U_i(t)(b_i + \, \alpha(\theta_i(t))(V_{E,i}(t) - V_{U,i}(t))) \\ &\quad - \sum \dot{E_i}(t)V_{E,\,i}(t) - \sum \dot{U_i}(t)V_{U,\,i}(t) \\ &\quad - \sum k_iV_i(t) \end{aligned}$$

If the labour force is exogenous, $L_i = E_i + U_i$. Hence, $\dot{E_i}(t) = -\dot{U_i}(t)$. Plugging this in the last expression, a term cancels out, namely:

$$(\,\alpha(\theta_i(t))E_i(t) - \phi_iE_i(t) + \dot{U_i}(t))(V_{E,i}(t) - V_{U,i}(t)) = 0$$

Finally,

$$\begin{aligned} W(t) &= F(E_n(t), E_m(t)) + \sum U_i(t)b_i \\ &\quad - \sum k_iV_i(t) \end{aligned}$$

where both E_i and V_i have to be related to tightness. In sum,

$$\mathbb{W}(t) = \int_t^{+\infty} e^{-r(T-t)} W(T) dT$$

with $W(t)$ as defined above. After a dynamic adjustment, $W(t)$ will reach a steady state. If $r \to 0$ and since I integrate over $[t, \pm\infty)$, I can neglect the adjustment path and consider that a planner should maximize $W(t)$ where t denotes the steady state.

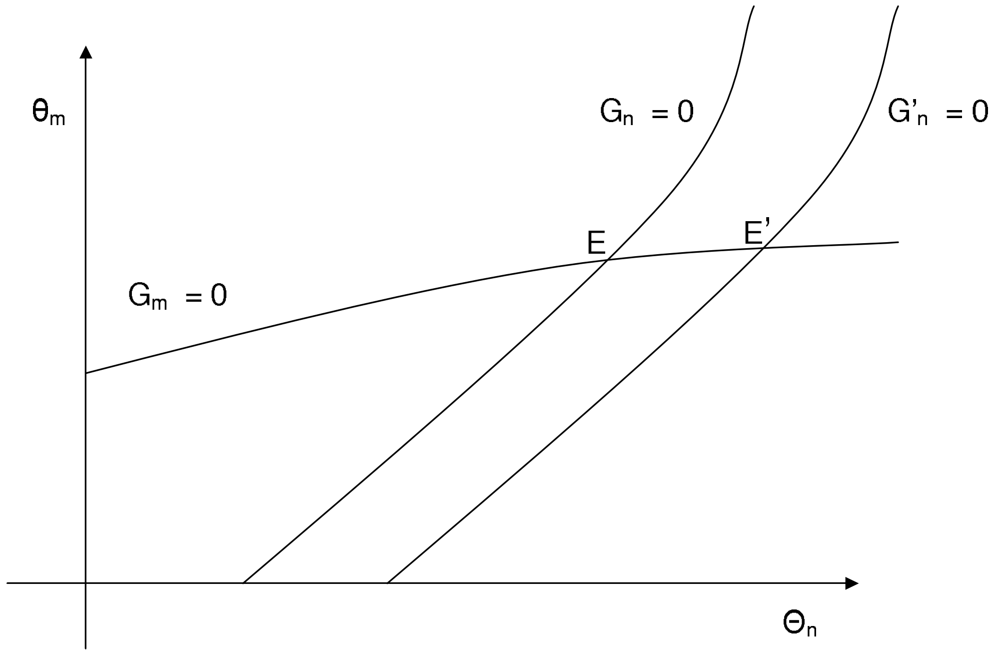

Figure 1: Existence and uniqueness of the steady-state equilibrium

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