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### Networks of Manufacturers and Retailers∗

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Abstract. We study the endogenous formation of networks between manufacturers of differentiated goods and multi-product retailers who interact in a successive duopoly. Joint consent is needed to establish and/or maintain a costly link between a manufacturer and a retailer. We find that only three distribution networks are stable for particular values of the degree of product differentiation and link costs: (i) the non-exclusive distribution & non-exclusive dealing network in which both retailers distribute both products is stable for intermediate degree of product differentiation and small link costs; (ii) the exclusive distribution & exclusive dealing network in which each retailer distributes a different product is stable for low degrees of product differentiation; (iii) the mixed distribution network in which one retailer distributes both products while the other retailer sells only one is stable for high degrees of product differentiation and large link costs. We show that the distribution networks that maximize social welfare are not necessarily stable. Thus, a conflict between stability and social welfare is likely to occur, even more if the degree of product differentiation is either low or high.

JEL Classification: C70, L13, L20, J50, J52

Key words: Networks, Retailers, Manufacturers.

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In various industries, such as e.g. automobiles, clothing, electronics, pharmaceuticals, and food, manufacturers develop networks of exchange both with input suppliers and retailers or wholesalers. Moreover, in the last few decades the importance of spot exchange in input or output procurement has decreased in favor of other methods such as manufacturer-supplier long-term contracting and manufacturer-retailer exchange networks. For automobiles, Nishiguchi (1994) has presented wide ranging evidence on the ways in which the Japanese industrial model has evolved from the traditional bargainingoriented manufacturer-supplier relationship to the current problem-solving-driven strategic industrial outsourcing. Firms rely more and more on a subset of suppliers with whom they maintain closed business ties. The number of direct suppliers to Japanese car manufacturers in 1988 was roughly one half of what it was for American or European manufacturers, for similar volumes of production. The manufacturer-retailer relationship for the European motor vehicle industry has also evolved over the last years. Until October 2002, only one type of distribution (a system of exclusive territories and selectivity) was permitted. The European Commission was not satisfied with the unexplained differences in prices across European countries and this motivated a legal change. The new regulation recently issued<sup>1</sup> is seeking for a change in the car distribution industry. As Commissioner Monti said: "The new rules that will become effective as of 1 October 2003 open the way to new distribution techniques, such as Internet sales and multi-branding - introducing more competition between different retail channels".2 Then it is expected that multi-branding dealers will appear and coexist with exclusive ones.

The literature on network formation has focused on the upstream part of the vertical chain, neglecting the analysis of the downstream part where manufacturers and retailers enter in long-term relationships.<sup>3</sup> Kranton and Minehart  $(2000a)$  have examined the emergence of buyer-seller networks when sellers have an outsourcing motivation in order to see whether networks of buyers and sellers can perform better than vertically integrated markets or spot exchange markets. Manufacturers can decide to build a dedicated

 $1^1$ Regulation 1400/2002 on the application of Article 81(3) of the Treaty to categories of vertical agreements and concerted practices in the motor vehicle sector.

<sup>2</sup>Extracted from "New rules for car sales and servicing" (September 2003) and European Commissioner Monti's speech "The new legal framework for car distribution" (February 2003). See http://www.europa.eu.int/comm/competition/ for more details.

<sup>&</sup>lt;sup>3</sup>The data in Betancourt (2004) suggests that there has been substantial forward vertical integration by manufacturers in the form of internalizing the wholesale function by selling directly to retailers. This process is most pronounced in the durable sectors: automobiles and other motor vehicles, 95.5 percent of sales; electronics, 70.9 percent of sales; toys and hobby goods, 95.6 percent of sales (US retail sector in 1987).

asset to produce their own inputs or, alternatively, they can invest in links to external sellers from which they will buy specialized inputs. They have established a connection between industrial structure and uncertainty in demand: outsourcing networks appear to be more efficient than vertically integrated structures when uncertainty in demand is substantial. Kranton and Minehart (2001) have focused on when the noncooperative formation of buyer-seller networks leads to the formation of efficient networks, while Kranton and Minehart (2000b) have examine the competitive equilibrium prices in buyer-seller networks. Wang and Watts (2003) have analyzed the formation of buyer-seller links when sellers can produce products of different quality.<sup>4</sup> In this paper, we are first to examine the emergence of manufacturer-retailer networks when both manufacturers and retailers decide the bilateral links they want to establish among them.

The literature on distribution systems initially addressed two questions: (i) whether manufacturers would prefer having a single common retailer rather than separate exclusive retailers;<sup>5</sup> and (ii) whether a manufacturer's brand is excluded from the market by use of exclusive contracts.<sup>6</sup> A series of papers have studied the distribution systems that arise when there is market power at both the manufacturing and retailing levels.<sup>7</sup> In particular, for the successive duopoly case, Chang(1992) and Dobson and Waterson (1997) have analyzed the distribution systems that arise by the joint maximization of the manufacturerretailer pair profits, allowing for side-payments if an exclusive contract is signed. Chang (1992) has found that manufacturer-retailer pairs always choose exclusive dealing. Once manufacturers and retailers are differentiated, Dobson and Waterson (1997) have shown

<sup>&</sup>lt;sup>4</sup>There is a vast literature devoted to analyze the important role played by network structures in determining the outcome of many other economic situations. For example, Hendricks, Piccione and Tang (1997) have shown that the structure of airline connections influences competition. Belleflamme and Bloch (2004), Goyal and Moraga (2001) and Goyal and Joshi (2003) have studied the formation of research and development networks and collusive alliances among corporations. Calvó-Armengol (2004) and Calvó-Armengol and Jackson (2004) have examined the role played by personal contacts in obtaining information about job opportunities.

<sup>&</sup>lt;sup>5</sup>An exclusive dealing agreement is a restriction of the retailer's behavior under which the retailer agrees not to buy from any other manufacturer. Similarly, an exclusive distribution agreement is a manufacturer's behavior restriction under which the manufacturer agrees not to sell to any other retailer. Lin (1990) and O'Brien and Shaffer (1993) have shown that exclusive dealership rather than common dealership is chosen to dampen competition between the manufacturers.

 $6$ In a setting with two manufacturers and only one retailer, O'Brien and Shaffer (1997) and Bernheim and Whinston (1998) have shown that vertical foreclosure is not an equilibrium. We refer to Rey and Tirole (2003) for a complete survey on vertical foreclosure.

 $7$ The possibility of a manufacturer hiring more than one retailer has been considered out of the successive duopoly structure. See Rey and Stiglitz (1995) for the case where manufacturers can hire several retailers in a perfect competition setting, and Besanko and Perry (1994) for the case of spatially differentiated retailers whose number is endogenously determined by free entry.

that the manufacturer-retailer pairs prefer non-exclusive dealing contracts when products and retailers are sufficiently differentiated. Moner-Colonques, Sempere-Monerris and Urbano (2004) have analyzed a successive duopoly where two manufacturers with asymmetric and differentiated brands choose strategically how many undifferentiated retailers to employ. When product differentiation is strong and brand asymmetry is moderate, both manufacturers distribute through both retailers. However, when both product differentiation and brand asymmetry are weak, exclusive dealing through a single retailer is used. There are also asymmetric equilibria in which one manufacturer distributes through both retailers but the other manufacturer distributes through one retailer. These equilibria can arise when both product differentiation and brand asymmetry are strong. Finally, Mycielski, Riyanto and Wuyts (2000) have studied manufacturers' choice of two types of vertical arrangement with retailers; exclusive dealing and exclusive territory. When products are less substitutable, in other words, the interbrand rivalry is weak, manufacturers prefer to sell brands to a large number of competitive retailers. When the interbrand rivalry is strong, exclusive territory with exclusive dealing is adopted by manufacturers.

In this paper we address the following questions:

- (i) What are the incentives of manufacturers and retailers to link and what is the architecture of "stable" networks of distribution when both manufacturers and retailers decide the bilateral links they want to establish among them?
- (ii) Are individual incentives to link adequate from a social welfare point of view?

In order to answer these questions we develop a three-stage game in a successive duopoly. Each manufacturer produces a differentiated product (brand) which is sold to retailers at a constant per unit price and retailers can be multiproduct sellers. In the first stage, the two manufacturers and the two retailers decide about bilateral relationships (or links) they want to establish among them. A link between a manufacturer and a retailer is necessary in order to sell the manufacturer's brand to consumers. The cost of a link is shared equally between the manufacturer and the retailer.<sup>8</sup> The collection of pairwise links between manufacturers and retailers defines a distribution network. In the second stage, both manufacturers choose simultaneously the terms of trade of their good to retailers (transfer prices). In the third stage, both retailers compete by setting simultaneously the quantity of each brand they are going to market.

<sup>8</sup> In the motor vehicle industry the "just-in-time" philosophy has been present for decades. It requires the coordination and collaboration across organizations and throughout the supply chain. This means that there should be a permanent relationship among them and this is costly. Also, it is becoming more common to find a supply channel management approach in different industries where independent members of the supply chain coordinate in the management of such a chain.

A simple way to analyze the networks that one might expect to emerge in the long run is to examine a sort of equilibrium requirement that agents not benefit from altering the structure of the network. A weak version of such condition is the pairwise stability notion defined by Jackson and Wolinsky (1996). A network is pairwise stable if no agent benefits from severing one of their links and no other two agents benefit from adding a link between them, with one benefiting strictly and the other at least weakly. While pairwise stability is natural and quite easy to work with, there are some limitations of the concept. First, it is a weak notion in that it only considers deviations on a single link at a time. For instance, it could be that an agent would not benefit from severing any single link but would benefit from severing several links simultaneously, and yet the network would still be pairwise stable. Second, pairwise stability considers only deviations by at most a pair of agents at a time. It might be that some group of agents could all be made better off by some complicated reorganization of their links, which is not accounted for under pairwise stability. A strongly stable network, whose definition is due to Jackson and van den Nouweland (2005), is a network which is stable against changes in links by any coalition of agents.9

Stable networks obtained from the joint consent of the agents involved might result in distribution networks that coincide with those resulting from both manufacturers and retailers signing exclusive dealing or exclusive distribution contracts. Thus, this model might be used by the competition authorities to distinguish whether exclusive dealing is agreed by all agents in the market (i.e. not imposed by one kind of agent), whether it is efficient and whether it is socially optimal. Exclusive dealing as many as other nonprice vertical restraints is challenged by competition authorities. The legal treatment of nonprice vertical restraints has not been uniform along the years. In the U.S. vertical restraints were initially considered as not per se illegal, then per se illegal and now a rule of reason is applied. The analytical justification for a rule of reason is the twofold effect of nonprice vertical restraints in general and exclusive dealing in particular.<sup>10</sup> Vertical restraints have a procompetitive effect when they are used to avoid the double marginalization inefficiency or to reduce the underprovision of services that affect the demand of the good. Vertical restraints have an anticompetitive effect when they are used to reduce or eliminate intrabrand competition (same brand is sold at different outlets), to dampen competition at the upstream levels, or to foreclose market access and prevent entry. The procompetitive effect is more likely to dominate the anticompetitive effect provided interbrand competition

 $9$ Jackson (2003, 2005) provides surveys of models of network formation.

 $10^1$ Caballero-Sanz and Rey (1996) and Dobson and Waterson (1997) have provided a detailed analysis of the economic evaluation of vertical restraints and the implications for competition policy.

is sufficiently strong.11



Figure 1: The six qualitatively different distribution networks.

In a successive duopoly, there are fifteen possible network architectures. But given the symmetry of products and retailers, there are only six qualitatively different distribution networks which are depicted in Figure 1. Depending on the distribution network, two kinds of competition may be eventually at place: interbrand competition and intrabrand competition. Therefore, the distribution network that will emerge is the result of the interplay of two effects: first, the one associated to the cost of implementing a particular network which depends on both the link cost size and the number of links; and a second one which is associated to the combination of inter and intrabrand competition that arises in each particular network. Given that agents act strategically and in their self-interest, the stable distribution network might differ from the one preferred by consumers or the<br><sup>11</sup>This point has been included in the EC Guidelines on vertical restraints: "The market position of

 $11$  This point has been included in the EC Guidelines on vertical restraints: "The market position of This point has been included in the EO Guidelines on vertical restraints. The market position of<br>the supplier and his competitors is of major importance, as the loss of intrabrand competition can be<br>problematic if interbra serious is the loss of intrabrand competition is limited. The stronger the position of the supplier, the more<br>serious is the loss of intrabrand competition...." (see Official Journal of the European Communities, C serious is the loss of intrabrand competition..." (see Official Journal of the European Communities, C distribution agreements between Telenor and Canal + Nordic, under which Telenor will have the exclusive<br>distribution agreements between Telenor and Canal + Nordic, under which Telenor will have the exclusive  $r_{\text{251}}$ , 19/10/2000). Recently, the European Commission has exempted for hve years certain exclusive<br>distribution agreements between Telenor and Canal + Nordic, under which Telenor will have the exclusive<br>wish to dist television platform Canal + Nordic's premium pay-TV channels in the Nordic region through its satellite<br>television platform Canal Digital. The argument of this exemption was the presence of a second satellite pay-TV distributor in the Nordic region, MTG/Viasat and that consumers would have available two distinct<br>pay-TV distributor in the Nordic region, MTG/Viasat and that consumers would have available two distinct pay-TV distributor in the Nordic region, MTG/Viasat and that consumers would have available two distinct<br>pay-TV brands at competitive prices, i.e., sufficient interbrand competition (see the IP/04/2, January 5, pay-TV brands at competitive prices, i.e. sufficient interbrand competition (spay-TV brands at competitive prices, i.e. sufficient interbrand competition (space). 2004 and the EC Commission Competition Policy Newsletter, Summer 2004).

one that maximizes social welfare.

We find that only three distribution networks are strongly stable for particular values of the degree of product differentiation and link costs. A first distribution network with four links, referred as *non-exclusive distribution* & *non-exclusive dealing*, in which both non-exclusive distribution & non-exclusive dealing, in which both<br>hydrotics is strongly stable for intermediate degree of product<br>all limit os its strangly stable for intermediate degree of product<br>all limit os test stran retailers distribute both products is strongly stable for intermediate degree of product differentiation and small link costs. In this distribution network, both interbrand and intrabrand competition are present in the market. A second distribution network with two links, referred as *exclusive distribution* & *exclusive dealing*, in which each retailer exclusive distribution & exclusive dealing, in which each recaller<br>nodes is strongly stable for low degrees of product differentiation.<br>work, no intrabrand competition appears in the market. A third<br>out differentiation an distributes a different product is strongly stable for low degrees of product differentiation. In this distribution network, no intrabrand competition appears in the market. A third distribution network with three links, referred as *mixed distribution system*, in which one mixed distribution system, in which one<br>tailer sells only one is strongly stable for<br>nk costs. Finally, for some values of the<br>olistribution network is strongly stable for<br>olistribution in the mough or intermediate,<br>*ling* retailer distributes both products while the other retailer sells only one is strongly stable for high degrees of product differentiation and large link costs. Finally, for some values of the degree of product differentiation and link costs, no distribution network is strongly stable. In particular, when the degree of product differentiation is high enough or intermediate, the non-exclusive distribution & non-exclusive dealing system will not emerge in the "longrun" while Mycielski, Riyanto and Wuyts (2000) and Moner-Colonques, Sempere-Monerris and Urbano (2004) have shown that it is a "short-run" equilibrium.<sup>12</sup>

non-exclusive distribution & non-exclusive dealing system will not emerge in the "non-<br>"which Myclooki, Riyanto and Wayus (2000) and Moner-Colongues, Sempere-Monerris<br>Urbano (2004) have shown that it is a "short-run" equi We also wonder whether the stable distribution network is efficient, in the sense that it generates the greatest surplus for the agents that integrate the network. We find that the three stable distribution networks can be efficient for particular values of the degree of product differentiation and link costs, but not necessarily for the values under which they are stable. Moreover, the distribution network, referred as *exclusive distribution* & non*exclusive dealing*, in which two manufacturers distribute their products using a single and identical retailer is never stable but it is efficient for low degrees of product differentiation. Thus, a conflict between stability and efficiency may occur.

exclusive distribution & non-<br>ir products using a single and<br>rees of product differentiation.<br>exclusive distribution & non-<br>, consumers are better off the<br>f in a market with interbrand<br>are is maximized by either of<br>gree of exclusive dealing, in which two manufacturers distribute their products using a single and<br>identical retailer is never stable but it is efficient for low degrees of product differentiation.<br>Thus, a conflict between stabil Since consumers do not account for link costs, the *non-exclusive distribution* & *non*non-exclusive distribution & non-<br>Thus, consumers are better off the<br>er off in a market with interbrand<br>welfare is maximized by either of<br>e degree of product differentiation<br>th, two distribution networks may<br>ers choose sim *exclusive dealing* system maximizes consumer surplus. Thus, consumers are better off the highest the level of competition. That is, they are better off in a market with interbrand and intrabrand competition in both produ *exclusive dealing* system maximizes consumer surplus. Thus, consumers are better off the highest the level of competition. That is, they are better off in a market with interbrand and intrabrand competition in both products. Social welfare is maximized by either of the four efficient distribution networks depending on the degree of product differentiation and on the link costs. When link costs are small enough, two distribution networks may

and on the link costs. When link costs are small enough, two distribution networks may<br> $\frac{12 \text{ In Mycielski, Riyanto and Wuyts (2000), the two manufacturers choose *simultaneously* among combi-  
ration of vertical approachs; evaluating distribution, non-exclusive distribution, excluding the  
equation of the two-dimensional system.$ non-exclusive dealing. In Moner-Colonques, Sempere-Monerris and Urbano (2004), two manufacturers<br>nations of vertical arrangements; exclusive distribution, non-exclusive distribution, exclusive dealing, and<br>non-exclusive de simultaneously whether to employ retailer one, retailer two, both or neither of them. Both papers<br>e *simultaneously* whether to employ retailer one, retailer two, both or neither of them. Both papers choose simultaneously whether to employ retailer one, retailer two, both or neither of them. Both papers use the subgame perfect Nash equilibrium to solve the game and assume that links are costless.

maximize social welfare. The non-exclusive distribution  $\&$  non-exclusive dealing system maximizes social welfare if the degree of product differentiation is high enough; otherwise, the *mixed distribution system* maximizes social welfare. When link costs become large, two other distribution networks may maximize welfare. The exclusive distribution  $\&$  exclusive *dealing* system maximizes welfare if the degree of product differentiation is high enough; otherwise, the exclusive distribution & non-exclusive dealing system maximizes welfare. Thus, a conflict between stability and social welfare is likely to occur, even more if the degree of product differentiation is either low or high. $^{13}$ 

The paper is organized as follows. The model is presented in Section 2. In Section 3 we analyze the stable distribution networks. In Section 4 we analyze the efficient networks and the networks that maximize consumer surplus and social welfare. Finally, Section 5 concludes.

non-exclusive distribution & non-exclusive dealing system<br>cologre of product differentialities in high coungle; otherwise,<br>axximizes social welfare. The exclusive distribution is exclusive<br>axximizes social welfare. The ex ment distribution system maximizes social welfare. When link costs become large, two movies and distribution system maximizes socialize the maximizes welfare. The exclusion distribution is sign emotion system maximizes we exclusive distribution & exclusive<br>etcluster differentiation is high enough;<br>ading system maximizes welfare.<br>likely to occur, even more if the<br>med in Section 2. In Section 3 we<br>we analyze the efficient networks<br>social wel dealing system maximizes settlement if the degree of product differentiation is high enough;<br>the reached system maximizes welfare in the degree of product differentiation is high enough.<br>Thus, a couldic between stelleilit exclusive distribution  $k$  non-exclusive distribution decays and the two controllers between each time of the two controllers and welfare is likely to occur, even more if the between eaching rand order when the system of We develop a three-stage game to study the formation of networks among manufacturers and retailers in a successive duopoly. To reach consumers manufacturers and retailers should form a product distribution network consisting of different bilateral relationships (or links) between them. In an initial stage, manufacturers and retailers decide the links they want to establish among them. A link between a manufacturer and a retailer is necessary in order to sell the manufacturer's brand to consumers. In the second stage, once the distribution network has been formed, manufacturers decide simultaneously the transfer prices to retailers. Finally, retailers decide simultaneously the quantity of each brand they are going to market.

The two manufacturers  $(M_1 \text{ and } M_2)$  produce their own branded good under constant returns to scale and incur a common unit cost c. The retailers  $(R_1 \text{ and } R_2)$  are supplied by the manufacturers at a constant unit price, the transfer price.<sup>14</sup> Let  $w_i$  denote the transfer

The two manufacturers  $(M_1 \text{ and } M_2)$  produce their own branded good under constant<br>rms to scale and incur a common unit cost c. The retailers  $(R_1 \text{ and } R_2)$  are supplied by<br>manufacturers at a constant unit price, the tran returns to scale and incur a common unit cost c. The retailers  $(R_1 \text{ and } R_2)$  are supplied by<br>the manufacturers at a constant unit price, the transfer price.<sup>14</sup> Let  $w_i$  denote the transfer<br><sup>13</sup>Mycielski, Riyanto and Wuy the manufacturers at a constant unit price, the transfer price.<sup>14</sup> Let  $w_i$  denote the transfer  $^{13}$ Mycielski, Riyanto and Wayts (2000) have studied the welfare implications of manufacturers' choices<br>of vertical arrang  $13$ Mycielski, Riyanto and Wuyts (2000) have studied the welfare implications of manufacturers' choices of vertical arrangements and its policy implications in a setting where retailers compete à la Bertrand. They have shown that, for a high degree of product differentiation, any policy measure to restrict vertical restraints is unnecessary. However, we get that such policies become necessary once retailers compete à la Cournot. Then, restricting exclusive distribution and exclusive dealing arrangements might have a positive impact on social welfare.

 $14$ We limit attention to linear contracts. Although the superiority of two-part tariff contracts over linear ones is usually established because with the former manufacturers have two instruments (the transfer price used to give the right incentives to retailers and the fixed fee used to extract all the rent generated by the selling of the good), linear contracts may turn appropriate if there are observability or renegotiation problems (see chapter 4 in Tirole 1988). Linear contracts are used in several industries. Iyer and Villas-Boas (2003) have reported that in sectors such as grocery retailing or department stores retailers do not

price set by manufacturer *i* for supplying brand *i*,  $i = 1, 2$ . We assume that retailers may be multi-product, in the sense that they are allowed to carry both products. We also assume that retailers are not differentiated in the sense that consumers get the same utility for consuming a brand no matter which retailer sells the brand to them. We denote by  $N = \{M_1, M_2, R_1, R_2\}$  the set of agents which are connected in a distribution network.

Let  $q_{ij}$  be the quantity of brand i that retailer j sells to consumers. In case both retailers distribute brand i, let  $Q_i = q_{i1} + q_{i2}$  denote the total amount produced of brand i. The retailing costs supported by the retailers are assumed to be zero. Inverse demand functions are given by

$$
p_1 = a - Q_1 - dQ_2
$$
  

$$
p_2 = a - Q_2 - dQ_1
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where  $a > c$  and  $0 < d < 1$  (own effects on prices are greater than cross effects). So, brands 1 and 2 are imperfect substitutes and parameter  $d$  measures the degree of interbrand rivalry, that is, how similar the brands are perceived by consumers. When  $d$  approaches <sup>1</sup> brands become closer substitutes (interbrand rivalry increases). Intrabrand rivalry, that is how similar retailers' services are perceived by consumers to be when selling the same brand, is maximal since retailers are not differentiated, they are perfect substitutes. Since retailers can be multi-product sellers, there may be in-store competition, which means interbrand rivalry in a retailer selling the two products.

i for supplying brand i,  $i = 1, 2$ . We assume that readients<br>he seare that the same that the same of that in they are allowed to carry both products. We<br>note the some of the forestial in the sease that consumes get the sa N= {M<sub>1</sub>, M<sub>2</sub>, R<sub>2</sub>, R<sub>2</sub>, R<sub>2</sub>, R<sub>2</sub>, R<sub>2</sub> the set of agents which are counted in a distribution network.<br>Let  $q_2$  be the quantity of brand *x* that connect  $\mu$  set to consumers. In case both<br>shee distribute brand *i* Let q<sub>u</sub> be the quantity of brand i data realizer  $\gamma$  sells to consumers. In each both that retailer given the quantity of brand is that realizes are resumed to be zero. Inverse denand the distinction are given by<br>The re recallers distribute brand is let  $Q_1 = q_1 = q_2$  denote the total amount produced of brand inferious are given by<br>  $p_1 = a - Q_2 = dQ_2$ <br>  $p_2 = -Q_2 = dQ_3$ <br>
where  $a > c$  and  $0 < d < c$  (own effects on prices are greater than errors e i. The retailing costs supported by the retailers are use unred to be zero. Inverse demand<br>functions are given by<br> $y_1 = a - Q_1 - dQ_2$ <br> $y_2 = a - Q_2 - dQ_1$ <br>where  $a > c$  and  $0 < d < 1$  (own effects on priors are generate than erces e  $p_1 = a - q_1 - a q_2$ <br>  $p_2 = a - Q_2 - dQ_1$ <br>
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m an existi The distribution network cannot be enforced. We assume that joint consent is needed to establish and/or maintain a link between a manufacturer and a retailer. The cost of maintaining such a distribution link for each agent is denoted by  $k \geq 0$ . In a distribution  $k \geq 0$ . In a distribution<br>d links indicate different<br>etwork g is simply a list<br>ner. If  $M_1$  is linked with<br>ral, if we are considering<br>that i and j are linked<br>to an existing network<br> $\kappa$  (i, j) from an existing<br>stribut network, manufacturers and retailers are the nodes in the graph and links indicate different bilateral relationships between the agents. Then, a distribution network  $g$  is simply a list g is simply a list<br>  $M_1$  is linked with<br>
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unit prices for beer<br>
tly se of which pair of manufacturers and retailers are linked to each other. If  $M_1$  is linked with of which pair of manufacturers and retailers are linked to each other. If  $M_1$  is linked with  $R_1$  and with  $R_2$ , we write  $(M_1, R_1) \in g$  and  $(M_1, R_2) \in g$ . In general, if we are considering a pair of agents *i* and *j*  $R_1$  and with  $R_2$ , we write  $(M_1, R_1) \in g$  and  $(M_1, R_2) \in g$ . In general, if we are considering *R*<sub>1</sub> and with *R*<sub>2</sub>, we write  $(M_1, R_1) \in g$  and  $(M_1, R_2) \in g$ . In general, if we are considering a pair of agents *i* and *j*, with *i*, *j* ∈ *N*, then  $(i, j) \in g$  indicates that *i* and *j* are linked under the networ a pair of agents i and j, with  $i, j \in N$ , then  $(i, j) \in g$  indicates that i and j are linked i and j, with  $i, j \in N$ , then  $(i, j) \in g$  indicates that i and j are linked<br>
che  $g$ . The network obtained by adding link  $(i, j)$  to an existing network<br>  $(i, j)$  and the network obtained by deleting link  $(i, j)$  from an existi under the network g. The network obtained by adding link  $(i, j)$  to an existing network g is denoted  $g + (i, j)$  and the network obtained by deleting link  $(i, j)$  from an existing network g is denoted  $g - (i, j)$ . Let G be the set of all possible distribution networks.

In what follows,  $g(12,0)$  represents the distribution network in which retailer  $R_1$  is selling brand 1 and brand 2 and retailer  $R_2$  sells no brand, while  $g(1,12)$  represents

g. The network obtained by adding link  $(i, j)$  to an existing network<br>
i, j) and the network obtained by deleting link  $(i, j)$  from an existing<br>
d  $g - (i, j)$ . Let G be the set of all possible distribution networks.<br>
g (12,0) g is denoted  $g + (i, j)$  and the network obtained by deleting link  $(i, j)$  from an existing<br>network g is denoted  $g - (i, j)$ . Let G be the set of all possible distribution networks.<br>In what follows,  $g(12, 0)$  represents the d g is denoted  $g$  − (i, j). Let G be the set of all possible distribution networks.<br>hat follows,  $g(12,0)$  represents the distribution network in which retailer R<br>rand 1 and brand 2 and retailer  $R_2$  sells no brand, while  $g(12,0)$  represents the distribution network in which retailer  $R_1$  is<br>brand 2 and retailer  $R_2$  sells no brand, while  $g(1,12)$  represents<br>fees to manufacturers. Sass (2005) has described the U.S. beer industry as a<br>s selling brand 1 and brand 2 and retailer  $R_2$  sells no brand, while  $g(1, 12)$  represents<br>seem to pay lump-sum fees to manufacturers. Sass (2005) has described the U.S. beer industry as a<br>three-tier system (brewers, dist seem to pay lump-sum fees to manufacturers. Sass (2005) has described the U.S. beer industry as a three-tier system (brewers, distributors and retailers) where brewers set constant per-unit prices for beer and do not charge distributors explicit franchise fees. Distributors in turn independently set simple linear wholesale prices to retailers.

the distribution network in which  $R_1$  is selling brand 1 and  $R_2$  sells brands 1 and 2; the distribution network in which R<sub>1</sub> is selling brand. Land R<sub>2</sub> sellin brands 1 and 2;<br>i.e.,  $g(12,0) = \frac{1}{2}(M_1/R_1)$ ,  $M_2 R_2$ ) and  $g(11,2) = \frac{1}{2}(M_1/R_1)$ ,  $M_2 R_2$  is brands 1 and 2;<br>i.e.,  $g(12,0) = \frac{1}{2}(M_1/R_1)$ , i.e.,  $g(12,0) = \{(M_1, R_1), (M_2, R_1)\}\$ and  $g(1,12) = \{(M_1, R_1), (M_1, R_2), (M_2, R_2)\}\.$ Given the symmetry of products and retailers, there are only six qualitatively different distribution networks out of fifteen. The six distribution networks we are going to analyze are  $g(1,0), g(1,1), g(12,0), g(1,2), g(12,1)$  and  $g(12,12)$  and are depicted in Figure 1. The distribution network  $g(1,0)$  is symmetric to  $g(0,1)$ ,  $g(0,2)$  and  $g(2,0)$ ;  $g(1,1)$  is symmetric to  $g(2,2)$ ;  $g(12,0)$  is symmetric to  $g(0,12)$ ;  $g(1,2)$  is symmetric to  $g(2,1)$ ; and  $g(12,1)$  is symmetric to  $g(12, 2)$ ,  $g(1, 12)$  and  $g(2, 12)$ .

g(12, 0) = {(M1, R1), (M2, R1), (M1, R1), (M1, R1), (M1, R1), (M1, R2), (M1, R2), (M1, R2), (M1, R2)). (M4, R2)<br>
between the symmetry of strokets and reasting, there are only at equalitatively different<br>theirs networks ca g(1, 0), g(1, 1), g(1, 2), g(2, 1)) and g(12, 12) and g(12, 12) and g(12, 12) and g(12, 12) and g(1, 2), g(1, 1) and g(1, 2), g(1, 1) is symmetric or g(1, 12); g(1, 2) is and g(1, 2); g(1, 2) is gymmetric or g(1, 2); g(1,  $g(1, 0)$  is symmetric to  $g(0, 1)$ ,  $g(0, 2)$  and  $g(2, 0)$ ;  $g(1, 1)$  is symmetric to  $g(2, 1)$ ; and  $g(1, 2)$  is symmetric to  $g(2, 1)$ ; and  $g(1, 2)$  is tymmetric to  $g(2, 1)$ ; and  $g(1, 2)$  is the equilibrium transfe g(2, 2); g(1, 2); g(1, 2)); g(1, 2) is symmetric to g(2, 1); and g(1, 2); g(1, 2); g(1, 2) is symmetric to g(2, 1); and g(1, 2); g(1, 2) is symmetric to g(2, 1); and g(1, 12); Hefore looking for the stability and effelome  $g(12, 2)$ ,  $g(1, 12)$  and  $g(2, 12)$ .<br>
king for the stability and effinate explore the stability and effinate work architecture, the equencer surplus and aggregate wet  $\Phi(g)$  be the sum of the intertwork  $g(12, 12) = \{(M_1$ Before looking for the stability and efficiency of distribution networks, we derive for each possible network architecture, the equilibrium transfer prices, quantities produced, profits, consumer surplus and aggregate welfare. We denote by  $\Pi_i(q)$  the profit of i in profits, consumer surplus and aggregate welfare. We denote by Πi(g) the profit of in<br>herewith  $g$ . Let  $\Phi(g)$  be the sum of the individual payoffs or profits. That is,  $\Phi(g) = \ln g_0(g) + \ln g_0(g) + \ln g_1(g)$ , for the sake of the network g. Let  $\Phi(g)$  be the sum of the individual payoffs or profits. That is,  $\Phi(g)$  = g. Let  $\Phi(g)$  be the sum of the individual payoffs or profits. That is,  $\Phi(g) = \Box_{\text{Mg}}(g) + \Box_{\text{Mg}}(g)$ , For the suke of the expotition we pose<br>tion network  $g(2, 12) = \{(M_1, R_1), (M_2, R_1), (M_2, R_2), (M_2, R_3)\}$ , in which ea  $\Pi_{M_1}(g) + \Pi_{M_2}(g) + \Pi_{R_1}(g) + \Pi_{R_2}(g)$ . For the sake of the exposition we present here the Π<sub>M1</sub> (g) + Π<sub>M2</sub> (g) + Π<sub>M2</sub> (g) + Π<sub>M2</sub> (g). For the sake of the exposition we present here the distribution network  $g(2, 12) = \{ \langle M_1, R_1 \rangle, \langle M_2, R_2 \rangle, \langle M_2, R_2 \rangle \}$ , in which exchines in the<br>word  $g(2, 12) = \{ \langle M_1,$ distribution network  $g(12, 12) = \{(M_1, R_1), (M_2, R_1), (M_1, R_2), (M_2, R_2)\}\$ , in which each  $g(12, 12) = \{(M_1, R_1), (M_2, R_1), (M_3, R_2), (M_2, R_2)\}$ , in which each<br>
and 2, referred as the *non-exclusive distribution* & *non-exclusive*<br>
functions in  $g(12, 12)$  are:<br>  $(g(12, 12)) = (w_1 - c)(q_{11} + q_{12}) - 2k$  (1)<br>  $(g(12, 12)) = ($ retailer sells brands 1 and 2, referred as the non-exclusive distribution & non-exclusive dealing system.<br>Agents objective functions in  $g(12, 12)$  are: dealing system.

$$
\Pi_{M_1}(g(12, 12)) = (w_1 - c)(q_{11} + q_{12}) - 2k \tag{1}
$$

$$
\Pi_{M_1}(g(12, 12)) = (w_1 - c)(q_{11} + q_{12}) - 2k
$$
\n
$$
\Pi_{M_2}(g(12, 12)) = (w_2 - c)(q_{21} + q_{22}) - 2k
$$
\n
$$
\Pi_{R_1}(g(12, 12)) = (p_1 - w_1)q_{11} + (p_2 - w_2)q_{21} - 2k
$$
\n(3)

$$
\Pi_{R_1}(g(12,12)) = (p_1 - w_1)q_{11} + (p_2 - w_2)q_{21} - 2k \tag{3}
$$

$$
\Pi_{R_1}(g(12, 12)) = (w_2 - c)(q_{21} + q_{22}) - 2k
$$
\n
$$
\Pi_{R_1}(g(12, 12)) = (p_1 - w_1)q_{11} + (p_2 - w_2)q_{21} - 2k
$$
\n
$$
\Pi_{R_2}(g(12, 12)) = (p_1 - w_1)q_{12} + (p_2 - w_2)q_{22} - 2k.
$$
\n(4)

In the last stage of the game, links and transfer prices are given. Under Cournot competition the retailers compete by choosing simultaneously the quantity of each brand they are going to market. The unique Nash equilibrium of this stage game is

First, the unique Nash equilibrium of this stage game is

\n
$$
q_{11}(g(12, 12)) = q_{12}(g(12, 12)) = \frac{a(1-d) - w_1 + dw_2}{3(1-d^2)}
$$
\n
$$
q_{21}(g(12, 12)) = q_{22}(g(12, 12)) = \frac{a(1-d) + dw_1 - w_2}{3(1-d^2)}.
$$

In the second stage, manufacturers decide simultaneously the transfer prices to retailers.<br>The unique Nash equilibrium of this stage game is<br> $w_1(g(12, 12)) = w_2(g(12, 12)) = w(g(12, 12)) = a - \frac{(a - c)}{(2 - d)}$ . The unique Nash equilibrium of this stage game is

sh equilibrium of this stage game is  
\n
$$
w_1(g(12, 12)) = w_2(g(12, 12)) = w(g(12, 12)) = a - \frac{(a - c)}{(2 - d)}
$$

Then, one can easily obtain the equilibrium profits:

$$
\Pi_{M_1}(g(12, 12)) = \Pi_{M_2}(g(12, 12)) = \frac{2(1-d)(a-c)^2}{3(1+d)(2-d)^2} - 2k
$$
\n
$$
\Pi_{R_1}(g(12, 12)) = \Pi_{R_2}(g(12, 12)) = \frac{2(a-c)^2}{9(1+d)(2-d)^2} - 2k.
$$
\n(6)

$$
\Pi_{R_1}(g(12,12)) = \Pi_{R_2}(g(12,12)) = \frac{2(a-c)^2}{9(1+d)(2-d)^2} - 2k.
$$
 (6)

To determine the efficient distribution network, we compute the sum of the individual equilibrium payoffs,  $\Phi(g(12, 12))$ . Then,

at distribution network, we compute the sum of the individual  
12, 12)). Then,  

$$
\Phi(g(12, 12)) = \frac{4(4-3d)(a-c)^2}{9(1+d)(2-d)^2} - 8k.
$$
 (7)

Let  $C(g(12, 12))$  denote the consumer surplus in case  $g(12, 12)$  is formed. The corresponding consumer surplus for this system of inverse linear demands is given by the expression Let  $C(g(12, 12))$  denote the consumer surplus in case  $g(12, 12)$  is formed. The corres<br>ing consumer surplus for this system of inverse linear demands is given by the expre<br> $\frac{1}{2}[(q_{11} + q_{12})^2 + (q_{21} + q_{22})^2]$ . Substitu

$$
\frac{1}{2}[(q_{11} + q_{12})^2 + (q_{21} + q_{22})^2].
$$
 Substituting for the equilibrium quantities, we obtain  

$$
C(g(12, 12)) = \frac{4(a-c)^2}{9(1+d)^2(2-d)^2}.
$$
(8)  
For any distribution network *g*, social or aggregate welfare is defined as the sum of con-

For any distribution network g, social or aggregate welfare is defined as the sum of consumer surplus and total equilibrium profits. Let  $W(g(12, 12))$  denote aggregate welfare in For any distribution network g (12, 12). Then,

m,  
\n
$$
W(g(12, 12)) = \frac{4(5+d-3d^2)(a-c)^2}{9(1+d)^2(2-d)^2} - 8k
$$
\n(9)

In the appendix we give the equilibrium profits, the sum of the individual equilibrium profits, the consumer surplus and the social welfare for each possible distribution network among the two manufacturers and the two retailers. The other relevant equilibrium variables  $q$ 's and  $w$ 's are available from the authors upon request.

#### 3 Stable distribution networks

A simple way to analyze the networks that one might expect to emerge in the long run is to examine a sort of equilibrium requirement that agents not benefit from altering the structure of the network. A weak version of such condition is the pairwise stability notion defined by Jackson and Wolinsky (1996). A network is pairwise stable if no agent benefits from severing one of their links and no other two agents benefit from adding a link between them, with one benefiting strictly and the other at least weakly.

**Definition 1** A network  $g$  is pairwise stable if

(i) for all  $(i, j) \in g$ ,  $\Pi_i(g) \geq \Pi_i(g - (i, j))$  and  $\Pi_j(g) \geq \Pi_j(g - (i, j))$ , and

(ii) for all  $(i, j) \notin g$ , if  $\Pi_i(g) < \Pi_i(g + (i, j))$  then  $\Pi_i(g) > \Pi_i(g + (i, j))$ .

Let us say that g' is adjacent to g if  $g' = g + (i, j)$  or  $g' = g - (i, j)$  for some  $(i, j)$ . A Let us say that g' is adjacent to g if  $g' = g + (i, j)$  or  $g' = g - (i, j)$  for some  $(i, j)$ . A network g' defeats g if either  $g' = g - (i, j)$  and  $\Pi_i (g') \ge \Pi_i (g)$ , or if  $g' = g + (i, j)$  with Let us say that  $g'$  is adjacent to  $g$  if  $g' = g + (i, j)$  or  $g' = g - (i, j)$  for some  $(i, j)$ . A network  $g'$  defeats  $g$  if either  $g' = g - (i, j)$  and  $\Pi_i (g') \ge \Pi_i (g)$ , or if  $g' = g + (i, j)$  with  $\Pi_i (g') \ge \Pi_i (g)$  and  $\Pi_j (g') \ge \Pi_j (g)$ stability is equivalent to saying that a network is pairwise stable if it is not defeated by another (necessarily adjacent) network.

While pairwise stability is natural and quite easy to work with, it is a concept with some limitations. First, it is a weak notion in that it only considers deviations on a single link at a time. For instance, it could be that an agent would not benefit from severing any single link but would benefit from severing several links simultaneously, and yet the network would still be pairwise stable. Second, pairwise stability considers only deviations by at most a pair of agents at a time. It might be that some group of agents could all be made better off by some complicated reorganization of their links, which is not accounted for under pairwise stability. The definition of strong stable networks is in that spirit, and is due to Jackson and van den Nouweland (2005). A strongly stable network is a network which is stable against changes in links by any coalition of agents.

A network  $g' \in G$  is obtainable from  $g \in G$  via deviations by  $S \subset N$  if

- (i)  $ij \in g'$  and  $ij \notin g$  implies  $ij \subset S$ , and
- (i)  $ij \in g'$  and  $ij \notin g$  implies  $ij \subset S$ , and<br>(ii)  $ij \in g$  and  $ij \notin g'$  implies  $ij \cap S \neq \emptyset$ . (ii)  $ij \in g$  and  $ij \notin g'$  implies  $ij \cap S \neq \emptyset$ .<br>The above definition identifies changes in a network that can be made by a coalition

S,<br>ew nt<br>my mand<br>of<br>of<br>b-te The above definition identifies changes in a network<br>without the need of consent of any agents outside of without the need of consent of any agents outside of  $S$ . Part (i) requires that any new hat can be made by a coalition  $S$ ,<br>  $S$ . Part (i) requires that any new<br>
This reflects the fact that consent<br>
ires that at least one agent of any<br>
gent in a link can unilaterally sever<br>  $S \subset N$ ,  $g'$  that is obtainable fr The above definition identifies changes in a networ<br>without the need of consent of any agents outside<br>links that are added can only be between agents in s that can be made by a coalition S,<br>of S. Part (i) requires that any new<br>S. This reflects the fact that consent<br>quires that at least one agent of any<br>r agent in a link can unilaterally sever<br> $\text{sup } S \subset N$ , g' that is obtai without the need of consent of any agents outside of  $S$ . Part (i) requires that any new<br>links that are added can only be between agents in  $S$ . This reflects the fact that consent<br>of both agents is needed to add a link. links that are add<br>of both agents is<br>deleted link be in deleted link be in  $S$ . This reflects the fact that either agent in a link can unilaterally sever ed can only be between agents in S. This reflects the fact that consent<br>needed to add a link. Part (ii) requires that at least one agent of any<br>S. This reflects the fact that either agent in a link can unilaterally sever<br> of both agents is<br>deleted link be in the relationship.

**Definition 2** A network g is strongly stable if for any  $S \subset N$ , g' that is obtainable from g is strongly stable if for any  $S \subset N$ , g' that is obtainable from<br>
d  $i \in S$  such that  $\Pi_i(g') > \Pi_i(g)$ , there exists  $j \in S$  such that<br>
les a powerful refinement of pairwise stability. The concept of<br>
also sense in smaller n g via deviations by S, and  $i \in S$  such that  $\Pi_i(g') > \Pi_i(g)$ , there exists  $j \in S$  such that  $\Pi_j(g') < \Pi_j(g).$  $g'$ )  $\langle \Pi_j(g) \rangle$ <br>Strong stability provides a powerful refinement of pairwise stability. The concept of

g via deviations by S, and  $i \in S$  such that  $\Pi_i(g')$ <br>  $\Pi_j(g') < \Pi_j(g)$ .<br>
Strong stability provides a powerful refineme:<br>
strong stability mainly makes sense in smaller ne<br>
stantial information about the overall structure a<br>
1  $> \Pi_i(g)$ , there exists  $j \in S$  such that<br>of pairwise stability. The concept of<br>ork situations where agents have sub-<br>objectual payoffs and can coordinate<br>objectual payoffs and can coordinate  $n_j$  (*g*)<br>S:<br>stron<br>stant  $\langle \Pi_j(g).$ <br>
ong stab<br>
stability<br>
l informa strong stability provides a powerful refinement of pairwise stability. The concept of<br>strong stability mainly makes sense in smaller network situations where agents have substrong stability mainly makes sense in smaller network situations where agents have substantial information about the overall structure and potential payoffs and can coordinate

their actions. $15$  That is, it makes sense to study the stability of distribution networks their actions.<sup>15</sup> That is, it makes sense to study the stabilers have a successive duopoly. between manufacturers and retailers in a successive duopoly. In order to characterize the strongly stable distribution networks<br>ween manufacturers and retailers in a successive duopoly.<br>In order to characterize the strongly stable distribution networks we first derive the

between manufacturers and retailers in a successive duopoly.<br>In order to characterize the strongly stable distribution networks we first derive the<br>pairwise stable network since a strongly stable network is pairwise stable In order to characterize the strongly stable distribution networks we first derive the pairwise stable networks since a strongly stable network is pairwise stable while the reverse is not true. To make meaningful compariso is not true. To make meaning<br> $k$  so that for each possible dis-<br>positive. The upper bound on positive. The upper bound on k is given in the next lemma and displayed in Figure  $2^{16}$ 



Figure 2: Bounds on the link cost

Lemma 1 All agents' equilibrium payoffs are positive in each possible distribution network if the link cost k is bounded above as follows,  $k \leq \frac{(a-c)^2}{36}$  if 0

pairwise stable network since a convertely stable to some is pairwise stable while the two stable distribution network any equilibrium output and payoffs are stable distribution network any equilibrium output and payoffs c/ عدد المساور المساور<br>المساور المساور المساو k is bounded above as follows,  $k \leq \frac{(a-c)^2}{36}$ <br>facilitated through industry associations which reg<br>dustry. For instance, the agri-food industry in Ca<br>ie Brewers Association of Canada whose role is to<br>generally between b  $< d \leq 0.779$  and<br>ms having some com-<br>as 31 industry associ-<br>and improve business<br>and the public in the<br>n provides services to<br>ssues, and the promo-<br>ucts Manufacturers of<br>er products manufac-<br>inada's independently<br>.agr.gc  $15$ Coordination may be facilitated through industry associations which regroup firms having some common interest within an industry. For instance, the agri-food industry in Canada has 31 industry associations. One of them is the Brewers Association of Canada whose role is to foster and improve business relations and cooperation generally between brewers in Canada and between them and the public in the furtherance and protection of their respective interests and welfare. The assocation provides services to industry including statistics on beer consumption, monitoring government and policy issues, and the promotion of responsible consumption. Another association is the Food and Consumer Products Manufacturers of Canada whose aim is to enhance growth and competitiveness of the food and consumer products manufacturing industry. There is also the Canadian Association of Independent Grocers which is a non-profit trade association founded in 1962 with the purpose of furthering the unique interests of Canada's independently owned and franchised supermarkets. More information can be found at http://www.agr.gc.ca/

<sup>&</sup>lt;sup>16</sup>In all figures that appear in the paper we have considered the case where  $(a - c) = 1$ .<br>
12

$$
k \le \frac{(1-d)(a-c)^2}{3(1+d)(2-d)^2}
$$
 if  $0.779 < d < 1$ .  
All proofs can be found in the

<sub>3(1+d)(2−d)</sub>-<br>2Ω<br>2Ω proofs All proofs can be found in the appendix. Denote the upper bound on

$$
\overline{k} \equiv \min \left\{ \frac{(a-c)^2}{36}, \frac{(1-d)(a-c)^2}{3(1+d)(2-d)^2} \right\}.
$$

The first term corresponds to the constraint on k that implies that  $\Pi_{R_1}(g(1,1))$  is positive while the second is the one that ensures  $\Pi_{M_1}(g(12, 12)) > 0$ . The following remarks are useful in understanding Figure 3 which displays the pairwise stable distribution networks.

- a) The distribution network  $g(12,12)$  is pairwise stable if and only if  $k < \min\{k_{(12,12)}^M, k_{(12,12)}^M, k_{(12,12)}$  $\{k_{(12,12)}^R\}$  since no agent wants to sever a link.
- b) The distribution network  $q(1,2)$  (and  $q(2,1)$ ) is pairwise stable if and only if  $k >$  $\{1,2\}$ ,  $k_{(1,2)}^R$  since no pair manufacturer-retailer wants to create a link and no agent wants to destroy a link.
- c) The distribution network  $g(12,1)$  (and  $g(12,2)$ ,  $g(1,12)$ ) and  $g(2,12)$ ) is pairwise stable if and only if  $k > \min\{k_{(12,12)}^M, k_{(12,12)}^R\}$  and  $k < \min\{k_{(1,2)}^M, k_{(1,2)}^R\}$  since no agent wants to sever a link and the pair manufacturer-retailer with only one link does not want to create another link.

Therefore, in the area C the only pairwise stable distribution network is  $g(1,2)$ ; in the area B both  $q(1,2)$  and  $q(12,12)$  are pairwise stable; in the area A only  $q(12,12)$  is pairwise stable; and in the area D  $q(12,1)$  is the only pairwise stable distribution network. The following proposition summarizes pairwise stability among distribution networks.

if  $\overline{k}$  is the one that<br>  $\overline{k}$  is the one that<br>  $\overline{k}$  is the one that<br>
anding Figure 3<br>
oution network govee no agent wan<br>
bution network govee no agent wan<br>
bution network govee the  $k_{(1,2)}^R$  since no s<br>
s to de k as k, where<br>
1,1)) is positi<br>
ng remarks a<br>
ution network<br>  $<$  min{ $k_{(12,1)}^M$ <br>
and only if k<br>
e a link and 1<br>
12)) is pairwi<br>  $k_{(1,2)}^R$  since 1<br>  $k_{(1,2)}^R$  since 1<br>  $k_{(1,2)}^R$  since 1<br>  $k_{(1,2)}^R$  since 1<br>  $k_{(1,2$  $\overline{k}$  = min  $\begin{cases} \frac{(a-c)^2}{36} \\ \frac{(a-c)^2}{36} \end{cases}$ <br>to the constraint<br>me that ensures  $\Pi$ <br>igure 3 which disp<br>twork  $g(12,12)$  is<br>ent wants to sever<br>twork  $g(12,1)$  (and<br>ince no pair manur<br>roy a link.<br>twork  $g(12,1)$  (and<br>,  $\frac{(1-d)(a-c)^2}{3(1+d)(2-d)}$ <br>
on k that implie<br>
on k that implie<br>
on k that implie<br>
pairwise stable if<br>
a link.<br>  $g(2,1)$  is pairwise<br>
pairwise stable if<br>
a link.<br>  $g(2,1)$  is pairwise stable if<br>
a link.<br>
d  $g(12,2)$ ,  $g(1)$ ,<br> .  $3(1 + d)(2 - d)^2$ <br>
on k that implies<br>  $\frac{1}{2}(g(12, 12)) > 0$ <br>
on k that implies<br>
invise stable if<br>  $\frac{1}{2}$  ink.<br>  $g(2, 1)$  is pairw<br>
cturer-retailer w<br>  $\frac{1}{2}(12, 2), g(1, 1)$ <br>  $\frac{1}{6}(12, 12), g(1, 1)$ <br>  $\frac{1}{6}(12, 12), g(1, 1)$ k that implies that  $\Pi_{R_1}(g(1,1))$  is positive  $g(12, 12) > 0$ . The following remarks are<br>the pairwise stable distribution networks.<br>wise stable if and only if  $k < \min\{k_{(12,12)}^M\}$ ,<br>ink.<br>2,1)) is pairwise stable if and o while the second is the one that ensures  $\Pi_{M_1}(q(2,12)) > 0$ . The following remarks are<br>second is the one that ensures  $\Pi_{M_1}(q(2,12))$  is pairwise stable directly and ensures are<br>all in understanding Figure 3 which displ  $g(12, 12)$  is pairwise stable if and only if  $k < \min\{k_{\text{II}}^M\}$ <br>unts to sever a link.<br> $g(1, 2)$  (and  $g(2, 1)$ ) is pairwise stable if and only if<br>pair manufacturer-retailer wants to create a link an<br>ink.<br> $g(12, 1)$  (and  $k(1)$  Th mi age Th stage do for here is  $(1, w)$  is  $n$  is tall  $r$  is  $k$  is tall  $r$  is  $n$   $g(1, 2)$  (and  $g(2, 1)$ ) is pairwise stable if and only if  $k >$ <br>pair manufacturer-retailer wants to create a link and no<br>ink.<br> $g(12, 1)$  (and  $g(12, 2)$ ,  $g(1, 12)$  and  $g(2, 12)$ ) is pairwise<br> $\min\{k_{(12)}^{N}(2, 12), k_{(12,$ min{ $k_{(1)}^{3m}$ <br>agent w.<br>The dis<br>stable if<br>agent w.<br>does not<br>fore, in h  $g(1,2)$ ; and in<br>ing prop<br>osition<br>(1,2). *T*<br>w values<br>ideal ing prop<br>is the values<br>ideal in region<br>is the values<br> $r \sinh \theta$  is the values<br>reare to Sand  $k_{(1)}^{(1)}$  is divided in a reduce  $k$  is divided in the  $\frac{1}{k}$  of  $\frac{1}{k}$  and  $\frac{1}{k}$  a  $g(12, 1)$  (and  $g(12, 2)$ ,  $g(1, 12)$  and  $g(2, 12)$ ) is pairwise<br>  $\inf\{k_{(12,12)}^{M}, k_{(12,12)}^{R}\}$  and  $k < \min\{k_{(1,2)}^{M}, k_{(1,2)}^{R}\}$  since no<br>
k and the pair manufacturer-retailer with only one link<br>
other link.<br>
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of *k close t.*<br> *nly pairwise*<br>
of *k close t.*<br> *igion,* kR  $g(1, 2)$ ; in the area<br>12, 12) is pairwise<br>ion network. The<br>networks.<br> $g(12, 12)$ ,  $g(12, 1)$ <br>In the first region,<br>he second, for low<br>stable network. In<br>the upper bound,<br>rmediate values of<br>networks are pair-<br>stem  $g(12, 1$  $g(1, 2)$  and  $g(12, 12)$  are pairwise stable; in the area A only  $g(12, 12)$  is pairwise<br>und in the area D  $g(12, 1)$  is the only pairwise stable distribution network. The<br>groposition summarizes pairwise stability among  $g(12, 1)$  is the only pairwise stable distribution network. The<br>aarizes pairwise stability among distribution networks.<br>distribution networks can be pairwise stable:  $g(12, 12)$ ,  $g(12, 1)$ <br>space  $(k, d)$  is partitioned in **Proposition 1** Only three distribution networks can be pairwise stable:  $g(12, 12)$ ,  $g(12, 1)$ <br>and  $g(1, 2)$ . The parameter space (k, d) is partitioned into four regions. In the first region, Only three distribution hectorics can be pairwise stable.  $y(12, 12)$ ,  $y(12, 11)$ <br>parameter space (k, d) is partitioned into four regions. In the first region,<br>intermediate values of k,  $g(12, 1)$  is the only pairwise st and  $y_1$ ,  $z_1$ . The parameter space ( $k$ ,  $\alpha$ ) is partitioned into four regions. In the first region, for low<br>values of d and intermediate values of  $k$ ,  $g(12, 1)$  is the only pairwise stable network. In<br>the third reg for low values of d,  $g(12, 12)$  is the only pairwise stable network. In the second, for low for low values of a,  $g(12, 12)$  is the only pairwise stable network. In the second, for low<br>values of d and intermediate values of k,  $g(12, 1)$  is the only pairwise stable network. In<br>the third region, for high and low values of d and intermediate values of k,  $g(12,1)$  is the only pairwise stable network. In values of d and intermediate values of k, g(12, 1) is the only pairwise stable network. In<br>the third region, for high and low values of d and values of k close to the upper bound,<br> $g(1,2)$  is the only pairwise stable netw the third region, for high and low values of  $d$  and values of  $k$  close to the upper bound,  $q(1,2)$  is the only pairwise stable network. In the fourth region, for intermediate values of d, there are two pairwise stable networks  $g(12,12)$  and  $g(1,2)$ .

It is interesting to note that when  $k = 0$ , only two distribution networks are pairwise stable:<sup>17</sup> the *non-exclusive distribution* & *non-exclusive dealing* system  $g(12, 12)$  and  $\frac{17}{17}$  For  $k = 0$  (assumption made in Mycielski, Riyanto and Wuyts (2000) and Moner-Colonques, Sempere-

the third region, the third region, for intermediate values of d, there are two pairwise stable network. In the fourth region, for intermediate values of d, there are two pairwise stable networks  $g(12,12)$  and  $g(1,2)$ .<br> g(1, 2) is the only pairwise stable network. In the fourth region, for intermediate catals of<br>d, there are two pairwise stable networks  $g(12, 12)$  and  $g(1, 2)$ .<br>It is interesting to note that when  $k = 0$ , only two distr d, there are two pairwise stable hetworks  $g(12, 12)$  and  $g(1, 2)$ .<br>It is interesting to note that when  $k = 0$ , only two distri-<br>wise stable:<sup>17</sup> the *non-exclusive distribution* & *non-exclusive* d<br><sup>17</sup>For  $k = 0$  (assum  $k = 0$ , only two distribution networks are pair-<br>tion & non-exclusive dealing system  $g(12, 12)$  and<br>tiyanto and Wuyts (2000) and Moner-Colonques, Sempere-<br>le distribution networks coincide exactly with the subgame<br>f Mone non-exclusive distribution & non-exclusive dealing system  $g(12, 12)$  and<br>ption made in Mycielski, Riyanto and Wuyts (2000) and Moner-Colonques, Sempere-<br>(2004)), the pairwise stable distribution networks coincide exactly <sup>17</sup> For  $k = 0$  (assumption made in Mycielski, Rivanto and Wuyts (2000) and Moner-Colonques, Sempere $p_{\text{inter}} = 0$  (assumption made in Mycletski, rityanto and Wuyts (2000) and Moner-Colonques, Sempere-Monerris and Urbano (2004)), the pairwise stable distribution networks coincide exactly with the subgame perfect Nash equi when its and critical (2004)<br>perfect Nash equilibrium dis when brands are symmetric:



Figure 3: Pairwise stability of distribution networks.

the exclusive distribution & exclusive dealing system  $g(1,2)$ ;  $g(12,12)$  is stable when the exclusive distribution & exclusive dealing system  $g(1, 2)$ ;  $g(12, 12)$  is stable when the<br>luttes are sufficiently differentiated  $d \in (0.692, 0.060)$ , and  $g(1, 2)$  is stable when products are<br>substitutes  $d \in (0.692, 0.9$ products are sufficiently differentiated  $d \in (0, 0.682)$ , both  $q(12, 12)$  and  $q(1, 2)$  are stable  $d \in (0, 0.682)$ , both  $g(12, 12)$  and  $g(1, 2)$  are stable<br>9), and finally,  $g(1, 2)$  is stable when products are<br>ea absence of link costs, the pairwise stability of a<br>ives of manufacturers and retailers to introduce infor intermediate values  $d \in (0.682, 0.909)$ , and finally,  $g(1,2)$  is stable when products are  $d \in (0.682, 0.909)$ , and finally,  $g(1, 2)$  is stable when products are<br>909, 1).<sup>18</sup> In the absence of link costs, the pairwise stability of a<br>strategic incentives of manufacturers and retailers to introduce in-<br>noe ther close substitutes  $d \in (0.909, 1)$  <sup>18</sup> In the absence of link costs, the pairwise stability of a  $d \in (0.909, 1).$ <br>
on the strategi<br>
tion once ther<br>
acturer and a 1<br>
and rivalry is n<br>
e distribution<br>
ion network  $g$ <br>
ould have no i<br>
and rivalry,  $d$ <br>  $l \in (0, 0.682),$ <br>
2) is defeated<br>
e values of inte<br>
sent is needed th network depends on the strategic incentives of manufacturers and retailers to introduce intrabrand competition once there is interbrand competition. Take the distribution network  $g(1,2)$ . A manufacturer and a retailer would like to form a link between them only if the degree of interbrand rivalry is not too high; that is, if and only if  $d \in (0, 0.682)$ . Thus, for  $d \in (0, 0.682)$ , the distribution  $q(1,2)$ . A manufacturer and a retailer would like to form a link between them only if the degree of interbrand rivalry is not too high; that is, if and only if  $d \in (0, 0.682)$ . Thus, for  $d \in (0, 0.682)$ . Thus, for<br>
entwork  $g(12, 2)$ . Take<br>
stable, the pair formed<br>
hem. But, given the low<br>
stablish a link between<br>
defeated by the network<br>
) is defeated by  $g(12, 12)$ <br>
682, 0.909).<br>
ween a manufacturer an  $d \in (0, 0.682)$ , the distribution network  $g(1,2)$  is defeated by the network  $g(12,2)$ . Take  $d \in (0, 0.682)$ , the distribution network  $g(1, 2)$  is defeated by the network  $g(12, 2)$ . Take<br>now the distribution network  $g(12, 2)$ . In order for  $g(12, 2)$  to be stable, the pair formed<br>by  $M_1$  and  $R_2$  should hav now the distribution network  $g(12,2)$ . In order for  $g(12,2)$  to be stable, the pair formed  $g(12, 2)$ . In order for  $g(12, 2)$  to be stable, the pair formed<br>interest in adding a link between them. But, given the low<br> $l \in (0, 0.682)$ , they also prefer to establish a link between<br>the distribution network  $g(1, 2)$ by  $M_1$  and  $R_2$  should have no interest in adding a link between them. But, given the low by  $M_1$  and  $R_2$  should have no interest in adding a link between them. But, given the low<br>degree of interbrand rivalry,  $d \in (0, 0.682)$ , they also prefer to establish a link between<br>them. Thus, for  $d \in (0, 0.682)$ , the degree of interbrand rivalry,  $d \in (0, 0.682)$ , they also prefer to establish a link between them. Thus, for  $d \in (0, 0.682)$ , the distribution network  $g(1,2)$  is defeated by the network  $q(12,2)$  and  $q(12,2)$  is defeated by  $q(12,12)$ . Observe that  $q(12,2)$  is defeated by  $q(12,12)$ ot only for these values of interbrand rivalry but also for  $d \in (0.682, 0.909)$ .<br><sup>18</sup>When joint consent is needed to establish and/or maintain a link between a manufacturer and a not only for these values of interbrand rivalry but also for  $d \in (0.682, 0.909)$ .

 $d \in (0, 0.682)$ , they also prefer to establish a link between<br>, the distribution network  $g(1, 2)$  is defeated by the network<br>d by  $g(12, 12)$ . Observe that  $g(12, 2)$  is defeated by  $g(12, 12)$ <br>terbrand rivalry but also  $d \in (0, 0.682)$ , the distribution network  $g(1, 2)$  is defeated by the network<br>
∴2) is defeated by  $g(12, 12)$ . Observe that  $g(12, 2)$  is defeated by  $g(12, 12)$ <br>
e values of interbrand rivalry but also for  $d \in (0.682,$  $g(12,2)$  and  $g(12,2)$  is defeated by  $g(12,12)$ . Observe that  $g(12,2)$  is defeated by  $g(12,12)$ <br>not only for these values of interbrand rivalry but also for  $d \in (0.682, 0.909)$ .<br><sup>18</sup>When joint consent is needed to est  $d \in (0.682, 0.909)$ .<br>ink between a manu<br>ble. In Moner-Colon<br>ifficiently large brand<br>ne when goods are str<br>n  $g(12, 2)$  as pairwise <sup>18</sup>When joint consent is needed to establish and/or maintain a link between a manufacturer and a retailer, the asymmetric distribution network  $g(12, 2)$  is no more stable. In Moner-Colonques, Sempereretailer, the asymmetric distribution network  $g(12,2)$  is no more stable. In Moner-Colonques, Sempere-Monerris and Urbano (2004) this network appears at equilibrium for sufficiently large brand asymmetry. In Mycielski, Riyanto and Wuyts (2000),  $g(12, 2)$  is an equilibrium outcome when goods are strong substitutes Mycielski, Riyanto and Wuyts (2000),  $g(12, 2)$  is an equilibrium outcome when goods are strong subut not perfect ones. Here only for positive link costs, one can sustain  $g(12, 2)$  as pairwise stable.

Once the formation of links is costly, the pairwise stability of a given distribution network also depends on the size of link costs. In such case, the incentives to add a link between a manufacturer and a retailer when the degree of interbrand rivalry is low can be offset by the negative effect of the costly link on profits. When it happens, the distribution networks  $g(12,2)$  and  $g(1,2)$  can be pairwise stable for low values of interbrand rivalry; see Figure 3.

Now, we turn to the characterization of strongly stable distribution networks. We already know that the only pairwise stable distribution networks are  $g(1,2)$ ,  $g(12,1)$  and  $g(12,12)$ . To check for strong stability we have to examine the incentives that a coalition of agents have to move from the pairwise stable networks to other networks. Specifically,

- a) In considering the strong stability of  $g(1,2)$ , we have to check for the incentives to move from  $g(1,2)$  to  $g(12,12)$ , next to  $g(12,0)$ , and then to  $g(1,1)$ .
- b) In considering the strong stability of  $q(12,12)$ , we have to check for the incentives to move from  $g(12, 12)$  to  $g(1, 2)$ , next to  $g(12, 0)$ , and then to  $g(1, 1)$ .
- c) In considering the strong stability of  $q(12,1)$ , we have to check for the incentives to move from  $g(12, 1)$  to  $g(1, 0)$ .

**Proposition 2** The distribution network  $g(1,2)$  is always strongly stable when it is pairwise stable. However, the distribution networks  $g(12,1)$  and  $g(12,12)$  are not necessarily strongly stable when they are pairwise stable.

g(12, 2) can be pairwise stable for low values of interbrand rivalry;<br>see tim to the characterizador of strongly stable distribution networks. We<br>we tim to the characterizador of strongly stable distribution intervalues.  $g(1, 2)$ ,  $g(12, 1)$  and<br>tives that a coalition<br>etworks. Specifically,<br>for the incentives to<br> $(1, 1)$ .<br>Ck for the incentives to<br> $g(1, 1)$ .<br>for the incentives to<br>table when it is pair-<br>i) are not necessarily<br>ibution netwo g(12, 12). To check for strong stability we have to examine the incentives that a coalition<br>of sgenths have to move from the pairwise stabile nearched to check for the incensives that a coalition<br>a) in considering the str  $g(1, 2)$ , we have to check for the incentives to  $o g(12, 0)$ , and then to  $g(1, 1)$ .<br>  $g(12, 12)$ , we have to check for the incentives<br>
to  $g(12, 0)$ , and then to  $g(1, 1)$ .<br>  $g(12, 1)$ , we have to check for the incentiv g(1, 2) to g(12, 12), next to g(12, 0), and then to g(1, 1).<br>
ing the strong stability of g(12, 12), we have to check for<br>
mg(12, 12) to g(1, 2), next to g(12, 0), and then to g(1,<br>
ing the strong stability of g(12, 1), w  $g(12, 12)$ , we have to check for the incentives<br>to  $g(12, 0)$ , and then to  $g(1, 1)$ .<br> $r(12, 1)$ , we have to check for the incentives to<br> $(1, 2)$  is always strongly stable when it is pair-<br>orks  $g(12, 1)$  and  $g(12, 12)$  $g(12, 12)$  to  $g(1, 2)$ , next to  $g(12, 0)$ , and then to  $g(1, 1)$ ,<br>the strong stability of  $g(12, 1)$ , we have to check for the<br>12, 1) to  $g(1, 0)$ .<br>e distribution network  $g(1, 2)$  is always strongly stable wh<br>r, the d g (12, 1), we have to check for the incentives to<br>  $y(1, 2)$  is always strongly stable when it is pair-<br>
vorks  $g(12, 1)$  and  $g(12, 12)$  are not necessarily<br>  $y(e,$ <br>  $k, d$  where the three distribution networks are<br>
restin  $g(12, 1)$  to  $g(1, 0)$ .<br>The distribution ne<br>ever, the distributi<br>hen they are pairwi<br>ays the areas in the<br>repairwise stable. It<br>in-exclusive dealing<br>entiated  $d \in (0.20$ <br> $(1, 2)$  is strongly sta<br>re some values of pr<br>been  $g(1, 2)$  is always strongly stable when it is pair-<br>works  $g(12, 1)$  and  $g(12, 12)$  are not necessarily<br>le.<br>(k, d) where the three distribution networks are<br>resting to note that for  $k = 0$ , the non-exclusive<br> $n g(12, 12)$  $g(12, 1)$  and  $g(12, 12)$  are not necessarily<br>where the three distribution networks are<br>g to note that for  $k = 0$ , the non-exclusive<br>, 12) is strongly stable when the products<br>d the *exclusive distribution* & *exclusive* Figure 4 displays the areas in the space  $(k, d)$  where the three distribution networks are Figure 4 displays the areas in the space  $(k, d)$  where the three distribution networks are<br>relative to pairwise stable. It is interesting to note that for  $k = 0$ , the non-crobisive<br>relative of anon-crobisine dealing system strongly stable or pairwise stable. It is interesting to note that for  $k = 0$ , the non-exclusive  $k = 0$ , the non-exclusive<br>table when the products<br>distribution & exclusive<br>y to the case of pairwise<br>h no network is strongly<br>r when the products are<br>f product differentiation<br>entiated,  $d \in (0, 0.202)$ ,<br>n each manufacture distribution & non-exclusive dealing system  $g(12, 12)$  is strongly stable when the products  $g(12, 12)$  is strongly stable when the products<br>, and the *exclusive distribution* & *exclusive*<br> $\in (0.682, 1)$ . Contrary to the case of pairwise<br>fferentiation for which no network is strongly<br>feated by  $g(1, 2)$  either are rather differentiated  $d \in (0.202, 0.510)$ , and the *exclusive distribution* & *exclusive*  $d \in (0.202, 0.510)$ , and the *exclusive distribution* & *exclusive*<br>rongly stable for  $d \in (0.682, 1)$ . Contrary to the case of pairwise<br>alues of product differentiation for which no network is strongly<br>use  $g(12, 12)$  is dealing system  $g(1,2)$  is strongly stable for  $d \in (0.682,1)$ . Contrary to the case of pairwise  $g(1,2)$  is strongly stable for  $d \in (0.682, 1)$ . Contrary to the case of pairwise<br>are some values of product differentiation for which no network is strongly<br>ppens because  $g(12, 12)$  is defeated by  $g(1, 2)$  either when stability, there are some values of product differentiation for which no network is strongly stable. This happens because  $q(12, 12)$  is defeated by  $q(1, 2)$  either when the products are  $g(12, 12)$  is defeated by  $g(1, 2)$  either when the products are 0.202)) or for intermediate levels of product differentiation act, when products are very differentiated,  $d \in (0, 0.202)$ , prefers to delete two links, one  $d \in (0, 0.202)$  or for intermediate levels of product differentiation<br>
In fact, when products are very differentiated,  $d \in (0, 0.202)$ ,<br>
ailers prefers to delete two links, one with each manufacturer, in<br>
ntiated duopoly very differentiated (for  $d \in (0, 0.202)$ ) or for intermediate levels of product differentiation (for  $d \in (0.510, 0.682)$ ). In fact, when products are very differentiated,  $d \in (0, 0.202)$ . d ∈ (0.510, 0.682)). In fact, when products are very differentiated,  $d \in (0, 0.202)$ , coalition of two retailers prefers to delete two links, one with each manufacturer, in to form a differentiated duopoly without intrabr the coalition of two retailers prefers to delete two links, one with each manufacturer, in order to form a differentiated duopoly without intrabrand rivalry (a situation close to two successive monopolies). For intermediate levels of  $d \in (0.510, 0.682)$  the coalition of  $d \in (0.510, 0.682)$  the coalition of<br>th each retailer, in order to form<br>d avoiding introducing intrabrand<br>n is high enough or intermediate, two manufacturers prefers to delete two links, one with each retailer, in order to form a differentiated duopoly in the distribution market and avoiding introducing intrabrand rivalry. Thus, when the degree of product differentiation is high enough or intermediate,



Figure 4: Strongly stable and pairwise stable distribution networks.

the non-exclusive distribution & non-exclusive dealing system will not emerge in the "longrun" while Mycielski, Riyanto and Wuyts (2000) and Moner-Colonques, Sempere-Monerris and Urbano (2004) have shown that it is a "short-run" equilibrium.

Once the formation of links is costly, the stability of a given distribution network also depends on the size of link costs. In such a case, the incentives to delete links by the coalition of two retailers when the degree of interbrand rivalry is very low, or by the coalition of the two manufacturers when the degree of interbrand rivalry is intermediate can be reinforced by the negative effect of the costly links on profits. There is a size of link  $g(12, 12)$  is no longer strongly stable. Higher size<br>
l rivalry make the distribution networks  $g(12, 1)$ <br>
to the case of pairwise stability, the possibility<br>
links with  $R_2$  and  $M_2$  moving to  $g(1, 0)$ , makes<br>
ne val costs from which the distribution network  $q(12, 12)$  is no longer strongly stable. Higher size of link costs and low degree of interbrand rivalry make the distribution networks  $g(12,1)$  $g(12, 1)$ <br>ssibility<br>, makes<br>pairwise<br>vent the<br>at least<br>and link<br>etworks<br>ould be and  $g(1,2)$  strongly stable. But contrary to the case of pairwise stability, the possibility  $g(1, 2)$  strongly stable. But contrary to the case of pairwise stability, the possibility  $M_1$  and  $R_1$  delete respectively their links with  $R_2$  and  $M_2$  moving to  $g(1, 0)$ , makes  $, 1$ ) no longer strongly stable f that  $M_1$  and  $R_1$  delete respectively their links with  $R_2$  and  $M_2$  moving to  $g(1,0)$ , makes that  $M_1$  and  $R_1$  delete respectively their links with  $R_2$  and  $M_2$  moving to  $g(1, 0)$ , makes  $g(12, 1)$  no longer strongly stable for some values of  $d$  and  $k$  for which it was pairwise stable. For high degree of  $g(12,1)$  no longer strongly stable for some values of d and k for which it was pairwise  $g(12, 1)$  no longer strongly stable for some values of d and k for which it was pairwise<br>stable. For high degree of interbrand rivalry, the fact of costly links does not prevent the<br>strong stability of  $g(1, 2)$ . In Figu stable. For high degree of interbrand rivalry, the fact of costly links does not prevent the strong stability of  $g(1,2)$ . In Figure 4, one can observe that while there are always at least  $g(1, 2)$ . In Figure 4, one can observe that while there are always at least<br>etwork, for some values of the degree of product differentiation and link<br>trongly stable network.<br>Ork is strongly stable we will observe a seque a pairwise stable network, for some values of the degree of product differentiation and link costs there is no strongly stable network.

When no network is strongly stable we will observe a sequence of distribution networks due to continuously profitable deviations. In terms of competition policy, it would be interesting to know which networks are likely to be visited by such sequence of profitable deviations. In fact we will show that some distribution networks will be visited at most once, while others will belong to a closed cycle and will be visited regularly. We now define what is meant by a closed cycle. A network  $g'$  strongly defeats g if (i)  $g'$  is obtainable from g via deviations by  $S \subset N$  and (ii)  $\Pi_i(g') \geq \Pi_i(g)$  for all  $i \in S$  and  $\Pi_j(g') > \Pi_j(g)$  for some  $j \in S$ . An improving path from a network g to a network g' is a finite sequence of graphs  $g_1, g_2, ..., g_K$  with  $g_1 = g$  and  $g_K = g'$  such that for any  $k \in \{1, ..., K-1\}$  we have  $g_{k+1}$  strongly defeats  $g_k$ . A set of networks  $\overline{G}$  form a cycle if for any  $g \in \overline{G}$  and  $g' \in \overline{G}$ there exists an improving path connecting  $g$  to  $g'$ . A cycle in  $\overline{G}$  lies on an improving path leading to a network that is not in  $\overline{G}$ . In characterizing the closed cycles (whose proof is given in the appendix) we distinguish two cases:

- ${R \choose (12,12)}, {k_{(1,2)}^M}, \ \max\{k_{s(12,1)}^M, k_{s(12,1)}^R\} > k > \min\{k_{s(1,2)}^M, {k_{s(1,2)}^R}\},$  then there is a unique closed cycle which consists of networks  $g(12, 12)$ ,  $g(1, 2)$ ,  $g(2, 1)$ ,  $q(12,1), q(12,2), q(1,12)$  and  $q(2,12)$ .
- $\{1,2\}, k_{(1,2)}^M\} > k > \max\{k_{s(12,1)}^M, k_{s(12,1)}^R\}$ , then there is a unique closed cycle which consists of all possible distribution networks.

g' strongly defeats g if (i) g' is obtainable from<br>  $\geq \Pi_i(g)$  for all  $i \in S$  and  $\Pi_j(g') > \Pi_j(g)$  for<br>  $\sqrt{x}$ worth g to a network g' is a finite sequence of<br>
y' such that for any  $k \in \{1, ..., K - 1\}$  we have<br>  $\sqrt{x}$  form a cycl g via deviations by  $S \subset N$  and (ii) Π<sub>i</sub>(g'<br>some  $j \in S$ . An improving path from a n<br>graphs  $g_1, g_2, ..., g_K$  with  $g_1 = g$  and  $g_K = g_{k+1}$  strongly defeats  $g_k$ . A set of networ<br>there exist an improving path connecting<br>in  $\overline$ ) ≥ Π<sub>i</sub>(g) for all  $i \in S$  and Π<sub>j</sub>(g'<br>etwork g to a network g' is a finit<br>etwork g to a network g' is a finit<br>g' such that for any  $k \in \{1, ..., K$ <br>ks  $\overline{G}$  form a cycle if for any  $g \in \{g$  to g'. A cycle  $\overline{G}$  is a c  $> \Pi_j(g)$  for<br>sequence of<br> $\cdot 1$ } we have<br>and  $g' \in \overline{G}$ <br>f no network<br>aracterizing<br>asse:<br> $R_{s(1,2)}$ }, then<br> $\cdot$ ,2),  $g(2,1)$ ,<br>closed cycle<br>the network<br>ble network<br>ly stable. In<br>ich consists<br>In area E<br>onsists of all<br> $g(1,$ j ∈ S. An improving path from a network g to a network g is a finite vequence of  $\eta$ . An improving g to a network g is a network g is a finite vequence of  $\eta$ ,  $\eta$ , graphs g<sub>2</sub>,  $y_1,..., y_N$  with gn – p and  $g_K = g'$  such that for any k  $\in [1,...,K-1]$  we have done as the sumply defect s  $g_k$ . A set of networks G form a cycle if for any  $g \in G$  and  $g' \in G$  also an improving puth connecting  $g$ gk-14 strongly defacts gk. A set of inclusions ( $E$  form a cycle if for any g ∈ G and g ∈ C and g ∈ G g to g'<br>
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in 17 G is a closed cycle if no network<br>t is not in  $\overline{G}$ . In characterizing<br>ve distinguish two cases:<br> $k > \min\{k_{s(1,2)}^M, k_{s(1,2)}^R\}$ , then<br>works  $g(12, 12), g(1, 2), g(2, 1),$ <br>n there is a unique closed cycle<br>ere is no strongly G lies on an improving path leading to a network that is not in G. In characterizing closed cycles (where proof is given in the appendix) we discinguish two cases:<br>
If min/min/left/2123,  $\kappa_{112}^{H}(2)$ , max( $\kappa_{212}^{H}(2$ **a)** If  $\min\{\min\{k_{(1)}^R\}$  there is a un  $g(12,1), g(12)$ <br>b) If  $\min\{k_{(1,2)}^R, k$  which consist Figure 5 displa; (areas A and E) area A no strongly of networks  $g(12, 1)$  area, respective the degree of differentiation is ob  $\binom{M}{1,2}$ , max { $k_{s(1,2)}^{M}$ }, max { $k_{s(2)}^{M}$  osed cycle which  $1, 12$  and  $g(2, 1)$ .<br>  $k > \max \{k_{s(1)}^{M}$  possible distril reas in the space some of the the network exists  $1, 2$ ,  $g(2, 1)$ ,  $g(2, 1)$ ,  $g(2, 1)$ ,  $g($  $k_{(1)}^{ex}$  and  $k_{(2)}^{ex}$  and  $(1, 1)$  is a proportional part of  $(1, 1)$  and  $k$  are  $k$  and  $k$  are  $k$  and  $k$  are  $k$  and  $s(12,1), k_{s(12,1)}^{R}$ ,  $k_{s(12,1)}^{R}$ ,  $k_{s(12,1)}^{R}$ ,  $k_{s(12,1)}^{R}$ ,  $k_{s(12,1)}^{R}$ ,  $k_{s(12,1)}^{R}$ ,  $k_{s(12,1)}^{R}$ , there distand the distribution 1<br>is and the distant parameter is a D the distrongly set on the distrib  $> k > \min{\kappa_{\delta(i)}^{\kappa_{\text{tot}}}}$ <br>tworks  $g(12, 12)$ <br>then there is a unique dosed cy<br>n networks are s<br>unique closed cy<br>n networks  $g(1, 12)$  and  $g(1, 12)$ <br>is networks  $k_s^{rel}(1,$  e chabely which is a chappen of  $l$  is a chappen of  $g(12, 12), g(1, 2), g(2, 1),$ <br>is a unique closed cycle<br>o strongly stable network<br>ks are strongly stable. In<br>osed cycle which consists<br>and  $g(2, 12)$ . In area E<br>ycle which consists of all<br>rks  $g(12, 12), g(1, 2)$  and<br>m network.  $g(12, 1)$ ,  $g(12, 2)$ ,  $g(1, 12)$  and  $g(2, 12)$ .<br>  $\min\{k_{(1,2)}^R, k_{(1,2)}^M\} > k > \max\{k_{s(12,1)}^M\}$ <br>
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c  $k_{(1)}^{ex}$  sts ays nd y s  $12$  and y s  $12$  and  $1$  a  $> k > \max\{k_{s(1)}^{out}\}$ <br>areas in the spa<br>areas in the spa<br>areas in the spa<br>areas ome of the t<br>e network exists<br>(1,2),  $g(2,1)$ ,  $g$ <br>ork exists and tl<br>reas B, C and D<br>*r*, the unique st<br>tion is high and<br>mediate and li<br>bution n  $s_{s(12,1)}^{R}$ ,  $k_{s(12,1)}^{R}$ ,  $k_{s(12,1)}^{R}$ ,  $k_{s(12,1)}^{R}$ , there distand the *g*(12, 1), there is a D the distrongly nd link cost is where *i* point *system* is and the distrongly nd link cost is where *i* point *sys* Figure 5 displays the areas in the space  $(k, d)$  where there is no strongly stable network Figure 5 displays the areas in the space  $(k, d)$  where there is no strongly stable network<br>set A and E) and where some of the three distribution networks are strongly stable. In<br>A A to strongly stable network exists and th (areas A and E) and where some of the three distribution networks are strongly stable. In area A no strongly stable network exists and there is a unique closed cycle which consists of networks  $g(12, 12)$ ,  $g(1, 2)$ ,  $g(2, 1)$ ,  $g(12, 1)$ ,  $g(12, 2)$ ,  $g(1, 12)$  and  $g(2, 12)$ . In area E  $g(12, 12)$ ,  $g(1, 2)$ ,  $g(2, 1)$ ,  $g(12, 1)$ ,  $g(12, 2)$ ,  $g(1, 12)$  and  $g(2, 12)$ . In area Etable network exists and there is a unique closed cycle which consists of all orbes. In areas B, C and D the distributions net no strongly stable network exists and there is a unique closed cycle which consists of all possible networks. In areas B, C and D the distributions networks  $g(12, 12)$ ,  $g(1, 2)$  and  $g(12, 12), g(1, 2)$  and<br>etwork. Thus, when<br>or when the degree<br>in Figure 5) we will<br>tion & non-exclusive<br>n & exclusive dealing<br>nd mixed distribution<br>Moreover, there is no<br>ill be visited at most<br>table deviations long  $g(12,1)$  are, respectively, the unique strongly stable distribution network. Thus, when  $g(12,1)$  are, respectively, the unique strongly stable distribution network. Thus, when<br>the degree of differentiation is high and link costs are not too high or when the degree<br>of differentiation is intermediate and link the degree of differentiation is high and link costs are not too high or when the degree of differentiation is intermediate and link costs are small (area A in Figure 5) we will observe a cycle of distribution networks where non-exclusive distribution & non-exclusive dealing will succeed to mixed distribution system, exclusive distribution & exclusive dealing will succeed to *non-exclusive distribution* & *non-exclusive dealing*, and *mixed distribution* system will succeed to exclusive distribution & exclusive dealing. Moreover, there is no cycle beside the closed one and networks outside the closed cycle will be visited at most once. So, from any other distribution networks all sequences of profitable deviations long enough go to the closed cycle.



Figure 5: Cycles and strongly stable networks.

#### 4 Efficiency, consumer surplus and social welfare

Some of the very central questions about network formation concern the conditions under which the networks which are formed by the players turn out to be efficient from an overall societal perspective. In order to discuss these issues we need to define what is meant by efficiency. The network structure is the key determinant of the level of productivity or utility to the society of players involved. In our case, a manufacturer's expected profit and a retailer's expected profit from establishing a link among them in order to sell the manufacturer's brand to consumers depend on how many links each of them has formed and on how many links the other manufacturer and retailer have established. Remember that  $\Phi$  is a function that assigns to each network q a value  $\Phi(q)$  that represents the overall total value of network  $g$  which is the sum of the equilibrium profits of the four agents.

g which is the sum of the equilibrium profits of the four agents.<br>
of efficiency is simply maximizing the overall total value ame<br>
This notion was referred to as strong efficiency by Jackson a<br>
ve will simply refer to it An obvious notion of efficiency is simply maximizing the overall total value among all possible networks. This notion was referred to as strong efficiency by Jackson and Wolinsky (1996), but we will simply refer to it as efficiency.

**Definition 3** A network g is efficient relative to  $\Phi$  if  $\Phi(g) \ge \Phi(g')$  for all

g a value  $\Phi(g)$  that represents the overall<br>equilibrium profits of the four agents.<br>aximizing the overall total value among<br>to as strong efficiency by Jackson and<br>s efficiency.<br> $\Phi \text{ if } \Phi(g) \ge \Phi(g') \text{ for all } g' \in G.$ <br>st one effi g is efficient relative to  $\Phi$  if  $\Phi(g) \ge \Phi(g)$ <br>vill always exist at least one efficient net<br>orks. A starting point is to examine efficient<br>18  $g' \in G$ .<br>ven that<br>nen the It is clear that there will always exist at least one efficient network, given that there is only a finite set of networks. A starting point is to examine efficiency when the cost of the links is negligible.

**Proposition 3** Suppose that links are costless,  $k = 0$ . Then, the efficient distribution network is  $g(12, 12)$  for  $0 < d \le 0.735$ ,  $g(12, 1)$  for  $0.735 < d \le 0.863$ , and  $g(12, 0)$  for  $0.863 < d < 1$ .

In the absence of costly links, the degree of product differentiation determines the stable networks but also the more profitable network from the manufacturers and retailers point of view. A low degree of product differentiation implies a more competitive environment and thus manufacturers (and retailers) prefer a distribution network without intrabrand competition. Since the transfer price under  $g(12,0)$  is smaller than the transfer price under  $q(1,2)$ , the distribution network with a unique multiproduct retailer  $q(12,0)$  is more efficient than the distribution network with two differentiated retailers  $g(1,2)$ . Higher degrees of product differentiation will give incentives to one manufacturer to use two retailers making  $q(12, 1)$  the new efficient network. Further increases in the degree of product differentiation will now give incentives to the second manufacturer to use two retailers too. Since the output expansion effect dominates the competition effect,  $q(12, 12)$ becomes the new efficient distribution network.

 $k = 0$ . Then, the efficient distribution<br>for 0.735  $< d \leq 0.863$ , and  $g(12,0)$  for<br>the that differentiation determines the stable<br>mplies a more competitive environment<br>multi-manufacturers and retailers point<br>multi-sup m g(12, 12) for 0 < d ≤ 0.735, g(12, 1) for 0.735 < d ≤ 0.863, and g(12, 0) for<br>1.1.<br>
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ransfer prices is compen-<br>  $e$  6 displays the efficient<br>  $i$  in the area A;  $g(1, 2)$  is<br>
work in the area B; and<br>
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nks are costly. Th aggregate profits than the exclusive distribution & non-exclusive dealing network  $q(12,0)$  $g(12, 0)$ <br>
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efficient<br>  $(1, 2)$  is<br>  $B$ ; and<br>
ne main<br>
ne com-<br>  $7$  of the<br>  $\frac{d}{dt}$ <br>  $\frac{d$ with in-store interbrand competition. Indeed, the difference in transfer prices is compensated by the increased degree of product differentiation. Figure 6 displays the efficient  $g(12, 12)$  is the efficient one in the area A;  $g(1, 2)$  is  $g(12, 1)$  is the efficient network in the area B; and e area D. The next proposition summarizes the main stribution networks when links are costly. The comdist distribution networks. The network  $q(12, 12)$  is the efficient one in the area A;  $q(1, 2)$  is the efficient network in the area C;  $g(12,1)$  is the efficient network in the area B; and  $g(12, 1)$  is the efficient network in the area B; and<br>area D. The next proposition summarizes the main<br>stribution networks when links are costly. The com-<br>distribution networks is given in Proposition 7 of the<br>rease cost  $g(12,0)$  is the efficient network in the area D. The next proposition summarizes the main g(12,0) is the efficient network in the area D. The next proposition summarizes the main<br>interesting features about efficient distribution networks when links are costly. The com-<br>plete characterization of the efficient d interesting features about efficient distribution networks when links are costly. The complete characterization of the efficient distribution networks is given in Proposition 7 of the appendix.

**Proposition 4** Suppose that links are costly,  $k > 0$ . Any distribution network marketing  $k > 0$ . Any distribution network marketing<br>tions on the degree of product differentiation<br>for low enough link cost and enough degree<br>ee the efficient distribution network is either<br>roduct differentiation, or  $g(12,1)$  for two products is efficient under particular conditions on the degree of product differentiation and the size of link costs.

The distribution network  $g(12, 12)$  is efficient for low enough link cost and enough degree  $g(12, 12)$  is efficient for low enough link cost and enough degree<br>As link costs increase the efficient distribution network is either<br>mediate degrees of product differentiation, or  $g(12, 1)$  for lower<br>19 of product differentiation. As link costs increase the efficient distribution network is either  $g(1,2)$ , for high and intermediate degrees of product differentiation, or  $g(12,1)$  for lower g(1,2), for high and intermediate degrees of product differentiation, or  $g(12,1)$  for lower<br>19

degrees of product differentiation. Further increases in link costs and intermediate degrees of product differentiation imply that the efficient distribution network is either  $q(1,2)$  or  $g(12,0)$ . Finally, for small enough degrees of product differentiation the efficient network is  $g(12,0)$ .



Figure 6: Efficient distribution networks.

There is no coincidence between the set of efficient networks and that of pairwise or strongly stable distribution networks. We find that the three stable networks can be efficient for particular values of the degree of product differentiation and link costs, but not necessarily for the values under which they are stable. Moreover, the network  $g(12,0)$ is efficient for large enough link costs and low enough product differentiation but it is never stable.

 $g(12, 0)$ <br>ut it is<br>stworks,<br>ibution<br>f intra-<br>network<br>ribution<br> $u$ Before analyzing the social welfare implications of the different distribution networks, it is worthy to study how consumer surplus is affected. Depending on the distribution network, one or two products are present in the market. Different combinations of intrabrand, interbrand and in-store competition can be present. Which distribution network will give the highest consumer surplus?

Proposition 5 The highest level of consumer surplus is achieved when the distribution network  $q(12, 12)$  is formed. Moreover, consumer surplus is increasing with the introduc $g(12, 12)$  is formed. Moreover, consumer surplus is increasing with the introduc-<br>roduct items in the outlets.<br>20 tion of product items in the outlets.

Consumers prefer that the three types of competition are present in both outlets, that is two product items in the two outlets. If this is not possible, they prefer one retailer with in-store competition and the other not, but with the presence of both inter and intrabrand competition, that is one outlet with one product item and the other with two products. If the latter two options are not possible, consumers prefer at least two product items, either two different products concentrated in one outlet for low enough product differentiation, for  $0.8597 < d < 1$ , or the same product item in each of the outlets for  $0.2826 < d < 0.8597$ . for  $0.4 \leq t \leq 4$ , or the same product item in each of the outlets for 0.2826 <  $d \leq 6 \leq 8597$ . The consumer varying for the same product item in each of the outlier degree and its distribution of the outlier the outli or finally, two different product items each one in a different outlet, for  $0 < d < 0.2826$ .  $\langle u \rangle \langle 0.2626\rangle$ .<br>
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d The consumer surplus depends on both the number of product items and its distribution among outlets. When the number of product items is the same, then its distribution also affects the degree of competition among manufacturers and among retailers. For example, in the distribution network  $g(12,0)$  although there is only one retailer the rivalry among  $g(12,0)$  although there is only one retailer the rivalry among<br>the transfer price is lower than the transfer price in a network<br>cecent in the market, as for example  $g(1,1)$ . In contrast, in the<br>the rivalry among retaile manufacturers implies that the transfer price is lower than the transfer price in a network where only one product is present in the market, as for example  $q(1,1)$ . In contrast, in the latter distribution network the rivalry among retailers is higher than in the former. The combination of both effects explains why  $g(12,0)$  generates a higher consumer surplus than  $g(1,1)$  when d is large enough.<sup>19</sup> Finally, the worst distribution network for consumers is the one with only one product item in a unique outlet. The complete characterization of consumers preferences over distribution networks at equilibrium can be found in the appendix.

 $g(1, 1)$ . In contrast, in the<br>than in the former. The<br>term consumer surplus than<br>non network for consumers<br>complete characterization<br>ium can be found in the<br>etworks on social welfare.<br>rization is relegated to the<br>ocial w  $g(12, 0)$  generates a higher consumer surplus than<br>y, the worst distribution network for consumers<br>a unique outlet. The complete characterization<br>on networks at equilibrium can be found in the<br>different distribution netw  $g(1, 1)$  when d is large enough.<sup>19</sup> Finally, the worst distribution network for consumers<br>is the one with only one product item in a unique outlet. The complete characterization<br>of of consumers preferences over distribu of consumers preferences over distribution networks at equilibrium can be found in the appendix.<br>
Finally, we analyze the effects of the different distribution networks on social welfare<br>
We give in the text the main feat appendix.<br>
Finally, we analyze the effects of the different distribution networks on social welfare.<br>
We give in the text the main feature while the complete characterization is released to the<br>experdix. The distribution Finally<br>We give in<br>appendix.<br> $g(12,0)$  on<br>and the si<br>can achiev<br>desirable<br>not necess<br>**Proposit**<br>always on<br>with<br>a)  $g(12,1$ <br>Rege<br> $\frac{19}{19}$ The equ<br> $w(g(1,2))$ Finally, we analyze the effects of the different distribution networks on social welfare. give in the text the main features while the complete characterization is relegated to the<br>endix. The distribution networks that attain the maximum social welfare are  $g(1,2)$  or<br>2,0) or  $g(12,11)$  or  $g(12,12)$ , dependin We give in the text the main features while the complete characterization is relegated to the appendix. The distribution networks that attain the maximum social welfare are  $g(1,2)$  or  $g(12,0)$  or  $g(12,1)$  or  $g(12,12)$ , depending on both the degree of product differentiation and the size of the link costs. Thus appendix. The distribution networks that attain the maximum social welfare are  $g(1,2)$  or appendix. The userstochand networks that attain the maximum social veltate g(1, 2) or  $g(12, 1)$  or  $g(12, 12)$ , depending on both the degree of product differentiation and the size of the link costs. Thus, only distribut  $q(12,0)$  or  $q(12,1)$  or  $q(12,12)$ , depending on both the degree of product differentiation g(12, 0) or g(12, 1) or g(12, 12), depending or local value differentiation<br>and the size of the link costs. Thus, only distribution networks that market both products<br>can achieve the highest social welfare. That is, inter and the size of the link costs. Thus, only distribution networks that market both products can achieve the highest social welfare. That is, interbrand competition is always socially<br>desirable in the market. However, intrabrand competition and in-store competition are<br>not necessarily socially desirable.<br>**Proposi** can achieve the highest social welfare. That is, interbrand competition is always socially desirable in the market. However, intrabrand competition and in-store competition are not necessarily socially desirable.

desirable in the market. However, intrabrand competition and in-store competition are<br>not necessarily socially desirable.<br>Proposition 6 The distribution network that attains the highest level of social welfare is<br>always o not necessarily socially desirable.<br> **Proposition 6** The distribution network that attains the highest level of social welfare is always one that markets both products. In particular the highest social welfare is obtained **Proposition 6** The distribution<br>always one that markets both pro-<br>with<br>a)  $g(12, 12)$  for high enough degrifiers are necessary and the size of k,<br> $\frac{19}{19}$ The equilibrum transfer price rank<br> $w(g(1,2)) < w(g(1,1)) = w(g(1,0)).$ **Proposition 6** The distribution network that attains the highest level of social welfare is that the distribution of the highest social welfare is obtained<br>high enough degrees of product differentiation and low enough link costs.<br>of the size of k,  $g(12, 12)$  maximizes social welfare for  $0 < d \le 0.0326$ .<br>transfer always one that markets both products. In particular the highest social welfare is obtained  $with$ 

with<br>a)  $g(12,12)$  for high enough degrees of product differentiation and low enough link costs.<br>Regardless of the size of k,  $g(12,12)$  maximizes social welfare for  $0 < d \le 0.0326$ .<br><sup>19</sup>The equilibrum transfer price ranki a)  $g$ <br> $\frac{19\pi}{1}$ <br> $w(g)$ a)  $g(12, 12)$  for high enough degrees of product differentiation and low enough link costs. Regardless of the size of k,  $g(12, 12)$  maximizes social welfare for  $0 < d \leq 0.0326$ .

 $g(12, 12)$  for high enough degrees of product digerentiation and low enough link costs.<br>  $Regardless of the size of k, g(12, 12) maximizes social welfare for 0 < d \le 0.0326$ .<br>
The equilibrum transfer price ranking is:  $w(g(12, 12)) = w(g(12, 0)) < w_2(g(12, 1)) < w_1(g(12, 1)) <$ <br> Regardless of the size of k,  $g(12, 12)$  maximizes social weight for  $0 < a \le 0.0326$ .<br>
ie equilibrum transfer price ranking is:  $w(g(12, 12)) = w(g(12, 0)) < w_2(g(12, 1)) < w_1(g(12, 1))$ <br>
2)  $> w(g(1, 1)) = w(g(1, 0)).$ <br>
21 <sup>19</sup>The equilibrum transfer price ranking is:  $w(g(12, 12)) = w(g(12, 0)) < w_2(g(12, 1)) < w_1(g(12, 1)) <$  $w(g(1, 2)) < w(g(1, 1)) = w(g(1, 0))$ 

- b)  $g(12,1)$  for low enough degrees of product differentiation and intermediate sizes of link costs. Regardless of the size of k,  $g(12,1)$  maximizes social welfare for  $d \geq 0.954$ .
- c)  $g(1,2)$  for intermediate acgress of product afferentiation,  $0.0520 \leq a \leq \frac{1}{3}$ sizes of link costs.
- $\alpha$ )  $g(12, 0)$  for intermediate acyrecs of product afferentiation,  $\frac{1}{3}$ sizes of link costs.

Figure 7 displays the distribution networks that maximize social welfare. The area A corresponds to the range of parameters where  $g(12, 12)$  maximizes social welfare; in the area C  $g(1,2)$  maximizes social welfare; in the area D  $g(12,0)$  maximizes social welfare, and in the area B  $q(12, 1)$  maximizes social welfare. Comparing Figure 5 with Figure 7 we observe that the distribution networks that firms will endogenously form following their own interest enter, in general, in contradiction with those that maximize welfare.



Figure 7: Social welfare maximizing distribution networks.

, and high For the particular case  $k = 0$ , we find that: (i) when  $g(12, 12)$  is strongly stable it is  $k = 0$ , we find that: (i) when  $g(12, 12)$  is strongly stable it is<br>ial welfare, but the reverse is not true; (ii)  $g(12, 1)$  maximizes<br>54,1) but it is not strongly stable. Once the formation of links<br>) also maximize soci the one that maximizes social welfare, but the reverse is not true; (ii)  $q(12,1)$  maximizes  $g(12, 1)$  maximizes<br>
i formation of links<br>
ies of the degree of<br>
rongly stable when<br>
r stable when they social welfare when  $d \in (0.954, 1)$  but it is not strongly stable. Once the formation of links  $d \in (0.954, 1)$  but it is not strongly stable. Once the formation of links<br>
l  $g(12,0)$  also maximize social welfare for some values of the degree of<br>
ion and of the link costs. While  $g(1,2)$  could be strongly stable whe is costly,  $g(1,2)$  and  $g(12,0)$  also maximize social welfare for some values of the degree of  $g(1, 2)$  and  $g(12, 0)$  also maximize social welfare for some values of the degree of differentiation and of the link costs. While  $g(1, 2)$  could be strongly stable when they izes social welfare, the networks  $g(12, 0)$ product differentiation and of the link costs. While  $g(1,2)$  could be strongly stable when  $g(1, 2)$  could be strongly stable when<br>d  $g(12, 1)$  are never stable when they it maximizes social welfare, the networks  $g(12,0)$  and  $g(12,1)$  are never stable when they  $g(12, 0)$  and  $g(12, 1)$  are never stable when they<br>22

reach the maximum welfare. Moreover, costly links increase the conflict between social welfare and strong stability with respect to the network  $q(12, 12)$ .

 $g(12, 12)$ .<br>2, 12),  $g($ <br>2, 12),  $g($ <br>ibution r<br>ibution r<br>ie sociall;<br>cations of<br>setting v<br>of interbi<br>h policies<br>contrast<br>or costless<br>differentia<br>ts might<br>both ret.<br>work  $g(12)$ <br>work  $g(12)$ <br>is of proces in stable for<br>t In general, there are distribution networks like  $g(12,12)$ ,  $g(12,0)$  and  $g(12,1)$  that  $g(12, 12)$ ,  $g(12, 0)$  and  $g(12, 1)$  that<br>ees on their own, as compared with<br>listribution network  $g(1,2)$  appears<br>ld be socially desirable. Mycielski,<br>mplications of manufacturers' choices<br>in a setting where retailers are less likely to arise when leaving the market forces on their own, as compared with  $g(1, 2)$  appears<br>ble. Mycielski,<br>cturers' choices<br>tailers compete<br>alry, any policy<br>necessary once<br>ones we obtain<br> $ve$  get that such<br>hen, restricting<br>positive impact<br>om  $g(12, 12)$  to<br>hich is the one<br>are values of the<br>work the socially desirable outcome. By contrast, the distribution network  $g(1,2)$  appears to be stable under more situations than what would be socially desirable. Mycielski, Riyanto and Wuyts (2000) have studied the welfare implications of manufacturers' choices of vertical arrangements and its policy implications in a setting where retailers compete à la Bertrand. They have shown that, for a low degree of interbrand rivalry, any policy measure to restrict vertical restraints is unnecessary. Such policies become necessary once there is a high degree of interbrand rivalry. Their results contrast with the ones we obtain in a setting where retailers compete à la Cournot. Even for costless links, we get that such policies are also necessary for high degrees of product differentiation. Then, restricting exclusive distribution and exclusive dealing arrangements might have a positive impact on social welfare. Impeding the profitable deviation of both retailers from  $g(12, 12)$  to  $q(1,2)$  would make strongly stable the distribution network  $q(12,12)$ , which is the one that maximizes social welfare.

### 5 Conclusion

 $g(12, 12)$  to<br>  $\mu$  is the one<br>  $\mu$  is the one<br>  $\mu$ <br>  $\mu$  is the *non*-<br>  $\mu$ <br>  $g(1, 2)$  would make strongly stable the distribution network  $g(12, 12)$ , which is the one<br>that maximizes social welfare.<br>
The that maximizes social welfare.<br>
The have analyzed the networks between two manufacturers of d We have analyzed the networks between two manufacturers of differentiated goods and two multi-product retailers that one might expect to emerge in the long run. We have found that only three distribution networks are strongly stable for particular values of the degree of product differentiation and link costs. A first distribution network, the nonexclusive distribution & non-exclusive dealing system, in which both retailers distribute exclusive distribution & non-exclusive dealing system, in which both retailers distribute<br>both products is strongly stable for intermediate degrees of product differentiation and<br>small link costs. In this distribution net both products is strongly stable for intermediate degrees of product differentiation and small link costs. In this distribution network, both interbrand and intrabrand competition are present in the market. A second distribution network, the exclusive distribution  $\&$  exexclusive distribution & ex-<br>ifferent product is strongly<br>ribution network, no intra-<br>ion, the *mixed distribution*<br>the other retailer sells only<br>tion and large link costs.<sup>20</sup><br>stry. The analysis made by Slade<br>FU.K. Monop *clusive dealing* system, in which each retailer distributes a different product is strongly *clusive dealing* system, in which each retailer distributes a different product is strongly stable for low degrees of product differentiation. In this distribution network, no intra-<br>brand competition appears in the mark stable for low degrees of product differentiation. In this distribution network, no intrabrand competition appears in the market. A third distribution, the mixed distribution system, in which one retailer distributes both products while the other retailer sells only one is strongly stable for high degrees of product differentiation and large link costs.20

retailer sells only<br>large link costs.<sup>20</sup><br>nalysis made by Slade<br>nopolies and Mergers<br>ic-house chains which<br>94, 54% of the public system, in which one retailer distributes both products while the other retailer sells only<br>one is strongly stable for high degrees of product differentiation and large link costs.<sup>20</sup><br><sup>20</sup>The mixed distribution system se  $20$ The mixed distribution system seems quite common in the beer industry. The analysis made by Slade (1998) for the U.K. beer industry reveals that one of the effects of the U.K. Monopolies and Mergers Commission report and the Beer Orders passed after 1989 was the formation of public-house chains which most often operate under exclusive purchasing contracts with major brewers. In 1994, 54% of the public

Finally, for some values of the degree of product differentiation and link costs, no distribution network is strongly stable. In particular, when the degree of product differentiation is high enough or intermediate and link costs are moderate, the *non-exclusive distribution* & non-exclusive dealing system will not emerge in the "long-run" in contrast with Mycielski, Riyanto and Wuyts (2000) and Moner-Colonques, Sempere-Monerris and Urbano (2004). However, we will observe a cycle among the above three distribution networks. This is consistent with the observation that the distribution chains organization differs across markets and industries over time.<sup>21</sup>

un" in contrast with My-<br>ere-Monerris and Urbano<br>vee distribution networks.<br>hains organization differs<br>thains organization differs<br>trabrand competition in<br>on & non-exclusive deal-<br>stribution network, when<br>enough, two dist non-exclusive dealing system will not emerge in the "long-run" in contrast with My-<br>Rhap and Wuye (2000) and Morecc-Golosquer, Samper-Moneris and Tchan<br>60). However, we will observe a cycle among the absent line of signif Consumers are better off in a market with interbrand and intrabrand competition in both products. Thus, they prefer the *non-exclusive distribution* & *non-exclusive deal*non-exclusive distribution & non-exclusive deal-<br>whether the stable distribution network, when<br>en link costs are small enough, two distribution<br>The non-exclusive distribution & non-exclusive<br>if the degree of product diffe ing system. We have also investigated whether the stable distribution network, when ing system. We have also investigated whether the stable distribution network, when the stable distribution intervals, when the stable distribution is the stable distribution intervals in exact the stable distribution int it exists, maximizes social welfare. When link costs are small enough, two distribution non-exclusive distribution & non-exclusive<br>e degree of product differentiation is high<br>tem maximizes social welfare. When link<br>tworks may maximize welfare. The exclu-<br>maximizes welfare if the degree of product<br>exclusive d networks may maximize social welfare. The non-exclusive distribution & non-exclusive *dealing* system maximizes social welfare if the degree of product differentiation is high dealing system maximizes social welfare if the degree of product differentiation is high<br>enough; oberwise, the wated distribution system maximizes social welfare. When link<br>osts become large, we oscher distribution networ mixed distribution system maximizes social welfare. When link<br>other distribution networks may maximize welfare. The ezelu-<br>usive dealing system maximizes welfare if the degree of product<br>coupl; otherwise, the exclusive di enough; otherwise, the *mixed distribution system* maximizes social welfare. When link costs become large, two other distribution networks may maximize welfare. The exclusive distribution & exclusive dealing system maximizes welfare if the degree of product differentiation is high enough; otherwise, the exclusive distribution  $\&$  non-exclusive deal*ing* system maximizes welfare. Thus, a conflict between stability and social welfare is likely to occur, even more if the degree of product differentiation is either low or high. This conflict is crucial from a competition policy perspective and is summarized in Table 1.

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sive distribution & exclusive dealing system maximizes welfare if the degree of product differentiation is high enough; otherwise, the exclusive distribution & non-serietisted deal<br>ing system maximizes welfare. Thus, a co exclusive distribution & non-exclusive deal-<br>flict between stability and social welfare is<br>roduct differentiation is either low or high.<br>cy perspective and is summarized in Table 1.<br>pare-Colonques and Xavier Wauthy for hel ing system maximizes welfare. Thus, a conflict between stability and social welfare is<br>likely to occur, even more if the degree of product differentiation is either low or high.<br>This conflict is crucial from a competition houses were owned by either national or regional and local brewers, 27% corresponded to free pubs while the new formed public-chains accounted for the 19% of 56675 total number of public houses. The free pubs sell the beer of several brewers while the tied pubs are exclusive. Thus, there is evidence that exclusive retailers compete with multi-brand ones in the U.K. beer industry. For the U.S. beer industry, Sass (2005) has found evidence of a mixed pattern of distribution. All major U.S. brewers employ a mix of exclusive and non-exclusive distributors for their products. In particular, 38.7%. of Anheuser-Busch distributors were exclusive, while 98,3% of Miller and Coors distributors were non-exclusive. So, U.S. beer industry is another example of coexistence of exclusive retailers with multi-brand ones.

<sup>&</sup>lt;sup>21</sup> Examples of industries where products are differentiated and are sold by several multi-product retailers are books, TV sets, cola carbonated drinks; where products are differentiated but sold mainly by exclusive retailers are cars; where products are not very differentiated and sold by exclusive retailers are gas; and where products are differentiated and sold by both exclusive and multi-brand retailers are beer, industrial machinery/equipment and electronic and electric equipment.



Table 1: The conflict between stable distribution networks and social welfare

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### Appendix

#### Manufacturers and retailers payoffs  $\mathbf{A}$

We give the payoffs of the different possible distribution networks between two manufacturers and two retailers. Given the symmetry of the model, only six different distribution networks are at play. Apart from the distribution network  $g(12, 12)$  already examined in Section 2, the other distribution networks are as follows.

 $g(12, 12)$  already examined in<br>distribution networks  $g(0, 1)$ ,<br>the market: one manufacturer<br>offs, its sum, consumer surplus<br> $u_2(g(0, 2))$  (10)<br> $u_2(g(0, 2))$  (11)<br> $y) = \frac{3(a - c)^2}{16} - 2k$  (12)<br> $y) = \frac{(a - c)^2}{32} - 2k$  (13)<br> $y,$ a) The distribution network  $q(1,0)$  is symmetric to the distribution networks  $q(0,1)$ ,  $g(1, 0)$  is symmetric to the distribution networks  $g(0, 1)$ ,<br>
is a successive monopoly in the market: one manufacturer<br>
a single retailer. Agents' payoffs, its sum, consumer surplus<br>  $(g(2, 0)) = \Pi_{M_2}(g(0, 1)) = \Pi_{M_2}(g(0,$  $g(2,0)$  and  $g(0,2)$ . There is a successive monopoly in the market: one manufacturer sells its product through a single retailer. Agents' payoffs, its sum, consumer surplus and social welfare are:

$$
\Pi_{M_1}(g(1,0)) = \Pi_{M_1}(g(2,0)) = \Pi_{M_2}(g(0,1)) = \Pi_{M_2}(g(0,2))
$$
  
= 
$$
\frac{(a-c)^2}{8} - k
$$
 (10)

$$
g(2,0) \text{ and } g(0,2). \text{ There is a successive monopoly in the market: one manufacturer\nsells its product through a single retailer. Agents' payoffs, its sum, consumer surplus\nand social welfare are:\n
$$
\Pi_{M_1}(g(1,0)) = \Pi_{M_1}(g(2,0)) = \Pi_{M_2}(g(0,1)) = \Pi_{M_2}(g(0,2))
$$
\n
$$
= \frac{(a-c)^2}{8} - k
$$
\n
$$
\Pi_{R_1}(g(1,0)) = \Pi_{R_1}(g(2,0)) = \Pi_{R_2}(g(0,1)) = \Pi_{R_2}(g(0,2))
$$
\n
$$
= \frac{(a-c)^2}{16} - k
$$
\n
$$
\Phi(g(1,0)) = \Phi(g(2,0)) = \Phi(g(0,1)) = \Phi(g(0,2)) = \frac{3(a-c)^2}{16} - 2k
$$
\n
$$
C(g(1,0)) = C(g(2,0)) = C(g(0,1)) = C(g(0,2)) = \frac{(a-c)^2}{32}
$$
\n
$$
W(g(1,0)) = W(g(2,0)) = W(g(0,1)) = W(g(0,2)) = \frac{7(a-c)^2}{32} - 2k
$$
\n(14)  
\n26
$$

$$
= \frac{(a-c)^2}{16} - k \tag{11}
$$
\n
$$
\Phi(g(1,0)) = \Phi(g(2,0)) = \Phi(g(0,1)) = \Phi(g(0,2)) = \frac{3(a-c)^2}{16} - 2k \tag{12}
$$
\n
$$
C(g(1,0)) = C(g(2,0)) = C(g(0,1)) = C(g(0,2)) = \frac{(a-c)^2}{32} \tag{13}
$$
\n
$$
W(g(1,0)) = W(g(2,0)) = W(g(0,1)) = W(g(0,2)) = \frac{7(a-c)^2}{32} - 2k \tag{14}
$$
\n
$$
26
$$

$$
C(g(1,0)) = C(g(2,0)) = C(g(0,1)) = C(g(0,2)) = \frac{(a-c)^2}{32}
$$
\n(13)

$$
\Phi(g(1,0)) = \Phi(g(2,0)) = \Phi(g(0,1)) = \Phi(g(0,2)) = \frac{3(a-c)}{16} - 2k \qquad (12)
$$
  
\n
$$
C(g(1,0)) = C(g(2,0)) = C(g(0,1)) = C(g(0,2)) = \frac{(a-c)^2}{32} \qquad (13)
$$
  
\n
$$
W(g(1,0)) = W(g(2,0)) = W(g(0,1)) = W(g(0,2)) = \frac{7(a-c)^2}{32} - 2k \qquad (14)
$$
  
\n
$$
26
$$

b) The distribution network  $g(1,1)$  is symmetric to  $g(2,2)$ .

$$
\Pi_{M_1}(g(1,1)) = \Pi_{M_2}(g(2,2)) = \frac{(a-c)^2}{6} - 2k
$$
\n
$$
\Pi_{M_1}(g(1,1)) = \Pi_{M_2}(g(1,1)) = \Pi_{M_1}(g(2,2)) = \Pi_{M_2}(g(2,2)) = \Pi_{M_1}(g(2,2))
$$
\n(15)

bution network 
$$
g(1,1)
$$
 is symmetric to  $g(2,2)$ .  
\n
$$
\Pi_{M_1}(g(1,1)) = \Pi_{M_2}(g(2,2)) = \frac{(a-c)^2}{6} - 2k
$$
\n
$$
\Pi_{R_1}(g(1,1)) = \Pi_{R_2}(g(1,1)) = \Pi_{R_1}(g(2,2)) = \Pi_{R_2}(g(2,2))
$$
\n
$$
= \frac{(a-c)^2}{36} - k
$$
\n
$$
\Phi(g(1,1)) = \Phi(g(2,2)) = \frac{2(a-c)^2}{9} - 4k
$$
\n
$$
C(g(1,1)) = C(g(2,2)) = \frac{(a-c)^2}{18}
$$
\n
$$
W(g(1,1)) = W(g(2,2)) = \frac{5(a-c)^2}{18}
$$
\n
$$
W(g(1,1)) = W(g(2,2)) = \frac{5(a-c)^2}{18} - 4k
$$
\n
$$
\Phi(g(1,2)) = \Pi_{M_2}(g(1,2)) = \Pi_{M_1}(g(2,1)) = \Pi_{M_2}(g(2,1))
$$
\n
$$
= \frac{2(2-d)(a-c)^2}{(2+d)(1-d)^2} - k
$$
\n
$$
\Pi_{R_1}(g(1,2)) = \Pi_{R_2}(g(1,2)) = \Pi_{R_1}(g(2,1)) = \Pi_{R_2}(g(2,1))
$$
\n
$$
= \frac{4(a-c)^2}{(2+d)^2(4-d)^2} - k
$$
\n
$$
\Phi(g(1,2)) = \Phi(g(2,1)) = \frac{4(6-d^2)(a-c)^2}{(2+d)^2(4-d)^2} - 4k
$$
\n
$$
C(g(1,2)) = C(g(2,1)) = \frac{4(6-d^2)(a-c)^2}{(2+d)^2(4-d)^2} - 4k
$$
\n
$$
C(g(1,2)) = \Phi(g(2,1)) = \frac{4(7-d^2)(a-c)^2}{(2+d)^2(4-d)^2} - 4k
$$
\n
$$
W(g(1,2)) = \Pi_{M_2}(g(1,0)) = \Pi_{M_1}(g(0,12)) = \Pi_{M_2}(g(0,12))
$$
\n
$$
= \frac{(1-d)(a-c
$$

$$
\Phi(g(1,1)) = \Phi(g(2,2)) = \frac{2(a-c)^2}{9} - 4k
$$
\n
$$
C(g(1,1)) = C(g(2,2)) = \frac{(a-c)^2}{18}
$$
\n(18)

$$
C(g(1,1)) = C(g(2,2)) = \frac{(a-c)^2}{18}
$$
\n(18)

$$
W(g(1,1)) = W(g(2,2)) = \frac{5(a-c)^2}{18} - 4k
$$
\n(19)

) The distribution network

$$
\Pi_{M_1}(g(1,2)) = \Pi_{M_2}(g(1,2)) = \Pi_{M_1}(g(2,1)) = \Pi_{M_2}(g(2,1))
$$
\n
$$
2(2-d)(a-c)^2
$$
\n(20)

$$
= \frac{2(2-a)(a-b)}{(2+d)(4-d)^2} - k
$$
  
\n
$$
= \Pi_{\text{D}}(a(1,2)) - \Pi_{\text{D}}(a(2,1)) - \Pi_{\text{D}}(a(2,1)) \tag{21}
$$

$$
\Pi_{R_1}(g(1,2)) = \Pi_{R_2}(g(1,2)) = \Pi_{R_1}(g(2,1)) = \Pi_{R_2}(g(2,1))
$$
\n
$$
= \frac{4(a-c)^2}{(2+d)^2(4-d)^2} - k
$$
\n(21)

$$
= \frac{(a-c)}{36} - k \qquad (16)
$$
\n
$$
\Phi(g(1,1)) = \Phi(g(2,2)) = \frac{2(a-c)^2}{9} - 4k \qquad (17)
$$
\n
$$
C(g(1,1)) = C(g(2,2)) = \frac{(a-c)^2}{18} \qquad (18)
$$
\n
$$
W(g(1,1)) = W(g(2,2)) = \frac{5(a-c)^2}{18} - 4k \qquad (19)
$$
\n
$$
W_1(g(1,2)) = W_{1/2}(g(1,2)) = \Pi_{M_1}(g(2,1)) = \Pi_{M_2}(g(2,1)) \qquad (20)
$$
\n
$$
= \frac{2(2-d)(a-c)^2}{(2+d)(4-d)^2} - k
$$
\n
$$
\mu_1(g(1,2)) = \Pi_{R_2}(g(1,2)) = \Pi_{R_1}(g(2,1)) = \Pi_{R_2}(g(2,1)) \qquad (21)
$$
\n
$$
= \frac{4(a-c)^2}{(2+d)^2(4-d)^2} - k
$$
\n
$$
\Phi(g(1,2)) = \Phi(g(2,1)) = \frac{4(6-d^2)(a-c)^2}{(2+d)^2(4-d)^2} - 4k \qquad (22)
$$
\n
$$
C(g(1,2)) = C(g(2,1)) = \frac{4(a-c)^2}{(2+d)^2(4-d)^2} - 4k \qquad (23)
$$
\n
$$
W(g(1,2)) = W(g(2,1)) = \frac{4(7-d^2)(a-c)^2}{(2+d)^2(4-d)^2} - 4k \qquad (24)
$$
\n
$$
W(g(1,2)) = W(g(2,0)) = \frac{4(7-d^2)(a-c)^2}{(2+d)^2(4-d)^2} - 4k \qquad (24)
$$
\n
$$
W(g(1,2)) = \Pi_{M_2}(g(12,0)) = \Pi_{M_1}(g(0,12)) = \Pi_{M_2}(g(0,12)) \qquad (25)
$$
\n
$$
= \frac{(1-d)(a-c)^2}{2(1+d)(2-d)^2} - k
$$
\n
$$
(g(12,0)) = \Phi(g(0,12)) = \frac{(3-2d)(a-c)^2}{2(1+d)(2-d)^2} - 4k
$$

$$
C(g(1,2) = C(g(2,1)) = \frac{4(a-c)^2}{(2+d)^2(4-d)^2}
$$
\n(23)

$$
W(g(1,2) = W(g(2,1)) = \frac{4(7-d^2)(a-c)^2}{(2+d)^2(4-d)^2} - 4k
$$
\n(24)

d) The distribution network  $g(12,0)$  is symmetric to  $g(0,12)$ .

$$
\Pi_{M_1}(g(12,0)) = \Pi_{M_2}(g(12,0)) = \Pi_{M_1}(g(0,12)) = \Pi_{M_2}(g(0,12))
$$
\n
$$
= \frac{(1-d)(a-c)^2}{2(1+d)(2-d)^2} - k
$$
\n(25)

$$
H_{R_1}(y(1,1)) = H_{R_2}(y(1,1)) - H_{R_3}(y(2,2)) - H_{R_2}(y(2,2))
$$
\n
$$
= \frac{(a-c)^2}{36} - k \qquad (17)
$$
\n
$$
G(g(1,1)) = \Phi(g(2,2)) = \frac{2(a-c)^2}{18} - 4k \qquad (17)
$$
\n
$$
G(g(1,1)) = U(g(2,2)) = \frac{5(a-c)^2}{18} - 4k \qquad (19)
$$
\n
$$
W(g(1,1)) = W(g(2,2)) = \frac{5(a-c)^2}{18} - 4k \qquad (19)
$$
\n
$$
= \frac{2(2-d)(a-c)^2}{18} - k \qquad (19)
$$
\n
$$
= \frac{2(2-d)(a-c)^2}{(2+d)(4-d)^2} - k
$$
\n
$$
\Pi_{R_1}(g(1,2)) = \Pi_{R_2}(g(1,2)) = \Pi_{R_1}(g(2,1)) = \Pi_{R_2}(g(2,1)) \qquad (20)
$$
\n
$$
= \frac{4(a-c)^2}{(2+d)^2(4-d)^2} - k
$$
\n
$$
\Phi(g(1,2)) = \Phi(g(2,1)) = \frac{4(6-d^2)(a-c)^2}{(2+d)^2(4-d)^2} - 4k \qquad (22)
$$
\n
$$
G(g(1,2)) = C(g(2,1)) = \frac{4(6-d^2)(a-c)^2}{(2+d)^2(4-d)^2} - 4k \qquad (22)
$$
\n
$$
W(g(1,2)) = U(g(2,1)) = \frac{4(7-d^2)(a-c)^2}{(2+d)^2(4-d)^2} - 4k \qquad (23)
$$
\n
$$
W(g(1,2)) = W(g(2,1)) = \frac{4(7-d^2)(a-c)^2}{(2+d)^2(4-d)^2} - 4k \qquad (24)
$$
\n
$$
H_{\text{
}}(g(12,0)) = \Pi_{M_2}(g(12,0)) = \Pi_{M_1}(g(0,12)) = \Pi_{M_2}(g(0,12)) \qquad (25)
$$
\n
$$
= \frac{(1-d)(a-c)^2}{2(1+d)(2-d)^2} - k
$$

$$
\Pi_{R_1}(g(12,0)) = \Pi_{R_2}(g(0,12)) = \frac{(a-c)^2}{2(1+d)(2-d)^2} - 2k
$$
\n
$$
\Phi(g(12,0)) = \Phi(g(0,12)) = \frac{(3-2d)(a-c)^2}{2(1+d)(2-d)^2} - 4k
$$
\n
$$
C(g(12,0)) = C(g(0,12)) = \frac{2(a-c)^2}{8(1+d)^2(2-d)^2}
$$
\n
$$
W(g(12,0)) = W(g(0,12)) = \frac{(7+2d-4d^2)(a-c)^2}{4(1+d)^2(2-d)^2} - 4k
$$
\n
$$
27
$$
\n(29)

$$
C(g(12,0)) = C(g(0,12)) = \frac{2(a-c)^2}{8(1+d)^2(2-d)^2}
$$
\n(28)

$$
\Phi(g(12,0)) = \Phi(g(0,12)) = \frac{(5-2a)(a-c)}{2(1+d)(2-d)^2} - 4k
$$
\n
$$
C(g(12,0)) = C(g(0,12)) = \frac{2(a-c)^2}{8(1+d)^2(2-d)^2}
$$
\n
$$
W(g(12,0)) = W(g(0,12)) = \frac{(7+2d-4d^2)(a-c)^2}{4(1+d)^2(2-d)^2} - 4k
$$
\n
$$
27
$$
\n(29)

#### e) The distribution network  $g(12, 1)$  is symmetric to  $g(12, 2)$ ,  $g(1, 12)$  and  $g(2, 12)$ .

$$
\Pi_{M_1}(g(12,1)) = \Pi_{M_1}(g(1,12)) = \Pi_{M_2}(g(12,2)) = \Pi_{M_2}(g(2,12))
$$
  
= 
$$
\frac{(1-d)(2-d)(2+d)(8+5d)^2(a-c)^2}{6(1+d)(16-7d^2)^2} - 2k
$$
 (30)

$$
\Pi_{M_2}(g(12,1)) = \Pi_{M_2}(g(1,12)) = \Pi_{M_1}(g(12,2)) = \Pi_{M_1}(g(2,12))
$$
  
= 
$$
\frac{(1-d)(8+4d-d^2)^2(a-c)^2}{2(1+d)(16-7d^2)^2} - k
$$
 (31)

$$
\Pi_{R_1}(g(12,1)) = \Pi_{R_1}(g(12,2)) = \Pi_{R_2}(g(1,12)) = \Pi_{R_2}(g(2,12))
$$
  
= 
$$
\frac{(52 + 28d - 7d^2 - d^3)(a - c)^2}{36(1 + d)(16 - 7d^2)} - 2k
$$
 (32)

$$
\Pi_{R_2}(g(12,1)) = \Pi_{R_2}(g(12,2)) = \Pi_{R_1}(g(1,12)) = \Pi_{R_1}(g(2,12))
$$
  
= 
$$
\frac{(8+3d-2d^2)^2(a-c)^2}{9(16-7d^2)^2} - k
$$
 (33)

$$
\Phi(g(12,1)) = \Phi(g(12,2)) = \Phi(g(1,12)) = \Phi(g(2,12))
$$
  
= 
$$
\frac{(a-c)^2}{36(1+d)(16-7d^2)^2}(3776+1280d-3232d^2 -1192d^3+509d^4+155d^5) - 6k
$$
 (34)

ribution network 
$$
g(12,1)
$$
 is symmetric to  $g(12,2)$ ,  $g(1,12)$  and  $g(2,12)$ .  
\n
$$
\Pi_{M_1}(g(12,1)) = \Pi_{M_2}(g(1,12)) = \Pi_{M_2}(g(12,2)) = \Pi_{M_2}(g(2,12))
$$
\n
$$
= \frac{(1-d)(2-d)(2+d)(16-7d)^2}{6(1+d)(16-7d^2)^2} - 2k \qquad (30)
$$
\n
$$
\Pi_{M_2}(g(12,1)) = \Pi_{M_2}(g(1,12)) = \Pi_{M_1}(g(12,2)) = \Pi_{M_1}(g(2,12))
$$
\n
$$
= \frac{(1-d)(8+4d-d^2)^2(a-c)^2}{2(1+d)(16-7d^2)^2} - k \qquad (31)
$$
\n
$$
\Pi_{R_1}(g(12,1)) = \Pi_{R_1}(g(12,2)) = \Pi_{R_2}(g(1,12)) = \Pi_{R_2}(g(2,12))
$$
\n
$$
= \frac{(52+28d-7d^2-d^2)(a-c)^2}{36(1+d)(16-7d^2)} - 2k \qquad (32)
$$
\n
$$
\Pi_{R_2}(g(12,1)) = \Pi_{R_2}(g(12,2)) = \Pi_{R_1}(g(1,12)) = \Pi_{R_1}(g(2,12))
$$
\n
$$
= \frac{(8+3d-2d^2)^2(a-c)^2}{9(16-7d^2)^2} - k \qquad (33)
$$
\n
$$
\Phi(g(12,1)) = \Phi(g(12,2)) = \Phi(g(1,12)) = \Phi(g(2,12))
$$
\n
$$
= \frac{(a-c)^2}{36(1+d)(16-7d^2)^2}(3776+1280d-3232d^2 -1192d^3 +509d^4 +155d^5) - 6k \qquad (34)
$$
\n
$$
C(g(12,1)) = C(g(1,22)) = C(g(1,12)) = C(g(2,12))
$$
\n
$$
= \frac{(a-c)^2}{72(1+d)^2(16-7d^2)^2}(1600
$$

$$
W(g(12,1)) = W(g(12,2)) = W(g(1,12)) = W(g(2,12))
$$
  
= 
$$
\frac{(a-c)^2}{72(1+d)^2(16-7d^2)^2}(9152+11968d-4016d^2 -9560d^3 -1493d^4 +1408d^5 +335d^6) -6k
$$
 (36)

#### $\bf{B}$ Several results and proofs

# **B.1 Bounds on**<br>Proof of Lemma 1

#### Proof of Lemma 1

C(g(12, 1)) = C(g(1, 12)) = C(g(1, 12)) = C(g(2, 12))<br>
=  $\frac{(a-c)^2}{72(1+d)^2(16-7d^2)^2}(1600+1856d-$ <br>  $-712d^3-127d^4+80d^5+25d^6)$ <br>  $W(g(12, 1)) = W(g(12, 2)) = W(g(1, 12)) = W(g(2, 12))$ <br>
=  $\frac{(a-c)^2}{72(1+d)^2(16-7d^2)^2}(9152+11968d-$ <br>  $-956$ =  $\frac{(a-c)}{72(1+d)^2(16-712d^3-127d)}$ <br>
=  $W(g(12,2)) =$ <br>
=  $\frac{(a-c)^2}{72(1+d)^2(16-72d^3-149d^2)}$ <br>
5 and proofs<br>
tion networks  $g(1)$ <br>
nost binding const<br>  $k < \frac{(a-c)^2}{36}$ . Simila<br>
oses  $\Pi_{R_1}(g(1,2))$ <br>
2.  $72(1 + d)^2(16 - 7d^2)^2$  (1600 + 1850a - 112d<sup>2</sup><br>  $-712d^3 - 127d^4 + 80d^5 + 25d^6$ )<br>  $W(g(12,2)) = W(g(1,12)) = W(g(2,12))$ <br>  $(a - c)^2$ <br>  $72(1 + d)^2(16 - 7d^2)^2$  (9152 + 11968d - 4016<br>  $-9560d^3 - 1493d^4 + 1408d^5 + 335d^6) - 6k$ <br> **nd proofs**<br>  $-712d^3 - 127d^4 + 80d^5 + 25d^6$  (35)<br>  $W(g(12,2)) = W(g(1,12)) = W(g(2,12))$ <br>  $(a-c)^2$ <br>  $\overline{72(1+d)^2(16-7d^2)^2}(9152+11968d-4016d^2$ <br>  $-9560d^3 - 1493d^4 + 1408d^5 + 335d^6) - 6k$  (36)<br> **nd proofs**<br>
and **proofs**<br>
an networks  $g(1,0)$  an W(g(12, 1)) = W(g(12, 2)) = W(g(1, 12)) = W(g(2, 12))<br>
=  $\frac{(a-c)^2}{72(1+d)^2(16-7d^2)^2}(9152+11968d-4$ <br>
-9560d<sup>3</sup> - 1493d<sup>4</sup> + 1408d<sup>5</sup> + 335d<sup>6</sup>) - 6.<br> **ral results and proofs**<br>
mds on k<br>
emma 1<br>
st the distribution networks  $=\frac{(a-c)^2}{72(1+d)^2(16-c)}$ <br>  $-9560d^3 - 149$ <br> **s and proofs**<br>
and **proofs**<br>
and **proofs**<br>  $g(1)$ <br>
and  $k < \frac{(a-c)^2}{36}$ . Simila<br>
oses  $\Pi_{R_1}(g(1,2))$ <br>  $2$  $72(1+d)^2(16-7d^2)^2(9152+11968d-4016d^2-9560d^3-1493d^4+1408d^5+335d^6)-6k$ <br> **nd proofs**<br>
11 networks  $g(1,0)$  and  $g(1,1)$ . From direct is<br>
binding constraint for k is the one which<br>  $\frac{(a-c)^2}{36}$ . Similarly, for  $g(1,2)$  −9560 $d^3$  − 1493 $d^4$  + 1408 $d^5$  + 335 $d^6$ ) − 6k (36)<br> **nd proofs** (36)<br> **nd proofs** (1,0) and  $g(1,1)$ . From direct inspection of<br>
binding constraint for k is the one which ensures that<br>  $\frac{(a-c)^2}{36}$ . Similarly, for Consider first the distribution networks  $g(1,0)$  and  $g(1,1)$ . From direct inspection of  $g(1,0)$  and  $g(1,1)$ . From direct inspection of<br>onstraint for k is the one which ensures that<br>milarly, for  $g(1,2)$  the most binding constraint<br>()) > 0, that is  $k < \frac{4(a-c)^2}{(4-d)^2(2+d)^2}$ . For  $g(12,0)$ ,<br>28 the agents' profits, the most binding constraint for  $k$  is the one which ensures that k is the one which ensures that<br>
(1,2) the most binding constraint<br>
is  $k < \frac{4(a-c)^2}{(4-d)^2(2+d)^2}$ . For  $g(12,0)$ ,  $\Pi_{R_1}(g(1,1)) > 0$ , that is,  $k < \frac{(a-c)^2}{36}$ <br>for k is the one that imposes  $\Pi_{R_1}(g)$ for k is the one that imposes  $\Pi_{R_1}(g(1,2)) > 0$ , that is  $k < \frac{4(a-c)^2}{(4-d)^2(2+d)^2}$ . For<br>28 . Similarly, for  $g(1,2)$  the most binding constraint<br>  $(1,2)$ ) > 0, that is  $k < \frac{4(a-c)^2}{(4-d)^2(2+d)^2}$ . For  $g(12,0)$ ,<br>
28 k is the one that imposes  $\Pi_{R_1}(g(1,2)) > 0$ , that is  $k < \frac{4(a-c)^2}{(4-d)^2(2+c)}$ <br>28  $g(12,0),$   $\Pi_{R_1}(g(12,0)) > 0$  or equivalently  $k < \frac{(a-c)^2}{4(2-d)^2(1+d)}$  is the most binding constraint. Comparing  $\frac{(a-c)^2}{36}$  with  $\frac{4(a-c)^2}{(4-d)^2(2+d)^2}$  and with  $\frac{(a-c)^2}{4(2-d)^2(1+d)}$ , it follows that the former expression is smalle paring  $\frac{(a-c)^2}{36}$  with  $\frac{4(a-c)^2}{(4-d)^2(2+1)}$ <br>is smaller than each of the<br>it follows that  $k < \frac{(a-c)^2}{36}$ <br> $\Pi_{R_1}(g(12,1)) > 0, \Pi_{R_2}(g(1$ paring  $\frac{(a-c)^2}{36}$  with  $\frac{4(a-c)^2}{(4-d)^2(2+d)^2}$  and with  $\frac{(a-c)^2}{4(2-d)^2(1+d)}$ <br>is smaller than each of the other two expressions is<br>it follows that  $k < \frac{(a-c)^2}{36}$  is a more binding con<br> $\Pi_{R_1}(g(12,1)) > 0$ ,  $\Pi_{R_2}(g(12,1$ is smaller than each of the other two expressions for  $1 > d$ . For  $g(12, 1)$  and  $g(12, 12)$ , it follows that  $k < \frac{(a-c)^2}{36}$  is a more binding condition for  $\frac{(1-a)(a-c)^2}{3(2-d)^2(1+d)}$ , is more binding<br>0. Therefore, by comparing<br>on k as a function of d preexpressions  $\frac{(a-c)^2}{36}$  and  $\frac{(1-d)(a-c)^2}{3(2-d)^2(1+d)}$  we have the upper bound on<br>sented in Lemma 1.■<br>**B.1.1** Pairwise stable distribution networks. sented in Lemma 1.

# B.1.1 Pairwise stable distribution networks. **B.1.1** Pairwise stable<br>Proof of Proposition 1

#### **Proof of Proposition 1**

Remember that the link cost parameter is bounded above as indicated by Lemma 1. We proceed by steps.

- First, the distribution networks where one manufacturer and one retailer are out of the market (i.e.  $q(1,0), q(2,0), q(0,1)$  and  $q(0,2)$ ) are not pairwise stable since the manufacturer present in the market and the retailer selling no product would have incentives to create a link between them (i.e.,  $g(1,0)$  is defeated by  $g(1,1)$ ).
- $\Pi_{R_1}(g(12,0)) > 0$  or equivalently  $k < \frac{16.26}{32.26}$ <br>
paring  $\frac{(4c-2)^2}{362.21}$  with  $\frac{4(4c-2)^2}{14.29^2(21,6)^2}$  and with  $\frac{12.66}{32.26^2}$ <br>
is maller than each of the other two expression<br>
it follows that  $k < \frac$ , it follows that the former expression<br>for  $1 > d$ . For  $g(12, 1)$  and  $g(12, 12)$ ,<br>dition for k than those imposed by<br>2,12) > 0 for  $1 > d$ . Similarly, the<br>t is  $k < \frac{d(2-d)(q-c)^2}{3(1+d)}$ , is more binding<br>12,1) > 0. Therefore, > d. For  $g(12, 1)$  and  $g(12, 12)$ ,<br>1 for k than those imposed by<br>
> 0 for 1 > d. Similarly, the<br>  $c < \frac{(1-d)(e-c)^2}{3(2-d)^2(1+d)}$ , is more binding<br>
> > 0. Therefore, by comparing<br>
> > 0. Therefore, by comparing<br>
1 > 0. Therefo  $k < \frac{(a-c)^2}{36}$ <br>  $0, \Pi_{R_2}(g($ <br>
that ensures<br>  $\text{ring } \Pi_{M_1}(g)$ <br>  $\frac{(1-\alpha)^2}{2}$  and  $\frac{(1-\alpha)^2}{3(2-\alpha)}$ <br>  $1. \blacksquare$ <br>
se stable (<br>
osition 1<br>
the link co<br>
s.<br>
stribution 1<br>
the link co<br>
s.<br>
stribution 1<br>
the link co<br>  $\frac{(1-\alpha)^2}{3$ k than those imposed by<br>
for  $1 > d$ . Similarly, the<br>  $\frac{-d}{(a-c)^2}$ , is more binding<br>  $\frac{-d}{(a-c)^2}$ , is more binding<br>  $\therefore$  Therefore, by comparing<br>
n k as a function of d pre-<br>
and one retailer are out of<br>
to pairwise stab ΠR<sub>1</sub> (g(12, 1)) > 0) H<sub>R2</sub> (g(12, 1)) > 0 so and ΠR<sub>2</sub> (g(12, 12)) > 0 (ex 1) o for 1 > d. Similarly, the constrained the more binding the expectation constrained in the constrained in the similar constrained in the sim k that ensures  $\Pi_{M1}(g(12, 12)) > 0$ , that is  $k < \frac{(1-6)(\alpha-1)^2}{8(2-\alpha)(1+\alpha)}$ <br>surving  $\Pi_{M1}(g(12, 1)) > 0$  and  $\Pi_{M2}(g(12, 1)) > 0$ . Therefore  $\frac{1}{36}$  and  $\frac{(1-d)(\alpha-1)^2}{4(2-\alpha)^2(1+\alpha)}$  we have the upper bound on  $k$  as a ma 1. that those ensuring ΠM<sub>1</sub> (g(12, 1)) > 0 and Π<sub>M2</sub>(g(12, 1)) > 0. Therefore, by comparing<br>happensions  $\frac{E_{\text{rad}}(2)}{2}$  and  $\frac{1}{32\pi\pi^2/(1+\alpha)}$  we have the upper bound on *k* as a function of *d* pre-<br>sented in Lemma 1 icated by Lemma 1. We<br>
done retailer are out of<br>
pairwise stable since the<br>
graphy interaction of d pre-<br>
eated by  $g(1,1)$ ).<br>
nere is one manufacture<br>
that no retailer wants to<br>
with two links wants to<br>
stable manufactur  $g(1,0)$ ,  $g(2,0)$ ,  $g(0,1)$  and  $g(0,2)$ ) are not pairwise stable since the<br>seent in the market and the retailer selling no product would have<br>to saint between them (i.e.,  $g(1,0)$  is defeated by  $g(1,1)$ ).<br>distribution  $g(1, 0)$  is defeated by  $g(1, 1)$ ).<br>
and  $g(2, 2)$  there is one manuf<br>
nks. Note that no retailer w.<br>
mufacturer with two links with<br>
contradicts the restriction of<br>
s to check whether any retailes<br>
f the market. Take fo b) Second, in case of distribution networks  $g(1,1)$  and  $g(2,2)$  there is one manufacturer  $g(1, 1)$  and  $g(2, 2)$  there is one manufacturer<br>
at two links. Note that no retailer wants to<br>
the manufacturer with two links wants to<br>  $\frac{3}{2}$  which contradicts the restriction on  $k <$ <br>
remains to check whether any out of the market and the other with two links. Note that no retailer wants to break its unique link. Furthermore, the manufacturer with two links wants to break one link if and only if  $k > \frac{(a-c)^2}{24}$  which contradicts the restriction on  $k > \frac{(a-c)^2}{24}$ <br>Then, it results that it is<br>unifactured  $\frac{((a-c)^2)}{7d^2} - 2$ <br>vays holds (2, 2) are rout of the<br>tworks  $g(1)$ <br>with two lie retailer<br> $\frac{c)^2}{2}$ . Since<br>Now we shot the marke 29 ants  $R_1$ .<br>ly if hen, and  $\alpha$  out  $\alpha$  its  $\alpha$  its  $\alpha$  its  $\alpha$  its  $\alpha$  is  $\alpha$  its  $\alpha$  i  $k \equiv \min\left\{\frac{(a-c)^2}{36}\right\}$ <br>to create a lin<br> $\Pi_{R_1}(g(12, 1)) = \frac{(6+2d+d^3)(a-c)^2}{6(1+d)(16-7d^2)}$ <br>we conclude th<br>retailer and the<br>ird, in case of<br>of the market<br>unique link. If<br>and only if k<br>tailer never brea link with the  $\overline{k} \equiv \min\{\frac{(a-c)^2}{36}, \frac{(1-d)(a-c)^2}{3(1+d)(2-d)^2}\}\.$  Then, it remains to check whether any retailer wants to create a link with the manufacturer out of the market. Take for example  $R_1$ . ,  $\frac{(1-d)(a-c)^2}{3(1+d)(2-d)}$ <br>k with the <br><br> $\frac{(52+28d-7d^2-36(1+d))}{36(1+d)}$ <br>> k, which<br>at  $g(1, 1)$  are manufacture distribution<br>and the oth urthermore<br>>  $\frac{(4+3d^2-d^3-16(1+d))}{16(1+d)}$ <br>eaks one line  $\frac{(1-a)(a-c)^2}{3(1+d)(2-d)^2}$ . Then, it remains to check whether any retailer wants<br>with the manufacturer out of the market. Take for example  $R_1$ .<br> $\frac{(52+28d-7d^2-d^3)(a-c)^2}{36(1+d)(16-7d^2)} - 2k > \Pi_{R_1}(g(1,1)) = \frac{(a-c)^2}{36} - k$  if and  $\frac{36}{10}$  −  $\frac{(6+2d+d^3)(a-c)^2}{6(1+d)(16-7d^2)}$ <br>we conclude the tailer and the irretailer and the irred, in case of  $\frac{6+2d+a^2}{6(1+d)(16-7d^2)}$  > k, which always holds since  $\frac{(6+2d+d^2)(d-c)^2}{6(1+d)(16-7d^2)}$  >  $\frac{(1-d)(d-c)^2}{3(1+d)(2-d)^2}$ . Then,<br>we conclude that  $g(1,1)$  and  $g(2,2)$  are not pairwise stable since one link between a<br>etailer and we conclude that  $g(1,1)$  and  $g(2,2)$  are not pairwise stable since one link between a retailer and the manufacturer out of the market will always be created.
- to create a link with the manufacturer out of the market. Take for example  $R_1$ .<br>  $\Pi_{R_1}(g(12,1)) = \frac{(52+28d-7d^2 d^2)}{36(1+d)(16-7d^2)} 2k > \Pi_{R_1}(g(1,1)) = \frac{(a-c)^2}{36} k$  if and only if  $\frac{(6+2d+d^2)(a-c)^2}{6(1+d)(16-7d^2)} > k$ , whic  $\Pi_{R_1}(g(12, 1)) = \frac{(52+28d-7d^2-d^3)(a-c)^2}{36(1+d)(16-7d^2)}$ <br>  $\frac{(6+2d+d^3)(a-c)^2}{6(1+d)(16-7d^2)} > k$ , which always hwe conclude that  $g(1, 1)$  and  $g(2, 2)$  retailer and the manufacturer out of ird, in case of distribution network  $\frac{2(28a - (a^2 - a^2)(a-c)^2}{36(1+d)(16-7d^2)} - 2k > \Pi_{R_1}(g(1, 1)) = \frac{(a-c)^2}{36}$ <br>which always holds since  $\frac{(6+2d+d^3)(a-c)^2}{6(1+d)(16-7d^2)} > \frac{1}{3}$ <br>1, 1) and  $g(2, 2)$  are not pairwise stable since conufacturer out of the market will k if and only if<br>  $\frac{d}{d}(a-c)^2$ . Then,<br>  $\frac{d}{d}(2-d)^2$ . Then,<br>
link between a<br>
ed.<br>
one retailer out<br>
tts to break its<br>
eak one link if<br>
than  $\overline{k}$  this re-<br>
vants to create<br>  $I_{M_1}(g(12,1)) =$ > k, which always holds since  $\frac{(6+2d+d^3)(a-c)^2}{6(1+d)(16-7d^2)}$ <br>at  $g(1,1)$  and  $g(2,2)$  are not pairwise stable si<br>emanufacturer out of the market will always<br>distribution networks  $g(12,0)$  and  $g(0,12)$  tl<br>and the other  $> \frac{(1-d)(a-c)^2}{3(1+d)(2-d)}$ <br>
nce one link b<br>
e created.<br>
ere is one ret<br>
ere is one ret<br>
ere wants to<br>
s to break or<br>
reater than  $\overline{k}$ <br>
turer wants<br>  $M_1$ .  $\Pi_{M_1}(g)$  $g(1, 1)$  and  $g(2, 2)$  are not pairwise stable since one link between a<br>anufacturer out of the market will always be created.<br>stribution networks  $g(12, 0)$  and  $g(0, 12)$  there is one retailer out<br>d the other with two l c) Third, in case of distribution networks  $g(12,0)$  and  $g(0,12)$  there is one retailer out  $g(12, 0)$  and  $g(0, 12)$  there is one retailer out<br>o links. No manufacturer wants to break its<br>er with two links wants to break one link if<br>nce this expression is greater than  $\overline{k}$  this re-<br>show that one manufacturer of the market and the other with two links. No manufacturer wants to break its unique link. Furthermore, the retailer with two links wants to break one link if and only if  $k > \frac{(4+3a^2-d^2)(a-c)^2}{16(1+d)(2-d)^2}$ . Since this expression is greater than tailer never breaks one link. Now we show that one manufacturer want a link with the retailer out of the market. Take for example  $M_1$ .  $k > \frac{(4+3d^2-d^3)(a-c)^2}{16(1+d)(2-d)^2}$ <br>breaks one link. No<br>che retailer out of the k this re-<br>to create<br> $(12, 1)$  = tailer never breaks one link. Now we show that one manufacturer wants to create a link with the retailer out of the market. Take for example  $M_1$ .  $\Pi_{M_1}(g(12, 1)) =$ <br>29

 $\frac{(1-d)(2-d)(2+d)(8+5d)^2(a-c)^2}{6(1+d)(16-7d^2)^2}$ <br>  $\frac{(1-d)(256-208d^2+64d^3+45d^4-c^2)}{6(2-d)^2(16-7d^2)^2}$ <br>
hand side of the inequali<br>
networks  $g(12,0)$  and  $g(0)$ 6(1) and  $\frac{d}{2\pi} \frac{\cos(\alpha - \alpha)}{\sin(\alpha - \alpha)}$  and  $\alpha$  (12, 2)) =  $\frac{d}{2\pi} \frac{\cos(\alpha - \alpha)}{\sin(\alpha - \alpha)}$ <br>
(6) and  $\frac{\cos(\alpha - \alpha)}{\cos(\alpha - \alpha)}$  and  $\frac{d}{2\pi} \frac{\cos(\alpha - \alpha)}{\cos(\alpha - \alpha)}$  and  $\alpha$  (12, 0)) are not particle with  $\bar{k}$ . We conclude the ineq  $2(1+d)(2-d)^2$ <br>ality always heconclude that<br>express the cone line<br>ays be created *k* if and only if<br>
lds since the left<br>
the distribution<br>
between a man-<br>
ote that no agent<br>
mder which both<br>
y  $R_1$  if and only<br>  $R_1$  if and only<br>  $\frac{2(2-d)(a-c)^2}{(2+d)(4-d)^2} - k$ <br>  $\frac{2(2-d)(a-c)^2}{(2+d)(4-d)^2} - k$ <br>  $\frac{25d^7}{(a-c)^2$  $\frac{(1-d)(256-208d^2+64d^3+45d^4-25d^5)(a-c)^2}{6(2-d)^2(16-7d^2)^2}$ <br>hand side of the inequality is greatworks  $g(12,0)$  and  $g(0,12)$  are represented that the retailer out of the suffacturer and the retailer out of the  $6(2-d)^2(16-7d^2)^2$ <br>of the inequal<br> $q(12,0)$  and  $q($ <br>and the retaile hand side of the inequality is greater than networks  $g(12,0)$  and  $g(0,12)$  are not pairwise stable since one link between a manufacturer and the retailer out of the market will always be created.

d) Fourth, consider the distribution networks  $g(1,2)$  and  $g(2,1)$ . First, note that no agent wants two break a link. Therefore we have to find the conditions under which both one manufacturer and one retailer want to create a link.

A manufacturer, say  $M_2$ , wants to create a link with a retailer, say  $R_1$  if and only  $\frac{(2+d)(4-d)^2}{(7)(a-c)^2}$ <br>is negative formation  $\equiv k_{(1,2)}^M$ . It is easy to check that  $k_{(1,2)}^M$ 

which is equivalent to  $k < \frac{(2-a)(1024-1088a^2-1120a^3-128a^3-52a^3+45)}{6(1+d)(2+d)(4-d)^2(16-7d^2)^2}$ <br>  $\equiv k_{(1,2)}^M$ . It is easy to check that  $k_{(1,2)}^M > \overline{k}$  for  $d \in (0,0.265)$ <br>  $d \in (0.682, 1)$ .<br>
Similarly,  $R_1$  wants to cre

 $\frac{(2+d)^2(4-d)^2}{(a-c)^2} \equiv k_{(1,2)}^R$ <br>
able if and on is equivalent to  $k < \frac{(1024+1152d+832d^2+176d^3-8d^3+68d^3-3d^3-d^3)(a-c)^2}{36(1+d)(16-7d^2)(2+d)^2(4-d)^2} \equiv k_{(1,2)}^R$ , where  $k_{(1,2)}^R > \overline{k}$  for  $d \in (0.295, 1)$ .<br> **d.i)** Thus, the distribution network  $g(1,2)$  and  $g(2,1)$  are  $k_{(1,2)}^R > \overline{k}$  for  $d \in (0.295, 1)$ .

**d.i)** Thus, the distribution network  $g(1,2)$  and  $g(2,1)$  are stable if and only if  $k >$  ${R \choose (1,2)}$ , where  $k^{R}_{(1,2)} < k^{M}_{(1,2)}$  if

e) Fifth, consider distribution network  $q(12,12)$ . Note that the only way to break the stability of this distribution network is by breaking a link.

A manufacturer, say  $M_1$ , wants to break a link if and only if  $\Pi_{M_1}(g(12,2)) =$  $\frac{(1-d)(8+4d-d^2)^2(a-c)^2}{2(1+d)(16-7d^2)^2}$ <br>to  $k > \frac{(1-d)(256-32)}{6}$ <br>d ∈ (0,0.344) and is<br>Similarly, a retailer<br>(8+3d-2d<sup>2</sup>)<sup>2</sup>(a-c)<sup>2</sup>  $2(1+d)(16-7d^2)^2$ <br>  $k > \frac{(1-d)(256-320d^2)}{6(1+6)}$ <br>  $2(0, 0.344)$  and is not<br>
allarly, a retailer, s<br>  $\frac{3d-2d^2)^2(a-c)^2}{9(16-7d^2)^2} - k > 0$ to  $k > \frac{(1-a)(256-320a^2-96a^2+86a^2+36a^2-3a^2)(a-c)^2}{6(1+d)(2+d)^2(16-7d^2)^2} \equiv k_{(12,12)}^M$ , where  $k_{(12,12)}^M$ 

> k. This inequality always holds since the left<br>
or than  $\overline{k}$ . We conclude that the distribution<br>
then invistes stable since one link hetween a man-<br>
narket will always be created.<br>
Since the since one link hetween a k. We conclude that the distribution<br>
e stable since one link between a man-<br>
will always be created.<br>
2) and  $g(2,1)$ . First, note that no agent<br>
find the conditions under which both<br>
cate a link.<br>
ink with a retailer, s  $g(12,0)$  and  $g(0, 12)$  are not pairwise stable since one link between a man-<br>
and the could of the mate will dowly to constel.<br>
Sistem the distribution networks  $g(1, 2)$  and  $g(2, 1)$ . First, note that no agent<br>
betwee  $g(1, 2)$  and  $g(2, 1)$ . First, note that no agent<br>we to find the conditions under which both<br>to create a link.<br>to create a link.<br> $\approx a$  link with a retailer, say  $R_1$  if and only<br> $\frac{1-c)^2}{2} - 2k > N_1 M_2(g(1, 2)) = \frac{2(2-d)(a-d)^$ A manufacturer, say  $M_2$ , wants to create a link with a retailer, say  $R_1$  if and only if  $\prod_{i=1}^{\infty} \frac{d_i}{2}$  ( $\prod_{i=1}^$ if Π<sub>M2</sub> (g(12, 2)) =  $\frac{(1-d)(2-d)(2+d)(8+5d)/(4-d)^2}{6(1+d)(16-7d^2)^2}$ <br>which is equivalent to  $k < \frac{(2-d)(1024-108d^2-164)}{(11+4)(18-7d^2)^2}$ <br> $d \in (0.682,1).$ <br>Similarly,  $R_1$  wants to create a link with  $M_2$ <br> $d \in (0.682,1).$ <br>Similarly,  $-\frac{\partial Q+4(0)^{1.4}(\sin^2\theta_0^2 - \cos^2\theta_0^2 - 2k \cos^2\theta_0^2$ or he we he int or me also he can be a series of the contraction of  $\alpha$  is contracted to the contraction of  $k < \frac{(2-\alpha)(1024-1088d-1120d^3-728d^2-16d-6)^2}{6(1+\alpha)(2+\alpha)(1-\alpha)^2(16-\alpha)^2}$ <br>
check that  $k_{1,2}^{M} > \bar{k}$  for  $d \in (0,0.265)$  and is nega<br>
create a link with  $M_2$  if and only if<br>  $\frac{a-\pi a^2-a^2}{4(1+\alpha)(1+\alpha)^2}(\frac{a-\alpha)^2}{2} - 2k > \Pi_{R_1}(g$  $k_{(1,1)}^M$  = (0 mila  $k_1$  = (0 mila  $k_2$  = (0 mila  $k_3$  = (0 mila  $\frac{d}{2}$  = (0 mila  $\frac{d}{2}$  = (0 mila  $\frac{d}{2}$  = (0 mila  $\frac{3d-6}{9}$  = 1 y, ex dc alw  $k_{(1,1)}^M$  is written in the same of  $k_{(1,2)}^M$  is  $\frac{32d^2}{1+d}$  is  $g$  of  $(1,2)$  is  $\frac{6d^5}{d^2)^2}$  if  $k$  is  $(1,2)$  is  $d \in \text{C}$  in  $M_2$  is  $M_2$  in  $M_2$  in  $M_2$ > *k* for  $d \in (0, 0.265)$  and is negative for  $M_2$  if and only if  $k > \Pi_{R_1}(g(1,2)) = \frac{4(a-c)^2}{(2+d)^2(4-d)^2} - k$  which  $\frac{6a^3 - 8d^4 + 68d^6 - 3d^6 - a^7/(a-c)^2}{(2+d)^2(4-d)^2} = k_{(1,2)}^R$ , where  $2)$  and  $g(2, 1)$  are stable if and on  $d \in (0.682, 1).$ <br>
Similarly,  $R_1$ <br>  $\Pi_{R_1}(g(12, 2))$ <br>
is equivalent<br>  $k_{(1,2)}^R > \overline{k}$  for  $d$ <br> **d.i)** Thus, the<br>
min{ $k_{(1,2)}^M$ ,  $k_{(1)}^R$ <br>
f(th, consider of<br>
stability of th<br>
A manufacture  $\frac{(1-d)(8+4d-d^2)^2}{2(1+d)(16-7d)}$ Similarly,  $R_1$  wants to create a link with  $M_2$  if and only if  $\Pi_{R_1}(g(12,2)) = \frac{(52+286-7d^2 - d^2)(6-1)^2}{36(1+d)(16-7d^2)^2} - 2k > \Pi_{R_1}(g(1,2))$ <br>is equivalent to  $k < \frac{(1024+1152d+9)^2d^2+176d^2 - 8d^4 + 368d^2 - 3d^6 - 3d^6 - 3d^$  $\Pi_{R_1}(g(12, 2)) = \frac{(52+28d-d-d^2)(a-c)^2}{36(1+d)(16-7d^2)}$ <br>
is equivalent to  $k < \frac{(1024+1152d+832-36(1+d)(16-7d^2))}{36(1+d)(16-7d^2)}$ <br>  $k_{(1,2)}^R > \overline{k}$  for  $d \in (0.295, 1)$ .<br> **d.i)** Thus, the distribution network<br>
min{ $k_{(1,2)}^M$ ,  $\frac{4(2-2)dx}{36(1+d)(16-7d^2)} - 2k > \Pi_{R_1}(g(1,2)) = \frac{4(2-6)^2}{(2+d)^2(4-6)^2}$ <br>  $\leq \frac{(1024+1152d+832d^2+176d^3-8d^4+68d^3-3d^6-d^6)(1a-c)^2}{36(1+d)(16-7d^2)(2+d)^2(4-d)^2}$  = 295, 1).<br>
Bibution network  $g(1,2)$  and  $g(2,1)$  are stable i k which<br>, where<br>y if  $k >$ <br>reak the<br>2,2)) =<br>uivalent<br> $> \overline{k}$  for<br> $> \overline{k}$  for<br> $\langle k_{(12,12)}^M \rangle$ <br>alent to<br> $< \overline{k}$  for<br> $\iota \{k_{(12,12)}^M \rangle$ <br>2). Take<br>que link.<br>at there out want  $k < \frac{(1024+1152d+832d^2+176d^2-8d^2+168d^2-3d^0-d^t)(a-6)^2}{36(1+d)(16-7d^2)(2+d^2)(4-d)^2}$ <br>
(0.295, 1).<br>
Sistribution network  $g(1,2)$  and  $g(2,1)$  are stable i<br>
, where  $k_{(1,2)}^R < k_{(1,2)}^M$  if  $d \in (0,0.269)$ .<br>
ribution network  $k_{(1)}^R$  do to  $(g$  is  $n_{(2)}^R$  and  $(2, 2)$  are  $(2, 2)$  are  $(2, 2)$  $k_{(1)}^R$  d.i mi th sta  $A_{(1-\text{ to } d\text{ s} \text{ is }k)$  all e.i  $k_{(1-\text{ cal } k)}^R$  is > k for  $d \in (0.295, 1)$ .<br>
Thus, the distribution<br>
c<sup>M</sup><sub>(1,2)</sub>,  $k_{(1,2)}^R$ , where  $k_{(2)}^R$ <br>
onsider distribution n<br>
ity of this distribution<br>
mufacturer, say  $M_1$ ,<br>  $\frac{8+4d-d^2}{d^2(a-c)^2} - k$ <br>
>  $\frac{(1-d)(256-320d^2-96d^3-6($  $g(1, 2)$  and  $g(2, 1)$  are stable if and only if  $k >$ <br>  $g_1$  if  $d \in (0, 0.269)$ .<br>
(12, 12). Note that the only way to break the<br>
is by breaking a link.<br>
break a link if and only if  $\Pi_{M_1}(g(12, 2)) =$ <br>  $2, 12$ )) =  $\frac{2(1$ min{ $k_{(1)}^M$ <br>fth, cons<br>stability<br>A manu<br> $\frac{(1-d)(8+d)}{2(1+d)}$ <br>to  $k > d \in (0,0$ <br>Similarl<br> $\frac{(8+3d-2d)}{9(16-d)}$ <br> $k > \frac{(1-d)}{4(12,12)}$ <br>hall  $d \in ($ <br>e.i) The<br> $k_{(12,12)}^R$ <br>hally, co:<br>for exan<br> $M_1$  does  $M_{(1,2)}, k_{(1,3)}^R$ <br>
onsider<br>
ty of the nufactu<br>  $\frac{1+4d-d^2}{d(16-7)}$ <br>  $> \frac{(1-d)}{d(16-7)}$ <br>  $\frac{2d^2}{d^2}$ <br>  $\frac{(1-d)}{6-7d^2}$ <br>  $\frac{2}{d^2}$ <br>  $\frac{1}{2}$ <br>  $\frac{d^2}{d^2}$ <br>  $\frac{1}{2}$ <br>  $\frac{1}{2}$ <br>  $\frac{1}{2}$ <br>  $\frac{1}{2}$ <br>  $\frac{1}{2}$ <br>  $\frac{$  $k_{(1)}^R$  and  $k_{(2)}$  and  $k_{(3)}$  and  $k_{(4)}$  and  $k_{(5)}$  and  $k_{(6)}$  and  $k_{(7)}$  and  $k_{(8)}$  and  $k_{(9)}$  and  $k_{(10)}$  and  $k_{(11)}$  and  $k_{(12)}$  and  $k_{(13)}$  and  $k_{(14)}$  and  $k_{(15)}$  and  $k_{(16)}$  and  $k_{(17)}$  and  $k_{(18)}$  $\langle k_{(1)}^M \rangle$ <br>ork g<br>twork g<br>twork to<br> $t_1(g(1))$ <br> $t_4(g(1))$ <br> $t_4(g(1))$ <br> $t_4(g(1))$ <br> $t_5(g(1))$ <br> $t_6(g(1))$ <br> $t_6(g(1))$ <br> $t_7(g(1))$ <br> $t_7(g(1))$ <br> $t_8(g(1))$ <br> $t_7(g(1))$ <br> $t_8(g(1))$ <br> $t_8(g(1))$ <br> $t_8(g(1))$ <br> $t_7(g(1))$ <br> $t_8(g(1))$ <br> $t_8(g(1))$ <br> $t_8(g(1))$ <br> $t_8$ d ∈ (0, 0.269).<br>
12). Note tha<br>
y breaking a l<br>
ak a link if  $\varepsilon$ <br>
(i) =  $\frac{2(1-d)(a-1)}{3(1+d)(2-1)}$  =  $k_{(1)}^{M}$ <br>
(09, 1).<br>
reak a link if<br>
=  $\frac{2(a-c)^2}{9(1+d)(2-d)^2}$  =  $k_{(12,1)}^R$ <br>
pairwise stable<br>
pairwise stable<br>
(12, 1)  $g(12, 12)$ . Note that the only way to break the<br>
k is by breaking a link.<br>
(o break a link if and only if  $\Pi_{M_1}(g(12,2)) =$ <br>  $(12, 12)) = \frac{2(1-d)(a-c)^2}{3(1+d)(2-d)^2} - 2k$  which is equivalent<br>  $\frac{6a^5 - 3a^6)(a-c)^2}{3(1+d)(2-d)^2} \equiv k_{($ A manufacturer, say  $M_1$ , wants to break a link if and only if Π<sub>M1</sub> (g(12, 2)) =  $\frac{(1-\alpha)(8+4d-\alpha^2)^2(6-\alpha^2)^2}{2(1+\delta)(16-\pi d^2)^2} = k$  iii  $\frac{1}{3(1+\delta)(2+\delta^2)^2} = k$  which is equivalent to  $k > \frac{(1-\delta)(2\delta-32\alpha^2-96d^2)(8\alpha^2+36\alpha^2$  $k > \Pi_{M_1}(g(12, 12)) = \frac{2(1-d)(a-c)^2}{3(1+d)(2-d)^2}$ <br>  $-\frac{96d^3+88d^4+36d^5-3d^6)(a-c)^2}{4(1+d)(2-d)^2} \equiv k_{(12, 12)}^{M(2+d)^2(16-7d^2)^2} = k_{(12, 12)}^{M(2)}$ <br>
gative for  $d \in (0.909, 1)$ .<br>
by  $R_1$  wants to break a link if an<br>  $\cdot \Pi_{R_1}(g(12, 1$  $\frac{2(1-a)(a-c)}{3(1+d)(2-d)^2} - 2k$  which is equivalent<br>  $\frac{2(1-a)(a-c)^2}{2(12,12)},$  where  $k_{(12,12)}^M > \overline{k}$  for<br>  $\therefore$  link if and only if  $\Pi_{R_1}(g(2,12)) =$ <br>  $\frac{(a-c)^2}{d(12-4)^2} - 2k$  which is equivalent to<br>  $\equiv k_{(12,12)}^R$ , where  $k > \frac{(1-d)(256-320d^2-96d^3+88d^4+36d^3-3d^6)(a-c)^2}{6(1+d)(2+d)^2(16-7d^2)^2}$ <br>
(0,0.344) and is negative for  $d \in (0.909, 1)$ .<br>
iilarly, a retailer, say  $R_1$  wants to break a<br>  $\frac{3d-2d^2}{9(16-7d^2)^2} - k > \Pi_{R_1}(g(12, 12)) = \frac{2(k-1)^$ 6(1+d)(2+d)<sup>2</sup>(16-7d<sup>2</sup>)<sup>2</sup> ≡<br>
is negative for  $d \in (0.909, 1)$ .<br>
r, say  $R_1$  wants to break a lin<br>  $k > \Pi_{R_1}(g(12, 12)) = \frac{2(a-c)}{9(1+d)(2-a)}$ <br>  $\frac{00d^2+28d^3-7d^4-20d^5+4d^6)(a-c)^2}{+d(2+d)^2(16-7d^2)} \equiv k$  $k_{(1)}^M$  if  $\frac{1}{(1)^2}$  if  $\frac{1}{(1)^2}$  ble  $2, 2$  wis ph.  $k_{(1)}^M$ <br> $\Pi_R$  is  $k_{(1)}^M$ <br> $\vdots$   $k_{(k)}^M$ <br> $\vdots$   $k_{(k)}$ <br> $\vdots$   $k_{(k)}$ <br> $\vdots$   $k_{(k)}$ > k for<br>  $k = 12$ ) = <br>
alent to<br>  $\langle k_{(12,12)}^M \rangle$ <br>
2). Take<br>
que link.<br>
at there<br>
oot want  $d \in (0, 0.344)$  and is negative for  $d \in (0.909, 1)$ .<br>
Similarly, a retailer, say  $R_1$  wants to break a<br>  $\frac{(8+3d-2d^2)^2(a-c)^2}{9(1+2)^2} - k > \Pi_{R_1}(g(12, 12)) = \frac{2(1+d)(256+64d-100d^2+28d^3-7d^4-20d^5+4d^6)(a-c)^2}{9(1+d)(2+d)^2(16-7d^2$ Similarly, a retailer, say  $R_1$  wants to break a link if and only if  $\Pi_{R_1}(g(2, 12)) = \frac{g(a-e)^2}{9(1+e^{-2t})^2} - k > \Pi_{R_1}(g(12, 12)) = \frac{g(a-e)^2}{9(1+d)(2-d)^2} - 2k$  which is equivalent to  $k > \frac{(1-d)(256+64d-100d^2+28d^9-7d^4-20d^5+4d^$  $\frac{(8+3d-2d^2)^2(a-c)^2}{9(16-7d^2)^2}$ <br>  $k > \frac{(1-d)(256+64d-1)}{9(1)}$ <br>
all  $d \in (0, 1)$ .<br>
e.i) The distribution  $\binom{R}{(12,12)}$ , where  $0 < k^{R}_{(12,12)}$ all  $d \in (0,1)$ .  $k >$ 

9(1+d)(2+d)<sup>2</sup>(16-7d<sup>2</sup>) ≡<br>
tion network  $g(12, 12)$  is pairwise<br>  $k_{(12,12)}^R < k_{(12,12)}^M$  if  $d \in (0, 0.480)$ .<br>
c distribution networks  $g(12, 1)$ The distribution network  $g(12,12)$  is pairwise stable if and only if  $k < \min\{k_{(12,12)}^M,$  $\{ \begin{array}{l} R\\(12,12) \end{array} \}$ , where  $k^R_{(12,12)} < k^M_{(12,12)}$  if

 $k > \Pi_{R_1}(g(12, 12)) = \frac{2(a-c)^2}{9(1+d)(2-c)}$ <br>  $\omega a^2 + 28a^3 - 7a^4 - 20a^5 + 4a^6)(a-c)^2 \equiv k \frac{1}{2}$ <br>  $(k \to a)(2+d)^2(16-7a^2) \equiv k \frac{1}{2}$ <br>  $k \to a \to b \to b \to b \to c \to a \to b \to c \to a \to b \to c \to c \to a \to b \to b \to b \to c \to c \to a \to b \to b \to b \$  $\frac{2(a-c)^2}{9(1+d)(2-d)^2} - 2k$  which is equivalent to<br>  $\frac{-c)^2}{2} \equiv k_{(12,12)}^R$ , where  $0 < k_{(12,12)}^R < \overline{k}$  for<br>
irwise stable if and only if  $k < \min\{k_{(12,12)}^M\}$ .<br>
1.480).<br>
2, 1),  $g(12,2)$ ,  $g(1,12)$  and  $g(2,12)$ . Take<br>  $k > \frac{(1-d)(256+64d-100d^2+28d^3-7d^4-20d^3+4d^6)(a-c)^2}{9(1+d)(2+d)^2(16-7d^2)}$ <br>all  $d \in (0,1)$ .<br>e.i) The distribution network  $g(12,12)$  is pairw:<br> $k_{(12,12)}^R$ , where  $k_{(12,12)}^R < k_{(12,12)}^M$  if  $d \in (0,0.48$ <br>nally, consider the  $k_{(1)}^R$ <br>tal $(12$ <br>ver it iddl  $\langle k_{(1)}^R \rangle$ <br>if  $k \langle k_{(1)} \rangle$ <br>nd  $g(k)$ <br>its abov  $\langle k \rangle$  for  $\{k_{(12,12)}^M\}$ . Take ue link.<br>at there want  $d \in (0, 1)$ .<br>
) The distribution of the distribution of the distribution of the example does not always and the distribution of the distribut  $g(12, 12)$  is pairwise stable if and only if  $k < \min\{k_{(1:1)}^M\}$  if  $d \in (0, 0.480)$ .<br>
1 networks  $g(12, 1), g(12, 2), g(1, 12)$  and  $g(2, 12)$ . T case  $M_2$  and  $R_2$  never want to break its unique lie link with  $R_2$  since  $k_{(1)}^R$  nal for  $M$  is  $k_{(1)}^R$  are  $(2, 1)$  and  $k_{\text{en}}$  $\langle k_{(1)}^M \rangle$ <br>bution this eak the create to create the create the create the create the create the create the create that  $\frac{1}{2}$  $d \in (0, 0.480)$ .<br>
orks  $g(12, 1)$ ,  $g$ <br>  $M_2$  and  $R_2$  ne<br>
with  $R_2$  since<br>
link between b<br>
30 **f**) Finally, consider the distribution networks  $q(12,1)$ ,  $q(12,2)$ ,  $q(1,12)$  and  $q(2,12)$ . Take hally, consider the distribution networks  $g(12, 1), g(12, 2), g(1, 12)$  and  $g(2, 12)$ . Take<br>for example  $g(12, 1)$ . In this case  $M_2$  and  $R_2$  never want to break its unique link.<br> $M_1$  does not want to break the link wit  $g(12, 1), g(12, 2), g(1, 12)$  and  $g(2, 12)$ . Take<br>nd  $R_2$  never want to break its unique link.<br> $R_2$  since it is proved in **c**) above that there<br>between both. Similarly,  $R_1$  does not want for example  $g(12, 1)$ . In this case  $M_2$  and  $R_2$  never want to break its unique link. g(12, 1). In this case  $M_2$  and  $R_2$  never want to break its unique link.<br>want to break the link with  $R_2$  since it is proved in c) above that there<br>incentive to create a link between both. Similarly,  $R_1$  does not wa  $M_1$  does not want to break the link with  $R_2$  since it is proved in c) above that there is always an incentive to create a link between both. Similarly,  $R_1$  does not want is always an incentive to create a link between both. Similarly,  $R_1$  does not want  $$\,30$ 

to break the link with  $M_2$  since it is proved in b) above that there is always an incentive to create a link between both. Considering the link between  $M_1$  and  $R_1$  it happens that  $M_1$  will break the link if and only if  $k > k_{(1,2)}^M$  while  $R_1$  will do it if and only if have incentives to do it, that is if and only if  $k < \min\{k_{(12,12)}^M, k_{(12,12)}^R\}$ .

f.i) Thus, the distribution network  $g(12,1)$  (and  $g(12,2)$ ,  $g(1,12)$ ) and  $g(2,12)$ ) is pairwise stable if and only if  $k > \min\{k_{(12,12)}^M, k_{(12,12)}^R\}$  and  $k < \min\{k_{(1,2)}^M, k_{(1,2)}^R\}$ . **f.i)** Thus, the distribution network  $g(12, 1)$  (and  $g$ ) pairwise stable if and only if  $k > \min\{k_{(12,12)}^M, k_{(12,12)}^R\}$ . Combining **d.i)**, **e.i)** and **f.i)** yields the proposition.

# B.1.2 Strong stable distribution networks **B.1.2** Strong stable d

#### Proof of Proposition 2

to break the link with  $M_2$  since it is proved in b) above that there is always an introduction to mean a link break the link (food only if  $k > \frac{16}{12}$ , with  $R_1$  will have the link of some that the link of some that c a link between both. Considering the link between  $M_1$  and  $M_1$  is the link if and only if  $k > k_{1/2}^2$ , while  $R_2$  will do is if  $M_1$  and R<sub>2</sub> will create a link if both of it, that is if and only if  $k < \min\{k_{1/2}^$ incentive to create a link between both. Considering the link between M1 and R1 it is easily proved the link between M1 and R1 it is easily respect to the link between M1 and R1 it is easily respect to do the link if so w happens that  $M_1$  will break the link if and only if  $k > k_1^M$ <br>and only if  $k > k_1^M$ , see d) above). Finally,  $M_2$  and  $Ra$ <br>have intentives to do it, that is if and only if  $k <$  min( $k_1^M$ ,<br>fi) Thus, the distribution ne  $\binom{M}{1,2}$  while  $R_1$  will do it if<br>  $\binom{1}{1,2}$  while  $R_1$  will do it if<br>  $\binom{1}{2,12}, k_{(12,12)}^R$ ,<br>  $\binom{1}{1,2}$  and  $g(2,12)$  is<br>  $\binom{1}{2}$   $\binom{1}{1,2}$  and  $g(2,12)$  is<br>  $\binom{1}{2}$   $\binom{1}{2}$  and  $g(2,12)$  is<br>  $k > k_{(1)}^R$ <br>
(ves to  $k_{(1)}^R$ <br>
(ves to  $k_{(1)}^R$ )<br>
(he dist<br>
ble if a<br>
d.i), e.<br>
stable<br>
sition<br>
at  $g(1)$ <br>
(ht to be<br>
facture<br>  $\cdot$  off wi<br>  $\cdot d^2$ )( $a-c$ )<br>  $\cdot d^2$ )( $a-c$ )<br>
and mo<br>  $\cdot$  The s<br>  $\cdot k_{s(1,2)}^R$ <br>
and by 0<br>  $k < \min\{k_{(1)}^M\}$ <br>
(and  $g(12, 2)$ ,  $k_{(12,12)}^R$  and  $g(12, 2)$ ,  $k_{(12,12)}^R$  and osition.  $\blacksquare$ <br>
ble when it if ect to  $g(12)$ , ust benefit f<br>
if  $k < \min\{k_{s(1,2)}^M\}$  hold is decreased here the shold for wo links and<br>  $\begin{array}{c}\nM & 0.44 \\
(12,12), k^2\n\end{array}$ <br>  $g(1,1)$ <br>  $g(1,1)$ <br>  $g(1,1)$ <br>  $h(k)$ <br>  $h(k)$ <br>  $h(\frac{4(4-10a)}{3(1+10)})$ <br>  $h(k)$  $g(12, 1)$  (and  $g(12, 2)$ ,  $g(1, 12)$  and  $g(2, 12)$ ) is<br> $1{k_{(12, 12)}^N$ ,  $k_{(12, 12)}^R$ ,  $k_{(12, 13)}^R$  and  $k < \min\{k_{(1, 2)}^R, k_{(1, 2)}^R\}$ .<br>the proposition.  $\blacksquare$ <br>works<br>maly stable when it is pairwise stable. In or-<br>w  $k > \min\{k_{(1)}^M\}$ <br>yields the p<br>yields the p<br>ion networ!<br>uys strongly<br>stable with<br>two retailer<br>2) if and on<br>irst term of<br>und such that<br>, 12). This t<br>m above is t<br>want to crea<br>g zero at  $d =$ <br>in $\{k_{s(1,2)}^M, k_{s(1,2)}^R\}$ ,  $\frac{M}{(12,12)}, \frac{R}{(12,12)}, \frac{R}{(12,12)}, \frac{R}{(12,12)}, \frac{R}{(12,12)}, \frac{R}{(12,12)}$  or proposition the about for  $k$  intersted two  $k = 0.202$ .<br>  $\frac{kR}{s(1,2)}$  der for  $g$  ( $\frac{kR}{s(1,2)}$  der for  $g$  (ion  $S = \{R_2 \text{ and prefers} \}$  strongly w  $k < \min\{k_{(1)}^M\}$ <br>pairwise sta<br>2), the coali<br>om the move<br> $\frac{4(4-10d+6d^2-4)}{3(1+d)(2+d)}$ <br>sion corresp<br>e manufactu<br>sing with d<br>: denoted by<br>move to g(1<br>it is easy to<br>ersect the r<br>b be strongly<br>must gain wi<br>new link bet<br>ce  $\$  $\frac{M}{(1,2)}, k_{(1,2)}^R$ <br>stable.<br>alition f<br>we. Therefore and the sponds<br>curers we and r<br>by  $k_{s(1,2)}^R$ <br>(12, 12)<br>to chec region<br>ly stabl with the setween<br> $\frac{-c)^2}{+d}$  is a o  $g(12,$ <br>nember<br>with the setween<br> $\frac{-c)^2}{+d}$  is a First we show that  $g(1,2)$  is always strongly stable when it is pairwise stable. In or $g(1,2)$  is always strongly stable when it is pairwise stable. In orcolonismed with respect to  $g(1,2,1)$ , the condition formed between the four strongly stable with respect to  $g(1,12)$  if and only if  $k < \min\{\frac{4(4-1)(4-4d$ der for  $q(1,2)$  not to be strongly stable with respect to  $q(12,12)$ , the coalition formed  $g(1,2)$  not to be strongly stable with respect to  $g(12, 12)$ , the coalition formed<br>two manufactures and the two retailers must benefit from the move. The four<br>tre-better off with  $g(12, 12)$  if und only if  $k < \min(\frac{32-10$ by the two manufacturers and the two retailers must benefit from the move. The four agents are better off with  $g(12, 12)$  if and only if  $k < \min\{\frac{4(4-10d+6d^2-4d^3+d^4)(a-c)^2}{3(1+d)(2+d)(8-6d+d^2)}\}$ g (12, 12) if and only if  $k < \min\{\frac{4(4-6)^{4}+d^{2}-10d+6d^{2}-4d^{2}+d^{2}V_{0}(-c)^{2}}{4d(1+6)(2+4d)(8-6d+2)}\}$ .<br>The first term of the above expression corresponds to the  $M_{\epsilon}(1,2)$ , and such that for  $k \leq k_{\delta(1,2)}^{M}$  the manufa  $\frac{-10a+6a^2-4a^3+a^3}{3(1+d)(2+d)(8-6d+d^2)},$ <br>3 $(1+d)(2+d)(8-6d+d^2),$ <br>on corresponds to the<br>manufacturers want to<br>mg with d and reaches<br>lenoted by  $k_{s(1,2)}^R$ , and  $\frac{-2(4-20d+d^2)(2+2d-d^2)(a-c)^2}{9(1+d)(2+d)^2(8-6d+d^2)^2}$ <br>threshold for k denoted by<br>create two links and move<br>zero at  $d = 0.510$ . The se  $\frac{1}{2\pi(2+\alpha)(2+\alpha)}$ . The first term of the above expression corresponds to the saloid for k denoted by  $k_{\alpha(1,2)}^M$ , and such that for  $k \leq k_{\alpha(1)}^M$ , and such that for  $k \leq k_{\alpha(1)}^M$ , and such that for  $k \leq k_{\alpha(1)}^M$ threshold for k denoted by  $k_{s(1,2)}^M$ , and such that for  $k \leq k_{s(1,2)}^M$  the manufacturers want to k denoted by  $k_{s(1)}^{M}$ <br>liks and move to<br>.510. The second<br> $k \leq k_{s(1,2)}^{R}$  retai<br>reases with d rea<br>fined by  $0 \leq k$ <br>wise stable (see I<br>12,0), each meml<br>evering the link b<br>s easily proved t<br> $\frac{(1-d)(a-c)^{2}}{2(2-d)^{2}(1+d)}$ , an  $\frac{M}{s(1,2)}$ , and such that for  $k \leq k_{s(1,2)}^{M}$ , and such that for  $k \leq k_{s(1,2)}^{M}$  and term above is the threshold ailers want to create two links eaching zero at  $d = 0.202$ . How  $k \leq \min\{k_{s(1,2)}^{M}, k_{s(1,2)}^{R}\}$  doe  $g(12, 12)$ . This threshold is decreasing with d and reaches<br>term above is the threshold for k denoted by  $k_{s(1,2)}^R$ , and<br>lers want to create two links and move to  $g(12, 12)$ . This<br>ching zero at  $d = 0.202$ . However, it create two links and move to  $g(12, 12)$ . This threshold is decreasing with d and reaches zero at  $d = 0.510$ . The second term above is the threshold for k denoted by  $k_{s(1,2)}^R$ , and  $d = 0.510$ . The second term above is the threshold for k denoted by  $k_{\delta}^{R}$ <br>the for  $k \leq k_{\delta(1,2)}^{R}$  retailers want to create two links and move to  $g(12, 1)$ ;<br>dincreases with  $d$  reaching zero at  $d = 0.202$ . However  $\frac{R}{s(1,2)}$ , and<br>12). This<br>theck that<br>ion where<br>table with<br>the move<br>een them.<br>is always<br>(12,0). In<br>oer of the<br>k between<br>d that  $R_2$ <br> $\downarrow g(1,2)$  is<br>ough that<br>ballition of  $k \nleq k_{s}^{R}$ <br>eases w<br>ined by<br>wise sta<br>2,0), ea<br>vering t<br>easily<br> $\frac{(1-d)(a-2)(1-d)^{2}(1$ such that for  $k \leq k_{s(1,2)}^R$  retailers want to create two links and move to  $g(12, 12)$ . This<br>y to check that<br>ne region where<br>ngly stable with<br>n with the move<br> $\frac{(a-c)^2}{(2+d)}$  is always<br>ito  $g(12,0)$ . In<br>member of the<br>he link between<br>proved that  $R_2$ <br>d then  $g(1,2)$  is<br>is enough that<br>any coalition threshold increases with d reaching zero at  $d = 0.202$ . However, it is easy to check that d reaching zero at  $d = 0.202$ . However, it is easy to check that  $\leq k \leq \min\{k_{\text{sl}(1,2)}^M, k_{\text{sl}(1,2)}^R\}$  does not intersect the region where (see Figure 4). In order for  $g(1,2)$  not to be strongly stable with member o  $k \n\leq \min\{k_{s(i)}^M\}$ <br>e Figure 4). I<br>mber of the c<br>s between  $M_2$ <br>d that  $M_2$  also<br>nd then  $g(1, 2)$ <br>strongly stabs<br>ist gain with<br>new link betw<br> $\frac{4(a-c)^2}{(4-d)^2(2+d)^2}$  i<br>to  $g(1, 1)$ .<br>tability of  $g$  (<br>a link each (<br>e, w  $\frac{M}{s(1,2)}, k_{s(}^{R}$ <br>In orde<br>coalitio<br> $I_2$  and  $I_3$ <br>always p<br>, 2) is st<br>able with the metrue the is always<br> $g(12, 12)$ <br>(the co<br>nde that<br>31 the region defined by  $0 \leq k \leq \min\{k_{s(1,2)}^M, k_{s(1,2)}^R\}$  does not intersect the region where  $g(1, 2)$  is pairwise stable (see Figure 4). In order for  $g(1, 2)$  not to be strongly stable with<br>respect to  $g(12, 0)$ , each member of the coalition  $S = \{M_2, R_1\}$  must gain with the move<br>that entails severing the link respect to  $g(12,0)$ , each member of the coalition  $S = \{M_2, R_1\}$  must gain with the move  $g(12, 0)$ , each member of the coalition  $S = \{M_2, R_1\}$  must gain with the move<br>s severing the link between  $M_2$  and  $R_2$  and creating a new link between them.<br>t is easily proved that  $M_2$  always prefers  $g(1, 2)$  sin that entails severing the link between  $M_2$  and  $R_2$  and creating a new link between them. that entails severing the link between  $M_2$  and  $R_2$  and creating a new link between them.<br>However, it is  $(\frac{1-\theta}{2})e^{-2}$  is always prefers  $g(1,2)$  since  $\frac{2(2-\theta)(a-c)^2}{(4-a)^3(2+a)^2}$  is always<br>greater than  $\frac{f(1-\theta)(a-c)^$ However, it is easily proved that  $M_2$  always prefers  $g(1, 2)$  since  $\frac{2(2-d)(a-c)^2}{(4-d)^2(2+d)}$ <br>greater than  $\frac{(1-d)(a-c)^2}{2(2-d)^2(1+d)}$ , and then  $g(1, 2)$  is strongly stable with respect to  $g$  order for  $g(1, 2)$  not to be  $\frac{(2-4)(d-c)^2}{(4-d)^2(2+d)}$  is always<br>spect to  $g(12,0)$ . In<br>each member of the<br>ing the link between<br>asily proved that  $R_2$ greater than  $\frac{(1-d)(a-c)^2}{2(2-d)^2(1+d)}$ <br>order for  $g(1,2)$  not to<br>coalition  $S = \{M_1, R_2\}$ <br> $M_2$  and  $R_2$  and creating greater than  $\frac{1(-d)(d-c)^2}{2(-d)^2(1+d)}$ , and then  $g(1, 2)$  is strongly stable with respect to  $g(1, 1)$ , each memh<br>coalition  $S = \{M_1, R_2\}$  must gain with the move that entails severing the linh<br> $M_2$  and  $R_2$  and creatin  $g(1, 2)$  is strongly stable with respect to  $g(12, 0)$ . In<br>stable with respect to  $g(1, 1)$ , each member of the<br>with the move that entails severing the link between<br>between them. However, it is easily proved that  $R_2$ <br>order for  $g(1,2)$  not to be strongly stable with respect to  $g(1,1)$ , each member of the  $g(1, 2)$  not to be strongly stable with respect to  $g(1, 1)$ , each member of the  $S = \{M_1, R_2\}$  must gain with the move that entails severing the link between  $R_2$  and creating a new link between them. However, it is e coalition  $S = \{M_1, R_2\}$  must gain with the move that entails severing the link between  $S = \{M_1, R_2\}$  must gain with the move that entails severing the link between  $R_2$  and creating a new link between them. However, it is easily proved that  $R_2$  efers  $g(1,2)$  since  $\frac{4(a-c)^2}{(4-d)^2(2+d)^2}$  is always grea  $M_2$  and  $R_2$  and creating a new link between them. However, it is easily proved that  $R_2$ <br>always prefers  $g(1,2)$  since  $\frac{4(a-e)^2}{(4-a^2)^2+d)^2}$  is always greater than  $\frac{(a-e)^2}{36}$ , and then  $g(1,2)$  is<br>strongly stabl strongly stable with respect to  $q(1,1)$ .

g (1, 2) since  $\frac{4(a-c)^2}{(4-d)^2(2+)}$ <br>with respect to g(1, 1<br>g the strong stability dide to break a link each.). Therefore, we con  $g(1,2)$  is<br>ugh that<br>alition of<br>for  $k >$  $g(1, 1)$ .<br>ility of<br>nk each<br>re conc Considering the strong stability of  $g(12, 12)$  with respect to  $g(1,2)$  it is enough that g (12, 12) with respect to  $g(1,2)$  it is enough that (the coalition S that deviates is any coalition of ude that  $g(12, 12)$  is not strongly stable for  $k >$ 31 two agents decide to break a link each (the coalition  $S$  that deviates is any coalition of S that deviates is any coalition of 12) is not strongly stable for  $k >$ cardinality two). Therefore, we conclude that  $q(12,12)$  is not strongly stable for  $k >$  $g(12, 12)$  is not strongly stable for  $k >$   $\min\{k_{s(1,2)}^M, k_{s(1,2)}^R\}$ . To see if  $g(12, 12)$  is strongly stable we have to check if the coalition  $S = \{M_1, M_2\}$  wants to move to  $g(12, 0)$ . This happens for k greater than  $\frac{(1-d)(a-c)^2}{6(2-d)^2(1+d)}$ . 6(2-d)<sup>2</sup>(1+d)<br>
ans that the<br>
<u>-d)(a-c)<sup>2</sup></u><br>
2-d)<sup>2</sup>(1+d) i<br>
(1, 1). The<br>
2. Retailer This later threshold is always greater than  $k_{(12,12)}^M$  (see Figure 3), which means that the intersection between the region where  $\frac{(1-a)(a-c)^2}{6(2-d)^2(1+d)}$  is<br>  $g(1,1)$ . The<br>  $M_2$ . Retailers<br>  $\frac{c)^2}{a}$ . But this<br>
rongly stable empty. Finally, we check the strong stability of  $g(12,12)$  with respect to  $g(1,1)$ . The coalition to be considered is  $S = \{R_1, R_2\}$  which has to break the links with  $M_2$ . Retailers gain in the move from  $g(12, 12)$  to  $g(1, 1)$  if and only if  $k > \frac{(4+3a^2-a^2)(a-c)^2}{36(2-a)^2(1-d)}$ . But this inequality is never satisfied since  $\overline{k} < \frac{(4+3d^2-d^3)(a-c)^2}{36(2-d)^2(1-d)}$ . Thus,  $g(12, 12)$  is strongly stable aga against  $g(1,1)$ . Summarizing,  $g(12,12)$  is strongly stable for  $k \le \min\{k_{s(1,2)}^M,$ 

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theref  $\frac{M}{s(1,2)}, \frac{R}{s(s+1,2)}, \frac{R}{s(s+2,2)}, \frac{R}{s(s+1,2)}$ <br>ater three ection be<br> $\therefore$  Finall on to be<br> $\therefore$  Finall on to be<br> $\therefore$  Finall on to be<br> $\frac{1}{s}$  at  $g(1,1)$  msiderin<br>ered is  $\frac{R}{s}$ <br> $\frac{R}{s}$  if  $k$  is<br> $\frac{R}{s}$  if  $k$ g (12, 12) is strongly stable we have to check if the coalition<br>is to g (12, 0). This happens for k granter than  $\frac{5}{62}$ -a)<br>is greater than  $k_{1/2}^{1/2}$  is jointwise stable and  $k > \frac{(41)(10-6)^2}{62-2(11+2)}$  is strong st S = {M<sub>1</sub>, M<sub>2</sub>} wants to move to g (12, 0). This happens for k greater than  $\frac{1}{2}$  wants to move to g (12, 0). This happens for k greater than  $\frac{1}{2}$  with  $\frac{1}{2}$  with the set of the set of the set of k greater t .  $k_{(1)}^M$  and  $k_{(2)}^M$  and  $k_{(3)}^M$  and  $k_{(4)}^M$  and  $k_{(5)}^M$  and  $k_{(6)}^M$  and  $k_{(7)}^M$  and  $k_{(8)}^M$  a g (12, 12) is pairwise stable and  $k > \frac{1-q_1}{6(2-\alpha)(2-\alpha)}$ <br>tability of  $g(12, 12)$  with respect to  $g(1, 1)$ .  $V_2$  by  $V_3$  which as to break the limits with  $M_2$ . Retail<br>1, 1, 1) if and only if  $k > \frac{(4+3\theta^2-\alpha^2)(1-\alpha^2)}{$ g (12, 12) with respect to g (1, 1). The<br>ss to break the links with  $M_2$ . Retailers<br>only if  $k > \frac{(4\pi 3d^2 - d^3)(a - c)^2}{36(2 - d)^2(1 - d)}$ . But this<br>only if  $k > \frac{(4\pi 3d^2 - d^3)(a - c)^2}{36(2 - d)^2(1 - d)}$ . But this<br> $\frac{c}{d\rho}$ . Thus, g  $S = \{R_1, R_2\}$  which has to break the links with  $M_2$ . Retailers<br>  $\sin 2x$ ,  $12$  for  $\sin 2x$  is  $\sin 2x$  is  $\cos 2x$ <br>  $\sin 2x$  is  $\cos 2x$ ,  $\sin 2x$  is  $\sin 2x$  is  $\cos 2x$ ,  $\sin 2x$  is  $\sin 2x$ <br>  $\sin x$ ,  $\int (12, 12)$  is  $\sin x \cos 2x\$ g (12, 12) to g (1, 1) if and only if k > (4+3d2−d3)(a−c)2 inequality is never satisfied since  $k < \frac{(4+3a^2-a^2)(a-c)^2}{36(2-d)^2(1-d)}$ . Thus,<br>against  $g(1,1)$ . Summarizing,  $g(12,12)$  is strongly stable for<br>Considering the strong stability of  $g(12,1)$  with respect to<br>considered is  $S =$  $\overline{k} < \frac{(4+3d^2-d^3)(a-c)^2}{36(2-d)^2(1-d)}$ <br>
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112d-464d<sup>2</sup>+208d<sup>3</sup>+73d<sup>4</sup><br>
24(16-7d<sup>2</sup>)<sup>2</sup>(1  $g(12, 12)$  is strongly stable<br>  $\leq \min\{k_{s(1,2)}^M, k_{s(1,2)}^R\}$ .<br>  $g(1,0)$ , the coalition to be<br>
to sever simultaneously its<br>  $I_1$  prefers to sever the link<br>  $I_1$  prefers to sever the link<br>  $I_2$  prefers to sever the l g(1, 1). Summarizing, g(12, 12) is strongly stable for  $k \leq \min\{k_{\text{eff}}^M\}$ <br>ideiring the strong stability of g(12, 1) with respect to g(1, 0), the<br>ded is  $S = \{M_1, H_1\}$ . Each agent in the coalition has to sever simple<br> $k_{s}^{R}$  and we due to the set of  $k_{s}$  and  $s(1,2)^f$ .<br>ition to<br>ition to<br>neously<br>er the ethe l<br> $l = 0.3$ <br> $s$  $s$  $d^3$  $(a-1)$ <br> $s$  asing  $\frac{s}{(1+d)}$ <br>gents is mind<br>satisf<br>and tl<br>satisf<br>and tl<br> $g(1,0)$ ,<br>tersec<br>sists<br>sists<br> $s^R_{s(1,2)}$ <br>de for pair<br> $g(1,2)$ <br>of pair<br> $g(1,2)$ <br> Considering the strong stability of  $g(12,1)$  with respect to  $g(1,0)$ , the coalition to be g (12, 1) with respect to g (1, 0), the coalition to be<br>nt in the coalition has to sever simultaneously its<br>is  $\frac{1}{24(16-rd^2)^2(4+2)}$ . Let us denote the link<br> $\frac{1}{64(16-rd^2)^2(4+2)}$ . Let us denote the later<br> $\frac{1}{24(16-rd$ considered is  $S = \{M_1, R_1\}$ . Each agent in the coalition has to sever simultaneously its  $S = \{M_1, R_1\}$ . Each agent in the coalition has to sever simultaneously its and  $M_2$ , respectively. It is easy to show that  $M_1$  prefers to sever the link graps that  $M_2$  and  $M_3$  is greater than  $\frac{8M}{2(123)}$ . Firs link with  $R_2$  and  $M_2$ , respectively. It is easy to show that M<sub>1</sub> prefers to sever the link<br>with R<sub>2</sub> if  $k$  is greater than  $228-128+288+288+11n$ <br>for the  $64$  there is the three index<br>Similarly,  $R_2$  profers to seve with  $R_2$  if k is greater than  $\frac{(256-512d-464d+2488d+248d-42d^2)(\alpha-0)^2}{24(16-7d^2)^2(1+d)}$ <br>threshold by  $k_{2(12,1)}^M$ . This threshold is decreasing with d and re<br>Similarly,  $R_1$  prefers to sever the link with  $M_2$  if k link with  $R_2$  and  $M_2$ , respectively. It is easy to show that  $M_1$  prefers to sever the link<br>with  $R_2$  if k is greater than  $\frac{(256-512d-464d^2+208d^3+73d^4-47d^5)(a-c)^2}{24(16-7d^2)^2(1+d)}$ . Let us denote the later<br>thres threshold by  $k_{s(12,1)}^M$ . This threshold is decreasing with  $k_{s(1)}^M$  prediction in the properties of  $k_{s(1)}^R$  and  $k_{s(1)}^R$  at independent values of  $\alpha$  or  $\alpha$  or  $\alpha$  or  $\alpha$  or  $\beta$  or  $g(0)$ d and reaches zero at  $d = 0.3892$ .<br>
ater than  $\frac{(63-32d+35d^2+59d^3)(a-c)^2}{144(16-i7d,1)(i+7d)}$ ,<br>
positive, initially decreasing with<br>
conclude that both agents in S<br>  $x{k_{s(12,1)}^M, k_{s(12,1)}^R}$ . Remind that<br>
e intersection Similarly, R<sub>1</sub> prefers to sever the link with M<sub>2</sub> if k is greater than  $\frac{163-35d-35d-1}{144(36-76)(11-6)}$ <br>We denote the later threshold by  $k_{a}^{(2)}(2,3)$ . It is always positive, initially decreasing with the move to  $g$ 144(16-7d<sup>2</sup>)(1+d)<br>
lly decreasing<br>
t both agents<br>  $\{a_{2,1}\}$ . Remind<br>
of the k satis We denote the later threshold by  $k_{s(12,1)}^R$ . It is always positive, initially decreasing with  $k_{s}^{R}$  at to reg atis and it reduces are more that  $k_{s}^{R}$  at to be asset in the same  $r$  at  $(1, s_{s})$  and  $(1, s_{s})$  and  $(1, s_{s})$  and  $(1, s_{s})$  $k_{s(1,1)}^{R}$ . It is always positive, initially decreasing with  $t d = 0.7490$ . We conclude that both agents in  $S$   $g(1,0)$  if  $k > \max\{k_{s(12,1)}^{R}, k_{s(12,1)}^{R}\}$ . Remind that gion defined by the intersection of the  $k$  sati  $S_{s(12,1)}^{R}$ . Remind that  $\binom{R}{(1,2)}$  and those satisfying  $k > k_{(12,12)}^R$  (see Figure 3). Figure 4 displays the relationship between the thresholds under which  $g(12, 1)$  is pairwise stable and those that imply that the agents in the coalition  $S = \{M_1, R_1\}$  gain with the move to  $g(1,0)$ . We therefore conclude that  $g(12, 1)$  is strongly stable in the region defined by the intersection of  $k < \min \{k_{(1,2)}^R, k_{s(12,1)}^M\}$  and  $k > k_{(12,12)}^R$ .

### Characterization of closed cycles when no strongly stable network exists We should consider two different cases.

. d and then increasing reaching k at d = 0.7490. We conclude that both agents in S<br>
=  $(N_1, N_1)$  gain with the move to  $g(1,0)$  if k > max(k<sup>2</sup><sub>0</sub>t<sub>0</sub>,  $n_2R_{N_1}^{(1)}$ ). Fermind that<br>  $g(12,1)$  is pairwise stable in the re  $= {M_1, R_1}$  gain with the move to  $g(1, 0)$  if  $k > \max\{k_{\delta}^M(z, 1)\}$  is pairwise stable in the region defined by the inte  $k < \min\{k_{(1,2)}^M, k_{(1,2)}^R\}$  and those satisfying  $k > k_{(12,12)}^R$  (see Ithe relationship betwee  $\frac{M}{s(12,1)}, k_{s(12,1)}^R$ <br>tersection<br>Figure :<br>Figure :<br>(1) is pair<br>n with the stable n<br>stable n<br> $\Rightarrow k > \text{mi}$ <br>observe a<br> $j(12,2),$ <br>twork  $g(1)$ <br>the coali<br>wo links n<br>know by  $g(12,1)$  is pairwise stable in the region defined by the intersection of the k satisfying  $k \leq k_{\text{min}}(k_{\text{H},2},k_{\text{min}}^2)$  and those satisfying  $k \geq k_{\text{H},2,1}^2$  (see Figare 3). Figure 4 displays  $k \geq k_{\text{H},2,1}^2$  (  $k < \min\{k_{(1)}^M\}$ <br>the relations:<br>that imply therefore con<br>of  $k < \min$  {<br>**Characteriz**<br>We should co<br>**1.** When mi this are<br>by the For suc<br>stable or both<br> $g(2, 1)$ )<br>and str<br> $g(2, 0)$ ,  $\frac{M}{(1,2)}, k_{(1)}^R$ <br>
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(), g(0,  $k > k_{(1)}^R$ <br>ler whic<br>= { $M_1$ <br>table ir<br> $\blacksquare$ <br>no st<br>no st<br>no st<br>no st<br>(12,1),  $g$ <br>now tha<br>anufact<br>ives to<br>een reads<br>ince the st<br>(2,2)), 3<br>32  $g(12, 1)$  is pairwise stable and those<br>  $\{c_1\}$  gain with the move to  $g(1, 0)$ . We<br>
the region defined by the intersection<br> **mgly stable network exists**<br>  $\{a_{r,1}\}\}\n> k > \min\{k_{s(1,2)}^M, k_{s(1,2)}^R\}$ . In<br>
a we observe a  $S = \{M_1, R_1\}$  gain with the move to  $g(1, 0)$ . We<br>
stable in the region defined by the intersection<br>  $\cdot$  **=**<br> **en no strongly stable network exists**<br>  $\frac{M}{s(12,1)}, \frac{R}{s(12,1)}, \frac{R}{s(12,1)}\} > k > \min\{k_{s(1,2)}^{M}, k_{s(1,2)}^{$  $g(12, 1)$  is strongly stable in the region defined by the intersection<br>  $g(12, 1)$  and  $k > k_{(12,12)}^R$ .  $\blacksquare$ <br>
closed cycles when no strongly stable network exists<br>
different cases.<br>  $h^{(12,12)}$ ,  $h^{(12,13)}$ ,  $h^{(12,11)}$  $k < \min \{k_{(1)}^R\}$ <br>
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When  $\min\{\text{this area} \text{ by the net for such a table }$ <br>
for such stable bee<br>
or both c<br>  $g(2,1))$ . (and stron<br>  $g(2,0)$ , g  $\frac{R}{(1,2)}, k_{s(1)}^M$ <br>ation o:<br>asider tv<br>asider tv<br> $\{\min\{k\}\}$ <br>a there i<br>networks<br>ecause t<br>coalitio<br>Once gl<br>ng stab:<br> $g(0,1)$ ,  $k > k_{(1)}^R$ <br>  $\{xyz\}$ , ycles<br>  $k_{(2)}^R$ , ma<br>  $k_{(2)}^R$ , ma<br>  $k_{(2)}^R$  and  $k_{(2)}^R$ <br>  $\{z\}$  costs<br>  $\{z, 1\}$  and  $\$ When min{min{ $k_{(1)}^R$ <br>this area there is<br>by the networks<br>For such values  $\alpha$ <br>stable because th<br>or both coalition<br> $g(2,1)$ ). Once  $g(\alpha)$ <br>and strong stabil<br> $g(2,0)$ ,  $g(0,1)$ ,  $g$  $\{^{R}_{(12,12)}, \, k^{M}_{(1,2)}\}, \, \max\{k^{M}_{s(12,1)}, \, k^{R}_{s(12,1)}\}\} > k > \min\{k^{M}_{s(1,2)}, \, k^{R}_{s(1,2)}\}.$  In  $k_{(1)}^M$ <br>troi 12<br>lin liti<br>liti<br>uld or  $\alpha$ <br>at  $\binom{M}{(1,2)}$ , max  $\{k_{s(i)}^{M}$ <br>
congly stable no<br>
(2),  $g(1,2)$ ,  $g(2)$ <br>
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t no coalition l<br>  $g(1,1)$  (or  $g$  $\frac{M}{s(12,1)}, \frac{k_s}{s(12,1)}, \frac{R_s}{s(12,1)}, \frac{R_s}{s(12,1)}, \frac{R_s}{s(12,1)}, \frac{R_s}{s(12,1)}, \frac{R_s}{s(12,1)}, \frac{R_s}{s(12,2)}, \frac{R_s}{s(12,1)}, \frac{R_s}{s(12,1)}, \frac{R_s}{s(12,1)}, \frac{R_s}{s(12,1)}, \frac{R_s}{s(12,1)}, \frac{R_s}{s(12,1)}, \frac{R_s}{s(12,1)}, \frac{R_s}{s(12,1)}, \frac{R_s}{s(12,1)}, \$ > k > min{ $k_{s(i)}^M$ <br>observe a close<br> $g(12, 2)$ ,  $g(1, 12)$ <br>etwork  $g(12, 12)$ <br>r the coalition of<br>wo links movin<br>know by the pr<br>io move to netw<br>0) (or  $g(0, 12)$ )  $k_{s}^{R}$ <br>  $\text{gcd}$  and  $\text{det}$   $\text{det}$   $\text{det}$   $\text{det}$   $\text{det}$   $\text{det}$   $\text{det}$   $\text{det}$   $\text{det}$ this area there is no strongly stable network and we observe a closed cycle formed by the networks  $g(12, 12)$ ,  $g(1, 2)$ ,  $g(2, 1)$ ,  $g(12, 1)$ ,  $g(12, 2)$ ,  $g(1, 12)$  and  $g(2, 12)$ .  $g(12, 12)$ ,  $g(1, 2)$ ,  $g(2, 1)$ ,  $g(12, 1)$ ,  $g(12, 2)$ ,  $g(1, 12)$  and  $g(2, 12)$ .<br>
f the link costs we know that the network  $g(12, 12)$  is not strongly<br>
e coalition of two manufacturers, or the coalition of two retai For such values of the link costs we know that the network  $q(12, 12)$  is not strongly  $g(12, 12)$  is not strongly<br>palition of two retailers,<br>ss moving to  $g(1,2)$  (or<br>by the proofs of pairwise<br>e to networks  $g(1,0)$  (or<br> $g(0,12)$ ). However, the stable because the coalition of two manufacturers, or the coalition of two retailers, or both coalitions, would have incentives to break two links moving to  $q(1,2)$  (or  $g(1,2)$  (or<br>of pairwise<br> $g(1,0)$  (or<br>wever, the  $g(2,1)$ ). Once  $g(1,2)$  (or  $g(2,1)$ ) has been reached, we know by the proofs of pairwise  $g(2, 1)$ ). Once  $g(1, 2)$  (or  $g(2, 1)$ ) has been reached, we know by the proofs of pairwise<br>and strong stability that no coalition has incentives to move to networks  $g(1, 0)$  (or<br> $g(2, 0)$ ,  $g(0, 1)$ ,  $g(0, 2)$ ),  $g(1,$ and strong stability that no coalition has incentives to move to networks  $g(1,0)$  (or  $g(1,0)$  (or<br>wever, the  $g(2,0), g(0,1), g(0,2), g(1,1)$  (or  $g(2,2)$ ), or  $g(12,0)$  (or  $g(0,12)$ ). However, the  $g(2, 0), g(0, 1), g(0, 2)), g(1, 1)$  (or  $g(2, 2)),$  or  $g(12, 0)$  (or  $g(0, 12)$ ). However, the<br>32 coalition formed by a manufacturer and a retailer would have incentives to form a link between them moving to  $g(12,2)$  (or  $g(12,1)$ ,  $g(1,12)$ ,  $g(2,12)$ ). Then, from  $g(12,2)$ , the coalition  $S = \{M_1, R_2\}$  would like to form the link between them moving to  $g(12, 12)$ . From  $g(12, 2)$  no coalition wants to move to any other network structure.

 $\{1,2\}, k_{(1,2)}^M\} > k > \max\{k_{s(12,1)}^M, k_{s(12,1)}^R\}$ . In this area there is no strongly stable network and we observe a closed cycle formed by all possible networks; i.e., there exists an improving path connecting any two network structures. For such values of the link costs we know that neither the network  $g(12,12)$  nor the network  $g(12,1)$  (or  $g(12,2)$ ,  $g(1,12)$ , and  $g(2,12)$ ) is strongly stable.

 $g(12, 2)$  (or  $g(12, 1)$ ,  $g(1, 12)$ ,  $g(2, 12)$ ). Then, from<br> $M_1, R_2$ ) would like to form the link between them<br> $(1, R_2)$  would like to form the link between them<br> $(2, 2)$  no coalition wants to move to any other networ g(12, 2), the coalition  $S = \{M_1, R_2\}$  would like to form the link between them<br>graving to  $y(12,12)$ . From  $y(12,2)$  are outhout wants to move to any other notwork<br>structure.<br>New min  $\{k_{1,21}^R, k_{1,21}^R\} > k > \max\{k_{1,1$  $g(12, 12)$ . From  $g(12, 2)$  no coalition wants to move to any other network<br>  $\binom{2}{12}$ ,  $k_{121}^2$ ),  $> k > \max\{k_{211}^N, k_{1221}^N\}$ . In this area there is no strongly<br>
with and we observe a closed cycle formed by all po When min{ $k_{(1)}^R$ <br>stable netwo<br>there exists<br>values of th<br> $g(12,1)$  (or<br>For  $k < k_{(1)}^R$ <br>facturers, or<br>to break tw<br>reached, the<br>tives to forr<br>From  $g(12,1)$ <br>moving to  $g$ <br>ously its lin<br>networks  $g$ <br> $g(2,2)$ ,  $g(12)$ <br> $g(12,$  $k_{(1)}^M$  and implies  $2, 2$  ,  $g_1$  and  $k \in \{2, 2, g_2\}$  ,  $g_2$  and  $k \in \{2, 2, g_3\}$  and  $k \in \{2, 2, g_4\}$  and  $k \in \{1, 2, 2\}$  and  $> k > \max\{k_{s(i)}^M\}$ <br>
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tween them mo<br>
lition  $S = \{M_1, \text{ut also coalition} \}$ <br>
(1, also coalit  $\frac{M}{s(12,1)}, \frac{R}{s(s(12,1)}, \frac{R}{s(s(s))})$ <br>osed cycl<br>mecting at neither<br>it neither<br>(2, 12)) is<br>ongly station (or  $g(2,$  manufac<br>noving to<br>(1,  $R_2$ } we on  $S = \{$ <br>ng to  $g(2)$  defeat t<br>ited by on<br>2),  $g(1, 1)$ <br>would lil<br>e  $g(1,$  $g(12, 12)$  nor the network<br>le.<br>ne coalition of two manu-<br>ns, would have incentives<br>1, 2) (or  $g(2, 1)$ ) has been<br>etailer would have incen-<br> $(12, 1)$ ,  $g(1, 12)$ ,  $g(2, 12)$ ).<br>m the link between them<br>d like to sever simul  $g(12, 1)$  (or  $g(12, 2)$ ,  $g(1, 12)$ , and  $g(2, 12)$ ) is strongly stable.<br>For  $k < k_{(12, 21)}^{(12, 12)}$  is not strongly stable because the<br>facturers, or the cosition of two residers, or both cositions;<br>facturers, or the co For  $k < k_{(12,12)}^R$ ,  $k < k_{(1)}^R$ <br>urers, o<br>reak tw<br>hed, th<br>is to form  $g(12)$ ,<br>is to form  $g(12)$ ,<br>is in got its line<br> $2$ ),  $g(12)$ ,<br> $k > k_{(1)}^R$ <br> $k > k_{(2)}^R$ <br>is the nd  $M_2$ <br> $(0, 1, 2)$ ,<br> $(2, 12)$ ,<br> $(2, 12)$ ,<br> $(2, 12)$ ,<br> $(2, 12)$ ,<br> $\therefore$ <br>**fficie**  $g(12, 12)$  is not strongly stable because the coalition of two manu-<br>coalition of two retailers, or both coalitions, would have incentives<br>s moving to  $g(1, 2)$  (or  $g(2, 1)$ ). Once  $g(1, 2)$  (or  $g(2, 1)$ ) has been<br>fiti facturers, or the coalition of two retailers, or both coalitions, would have incentives to break two links moving to  $q(1,2)$  (or  $q(2,1)$ ). Once  $q(1,2)$  (or  $q(2,1)$ ) has been  $g(1, 2)$  (or  $g(2, 1)$ ). Once  $g(1, 2)$  (or  $g(2, 1)$ ) has been<br>by a manufacturer and a retailer would have incen-<br>new into the g(12, 2) (or  $g(12, 1)$ ,  $g(1, 12)$ ,  $g(2, 12)$ ).<br> $= \{M_1, R_2\}$  would like to form the link reached, the coalition formed by a manufacturer and a retailer would have incentives to form a link between them moving to  $g(12,2)$  (or  $g(12,1)$ ,  $g(1,12)$ ,  $g(2,12)$ ). From  $g(12, 2)$ , the coalition  $S = \{M_1, R_2\}$  would like to form the link between them moving to  $g(12, 12)$ , but also coalition  $S = \{M_2, R_1\}$  would like to sever simultaneously its link with  $R_2$  and  $M_1$  moving to  $g(2,0)$ . Once  $g(2,0)$  has been reached, the networks  $g(2,2)$ ,  $g(12,0)$  and  $g(2,1)$  defeat the network  $g(2,0)$ . Next, the networks  $g(2,2), g(12,0)$  and  $g(2,1)$  are defeated by one of the asymmetric networks  $g(12,2)$ .  $g(12,1), g(1,12),$  or  $g(2,12)$ . And so on.

 $g(12, 2)$  (or  $g(12, 1), g(1, 12), g(2, 12)$ ).<br>
uld like to form the link between them<br>  $M_2, R_1$ } would like to sever simultane-<br>
0). Once  $g(2, 0)$  has been reached, the<br>
ne network  $g(2, 0)$ . Next, the networks  $g(12, 2)$  $g(12, 2)$ , the coalition  $S = \{M_1, R_2\}$  would like to form the link between them<br>g to  $g(12, 12)$ , but also coalition  $S = \{M_2, R_1\}$  would like to sever simultanes<br>its link with  $H_2$  and  $M_1$  moving to  $g(2, 0)$ . Onc  $g(12, 12)$ , but also coalition  $S = \{M_2, R_1\}$  would like to sever simultane-<br>ik with  $R_2$  and  $M_1$  moving to  $g(2, 0)$ . Once  $g(2, 0)$  has been reached, the<br>2, 2),  $g(12, 0)$  and  $g(2, 1)$  deread the network  $g(2, 0)$ onsly its link with  $H_2$  and  $M_1$  moving to  $g(2, 0)$ . Once  $g(2, 0)$  has been reached, the<br>networks  $g(2, 2)$ ,  $g(12, 0)$  and  $g(2, 1)$  decat the network  $g(2, 0)$ . Next, the networks<br> $g(12, 1)$ ,  $g(1, 12)$ , or  $g(2, 1$  $g(2, 2)$ ,  $g(12, 0)$  and  $g(2, 1)$  defeat the network  $g(2, 0)$ . Next, the networks  $2(2, 0)$  and  $g(2, 1)$  are defeated by one of the asymmetric networks  $g(12, 2)$ ,  $g(1, 2)$ ,  $M$ s ot on.<br>  $V_1^5(1, 2, 1)$  or  $g(1, 2,$  $g(2, 2)$ ,  $g(12, 0)$  and  $g(2, 1)$  are defeated by one of the asymmetric networks  $g(12, 2)$ ,  $g(12, 1)$ ,  $g(1, 12)$ , or  $g(2, 12)$ . And so on.<br>  $g(12, 1)$ ,  $g(12, 1)$ ,  $g(12, 1)$ ,  $(g(12, 2)$ ,  $g(1, 1)$ ,  $(g(12))$ , is not  $g(12,1)$ ,  $g(1, 12)$ , or  $g(2, 12)$ . And so on.<br>For  $k > k_{(12,12)}^R$ ,  $g(12,1)$  (or  $g(12,2)$ ,  $g$ <br>because the coalition  $S = \{M_1, R_1\}$  wou<br> $R_2$  and  $M_2$  moving to  $g(1,0)$ . Once  $g(1$ <br> $g(12,0)$  and  $g(1,2)$  defeat the  $k > k_{(1)}^R$ <br>use the<br>ind  $M_2$ <br>,0) and<br> $g(1,2)$  a<br> $(g(1,2)$ .<br> $(2,12)$ .<br> $(2,12)$ .<br>ifficient<br>ully change follow<br>signal contract  $3$ <br> $\frac{1}{2}$  follow<br> $d \leq 0$ For  $k > k_{(12,12)}^R$ ,  $g(12, 1)$  (or  $g(12, 2)$ ,  $g(1, 12)$ , and  $g(2, 12)$ ) is not strongly stable<br>ion  $S = \{M_1, R_1\}$  would like to sever simultaneously its link with<br>g to  $g(1, 0)$ . Once  $g(1, 0)$  has been reached, the networks  $g(1, 1)$ ,<br>2 because the coalition  $S = \{M_1, R_1\}$  would like to sever simultaneously its link with  $S = \{M_1, R_1\}$  would like to sever simultaneously its link with<br>  $g(1,0)$ . Once  $g(1,0)$  has been reached, the networks  $g(1,1)$ ,<br>  $g$ feat the network  $g(1,0)$ . Next, the networks  $g(1,1)$ ,  $g(12,0)$ <br>
d by one of the asy  $R_2$  and  $M_2$  moving to  $g(1,0)$ . Once  $g(1,0)$  has been reached, the networks  $g(1,1)$ ,  $R_2$  and  $M_2$  moving to  $g(1,0)$ . Once  $g(1,0)$  has been reached, the networks  $g(1,1)$ ,  $g(12,0)$  and  $g(1,2)$  defeat the network  $g(1,0)$ . Next, the networks  $g(1,1)$ ,  $g(12,0)$  and  $g(1,2)$  are defeated by one of the  $g(12,0)$  and  $g(1,2)$  defeat the network  $g(1,0)$ . Next, the networks  $g(1,1)$ ,  $g(12,0)$ g(12, 0) and  $g(1, 2)$  defeat the network  $g(1, 0)$ . Next, the networks  $g(1, 1)$ ,  $g(12, 0)$  and  $g(1, 2)$  are defeated by one of the asymmetric networks  $g(12, 2)$ ,  $g(12, 1)$ ,  $g(1, 12)$ , or  $g(2, 12)$  is defeated by a and  $g(1,2)$  are defeated by one of the asymmetric networks  $g(12,2), g(12,1), g(1,12)$ .  $g(1, 2)$  are defeated by one of the asymmetric networks  $g(12, 2), g(12, 1), g(1, 12),$ <br>  $(2, 12)$ . But the networks  $g(1, 1)$  and  $g(12, 0)$  are also defeated by  $g(12, 12)$ , and<br>  $(1, 12)$  is defeated by any of the asymmetr or  $g(2, 12)$ . But the networks  $g(1, 1)$  and  $g(12, 0)$  are also defeated by  $g(12, 12)$ , and  $g(2, 12)$ . But the networks  $g(1, 1)$  and  $g(12, 0)$  are also defeated by  $g(12, 12)$ , and  $2, 12$ ) is defeated by any of the asymmetric networks  $g(12, 2)$ ,  $g(12, 1)$ ,  $g(1, 12)$ ,  $g(2, 12)$ . And so on.<br> **Efficient dis**  $g(12,12)$  is defeated by any of the asymmetric networks  $g(12,2), g(12,1), g(1,12)$ .  $g(12, 12)$  is defeated by any of the asymmetric networks  $g(12, 2)$ ,  $g(12, 1)$ ,  $g(1, 12)$ ,<br>or  $g(2, 12)$ . And so on.<br>Efficient distribution networks<br>st fully characterize the efficient distribution networks. See Figure or  $g(2, 12)$ . And so on.<br>
B.1.3 Efficient distribution networks

#### Efficient distribution networks

We first fully characterize the efficient distribution networks. See Figure 6. We first fully check that the proposition 7

 $g(2, 12)$ . And so on.<br> **Efficient distribut**<br>
fully characterize the tition 7 The efficien<br>
as follows<br>  $0 < d \leq 0.2156$  and **Proposition 7** The efficient distribution network with positive link costs depends on both Proposition 7 *T*<br>d and k as follows<br>a)  $For 0 < d \leq 0.2$  $d$  and  $k$  as follows  $s$  follows<br>  $\langle d \leq 0.2156 \text{ and}$ 

a) For  $0 < d \le 0.2156$  and

**a.1)** 
$$
0 < k < k^B(d) < \frac{(a-c)^2}{36}
$$
 is  $g(12, 12)$   
\n**a.2)**  $0 < k^B(d) < k < \frac{(a-c)^2}{36}$  is  $g(1, 2)$   
\nfor  $0.2156 < d \le \frac{2}{3}$  and

 $\overline{a}$ .

\n- (a) For 
$$
0.2156 < d \leq \frac{2}{3}
$$
 and
\n- (b.1)  $0 < k < k^A(d) < k^C(d) < \frac{(a-c)^2}{36}$  is  $g(12, 12)$
\n- (c)  $k^A(d) < k < k^C(d) < \frac{(a-c)^2}{36}$  is  $g(12, 1)$
\n- (d)  $k^A(d) < k^C(d) < k < \frac{(a-c)^2}{36}$  is  $g(1, 2)$
\n- (e) For  $\frac{2}{3} < d \leq 0.735$  and
\n

 $\ddot{\phantom{0}}$ 

For 
$$
\frac{2}{3} < d \le 0.735
$$
 and  
\nc.1)  $0 < k < k^A(d) < k^E(d) < \frac{(a-c)^2}{36}$  is  $g(12, 12)$   
\nc.2)  $0 < k^A(d) < k < k^E(d) < \frac{(a-c)^2}{36}$  is  $g(12, 1)$   
\nc.3)  $0 < k^A(d) < k^E(d) < k < \frac{(a-c)^2}{36}$  is  $g(12, 0)$ 

d) For  $0.735 < d \le 0.863$  and

$$
or\ 0.735 < d \le 0.863 \ and
$$
\n
$$
d.1) \ 0 < k < k^E(d) < \min\{\frac{(a-c)^2}{36}, \frac{(1-d)(a-c)^2}{3(1+d)(2-d)^2}\} \text{ is } g(12, 1)
$$
\n
$$
d.2) \ 0 < k^E(d) < k < \min\{\frac{(a-c)^2}{36}, \frac{(1-d)(a-c)^2}{3(1+d)(2-d)^2}\} \text{ is } g(12, 0)
$$
\n
$$
or\ 0.863 < d < 1 \text{ and } 0 < k < \frac{(1-d)(a-c)^2}{3(1+d)(2-d)^2} \text{ is } g(12, 0)
$$
\n
$$
for \text{ Proposition 7 and Proposition 4 (main text)}
$$

### Proof of Proposition 7 and Proposition 4 (main text).

< k < k<sup>B</sup>(d) < k <  $\frac{(a-c)^2}{36}$ <br>
< k<sup>B</sup>(d) < k <  $\frac{(a-c)^2}{36}$ <br>
6 < d ≤  $\frac{2}{3}$  and<br>
< k < k<sup>A</sup>(d) < k < k<sup>C</sup>(d)<br>
< k<sup>A</sup>(d) < k ≤ k<sup>C</sup>(d)<br>
<  $\frac{(a-c)^2}{36}$  is  $\frac{(1-d)(a-c)^2}{3(1+d)}$ ,  $\frac{(1-d)(a-c)^2}{3(1+d)(2-d)}$  sition 4 <  $k^B(d) < k < \frac{(a-c)^2}{36}$ <br>
6 <  $d \leq \frac{2}{3}$  and<br>  $k < k^A(d) < k^C(d)$ <br>  $k^A(d) < k^C(d)$ <br>  $k^A(d) < k^C(d) < k^C(d)$ <br>  $k^A(d) < k^C(d) < k^C(d)$ <br>  $k^A(d) < k^C(d) < k^E(d)$ <br>  $k^A(d) < k^A(d) < k^E(d)$ <br>  $k^A(d) < k^A(d) < k^E(d) < k^A(d) < k^A(d) < k^A(d) < k^A(d) < k^B(d) < k^A(d) < k$  $\frac{(a-c)^2}{36}$ <br>  $\frac{(1-d)(a-c)}{36}$ <br>  $\frac{(1-d)(a-c)}{3(1+d)(2-c)}$ <br>
sition<br>  $>0$  ii<br>  $\frac{(1-d)(c-c)}{3(1+d)}$ <br>  $g(1, 1)$ <br>
me nur<br>  $g(1, 2$ **a.2)**  $0 < k^B(d)$ <br>  $For 0.2156 < d \leq \frac{2}{3}$ <br> **b.1)**  $0 < k < k^A(d)$ <br> **b.2)**  $0 < k^A(d)$ <br> **b.3)**  $0 < k^A(d)$ <br> **c**)  $For \frac{2}{3} < d \leq 0.75$ <br> **c.1)**  $0 < k < k^A(d)$ <br> **c.2)**  $0 < k^A(d)$ <br> **c.2)**  $0 < k^A(d)$ <br> **c.3)**  $0 < k^A(d)$ <br> **c.3)**  $0 < k^A(d)$ <br> **c.3** < k < k<sup>A</sup>(d) < k<sup>C</sup>(d) < k<sup>a</sup>=0<sup>2</sup><br>
< k<sup>A</sup>(d) < k < k<sup>C</sup>(d) <  $\frac{(a-c)^2}{36}$ <br>
< k<sup>A</sup>(d) < k < k<sup>C</sup>(d) < k <  $\frac{(a-c)^2}{36}$ <br>
< k<sup>A</sup>(d) < k<sup>C</sup>(d) < k <  $\frac{(a-c)^2}{36}$ <br>
< k < k<sup>A</sup>(d) < k<sup>C</sup>(d) < k <  $\frac{(a-c)^2}{36}$ <br>
< k < k<sup>A</sup>(  $g(12, 12)$ <br>  $g(12, 1)$ <br>  $g(12, 1)$ <br>  $g(12, 12)$ <br>  $g(12, 1)$ <br>  $g(12, 0)$ <br>  $\frac{a-c)^2}{(2-a)^2}$  } is <br>  $\frac{a-c)^2}{(2-a)^2}$  } is <br>  $\frac{a-c)^2}{(2-a)^2}$  } is <br>  $\frac{(a-c)^2}{(2-a)^2}$  } since<br>  $\left(\frac{(a-c)^2}{a}\right)$ <br>  $\frac{2}{(1+a)^2}$ <br>  $\frac{(11-18d-18d)}{$  $k < k^A(d) < k < k^C(d) < \frac{(a-c)^2}{36}$ <br>  $k < k^A(d) < k^C(d) < k < \frac{(a-c)^2}{36}$ <br>  $k < k^A(d) < k^C(d) < \frac{(a-c)^2}{36}$ <br>  $k < k^A(d) < k < k^B(d) < \frac{(a-c)^2}{36}$ <br>  $k < k^A(d) < k < k^E(d) < \frac{(a-c)^2}{36}$ <br>  $k < k^A(d) < k^E(d) < k < \frac{(a-c)^2}{36}$ <br>  $k < k^E(d) < \min\{\frac{(a-c)^2}{36}, \frac{(a-c)^2}{36}\}$  $g(1, 2)$ <br>  $g(1, 2)$ <br>  $g(12, 1)$ <br>  $g(12, 0)$ <br>  $\frac{(a-c)^2}{(2-a)^2}$ <br>  $\frac{(a-c)^2}{(2-a)^2}$ <br>  $\frac{(a-c)^2}{(2-a)^2}$ <br>  $\frac{(a-c)^2}{(a-c)^2}$ <br>  $\frac{(a-c)^2}{(a-c)^2}$ <br>  $\frac{(a-c)^2}{(a-c)^2}$ <br>  $\frac{(a-c)^2}{(a-c)^2}$ <br>  $\frac{(a-c)^2}{(a-c)^2}$ <br>  $\frac{(a-c)^2}{(a-c)^2}$ <br>  $\frac{(a-c)^2}{(a-c)^$ < k<sup>A</sup>(d) < k<sup>C</sup>(d) < k <  $\frac{(a-0)^2}{36}$ <br>
< d ≤ 0.735 and<br>
< k < k<sup>A</sup>(d) < k<sup>E</sup>(d) <  $\frac{(a-e)^2}{36}$ <br>
< k<sup>A</sup>(d) < k < k<sup>E</sup>(d) <  $\frac{(a-e)^2}{36}$ <br>
< k<sup>A</sup>(d) < k < k<sup>E</sup>(d) < k <  $\frac{(a-e)^2}{36}$ <br>
< k<sup>A</sup>(d) < k ≤ k<sup>E</sup>(d) < k <  $\frac{($  $g(12, 1)$ <br>  $g(12, 1)$ <br>  $g(12, 1)$ <br>  $g(12, 0)$ <br>  $\frac{a-c)^2}{(2-a)^2}$ <br>  $\frac{a-c}{(2-a)^2}$ <br>  $\frac{a-c}{(2-a)^2}$ <br>  $\frac{a}{(2-a)^2}$ <br>  $\frac{(a-c)^2}{(2-a)^2}$ <br>  $\frac{1}{(2-a)^2}$ <br>  $\frac{(11-1)^2}{2}$ <br>  $\frac{d(-2+1)^2}{2}$ <br>  $\Phi(g(12, 9)(12, 9)(12, 9)(12, 9)(12, 9)(12, 9)(1$  $k^A(d) < k^C$ <br>  $d \leq 0.735$  and<br>  $d \leq k^A(d) < k^C$ <br>  $d \leq k^A(d) < k^C$ <br>  $d \leq k^A(d) < k^E$ <br>  $d \leq 0.863$  an<br>  $d \leq k^C(d) < k^C$ <br>  $d \leq 0.863$  an<br>  $d \leq k^C(d) < k^C$ <br>  $d \leq 1$  and  $0$ <br>  $d \leq k^E(d) < k^C$ <br>  $d \leq 1$  and  $0$ <br>  $d \leq k^C(d)$ <br>  $d \leq$ < k < k<sup>A</sup>(d) < k<sup>E</sup>(d) < k<sup>E</sup>(d) <  $\frac{(a-c)^2}{36}$ <br>
< k<sup>A</sup>(d) < k < k<sup>E</sup>(d) <  $\frac{(a-c)^2}{36}$ <br>
< k<sup>A</sup>(d) < k ≤ (d) < k <  $\frac{(a-c)^2}{36}$ <br>
i < d ≤ 0.863 and<br>
< k < k<sup>E</sup>(d) < min{ $\frac{(a-c)^2}{36}$ ,  $\frac{(a-c)^2}{36}$ <br>
< k < k<sup>E</sup>(d) < mi g(12, 12) < k<sup>A</sup>(d) < k < k<sup>E</sup>(d) < k <sup>∈</sup> c<sup>0</sup> = <sup>(a</sup> = 0<sup>2</sup>)<br>
< k<sup>A</sup>(d) < k<sup>E</sup>(d) < k <  $\frac{(a-c)^2}{36}$ <br>
i < d ≤ 0.863 and<br>
< k < k<sup>E</sup>(d) < min{ $\frac{(a-c)^2}{36}$ ,  $\frac{(10-c)^2}{36}$ ,  $\frac{(10-c)^2}{36}$ ,  $\frac{(10-c)^2}{36}$ ,  $\frac{(10-c)^2}{36}$ ,  $\frac{(10-c$  $g(12, 1)$ <br>  $g(12, 0)$ <br>  $\frac{(a-c)^2}{(2-d)^2}$  }<br>  $\frac{(a-c)^2}{(2-d)^2}$  }<br>  $is \ g(12)$ <br>
(main  $\lt \frac{(a-c)^2}{(a-c)^2}$  } sin<br>  $\lt \frac{(a-c)^2}{a^2}$  } sin<br>
Therefo<br>  $=\frac{(11-18)}{1}$ <br>
er of lir<br>  $\frac{d(-2+3)}{2(1)}$ <br>
efore, w<br>  $\Phi(g(12, 1))$  $k^A(d) < k^E(d) < k < \frac{(a-c)^2}{36}$ <br>  $i < d \le 0.863$  and<br>  $k < k^E(d) < \min\left\{\frac{(a-c)^2}{36}, \frac{1}{36}\right\}$ <br>  $k < k^E(d) < k < \min\left\{\frac{(a-c)^2}{36}, \frac{1}{36}\right\}$ <br>  $k < d < 1$  and  $0 < k < \frac{(1-d)(a-c)^2}{3(1+d)(2)}$ <br> **oposition 7 and Propositio**<br>  $\Phi(g(12,0)) - \Phi(g(1,0)) > 0$  $\frac{(a-c)^2}{(2-d)^2}$  }<br> $\frac{(a-c)^2}{(2-d)^2}$  }<br> $is \ g(12)$ <br> $(s \ g(12)$ <br> $(s \ g(12)$ <br> $\frac{(a-c)^2}{(d)^2}$  } sin<br>Therefo<br> $= \frac{(11-18)}{1}$ <br> $= \frac{d(-2+3)}{2(1)}$ <br> $= 6$  for ey  $\Phi(g(12)$ <br> $= g(12, 1)$ **c.3)**  $0 < k^A(d) < k^E(d)$ <br>
For  $0.735 < d \le 0.863$  and<br> **d.1)**  $0 < k < k^E(d) < 1$ <br> **d.2)**  $0 < k^E(d) < k < 1$ <br> **)** For  $0.863 < d < 1$  and  $0 <$ <br> **roof of Proposition 7** and<br>
first note that  $\Phi(g(12,0)) - 4$ <br>
on on k milder than  $\overline{k} \equiv \text{min$  $k < k^E(d) < \min\{\frac{(a-c)^2}{36}\}\n$   $\lt k^E(d) < k < \min\{\frac{(a-c)^2}{36}\}\n$   $\lt d < 1 \text{ and } 0 < k < \frac{(1-q)^2}{3(1+1)}\n$ oposition 7 and Proposition 7 ( $\text{d}P(q(12,0)) - \Phi(q(1,0))$ ) is der than  $\overline{k} \equiv \min\{\frac{(a-c)^2}{36}, \frac{(a-c)^2}{36}\}\n$ solow that ,  $\frac{(1-d)(a-c)^2}{3(1+d)(2-d)}$ ,<br>
,  $\frac{(1-d)(a-c)^2}{3(1+d)(2-d)}$ ,<br>
,  $\frac{(1-d)(a-c)^2}{4(2-d)}$  is g<br>  $\frac{(1-d)(a-c)^2}{4(2-d)^2}$  is g<br>
tion 4 (ma<br>
> 0 if k <  $\frac{(a-c)^2}{1+d(2-d)^2}$ } is<br>
tive. Therefore<br>
1, 1)) =  $\frac{d(-2)}{1+d(2)}$ . Therefore<br>
inates  $\Phi(g(1$  $g(12, 1)$ <br>  $g(12, 0)$ <br>  $\Rightarrow$ **xt)**.<br>  $\frac{2-16d+9}{2-d}$ <br>  $\frac{2-16d+9}{4d}$ <br>  $\Phi(g(12, 12d^2-4d^3))$ <br>  $\Rightarrow$  and  $(1$ <br>  $\frac{16-14d-2}{(4-d)^2(2)}$ <br>
will pro<br>
) and tl<br>
or  $g(12)$  $k \leq k^E(d) < k < \min\{\frac{(a-c)^2}{36}\}$ <br>  $d < d < 1$  and  $0 < k < \frac{(1-d)^2}{3(1+d)}$ <br>
oposition 7 and Proposition  $\Phi(g(12,0)) - \Phi(g(1,0))$ <br>
der than  $\overline{k} \equiv \min\{\frac{(a-c)^2}{36}, \frac{(a-c)^2}{36}\}$ <br>
now that  $\Phi(g(12,0)) - \Phi(g(10))$ <br>
now that  $\Phi(g(12,0)) - \Phi(g(10))$ <br> ,  $\frac{(1-d)(a-c)^2}{3(1+d)(2-d)}$ <br>  $\frac{(b)(a-c)^2}{d)(2-d)^2}$  is g<br>
tion 4 (ma<br>
> 0 if k <  $\frac{(d-a)(a-c)^2}{1+d)(2-d)^2}$ } s<br>
tive. There<br>
1, 1)) =  $\frac{(11-c)^2}{(1+d)(2-d)^2}$ <br>
∴ Therefore<br>
inates  $\Phi(g(1,2))$ , or g(1:<br>
34 xt).<br>
xt).<br>  $\frac{(-16d+9d)}{(d-1)^2(1+d)}}$ <br>
he diffe<br>  $\Phi(g(12, \frac{2d^2-4d^3)}{(d-1)^2(2-d)^2)}$ <br>
and (11<br>  $\frac{6-14d-3d}{(d-1)^2(2-d)}$ <br>
and th<br>
or  $g(12, \frac{d}{2})$ **d.2)**  $0 < k^E(d) < k < \min\left\{\frac{(a-c)^2}{36}, \frac{(1-d)}{3(1+d)(2-d)}\right\}$ <br>
For 0.863 <  $d < 1$  and  $0 < k < \frac{(1-d)(a-c)^2}{3(1+d)(2-d)}$ <br> **Proof of Proposition 7 and Proposition** 4<br>
first note that  $\Phi(g(12,0)) - \Phi(g(1,0)) > 0$  if *l* on on *k* milder than  $\frac{(a-c)^2}{36}, \frac{(1-d)(a-c)^2}{3(1+d)(2-d)^2}$  is<br>
3(1+d)(2−d)<sup>2</sup> is<br> **position 4 (r**<br>
(0)) > 0 if k <  $g(12, 0)$ <br> **ain te**<br>  $\frac{(a-c)^2(12)}{32(2)}$ <br>
since t<br>
refore  $\frac{1-18d+12}{18(1+1)}$ <br>
of links<br>  $\frac{-2+3d}{2(1+d)}$ <br>
e, we w<br>  $g(12, 0)$ <br>  $12, 1$ , c First note that Φ(g(12, 0)) − Φ(g(1, 0)) > 0 if k < (a−c)2(12−16d+9d2−3d3)  $\frac{2(12-16a+9a^2-3a^2)}{32(2-d)^2(1+d)}$  which is a condi-<br>
ce the difference  $(12-16d+9d^2-$ <br>
re  $\Phi(g(12,0)) > \Phi(g(1,0))$  for all<br>  $\frac{d+12d^2-4d^3}{8(1+d)(2-d)^2} > 0$  for all k since tion on k milder than  $k \equiv \min\{\frac{(a-\epsilon)}{36}\}$ 

k milder than  $\overline{k} \equiv \min\left\{\frac{(a-c)^2}{36} - \frac{(a-c)^2}{-d^2(1+d)} - \frac{(a-c)^2}{36}\right\}$  is always  $\left[\cdot\right]$ .<br>we show that  $\Phi(g(12,0)) - \Phi$ <br>tribution networks have the s<br>d.<br>he difference  $\Phi(g(12,0)) - \Phi$ <br>k and positive as long as d:<br> $0 < d < \frac{2$ ,  $\frac{(1-d)(a-c)^2}{3(1+d)(2-d)}$ ;<br>  $\cositive$ . Tl<br>  $(g(1,1)) = \frac{3}{2}$ <br>
ame number<br>  $(g(1,2)) = \frac{2}{3}$ . Theref<br>
lominates  $\Phi$ <br>  $g(1,2)$ , or  $\frac{2}{3}$ . ( $\frac{(1-2)(a-c)^2}{3(1+d)(2-d)^2}$ ) since the difference  $(12-16d+9d^2-3d)(d-16d+9d^2-3d)(d-16d+16d^2-4d^2)(d-16d^2-4d^2)(d-16d^2-4d^2)(d-16d^2-4d^2)(d-16d^2-4d^2) > 0$  for all k since me number of links and  $(11-18d+12d^2-4d^3) > 0$ <br>((1  $3d^3$ )  $\frac{(a-c)^2}{32(2-d)^2(1)}$ <br>  $k \in [0, \overline{k}]$ .<br>
Second, we si<br>
both distribu<br>
for  $1 > d$ .<br>
Third, the di<br>
dent of k ancases:<br>
case a) 0 < distribu  $\frac{(a-c)^2}{32(2-d)^2(1+d)} - \frac{(a-c)^2}{36}$ <br>
[0,  $\overline{k}$ ].<br>
Ind, we show that  $\Phi$ (<br>
distribution network<br>
> d. <sup>26</sup> is always positive. Therefore  $\Phi(g(12,0)) > \Phi(g(1,0))$  for all  $\Phi(g(12,0)) - \Phi(g(1,1)) = \frac{(11-18d+12d^2-4d^3)(a-c)^2}{18(1+d)(2-d)^2} > 0$  for all *k* since vorks have the same number of links and  $(11-18d+12d^2-4d^3) > 0$ <br> $\Phi(g(12,0))$  $k \in [0, k]$ .<br>Second, w<br>both disti<br>for  $1 > d$ .<br>Third, th<br>dent of  $k$ <br>cases:<br>case a) (dist 18(1+*d*)(2−*d*)<sup>2</sup><br>nks and (11<br> $\frac{3d}{(16-14d-3)}$ <br> $\frac{1+d}{(4-d)^2(2-4d)}$ <br>we will prod for  $1 > d$ .

Second, we show that  $\Phi(g(12, 0)) - \Phi(g(1, 1)) = \frac{(11-18d+12d^2-4d^2)(a-c)^2}{18(1+d)(2-d)^2}$ <br>both distribution networks have the same number of links and  $(11 - 18$ <br>for  $1 > d$ .<br>Third, the difference  $\Phi(g(12, 0)) - \Phi(g(1, 2)) = \frac{d(-2+3d)(16 > 0$  for all k since<br>  $(4+12d^2-4d^3) > 0$ <br>  $\frac{((a-c)^2)}{d^2}$  is indepen-<br>
y considering two<br>
e possible efficient<br>
e possible efficient both distribution networks have the same number of links and  $(11 - 18d + 12d^2 - 4d^3) > 0$ <br>for  $1 > d$ .<br>Third, the difference  $\Phi(g(12, 0)) - \Phi(g(1, 2)) = \frac{d(-2+3d)(16-14d-3d^2+2d^3)(a-c)^2}{2(1+d)(4-d^2(2-d)^2(2-d)^2)^2+ d^2}$  is independent of  $> d.$ <br>  $\downarrow$ , th of k<br>  $\downarrow$ <br>  $\downarrow$ Third, the difference  $\Phi(g(12, 0)) - \Phi(g(1, 2)) = \frac{d(-2+3d)(16-14d-3d^2+2d^3)(a-c)^2}{2(1+d)(4-d)^2(2-d)^2(2+d)^2}$ <br>dent of k and positive as long as  $d > \frac{2}{3}$ . Therefore, we will proceed by con<br>cases:<br>**case a)**  $0 < d < \frac{2}{3}$  where  $\Phi(g($ Third, the difference  $\Phi(g(12,0)) - \Phi(g(1,2)) = \frac{d(-2+3d)(16-14d-3d^2+2d^3)(a-c)^2}{2(1+d)(4-d)^2(2-d)^2(2+d)^2}$  is independent of k and positive as long as  $d > \frac{2}{3}$ . Therefore, we will proceed by considering two cases: cases:

k and positive as long as  $d > \frac{2}{3}$ <br>  $0 < d < \frac{2}{3}$  where  $\Phi(g(1, 2))$  dom<br>
stribution networks are either  $g($  $\frac{2}{3}$ . Therefore, we will proceed by considering two<br>ninates  $\Phi(g(12,0))$  and then, the possible efficient<br>(1,2), or  $g(12,1)$ , or  $g(12,12)$ .<br>34  $\overline{a}$  $\langle d \rangle \leq \frac{2}{3}$ <br>bution 1  $\frac{2}{3}$  where  $\Phi(g(1, 2))$  dominates  $\Phi(g(12, 0))$  and then, the possible efficient networks are either  $g(1, 2)$ , or  $g(12, 1)$ , or  $g(12, 12)$ .<br>34 distribution networks are either  $g(1,2)$ , or  $g(12,1)$ , or  $g(12,12)$ .  $g(1, 2)$ , or  $g(12, 1)$ , or  $g(12, 12)$ .<br>34

- case b)  $\frac{2}{3}$  $\overline{a}$ distribution networks are either  $q(12,0)$ , or  $q(12,1)$ , or  $q(12,12)$ .  $\frac{1}{\sin \theta}$  and  $\frac{1}{\sin \theta}$
- $0 < d < \frac{2}{3}$  and  $0 \le k < \frac{(a 6)}{36}$
- c d 1 where  $\Phi(g(1,2))$  dominates  $\Phi(g(1,2))$  and the possible efficient<br>  $x \, d \leq 1$  where  $\Phi(g(1,0), \alpha \cdot g(12,1), \alpha \cdot g(12,12)$  is more  $g(12,12)$  is more  $g(12,12)$  by the  $g(12,12)$  by the  $g(12,12)$  by the  $g(12,12)$  by th  $g(12, 0)$ , or  $g(12, 1)$ , or  $g(12, 12)$ .<br>  $\therefore$ <br>  $g(10, 24)$ . The distribution network  $\frac{80-1024d+14d^8+128d^4+14d^8+128d^4+14d^8+128d^4+14d^8+128d^4+14d^8+128d^4+14d^8+128d^4+128d^4+128d^4+128d^4+128d^4+128d^4+128d^4$  $d < d < \frac{2}{3}$ <br>
ompare<br>  $d = 4$ <br>  $d = 2$ <br>
ompare<br>  $d = 4$ <br>  $d = 4$ <br>  $e = 4$ <br>  $e = 4$ <br>  $f = 4$ <br>  $g = 2$ <br>  $g = 2$ <br>  $h = 2$  $k < \frac{(a-c)^2}{36}$ <br>with  $g(1$ <br>if  $\frac{(1-d)(12)}{36}$ <br>with  $g(12)$ <br>y when  $\frac{72}{72}$ <br>> k. We<br> $k^A(d)$  is po<br> $k^A(d)$  is po<br> $k^A(d)$ <br> $\Phi(g(12, 12))$  is and  $0 < d$ <br>with  $g(1, 2)$ <br>and  $0 < d$ <br>with  $g(1, 2)$ <br> $\Rightarrow \Phi(g(1, 2))$ <br> $\Rightarrow \Phi(g(1, 2))$ <br> $\Rightarrow \$ . **a.1)** We compare  $g(12, 12)$  with  $g(12, 1)$ . The distribution network  $g(12, 12)$  is more  $g(12, 12)$  with  $g(12, 1)$ . The distribution network  $g(12, 12)$  is more  $\frac{(\sqrt{12}, 1)^2}{86(15\sqrt{3} - 2\sqrt{3})}$  ( $\frac{(\sqrt{12}, 1)^2}{8(15\sqrt{3} - 2\sqrt{3})}$  ( $\frac{(\sqrt{12}, 1)^2}{8(15\sqrt{3} - 2\sqrt{3})}$  is more  $\frac{1}{2}$  ( $\frac{(\sqrt{12}, 1)^2}{8(15\$ efficient than  $g(12, 1)$  if  $\frac{(1-d)(1280-1024d-1088d^2+224d^3-212d^4+44d^5+155d^6)(a-c)^2}{36(1+d)(2-d)^2(16-7d^2)^2}$  - 2k is positive, or equivalently when  $\frac{(1-d)(a-c)^2}{72(1+d)(2-d)^2(16-7d^2)^2}$  (1280 - 1024d - 1088d<sup>2</sup> + 224d<sup>3</sup>  $g(12, 1)$  if  $\frac{(1-d)(1280-10842-12847-21867+15667)(2-67924-168867-16216467-16216867-16216867-16216867-16216867-16216867-16216867-16216867-16216867-16216867-16216867-16216867-1621688-1621688-1621688-1621688-1621688-1621688$ positive, or equivalently when  $\frac{(1-d)(a-c)^2}{72(1+d)(2-d)^2(16-7d^2)^2}(1280-1024d-1088d^2+224d^3-212d^4+44d^5+155d^6) > k$ . We denote the left part of the previous expression by  $k^A(d)$ . It follows that  $k^A(d)$  is positive for  $0 <$ (ii)  $\Phi(g(12,1)) > \Phi(g(12,12))$  if and only if  $k^A(d) < k < \frac{(a-c)^2}{36}$ . Thus, for the particular case,  $k = 0$  and  $0 < d < \frac{2}{3}$ , we have  $\Phi(g(12, 12)) > \Phi(g(12, 1)).$
- 1008<sup>a-1</sup>/2140<sup>2</sup>-114a<sup>4</sup> +14a<sup>4</sup> +14a<sup>4</sup> +14a<sup>4</sup> +14a<sup>4</sup> +14a<sup>4</sup> +14a<sup>4</sup> +14a<sup>4</sup> +14a<sup>4</sup> +16b (a-td)<sup>2</sup><br>(a-td)<sup>2</sup> +14a<sup>4</sup> +14a<sup>4</sup> +14a<sup>4</sup> +16a<sup>4</sup> +224d<sup>3</sup> 46a<sup>4</sup> +224d<sup>3</sup> 46a<sup>4</sup> +224d<sup>3</sup> 46a<sup>4</sup> +16a<sup>4</sup> +16a<sup>4</sup> +1 72(1+d)(2−d)2=d)(16−7d2)2 (1280 – 1024d – 1088d<sup>2</sup> + 224d<sup>2</sup> – 6<br>
(ed chote the left part of the previous expression by<br>
(ed chote the left part of the previous expression by<br>
(of the first of the protous expression by 212d<sup>2</sup> + 44d<sup>2</sup> - 155d<sup>2</sup> > k. We denote the left part of the products expression by<br>  $212d^2 + 44d^3 - 15d^2$  ( $\frac{1}{2}$ , We denote the left part of the products of the previous expression by<br>
1. We also have that  $0 < k^4(d$  $k^A(a)$ . It follows that  $k^A(a)$  is positive for  $0<sup>2</sup> \leq d < 0.735$  and negative for 0.735  $< d < 3$ . We show that  $0 < k \leq 1$ . We show that  $0 < k \leq 4$  conclusion follows: (i)  $\Phi(q/(2, 2)) > \Phi(q/(2, 2))$  if and only if  $a^A(a) < k$  $\langle k^A(d) \rangle \frac{(a-c)^2}{36}$ <br>  $\Phi(g(12, 12)) > \Phi$ <br>  $\Phi(g(12, 12)) > \Phi$ <br>  $\Phi(g(12, 12))$  if and o<br>  $\Phi(d) < d < \frac{2}{3}$ , we<br>  $\Phi(g(1, 2))$  if and<br>  $\Phi(g(1, 2))$ . W  $d < d < \frac{2}{3}$ <br>if and c<br>if and c<br> $d < k$ <br> $g(12, 12)$ <br> $g(12, 12)$ <br> $\therefore$  the sar<br> $\leq k < k$ <br>where  $k^2$ <br>is posit:<br> $\frac{(a-c)^2}{36}$  for that<br>plies the plies the plies the complex of  $g\left(\frac{g(12, 1)}{2(2+d)^2(1)}\right)$ <br> $\frac{g(12, 1)}$ conclusion follows: (i)  $\Phi(g(12,12)) > \Phi(g(12,1))$  if and only if  $0 \le k < k^A(d)$  and  $\Phi(g(2,1)) > \Phi(g(12,1)) > \Phi(g(12,1))$  if end only if  $\Phi(\alpha|l \le k, k \le \frac{100}{340})$ . Then, for the period period of  $d \le \frac{1}{3}$ , we have  $\Phi(g(12,12)) > \Phi(g$ (ii)  $\Phi(g(12, 1)) > \Phi(g(12, 12))$  if and only if  $k^A(d) < k < \frac{(a-6)^2}{360}$ <br>particular case,  $k = 0$  and  $0 < d < \frac{2}{3}$ , we have  $\Phi(g(12, 12)) > \Phi(g(12, 12))$  if and only if  $0 < |g(12)$ ,  $k = 0$  and  $0 < d < \frac{2}{3}$ <br>
2, 12) with  $g(1, 2)$ .<br>  $(12)$   $> \Phi(g(1, 2))$  if<br>  $\Phi(g(1, 2))$ <br>  $\Phi(g(1, 2))$ . It follows<br>  $\Phi(g(1, 2))$ . It follows<br>  $\Phi(g(1, 2))$ . Where <sup>2</sup><sub>3</sub>, we have  $\Phi(g(12, 12)) > \Phi(g(12, 1))$ .<br>
Proceeding in the same way as before fand only if  $0 \le k < k^B(d)$  and (ii)  $\Phi$   $k < \frac{(a-c)^2}{36}$ , where  $k^B(d) = (20 - 22)$  ws that  $k^B(d)$  is positive for  $0 < d < 0 < k^B(d) < \frac{(a-c)^2}{36}$  f **a.2)** We compare  $q(12, 12)$  with  $q(1, 2)$ . Proceeding in the same way as before we have  $g(12, 12)$  with  $g(1, 2)$ . Proceeding in the same way as before we have<br>  $g(12, 12)$ )  $\triangleright \Phi(g(1, 2))$  if and only if  $k^2(d)$  and (ii)  $\Phi(g(1, 2))$ <br>
if and only if  $k^2(d)$   $\le k \le \frac{(k-\alpha)^2}{86}$ , where  $k^2(d) = (20 - 22d - 4d^2 +$ that: (i)  $\Phi(g(12, 12)) > \Phi(g(1, 2))$  if and only if  $0 \le k < k^B(d)$  and (ii)  $\Phi(g(1, 2)) >$ that: (i)  $\Phi(g(1,2, 12)) > \Phi(g(1,2))$  if and only if  $\Phi \subseteq k < k^3(d)$  and (ii)  $\Phi(g(1,2)) \ge \Phi(g(1,2, 2))$  if and only if  $k^2(d)$   $\propto k < \frac{4a^2}{8a^2}$ , where  $k^2(d) = (20 - 22d - 4d^2 + 4d^2 + 8d^2 + 8d$  $\Phi(g(12, 12))$  if and only if  $k^B(d) < k < \frac{(a-c)^2}{36}$ , where  $k^B(d) = (20 - 22d - 4d^2 + 3d^3)\frac{(2-d+2d^2)(a-c)^2}{9(1+d)(4-d)^2(2-d)^2(2+d)^2}$ . It follows that  $k^B(d)$  is positive for  $0 < d < 0.861$  and  $\Phi(g(12, 12))$  if and only if  $k^B(d) < k < \frac{(a-c)^2}{36}$ <br>  $3d^3 \frac{(2-a/42^B)(a-c)^2}{9(1+d)(4-d)^2(2-d)^2(2+d)^2}$ . It follows that  $k^B$ <br>
negative for  $0.861 < d < 1$  and that  $0 < k^B(d)$ <br>
the particular case,  $k = 0$  and  $0 < d < \frac{2}{3}$ , it for<br>
T  $k^B(d) = (20 - 22d - 4d^2 +$ <br>itive for  $0 < d < 0.861$  and<br>for all  $0 < d < \frac{2}{3}$ . Thus, for<br>it  $\Phi(g(12, 12)) > \Phi(g(1, 2))$ ,<br>that the efficient network is<br> $g(12, 1)) > \Phi(g(1, 2))$  if and<br>1)) if and only if  $k^C(d) <$ <br> $\frac{2}{(16-7d^2)^2} (20480$  $\frac{(2-a+2a^2)(a-c)^2}{9(1+d)(4-d)^2(2-d)^2(2+d)^2}$ . It follows that<br>tive for  $0.861 < d < 1$  and that  $0 < k^L$ <br>particular case,  $k = 0$  and  $0 < d < \frac{2}{3}$ ,<br>ther with the above conclusion for k<br>, 12) for  $0 < d < \frac{2}{3}$ .  $\frac{-c_1}{36}$  for all  $0 < d < \frac{2}{3}$ . Thus, for the particular case,  $k = 0$  and  $0 < d < \frac{2}{3}$ , it follows that  $\Phi(g(12, 12)) > \Phi(g(1, 2)).$ Together with the above conclusion for  $k = 0$  implies that the efficient network is  $g(12,12)$  for  $0 < d < \frac{2}{3}$
- $3d^3$ )  $\frac{(2-d+2d^2)(a-c)^2}{9(1+d)(4-d)^2(2-d)^2(2)}$ <br>negative for  $0.861 < d$ <br>the particular case, k<br>Together with the about  $g(12, 12)$  for  $0 < d < \frac{2}{3}$ <br>We compare  $g(12, 1)$  w<br>only if  $0 < k < k^C$ <br> $k < \frac{(a-c)^2}{36}$ , where  $k^C$ <br>19200 $k^B(d)$  is positive for  $0 < d < 0.861$  and<br>  $(d) < \frac{(a-c)^2}{36}$  for all  $0 < d < \frac{2}{3}$ . Thus, for<br>
t follows that  $\Phi(g(12, 12)) > \Phi(g(1, 2))$ .<br>
= 0 implies that the efficient network is<br>
that: (i)  $\Phi(g(12, 1)) > \Phi(g(1, 2))$  if and<br>  $\Rightarrow$ negative for  $0.861 < d < 1$  and that  $0 < k^B(d) < \frac{(a-c)^2}{36}$ <br>the particular case,  $k = 0$  and  $0 < d < \frac{2}{3}$ , it follows th<br>Together with the above conclusion for  $k = 0$  implies<br> $g(12, 12)$  for  $0 < d < \frac{2}{3}$ .<br>We compare  $g(12,$  $< d < \frac{2}{3}$ , (12)) ><br>efficient<br>>  $\Phi(g(1))$ <br> $\Phi(g(1))$ <br> $(20480 - 111d^8 + 0.893 <$ <br> $\Phi(g(20480 - 111d^8 + 0.893 < 2))$ <br> $\Phi(g(2))$ ,  $\Phi(g(2))$ ,  $\Phi(g(2))$  $k = 0$  and  $0 < d < \frac{2}{3}$ <br>bove conclusion for  $l$ <br> $\frac{2}{3}$ .<br>with  $g(1, 2)$ . We have  $\frac{2}{3}$ .<br>with  $g(1, 2)$ . We have  $\frac{2}{3}$ .<br> $\frac{2}{3}$  with  $g(1, 2)$ . We have  $\frac{2}{3}$ .<br> $\frac{2}{3}$  and  $\frac{2}{3}$  are  $\frac{2}{3}$  are  $\frac{$ <sup>2</sup>/<sub>3</sub>, it follows that  $\Phi(g(12, 12)) > \Phi(g(1, 2))$ .<br>  $k = 0$  implies that the efficient network is<br>
we that: (i)  $\Phi(g(12, 1)) > \Phi(g(1, 2))$  if and<br>
2))  $> \Phi(g(12, 1))$  if and only if  $k^C(d) < \frac{(1-d)(a-c)^2}{(1+d)(4-d)^2(16-td)^2(16-td)^2}$  (20  $k = 0$  implies that the efficient network is<br>
ve that: (i)  $\Phi(g(12,1)) > \Phi(g(1,2))$  if and<br>
2))  $> \Phi(g(12,1))$  if and only if  $k^C(d) < \frac{(1-d)(a-c)^2}{(1+d)(4-d)^2(2+d)^2(16-7d^2)^2} (20480 - 18432d + \frac{1}{7}7444d^6 + 1968d^7 - 111d^8 + 155d^9)$ . I  $g(12, 12)$  for  $0 < d < \frac{2}{3}$ <br>
We compare  $g(12, 1)$  w<br>
only if  $0 < k < k^C$ <br>  $k < \frac{(a-c)^2}{36}$ , where  $k^C$ <br>  $19200d^2 + 20224d^3 - 6$ <br>
follows that  $k^C(d)$  is p<br>
that  $0 < k^C(d) < \frac{(a-c)}{36}$ <br>
are final step before p<br>  $k^B(d)$ , and **a.3)** We compare  $g(12, 1)$  with  $g(1, 2)$ . We have that: (i)  $\Phi(g(12, 1)) > \Phi(g(1, 2))$  if and g(12, 1) with g(1, 2). We have that: (i)  $\Phi(g(12, 1)) > \Phi(g(1, 2))$  if and<br>  $k < k^C(d)$  and (ii)  $\Phi(g(1, 2)) > \Phi(g(12, 1))$  if and only if  $k^C(d) <$ <br>
where  $k^C(d)$  is equal to  $\frac{72(1+d)(4-d)^2(2+d)^2(16-7d^2)^2}{(20480-18432d} +$ <br>  $2224d^3$ only if <sup>0</sup>  $\langle k \rangle \langle k^C(d) \rangle$  and (ii)  $\Phi(g(1, 2)) > \Phi(g(12, 1))$  if and only if  $k^C(d) < \frac{1}{\epsilon}$ , where  $k^C(d)$  is equal to  $\frac{(1-d)(a-c)^2}{72(1+d)^3(1d-c)^2(2+d)^3(16-cd^3)^2}$  (20480 – 18432*d* + 20224*d*<sup>8</sup> – 42720*d*<sup>4</sup> – 19872*d*<sup>5</sup> + 7444*d*<sup>6</sup>  $\frac{-c_1}{36}$ , where follows that  $k^C(d)$  is positive for  $0 < d < 0.893$  and negative for  $0.893 < d < 1$  and that  $0 < k^C(d) < \frac{(a-c)^2}{36}$  for all  $0 < d < \frac{2}{3}$ .

 $k < \frac{(a-c)^2}{36}$ <br>
19200 $d^2$  +<br>
follows tha<br>
that  $0 < k^6$ <br>
ne final store<br>  $k^B(d)$ , a<br>
for  $0 < d$ <br>
or  $d = 0$  and<br>
cor  $d = 0$  an  $k^C(d)$  is equal to  $\frac{(1-d)(a-c)^2}{72(1+d)(4-d)^2(2+d)^2}$ <br>  $- 42720d^4 - 19872d^5 + 7444d^6 + 19$ <br>
s positive for  $0 < d < 0.893$  and neg:<br>  $\frac{-c)^2}{36}$  for all  $0 < d < \frac{2}{3}$ .<br>
proving the proposition is to con<br>
) for  $0 < d < \frac{2}{3}$ . It  $72(1+d)(4-d)^2(2+d)^2(16-7d^2)^2(20+60) = 16452d + d^5 + 7444d^6 + 1968d^7 - 111d^8 + 155d^9)$ . It  $l < 0.893$  and negative for  $0.893 < d < 1$  and  $\frac{2}{3}$ .<br>
(b)  $l < \frac{2}{3}$ .<br>
(c)  $0.893$  and negative for  $0.893 < d < 1$  and  $\frac{2}{3}$ .<br>
(c 19200 $d^2 + 20224d^3 - 42720d^4 - 19872d^5 + 7444d^6 + 1968d^7 - 111d^8 + 155d^9$ ). It<br>follows that  $k^C(d)$  is positive for  $0 < d < 0.893$  and negative for  $0.893 < d < 1$  and<br>that  $0 < k^C(d) < \frac{(a-c)^2}{36}$  for all  $0 < d < \frac{2}{3}$ .<br>ne fin  $k^C(d)$  is positive for  $0 < d < 0.893$  and negative for  $0.893 < d < 1$  and<br>  $d/d < \frac{(a-c)^2}{36}$  for all  $0 < d < \frac{2}{3}$ .<br>
before proving the proposition is to compare the three thresholds<br>  $k^C(d)$  for  $0 < d < \frac{2}{3}$ . It is easy to  $k > k^C(d)$   $\leq \frac{(a-c)^2}{36}$ <br>1 step before pr<br>2), and  $k^C(d)$  for  $d < 0.2156$  and 0 and  $d = 0.215$ <br>ends on the size<br> $d < 0.2156$  and  $d < k < k^C(d)$  $d < d < \frac{2}{3}$ <br>c propos<br> $\frac{2}{3}$ . It<br> $k^B(d)$   $\leq$ <br> $k^B(d)$ <br> $\leq$ <br> $k^B$ <br> $d)$   $\leq$   $k^B$ <br> $k^A(d)$  th<br>35 The final step before proving the proposition is to compare the three thresholds  $k^A(d)$ ,  $k^B(d)$ , and  $k^C(d)$  for  $0 < d < \frac{2}{3}$ <br>  $k^C(d)$  for  $0 < d < 0.2156$  and  $k^A(d) < k^B$ <br>
that for  $d = 0$  and  $d = 0.2156$  all of them<br>
network depends on the size of  $k$  and the<br> **case a.i)**  $0 < d < 0.2156$  and  $0 < k^C(d) <$  $k^A(d)$ ,  $k^B(d)$ , and  $k^C(d)$  for  $0 < d < \frac{2}{3}$ . It is easy to check that  $k^A(d) > k^B(d) > k^C(d)$  for  $0 < d < 0.2156$  and  $k^A(d) < k^B(d) < k^C(d)$  for  $0.2156 < d < \frac{2}{3}$ . It also happens  $k^A(d) > k^B(d) >$ <br>  $\langle \frac{2}{3} \rangle$ . It also happens<br>
fficient distribution<br>
(2),  $\Phi(g(12, 12)) >$  $k^C(d)$  for  $0 < d < 0.2156$  and  $k^A(d) < k^B(d) < k^C(d)$  for  $0.2156 < d < \frac{2}{3}$ <br>that for  $d = 0$  and  $d = 0.2156$  all of them are equal. Therefore, the efficient<br>work depends on the size of k and the value of d as follows:<br>case a. that for  $d = 0$  and  $d = 0.2156$  all of them are equal. Therefore, the efficial . It also happens<br>ient distribution<br>),  $\Phi(g(12,12)) >$ network depends on the size of  $k$  and the value of  $d$  as follows: network of<br>
case a.i)

 $d = 0$  and  $d = 0.2156$  all of them are equal. Therefore, the efficient distribution<br>depends on the size of  $k$  and the value of  $d$  as follows:<br>)  $0 < d < 0.2156$  and  $0 < k^C(d) < k^B(d) < k^A(d) < \frac{(a-c)^2}{36}$ .<br>If  $0 < k < k^C(d) < k^B(d) < k^$ k and the value of d as follows:<br>  $\langle k^C(d) \rangle \langle k^B(d) \rangle \langle k^A(d) \rangle \langle k^B(d) \rangle$ <br>  $\langle k^A(d) \rangle$  then  $\Phi(g(12, 1))$   $>$ <br>
35 0 36 0 366 0 366 0 366 1 370 1 386 1 386 1 386 1 386 1 386  $d < d < 0.2156$  and  $0 < k^C(d) < k^B(d) < k^A(d) < \frac{(a-c)^2}{36}$ <br>  $0 < k < k^C(d) < k^B(d) < k^A(d)$  then  $\Phi(g(12, 1)) > \Phi(g)$ <br>  $35$ (1) If  $0 < k < k^C(d) < k^B(d) < k^A(d)$  then  $\Phi(g(12,1)) > \Phi(g(1,2)), \Phi(g(12,12)) >$  $\langle k \times k^C(d) \times k^B(d) \times k^A(d) \text{ then } \Phi(g(12, 1)) > \Phi(g(1, 2)), \Phi(g(12, 12)) >$ <br>35

 $\Phi(g(1,2))$ , and  $\Phi(g(12,12)) > \Phi(g(12,1))$ . Therefore,  $g(12,12)$  is the efficient distribution network.

(2) If  $0 < k^C(d) < k < k^B(d) < k^A(d)$  then  $\Phi(g(12,1)) < \Phi(g(1,2)), \Phi(g(12,12)) > \Phi(g(1,2)),$  and  $\Phi(g(12,12)) > \Phi(g(12,1))$ . As before,  $g(12,12)$  is the efficient distribution network.

(3) If  $0 < k^C(d) < k^B(d) < k < k^A(d)$  then  $\Phi(g(12, 1)) < \Phi(g(1, 2))$ ,  $\Phi(g(12, 12)) < \Phi(g(1, 2))$ , and  $\Phi(g(12, 12)) > \Phi(g(12, 1))$ . Therefore,  $g(1, 2)$  is the efficient distribution ution network.

(4) If  $0 < k^C(d) < k^B(d) < k^A(d) < k < \frac{(a-c)^2}{36}$  then  $\Phi(g(12,1)) < \Phi(g(1,2))$ ,<br>  $\Phi(g(12,12)) < \Phi(g(1,2))$ , and  $\Phi(g(12,12)) > \Phi(g(12,1))$ . Hence,  $g(1,2)$  is the efficient distribution network.  $\frac{1}{9}$  cient

 $\frac{2}{3}$  and  $0 < k^A(d) < k^B(d) < k^C(d) < \frac{(a-c)^2}{36}$ . Following the same reasoning as before we have that either,  $g(12, 12)$  is the efficient distribution network for  $0 \lt k \lt k^A(d) \lt k^B(d) \lt k^C(d)$ , or  $g(12,1)$  for  $0 \lt k^A(d) \lt k \lt k^C(d)$ , or  $g(1,2)$  for  $0 < k^C(d) < k < \frac{(a-c)^2}{36}$ .

Figure 8 summarizes the above result. The area A corresponds to the area where  $q(12, 12)$  is the efficient distribution network, the area B is the one where  $q(1, 2)$  is efficient, and the uncolored area corresponds to the area where  $g(12, 1)$  is the efficient distribution network.

distr $\cose~\mathrm{b)}~\frac{2}{3}$  $\cdot$ 

- **b.1)** We compare  $g(12, 12)$  with  $g(12, 0)$ . We conclude that: (i)  $\Phi(g(12, 12)) > \Phi(g(12, 0))$ if and only if  $0 < k < k^D(d)$  and (ii)  $\Phi(g(12,0)) > \Phi(g(12,12))$  if and only if  $k^D(d) < k < \overline{k}$ , where  $k^D(d) = \frac{(5-6d)(a-c)^2}{72(1+d)(2-d)^2}$ . It follows that  $k^D(d)$  is positive for  $\frac{(5-6d)(a-c)^2}{72(1+d)(2-d)^2}$ . It follows that<br>  $d < 1$ , and that  $0 < k^D(d) <$ <br>
(ii) are possible, while for  $\frac{1}{6}$ <br>
(ii) are possible, while for  $\frac{1}{6}$  $0 < d < \frac{5}{6}$  and negative for  $\frac{5}{6}$ Therefore, for  $\frac{2}{3} < d < \frac{5}{6}$  (i) and (ii) are possible, while for  $\frac{5}{6}$ possible.
- $\Phi(g(t), 2))$ , and  $\Phi(g(t), 2))$ ). Therefore,  $g(12, 12)$  is the efficient distribution distribution distribution distribution distribution distribution distribution distribution of  $\Phi(g(t), 2))$ ,  $\Phi(g(t), 2)$ ,  $\Phi(g(t), 2)$ ,  $\Phi(g(t),$  $\epsilon \, k^{C}(d) < k < \bar{h}^{d}(d) < k^{A}(d) \text{ then } \Phi(g(12, 1)) \leq \Phi(g(1, 2)), \Phi(g(12, 1)) \geq 0$ <br>
(a)), and  $\Phi(g(12, 12)) \geq \Phi(g(12, 1)).$  As before,  $g(12, 12) \geq 0$  in the efficient distribution  $\epsilon k^{G}(d) < k < k^{A}(d)$  then  $\Phi(g(12, 1)) \leq \$ φ(*n*(12)), and  $\Phi(g(12,12))$  > Φ(g(12, 12)). As before, g(12, 12) is the efficient distribution of the effect of  $\Phi(g(12,1))$ ,  $\Phi(g(12,1))$ ,  $\Phi(g(12,1))$ ,  $\Phi(g(12,2))$ ,  $\Phi(g(12,2))$ ,  $\Phi(g(12,2))$ ,  $\Phi(g(12,2))$ ,  $\Phi(g(12,2))$ ,  $\Phi$  $\langle k'(d) \leq k''(d) \leq k \leq k^{d}(d)$  then  $\Phi(g(1,2)) \leq \Phi(g(1,2))$ ,  $\Phi(g(1,2)) \geq 0$ ,  $\langle k'(d) \leq k \leq k^{d}(d) \leq 0$ ,  $\Phi(g(2,12)) \geq \Phi(g(12,1))$ . Therefore,  $g(1,2) \geq 0$  is the efficient distribution is event.<br>  $\langle k'(d) \leq k^{d}(d) \leq k^{d}(d) \le$  $\Phi(g(1, 2);$  and  $\Phi(g(12, 12)) > \Phi(g(12, 1))$ . Therefore,  $g(1, 2)$  is the efficient distrib-<br>(i) If 0 < k<sup>t</sup>(d) < k<sup>2</sup>(d) < k<sup>2</sup>(d) < k<sup>2</sup>(d)  $k$  is  $\frac{\ln 2\alpha^2}{\ln 2}$  then  $\Phi(g(12, 1)) <sup>2</sup>$  θ $(g(12, 2))$ ), and  $\Phi(g(12, 2)) >$  $\langle k^C(d) \rangle \langle k^B(d) \rangle \langle k^A(d) \rangle \langle k^C(d) \rangle$ <br>  $\langle k^C(d) \rangle \langle k^C(d) \rangle$  and  $\Phi(g(12,12))$   $\geq \Phi(g(1,2))$ , and  $\Phi(g(12,12))$   $\geq \Phi(g(1))$ <br>
tribution network.<br>
156  $\langle d \rangle \langle k^B(d) \rangle \langle k^B(d) \rangle \langle k^C(d) \rangle$  as before we have that either,  $g(12,1$  $\frac{2}{36}$  then Φ(g(12, 1)) < Φ(g(1, 2)),<br>
(g(12, 1)). Hence, g(1, 2) is the effi-<br>  $k^C(d) < \frac{(a-c)^2}{36}$ . Following the same<br>
is the efficient distribution network<br>
1) for 0 <  $k^A(d) < k < k^C(d)$ , or<br>
a A corresponds to the  $\Phi(g(12,12)) < \Phi(g(1,2))$ , and  $\Phi(g(12,12)) > \Phi(g(12,1))$ . Hence,  $g(1,2)$  is the efficient distribution network<br>circulation network  $\Phi^A(g) < k^G(g) < k^G(g) < k^G(g) < \frac{(k-2)^2}{86}$ . Editioning the same<br>massining as before we have that 0.2156 <  $d < \frac{2}{3}$ <br>soning as before v<br>0 <  $k < k^A(d)$ <br>and  $(2)$  for  $0 < k^C(d)$ <br>are 8 summarize<br>2,12) is the effici<br>cient, and the unceribution network<br> $\frac{2}{3} < d < 1$  and 0<br>compare  $g(12, 12)$ <br>nd only if  $0 <$ <br> $(d) < k < \overline{k}$ ,  $\langle k^A(d) \rangle \langle k^B(d) \rangle \langle k^C(d) \rangle \langle \frac{(a-c)^2}{36}$ <br>that either,  $g(12, 12)$  is the efficient<br> $l \rangle \langle k^C(d)$ , or  $g(12, 1)$  for  $0 \langle k^A(\frac{(a-c)^2}{36})$ <br>bove result. The area A correspon<br>tribution network, the area B is that<br>area corres  $g(12, 12)$  is the efficient distribution network<br>or  $g(12, 1)$  for  $0 < k^A(d) < k < k^C(d)$ , or<br>The area A corresponds to the area where<br>work, the area B is the one where  $g(1, 2)$  is<br>nods to the area where  $g(12, 1)$  is the ef  $\langle k \rangle \langle k \rangle \langle d \rangle \langle k^B \langle d \rangle \langle k^B \langle d \rangle \langle k^C \langle d \rangle$ , or  $g(12,1)$  for  $0 \langle k^A \rangle \langle d \rangle \langle k \rangle \langle k^C \langle d \rangle$ , or<br>  $\langle k^C \rangle \langle d \rangle \langle k \rangle \langle k \rangle \langle d \rangle$ <br>  $\langle k \rangle \langle k \rangle \langle d \rangle \langle k \rangle \langle k \rangle$ <br>  $\langle k \rangle \langle d \rangle \langle k \rangle \langle d \rangle \langle k \rangle$ <br>  $\langle k \rangle \langle d \rangle \langle d \rangle \langle d \rangle$  for  $0 \$  $g(1, 2)$  for  $0 < k^C(d) < k < \frac{(a-c)^2}{36}$ <br>
Figure 8 summarizes the above 1<br>  $g(12, 12)$  is the efficient distributi<br>
efficient, and the uncolored area condistribution<br>
absorbed area condistribution<br>
b)  $\frac{2}{3} < d < 1$  and  $0 < k$  $g(12, 12)$  is the efficient distribution network, the area B is the one where  $g(1, 2)$  is difficient distribution network.<br> **b)**  $\frac{2}{3} < d < 1$  and  $0 < k < \overline{k}$ .<br>
We compare  $g(12, 12)$  with  $g(12, 0)$ ). We conclude that  $g(12, 1)$  is the efficient<br>  $(12, 12)) > \Phi(g(12, 0))$ <br>  $(2, 12))$  if and only if<br>  $k^D(d)$  is positive for<br>  $k^D(d)$  is positive for<br>  $k^E(d) <$ <br>  $g(12, 1)) > \Phi(g(12, 0))$ <br>  $\Phi(g(12, 0))$ <br>  $\Phi(\Phi^{-1})$ <br>  $\Phi(\Phi^{-1})$ . It follows<br>  $(63 < d < 1$  a  $d < d < 1$  and  $0 < k < k$ .<br>
Sumpare  $g(12, 12)$  with  $g(1)$ <br>  $d < k < \overline{k}$ , where  $k^D(d)$ <br>  $d < k < \overline{k}$ , where  $k^D(d)$ <br>  $d < \frac{5}{6}$  and negative for if<br>
offore, for  $\frac{2}{3} < d < \frac{5}{6}$  (i)<br>  $d < k$ <br>  $d < k$ <br>  $g(12, 12)$  with  $g(12, 0)$ ). We conclude that: (i)  $\Phi(g(12, 12)) > \Phi(g(12, 0))$ <br>
if  $0 < k < k^D(d)$  and (ii)  $\Phi(g(12, 0)) > \Phi(g(12, 12))$  if and only if<br>  $\overline{\kappa}$ , where  $k^D(d) = \frac{(5-6d)(6-6)^2}{72(1+d)(2-d)^2}$ . It follows that  $k^D(d)$  is  $\langle k \rangle \langle k \rangle \langle k \rangle$  and (ii)  $\Phi(g(12,0)) > \Phi(g(12,12))$  if and only if<br>where  $k^D(d) = \frac{(5-6a)(a-c)^2}{72(1+d)(2-d)^2}$ . It follows that  $k^D(d)$  is positive for<br>egative for  $\frac{5}{6} < d < 1$ , and that  $0 < k^D(d) < \overline{k}$  for all  $0 < d < 1$ .<br> $\langle d \$  $k^D(d) < k < \overline{k}$ , where  $k^D(d) = \frac{(5-6d)(a-c)^2}{72(1+d)(2-d)^2}$ <br>  $0 < d < \frac{5}{6}$  and negative for  $\frac{5}{6} < d < 1$ , and<br>
Therefore, for  $\frac{2}{3} < d < \frac{5}{6}$  (i) and (ii) are po<br>
possible.<br>
We compare  $g(12,1)$  with  $g(12,0)$ ). We conc  $k^D(d)$  is positive for<br>  $\overline{k}$  for all  $0 < d < 1$ .<br>  $< d < 1$  only (ii) is<br>  $(12, 1)) > \Phi(g(12, 0))$ <br>
f and only if  $k^E(d) <$ <br>  $\frac{5d^6|(a-c)^2}{3}$ . It follows<br>  $3 < d < 1$  and that<br>
and (ii) are possible,  $< d < \frac{2}{6}$ <br>herefore,<br>bssible.<br>le compa<br>and only<br> $< \overline{k}$ , wlat  $k^E(d)$ <br> $< k^E(d)$ <br>hile for 0<br>e **a.1**) for  $d < d < 1$ , and that  $0 < k^D(d) < \overline{k}$  for all  $0 < d < 1$ .<br>
and (ii) are possible, while for  $\frac{5}{6} < d < 1$  only (ii) is <br>
(ii) are possible, while for  $\frac{5}{6} < d < 1$  only (ii) is <br>
(ii)  $\Phi(g(12,0)) > \Phi(g(12,1))$  if and only if  $k^$  $\langle d \rangle \langle \frac{1}{6} \rangle$ <br>2, 1) with  $k \langle k^E(\frac{1}{6}) \rangle = 1$ <br>sitive for all 0  $\langle d \rangle$  1 or<br>comparis  $d < d < 1$  only (ii) is<br>  $(12, 1)) > \Phi(g(12, 0))$ <br>
and only if  $k^E(d) < \frac{d^6}{(a-c)^2}$ . It follows<br>  $3 < d < 1$  and that<br>
and (ii) are possible, **b.2)** We compare  $g(12, 1)$  with  $g(12, 0)$ . We conclude that: (i)  $\Phi(g(12, 1)) > \Phi(g(12, 0))$  $g(12, 1)$  with  $g(12, 0)$ ). We conclude that: (i)  $\Phi(g(12, 1)) > \Phi(g(12, 0))$ <br>  $0 < k < k^E(d)$  and (ii)  $\Phi(g(12, 0)) > \Phi(g(12, 1))$  if and only if  $k^E(d) <$ <br>  $e k^E(d) = \frac{(1280 - 2048d - 128d^2 + 1504d^3 - 578d^4 - 266d^5 + 155d^6)(a - c)^2}{72(2$ if and only if <sup>0</sup>  $\langle k \times k^E(d) \rangle$  and (ii)  $\Phi(g(12,0)) > \Phi(g(12,1))$  if and only if  $k^E(d) < k^E(d) = \frac{(1280 - 2048d - 128d^2 + 1504d^3 - 578d^4 - 266d^5 + 155d^6)(a - c)^2}{72(2 - d)^2(16 - 7d^2)^2}$ . It follows ositive for  $0 < d < 0.863$  and negative for  $0.863 < d <$ k < k, where kE(d) =  $\frac{(1280-2048d-128d^2+1504d^3-578d^4-266d^3+155d^6)(a-c)^2}{72(2-d)^2(16-7d^2)^2}$ <br>that kE(d) is positive for  $0 < d < 0.863$  and negative for  $0.863 < d < 0 < kE(d) < \overline{k}$  for all  $0 < d < 1$ . Then, for  $\frac{2}{3} < d < 0.8$  $72(2-d)^2(16-7d^2)^2$  . It follows<br>
72(2−d)<sup>2</sup>(16−7d<sup>2</sup>)<sup>2</sup><br>
3 and negative for 0.863 < d < 1 and that<br>
for  $\frac{2}{3}$  < d < 0.863 (i) and (ii) are possible,<br>
9.12) and  $g(12, 1)$ that  $kE(d)$  is positive for  $0 < d < 0.863$  and negative for  $0.863 < d < 1$  and that  $x^E(d) < \overline{k}$  for all  $0 < d < 1$ . Then, for  $\frac{2}{3} < d < 0.863$  (i) and (ii) are possible, for  $0.863 < d < 1$  only (ii) is possible.<br>
..1) for the comp  $0 \lt \kappa$   $(u) \lt \kappa$  for all  $0 \lt u \lt 1$ . Then, for  $\frac{1}{3}$  $\langle kE(d) \rangle \langle k\bar{k}$  for all  $0 \langle d \rangle \langle 1$ . Then, for  $\frac{2}{3}$  hile for  $0.863 \langle d \rangle \langle 1$  only (ii) is possible.<br>
ee **a.1)** for the comparison between  $g(12, 12)$ <br>
36  $< d < 0.863$  (i) and (ii) are possible, and  $g(12, 1)$ . while for  $0.863 < d < 1$  only (ii) is possible.<br> **b.3)** See **a.1)** for the comparison between  $q(12, 12)$  and  $q(12, 1)$ .
- while for  $0.863 < d < 1$  only (ii) is possible.<br>See **a.1)** for the comparison between  $g(12, 13)$ <br>36  $g(12, 12)$  and  $g(12, 1)$ .<br>36



Figure 8: Efficient distribution networks when  $0 < d < \frac{2}{3}$ .

Putting together b.1), b.2) and b.3) it follows that  $k^A(d) < k^D(d) < k^E(d) < \overline{k}$ Four different cases can be distinguished:

- $d < d < \frac{2}{3}$ <br>  $d < k^D$  ( $d$ )<br>  $g(12, 1)$  ion tha<br>  $g(12, 1)$  ion tha<br>  $g(12, 1)$  ion tha<br>  $g(12, 1)$  ion tha<br>  $g(12, 1)$  ion tha  $k^A(d) < k^D(d) < k^E(d) < \overline{k}$ .<br>  $> \Phi(g(12, 1)), \Phi(g(12, 12)) >$ <br>
sonclusion that  $g(12, 12)$  is the<br>  $< \Phi(g(12, 1)), \Phi(g(12, 12)) >$ <br>
conclusion that  $g(12, 1)$  is the<br>  $< \Phi(g(12, 1)), \Phi(g(12, 12)) <$ <br>
conclusion that  $g(12, 1)$  is the<br>  $< \Phi(g($ (i)  $0 < k < k^A(d) < k^D(d) < k^E(d) < \overline{k}$ , where  $\Phi(g(12, 12)) > \Phi(g(12, 1)), \Phi(g(12, 12)) >$  $\langle k < k^A(d) < k^D(d) < k^E(d) < \overline{k}, \text{ where } \Phi(g(12, 12)) > \Phi(g(12, 11)), \Phi(g(12, 12)) \geq \Phi(g(12, 0)) \text{ and } \Phi(g(12, 11)) > \Phi(g(12, 0)), \text{ with the conclusion that } g(12, 12) \text{ is the efficient distribution network.}$ <br>  $A(d) < k < k^D(d) < k^E(d) < \overline{k}, \text{ where } \Phi(g(12, 12)) < \Phi(g(12, 1)), \Phi(g(12, 12))$  $\Phi(g(12,0))$  and  $\Phi(g(12,1)) > \Phi(g(12,0))$ , with the conclusion that  $g(12,12)$  is the efficient distribution network.
- (ii)  $k^A(d) < k < k^D(d) < k^E(d) < \overline{k}$ , where  $\Phi(g(12, 12)) < \Phi(g(12, 1)), \Phi(g(12, 12)) >$  $\Phi(g(12,0))$  and  $\Phi(g(12,1)) > \Phi(g(12,0))$ , with the conclusion that  $g(12,1)$  is the efficient distribution network.
- $\Phi(g(12,0))$  and  $\Phi(g(12,1)) > \Phi(g(12,0))$ , with the conclusion that  $g(12,12)$  is the efficient distribution network.<br>  ${}^A(d) < k < k^D(d) < k^E(d) < \overline{k}$ , where  $\Phi(g(12,12)) < \Phi(g(12,1))$ ,  $\Phi(g(12,12)) > \Phi(g(12,0))$  and  $\Phi(g(12,1)) > \Phi(g(12,0))$  $k^A(d) < k < k^D(d) < k^E(d) < \overline{k}$ , where  $\Phi(g(12,12)) < \Phi(g(12,11))$ ,  $\Phi(g(12,12)) > \Phi(g(12,0))$  and  $\Phi(g(12,1)) > \Phi(g(12,0))$ , with the conclusion that  $g(12,1)$  is the efficient distribution network.<br>  $k^A(d) < k^D(d) < k < k^E(d) < \overline{k}$ , where  $\Phi(g(12,0))$  and  $\Phi(g(12,1)) > \Phi(g(12,0))$ , with the conclusion that  $g(12,1)$  is the efficient distribution network.<br>  $k^A(d) < k^D(d) < k < k^E(d) < \overline{k}$ , where  $\Phi(g(12,12)) < \Phi(g(12,1))$ ,  $\Phi(g(12,12)) < \Phi(g(12,0))$  and  $\Phi(g(12,1)) > \Phi(g(12,0))$ (iii)  $k^A(d) < k^D(d) < k < k^E(d) < \overline{k}$ , where  $\Phi(g(12, 12)) < \Phi(g(12, 1)), \Phi(g(12, 12))$  $k^A(d) < k^D(d) < k < k^E(d) < \overline{k}$ , where  $\Phi(g(12, 12)) < \Phi(g(12, 11))$ ,  $\Phi(g(12, 12)) < \Phi(g(12, 0))$  and  $\Phi(g(12, 1)) > \Phi(g(12, 0))$ , with the conclusion that  $g(12, 1)$  is the efficient distribution network.<br>  $k^A(d) < k^D(d) < k^E(d) < k < \overline{k}$ ,  $\Phi(g(12,0))$  and  $\Phi(g(12,1)) > \Phi(g(12,0))$ , with the conclusion that  $g(12,1)$  is the efficient distribution network.
- $\Phi(g(12,0))$  and  $\Phi(g(12,1)) > \Phi(g(12,0))$ , with the conclusion that  $g(12,1)$  is the efficient distribution network.<br>  $k^A(d) < k^D(d) < k^E(d) < k < \overline{k}$ , where  $\Phi(g(12,12)) < \Phi(g(12,1))$ ,  $\Phi(g(12,12)) < \Phi(g(12,0))$  and  $\Phi(g(12,1)) < \Phi(g(12,0))$ (iv)  $k^A(d) < k^D(d) < k^E(d) < k < \overline{k}$ , where  $\Phi(g(12, 12)) < \Phi(g(12, 1)), \Phi(g(12, 12)) <$  $k^A(d) < k^D(d) < k^E(d) < k < \overline{k}$ , where  $\Phi(g(12, 12)) < \Phi(g(12, 11))$ ,  $\Phi(g(12, 12)) < \Phi(g(12, 0))$  and  $\Phi(g(12, 1)) < \Phi(g(12, 0))$ , with the conclusion that  $g(12, 0)$  is the efficient distribution network.<br>37  $\Phi(g(12,0))$  and  $\Phi(g(12,1)) < \Phi(g(12,0))$ , with the conclusion that  $g(12,0)$  is the  $\Phi(g(12,0))$  and  $\Phi(g(12,1)) < \Phi(g(12,0))$ , with the conclusion that  $g(12,0)$  is the efficient distribution network.<br>37 efficient distribution network.

Figure 9 displays the efficient distribution networks for case b)  $\frac{2}{3}$ B corresponds to the area where  $g(12, 12)$  is efficient, the uncolored area corresponds to the area where  $g(12, 1)$  is efficient, and finally the area A corresponds to the area where  $g(12,0)$  is efficient.



Figure 9: Efficient distribution networks when  $\frac{2}{3}$ 

 $d < d < 1.$ <br>
we have  $d) < k^D$ <br>  $d) < k^D$ <br>  $k$  for 0.7<br>  $n$  nally  $g(1)$ <br>  $d$  (**iv**)  $i$ <br>  $d$  nd (**iv**)  $i$ <br>  $d$ <br>  $k$  *size*  $o$ <br>  $k$ <br>  $h$ ,  $1)$ )  $>$   $C$ <br>  $d$ ,  $l$ ,  $1$ ))  $>$   $C$ As a corollary, for the particular case  $k = 0$  and  $\frac{2}{3} < d < 1$ , we have that  $g(12, 12)$  is As a corollary, for<br>the efficient distribution<br>and (i) above applies;  $k = 0$  and  $\frac{2}{3}$ <br>  $l < 0.735$  sin<br>
ent distribu<br>
d (iii) above<br>
1 since  $k^E(d)$ <br>
Proposition<br>
nking is a fu<br>  $0 > C(g(12,0))$ <br>  $>C(g(1,1))$ <br>
38  $d < d < 1$ , we have that  $g(12, 12)$  is<br>  $\cos 0 < k^A(d) < k^D(d) < k^E(d) < \overline{k}$ <br>
(ion network for 0.735  $d < d < 0.863$ <br>
applies; finally  $g(12, 0)$  is the effi-<br>  $d < 0 < \overline{k}$  and (iv) above applies.<br>
(i) above applies.<br>
(i) above app the efficient distribution network for  $\frac{2}{3} < d < 0.735$  sin Heient distribution network for  $\frac{2}{3} < d < 0.735$  since  $0 < k^A(d) < k^B(d) < k^E(d) < \overline{k}$ <br>
(i) above applies;  $g(12, 1)$  is the efficient distribution network for 0.735  $< d < 0.863$ <br>  $k^A(d) < 0 < k^B(d) < k^E(d) < \overline{k}$  and (iii) above  $d < d < 0.735$  since  $0 < k^A(d) < k^D(d) < k^E(d) < \overline{k}$ <br>flicient distribution network for 0.735  $d < d < 0.863$ <br>and (iii) above applies; finally  $g(12,0)$  is the effi-<br> $d < 1$  since  $k^E(d) < 0 < \overline{k}$  and (iv) above applies.  $\blacksquare$ <br>of P and (i) above applies;  $g(12, 1)$  is the efficient distribution network for  $0.735 < d < 0.863$ and (i) above applies;  $g(12,1)$  is the efficient distribution network for 0.735 <  $d < 0.8$ <br>since  $k^A(d) < 0 < k^B(d) < k^E(d) < \overline{k}$  and (iii) above applies; finally  $g(12,0)$  is the electric distribution network for 0.863 <  $d$  $g(12, 1)$  is the efficient distribution network for  $0.735 < d < 0.863$ <br>  $d) < k^E(d) < \overline{k}$  and (iii) above applies; finally  $g(12, 0)$  is the effi-<br>
ork for  $0.863 < d < 1$  since  $k^E(d) < 0 < \overline{k}$  and (iv) above applies.  $\blacksquare$ <br> since  $k^A(d) < 0 < k^D(d) < k^E(d) < \overline{k}$  and (iii) above applies; finally  $g(12,0)$  is the efficient distribution network for 0.863 <  $d < 1$  since  $k^E(d) < 0 < \overline{k}$  and (iv) above applies.

## B.1.4 Consumer surplus analysisConsumer surplus analysis

We give the complete characterization of Proposition 5 in the main text and its proof. We give the complete characterization of Proposition 5 in the main text a<br>Proposition 8 The consumer surplus ranking is a function of the size of

**Proposition 8** The consumer surplus ranking is a function of the size of d as follows.

- $\mathbf{a}$ jor o 12)) >  $C(g(12))$ <br>  $d \leq 0.1413$ ,<br>
12)) >  $C(g(12))$ <br>
1413 <  $d \leq 0$ .
- d its proof.<br> *d as follows.*<br>  $(g(1,0))$ <br>  $(g(1,0))$  $C(g(12, 12)) > C(g(12, 1)) > C(g(1, 2)) > C(g(1, 2)) \geq C(g(1, 1)) > C(g(1, 0))$ <br>for  $0 < d \leq 0.1413$ ,<br> $C(g(12, 12)) > C(g(12, 1)) > C(g(1, 2)) > C(g(1, 1)) \geq C(g(12, 0)) > C(g(1, 0))$ <br>for  $0.1413 < d \leq 0.2826$ ,<br>38  $(12, 12)$ ) >  $C(g(12, 1))$ <br>r 0.1413 <  $d \le 0.2826$ , **b)**  $C(g(12, 12)) > C(g(12, 1)) > C(g(1, 2)) > C(g(1, 1)) \ge C(g(12, 0)) > C(g(1, 0))$  $C(g(12, 12)) > C(g(12, 1)) > C(g(1, 2)) > C(g(1, 1)) \le C(g(12, 0)) > C(g(1, 0))$ <br>for 0.1413 <  $d \le 0.2826$ , for  $0.1413 < d \leq 0.2826$ ,
- $\overline{\phantom{a}}$  $\frac{1}{2}$ , for  $0.2826 < d \leq \frac{2}{3}$ ,
- $\ddot{\phantom{0}}$  $for \frac{2}{3}$  $\ddot{\phantom{0}}$
- e)  $C(g(12, 12)) > C(g(12, 1)) > C(g(12, 0)) > C(g(1, 1)) > C(g(1, 2)) > C(g(1, 0))$  $for\,0.8597 < d < 1.$

**Proof:** All expressions of the consumer surplus corresponding to the different distribution networks are multiplied by the factor  $(a-c)^2$ . Then, the comparisons are independent of this factor and we will ignore it throughout this proof.

- $y \circ (y(1, 1)) > \circ (y(1, 0))$  is and only if  $288b$  $\frac{1}{2}$
- $\ddot{\phantom{a}}$ holds since  $1 > d$ .
- $\mathbf{v}$ since  $1 > d$ .
- $\mathcal{L}(\cdot)$ 0, which always holds because  $1 > d$ .
- $\mathcal{O}(\mathcal{V})$  $45d^5 + 25d^6 > 0$ , which always holds since  $1 > d$ .
- (vi)  $C(g(12,1)) > C(g(1,2))$  if and only if  $28672+22528d+23808d^2+51200d^3+15008d^4 16320d^5 - 5692d^6 - 364d^7 - 747d^8 - 20d^9 + 25d^{10} > 0$ , which always holds given that  $1 > d$
- (vii)  $C(g(12, 12)) > C(g(12, 1))$  if and only if  $1792 1024d 896d^2 + 544d^3 660d^4$  $116d^5 + 347d^6 + 20d^7 - 25d^8 > 0$ , which always holds because  $1 > d$ .<br>Using (i) to (vii) we have that  $C(g(1,0))$  is last in the ranking, and  $C(g(12, 12))$  and

C(g(12, 12))  $> C(g(1,1))$   $> C(g(1,1))$   $> C(g(1,1))$   $> C(g(1,2))$   $> C(g(1,2))$   $> C(g(1,2))$   $> C(g(1,2))$   $)$   $(c(g(1,2)))$   $(c(g(1,2)))$   $> C(g(1,2)))$   $> C(g(1,2))$   $> C(g(1,2))$   $)$   $(c(g(1,2)))$   $(c(g(1,2)))$   $)$   $(c(g(1,2)))$   $c(g(1,2))$   $)$   $(c(g(1,2)))$   $(c(g(1,2)))$   $)$  $r \ 0.2826 < d \leq \frac{2}{3}$ <br>  $(12, 12)) > C(g(1)$ <br>  $r \frac{2}{3} < d \leq 0.8597$ <br>  $(12, 12)) > C(g(1)$ <br>  $r \ 0.8597 < d < 1$ <br>
All expressions c<br>
is are multiplied<br>
for and we will ig<br>  $(1, 1)) > C(g(1, 1))$ <br>  $g(1, 2)) > C(g(1, 1))$ <br>  $g(1, 2)) > C(g(1, 1))$ <br> 12)) > C(g(12<br>
< d ≤ 0.8597,<br>
12)) > C(g(12<br>
< d ≤ 0.8597,<br>
12)) > C(g(12<br>
8597 < d < 1.<br>
expressions of<br>
re multiplied b<br>
and we will ig:<br>
1)) > C(g(1, 0<br>
2)) > C(g(1, 0<br>
2)) > C(g(1, 0<br>
12, 0)) > C(g(1<br>
12, 0)) > C(g(1 C(g(12, 12))  $> C(g(1, 1))$  if and only if  $\frac{1}{2}$  and  $\frac{1}{2}$  and  $\frac{1}{2}$  b)  $\frac{1}{2}$  c(g(12, 12))  $> C(g(1, 0))$  for  $C(g(1, 1)) > C(g(1, 1)) > C(g(1, 2)) > C(g(1, 0))$  for  $0.8597 < a < 1$ .<br>
Cfo (12, 12)  $> C(g(1, 0)) > C(g(1, 0)) > C(g(1, 1)) > C$  $(12, 12)) > C(g(12, 12)) > C(g(12, 12)) > C(g(12, 12)) > C(g(12, 12))$ <br>so are multiplied by an are multiplied by an are multiplied by an are multiplied by  $(1, 1)) > C(g(1, 0, 12)) > C(g(1, 12, 0)) > C(g(12, 1)) > C(g(12, 1)) > C(g(12, 1)) > C(g(12, 12)) > C(g(12,$ C(g(12, 1)) > C(g(12, 1)) of the constant  $\pi$  and  $\pi$  and networks are multiplied by the factor (a − c)<sup>2</sup>. Then, the comparisons are independent of<br>this factor and we will ignore it throughout this proof.<br>
(i)  $C(g(1,0)) > C(g(1,0))$  if and only if  $\frac{1}{2880} > 0$  which always holds.  $C(g(1, 1)) > C(g(1, 0))$  if and only if  $\frac{1}{28}$ <br>  $C(g(1, 2)) > C(g(1, 0))$  if and only if 6<br>
holds since  $1 > d$ .<br>  $C(g(12, 0)) > C(g(1, 0))$  if and only if<br>
since  $1 > d$ .<br>  $C(g(12, 1)) > C(g(1, 1))$  if and only if 1<br>
0, which always holds beca > 0 which always holds.<br>  $-32d + 12d^2 + 4d^3 - d^4$ <br>  $-4d + 3d^2 + 2d^3 - d^4 > 0$ ,<br>  $2-64d - 80d^2 + 360d^3 + 19$ <br>  $92-768d - 1472d^2 + 928d$ <br>  $ce 1 > d$ .<br>  $372+22528d + 23808d^2 +$ <br>  $9+25d^{10} > 0$ , which always holds because 1<br>
is last i C(g(1, 2)) > C(g(1, 0)) if and only if  $(44 - 32d + 12d^2 + 4d^2 - d^4) > 0$ , which always holds since  $1 > d$ .<br>
looks since  $1 > d$ .<br>
C(g(12, 0)) > C(g(1, 0)) if and only if  $44 - 4d + 3d^2 + 2d^3 - d^4 > 0$ , which always holds<br>
since  $> d.$ <br>  $C(g \ C(g$ <br>  $C(g \ s)$ <br>  $C(g(\ s \cdot 0, \cdot))$ <br>  $C(g \ a^6)$ <br>  $>C(g \ a^6)$ <br>  $C(g \$  $C(g(12, 0)) > C(g(1, 0))$  if and only if  $1-4d-3d^2+2d^3-d^4>0$ , which always holds<br>
since  $1>d$ .<br>  $C(g(12, 1)) > C(g(1, 0))$  if and only if  $192-64d-80d^2+360d^3+191d^4-104d^5-57d^6>0$ , which always holds because  $1 > d$ .<br>  $U_g(t(2, 1)) > C$  $> d.$ <br>1)) :<br>h alv<br>1)) ><br>25 $d^6$ <br>1)) :<br>25 $d^6$ <br>1) :<br> $-5$ <br>12) 347<br>to (e fir<br> $(g(1 - C(g$ <br>f 1 -<br> $C(g$  $C(g(12, 1)) > C(g(1, 1))$  if and only if 192−64d–80d<sup>2</sup>+360d<sup>2</sup>+191d<sup>4</sup>−104d<sup>3</sup>−57d<sup>6</sup> ><br>0, which always holds because  $1 > d$ .<br>7(g(12,1)) >  $C(g(12, 0))$  if and only if 1792−768d−1472d<sup>2</sup>+928d<sup>2</sup>+418d<sup>4</sup>−3802d<sup>2</sup><br>45d<sup>9</sup>+25d<sup>6</sup> >  $> d.$ <br>
aly if olds<br>
ly if olds<br>
ly if  $s^3 - 2$ <br>
d on  $s$ , wh<br>  $g(1, s)$ <br>
anki<br>  $C(g - d^4)$ <br>
if 0<br>  $2d^4$ <br>
e co.  $C(g(12, 1)) > C(g(12, 0))$  if and only if 1792–768d – 1472d<sup>2</sup>+928d<sup>3</sup>+418d<sup>4</sup> – 302d<sup>5</sup><br>
45d<sup>5</sup> + 25d<sup>6</sup> > 0, which always holds since  $1 > d$ .<br>  $C(g(12, 1)) > C(g(1, 2))$  if and only if  $28672+22528d+23808d^2+51200d^3+15008d^4-16320$  $45d^5 + 25d^6 > 0$ , which always holds since  $1 > d$ .<br>  $C(g(12, 1)) > C(g(1, 2))$  if and only if  $28672+2252$ <br>  $16320d^5 - 5692d^6 - 364d^7 - 747d^8 - 20d^9 + 25d^{10}$ <br>  $1 > d$ .<br>  $C(g(12, 12)) > C(g(12, 1))$  if and only if  $1792 - 116d^5 + 347d^$  $C(g(12, 1)) > C(g(1, 2))$  if and only if  $28672+22528d+23808d^2+51200d^3+15008d^4-16320d^5-5692d^6-364d^7-747d^8-20d^9+25d^{10} > 0$ , which always holds given that  $1 > d$ .<br>  $C(g(12, 12)) > C(g(12, 1))$  if and only if  $1792 - 1024d - 896d$ 16320 $d^5$  − 5692 $d^6$  − 364 $d'$  − 747 $d^8$  − 20 $d^9$  + 25 $d^{10}$  > 0, which always holds given that<br>
1 >  $d$ .<br>  $C(g(12, 12)) > C(g(12, 1))$  if and only if 1792 − 1024 $d$  − 896 $d^2$  + 54 $d^3$  − 660 $d^4$  − 116 $d^5$  + 347 $d^6$  +  $> d.$ <br>  $\binom{r}{g(1 + 6d^5)}$ <br>  $\begin{bmatrix} 1 & 1 \end{bmatrix}$ <br>  $\begin{bmatrix} 1 & 1 \end{bmatrix}$ <br>  $\begin{bmatrix} 0 & 1 \end{bmatrix}$ <br>  $\begin{bmatrix} 0 & 1 \end{bmatrix}$ <br>  $> 0$ <br>  $\begin{bmatrix} 2 & 1 \end{bmatrix}$ <br>  $> 0$ <br>  $\begin{bmatrix} 2 & 1 \end{bmatrix}$ <br>  $\begin{bmatrix} 2 & 1 \end{bmatrix}$ <br>  $\begin{bmatrix} 1 & 0 \end{bmatrix}$  $C(g(12, 12)) > C(g(12, 1))$  if and only if 1792 − 1024d – 896d<sup>2</sup> + 544d<sup>3</sup> – 660d<sup>4</sup> – 116d<sup>5</sup> + 347d<sup>6</sup> + 20d<sup>7</sup> – 25d<sup>8</sup> > 0, which always holds because 1 > d.<br>
ing (i) to (vii) we have that  $C(g(1, 0))$  is last in the rankin 116d<sup>5</sup> + 347d<sup>6</sup> + 20d<sup>7</sup> - 25d<sup>8</sup> > 0, which always holds because 1 > d.<br>
sing (i) to (vii) we have that  $C(g(1,0))$  is last in the ranking, and C<br>
2, 1)) are first and second in the ranking, respectively. It remains to s  $C(g(1, 0))$  is last in the ranking, and  $C(g(12, 12))$  and<br>
ae ranking, respectively. It remains to specify the rank-<br>
and  $C(g(1, 2))$ . It follows that  $C(g(1, 2)) > C(g(1, 1))$ <br>  $B^3 - d^4 > 0$ , that is for  $0 < d < 0.2826$ . Similarly,<br>  $C(q(12, 1))$  are first and second in the ranking, respectively. It remains to specify the rank- $C(g(12, 1))$  are first and second in the ranking, respectively. It remains to specify the ranking among  $C(g(12, 0))$ ,  $C(g(1, 1))$  and  $C(g(1, 2))$ . It follows that  $C(g(1, 2)) > C(g(1, 1))$  if and only if  $1 - 32d + 12d^2 + 4d^3 - d^4 > 0$ ing among  $C(g(12,0)), C(g(1,1))$  and  $C(g(1,2))$ . It follows that  $C(g(1,2)) > C(g(1,1))$  $C(g(12,0)), C(g(1,1))$  and  $C(g(1,2))$ . It follows that  $C(g(1,2)) > C(g(1,1))$ <br>if  $1 - 32d + 12d^2 + 4d^3 - d^4 > 0$ , that is for  $0 < d < 0.2826$ . Similarly,<br> $> C(g(12,0))$  if and only if  $0 < d < \frac{2}{3}$ . Finally,  $C(g(12,0)) > C(g(1,1))$ <br>if  $1 - 8d + 6d$ if and only if  $1 - 32d + 12d^2 + 4d^3 - d^4 > 0$ , that is for  $0 < d < 0.2826$ . Similarly, if and only if  $1 - 32d + 12d^2 + 4d^3 - d^4 > 0$ , that is for  $0 < d < 0.2826$ . Similarly,<br>  $C(g(1,2)) > C(g(12,0))$  if and only if  $0 < d < \frac{2}{3}$ . Finally,  $C(g(12,0)) > C(g(1,1))$ <br>
if and only if  $1 - 8d + 6d^2 + 4d^3 - 2d^4 > 0$ , that is for bo  $C(g(1,2)) > C(g(12,0))$  if and only if  $0 < d < \frac{2}{3}$ . Finally,  $C(g(12,0)) > C(g(1,1))$ <br>if and only if  $1 - 8d + 6d^2 + 4d^3 - 2d^4 > 0$ , that is for both  $0 < d < 0.1413$  and  $C(g(1, 2)) > C(g(12, 0))$  if and only if  $0 < d < \frac{2}{3}$ <br>if and only if  $1 - 8d + 6d^2 + 4d^3 - 2d^4 > 0$ , tha<br>0.8587  $< d < 1$ . Combining the three conditions a<br>stance, for  $0 < d \le 0.1413$ , it follows that  $C(g(1, 2)) >$ <br>39  $C(g(1, 1))$ <br>  $\text{ch } 0 < d < 0.1413 \text{ and}$ <br>
the proposition. For in-<br>  $C(g(1, 2)) > C(g(12, 0))$ if and only if  $1 - 8d + 6d^2 + 4d^3 - 2d^4 > 0$ , that is for both  $0 < d < 0.1413$  and  $0.8587 < d < 1$ . Combining the three conditions above yields the proposition. For instance, for  $0 < d \le 0.1413$ , it follows that  $C(g(1,2)) > C(g(1,1$  $0.8587 < d < 1$ . Combining the three conditions above yields the proposition. For in-0.8587  $\lt d \lt 1$ . Combining the three conditions above yields the proposition. For in-<br>stance, for  $0 \lt d \le 0.1413$ , it follows that  $C(g(1,2)) > C(g(1,1)), C(g(1,2)) > C(g(12,0))$ <br>39 stance, for  $0 < d \le 0.1413$ , it follows that  $C(g(1,2)) > C(g(1,1)), C(g(1,2)) > C(g(12,0))$  $d < d \leq 0.1413$ , it follows that  $C(g(1, 2)) > C(g(1, 1)), C(g(1, 2)) > C(g(12, 0))$ <br>39 and  $C(g(12,0)) > C(g(1,1))$  yielding a) in the proposition.

## B.1.5 Social welfare analysisSocial welfare analysis

We give the complete characterization of Proposition 6 in the main text and its proof. We give the complete characterization of Proposition 6 in the main text a<br>Proposition 9 The distribution network that maximizes social welfare is

- $\binom{q(12,1)}{q(12,1)}$ 
	- a.1*)* v
	- (12,12)−(1,2), or<br>
	(12,12)−(1,2), or<br>
	(12,12)−(1,2), or<br>
	(13,184824 < d ≤ 0.9535 and  $0 \le k < k_{(12,12)- (12,1)}^{sw}$ , or
	-

 $\int q(12,1)$ 

- $\begin{split} k_{(12,12)-(1,2)}^{sw}\ s_w\ \frac{(12,12)-(12,1)}{(12,12)-(12,1)}\ \frac{k}{k} &< k_{(12,1)-1}^{sw}\ \end{split}$  $\frac{2}{3}$  and  $k_{(12,12)-(12,1)}^{sw} \le k < k_{(12,1)-(1,2)}^{sw}$ , or<br>
and  $k_{(12,12)-(12,1)}^{sw} \le k < k_{(12,1)-(12,0)}^{sw}$ , or<br>
9535 and  $k_{(12,12)-(12,1)}^{sw} \le k < \overline{k}$ , or<br>
and  $0 \le k < \overline{k}$ .  $k^{sw}_{(12,1)-(1,2)}, \omega$ <br>  $\substack{sw\\(12,1)-(12,0)}, \omega$ <br>  $k < \overline{k}, \omega r$ b.3)
- (12,12)−(12,1) <sup>≤</sup><br>  $w$ <br>
(2,12)−(12,1) <sup>≤</sup><br>  $d k_{(12,12)- (12,1)}$ <br>  $k < \overline{k}$ . b.2)  $\frac{2}{3}$
- $\frac{3}{3}$  <  $a \le 0.67395$  and  $\kappa_{(12,12)-(12,1)}$  ≥<br>0.87953 <  $d < 0.9535$  and  $k_{(12,12)-(12)}^{sw}$ <br>0.9535 <  $d < 1$  and  $0 \le k < \overline{k}$ .<br>then either
- 

 $q(1,2)$ 

**b.4)** 0.9535 <  $d$  < 1 and  $0 \le k < \overline{k}$ .<br>
(1,2) when either<br> **c.1)** 0.032569 <  $d \le 0.184824$  and  $k_{(12,12)-(1,2)}^{sw} \le k < \overline{k}$ , or  $(12,12)-(1,2)$   $\leq$ <br>  $(2,2)$   $\leq$   $k$   $\lt k$   $\overline{k}$ .<br>  $(2,1)-(12,0)$   $\leq$   $k$ <br>
n networks  $g$  $0.1848$ <br> $\frac{2}{3}$  and

 $3^{(--)}$ 

(12,1)−(1,2) ≤<br>
and  $k_{(12,1)-}^{sw}$ <br>
tribution ne<br>
some other (12,1)−(12,0) ≤<br>
ion networks<br>
other distrib<br>  $\frac{7(a-c)^2}{288} - 2k$ <br>
i not attain **Proof:** We first show that the distribution networks  $g(1,0)$  and  $g(1,1)$  are always dominated in social welfare terms by some other distribution network. First note that the difference  $\frac{17(a-c)^2}{576} > \overline{k}$ , and therefore  $g(1,0)$  does no<br>  $W(g(1,2)) - W(g(1,1)) = \frac{(184 - 160d - 12d^2 + 20d)}{18(4-d)^2(2)}$ <br>
fore the distribution network  $g(1,1)$  does no<br>
Next, the difference  $W(g(1,2)) - W(g(1))$ 576 fore the distribution network  $g(1,1)$  does not maximize social welfare.

 $C(g(12,0)) > C(g(1, 1))$  yielding a) in the proposition.<br>
5 Social welffire analysis<br>
give the complete characterization of Proposition 6 in t<br>
position 9 The distribution network that maximizes s<br>  $(12, 12)$  when either<br>
a.1) **oposition 9** The dis<br>  $g(12, 12)$  when either<br> **a.1)**  $0 < d \le 0.032$ <br> **a.2)**  $0.032569 < d$ <br> **a.3)**  $0.184824 < d$ <br>  $g(12, 1)$  when either<br> **b.1)**  $0.184824 < d$ <br> **b.2)**  $\frac{2}{3} < d \le 0.87$ <br> **b.3)**  $0.87953 < d$ <br> **b.4)**  $0.9535 < d <$  $\begin{split} & \textit{when either} \\ & < d \leq 0.032569 \ \textit{and} \ 0 \leq k < \overline{k}, \ \textit{or} \\ & 032569 < d \leq 0.184824 \ \textit{and} \ 0 \leq k \\ & 032569 < d \leq 0.9535 \ \textit{and} \ 0 \leq k \\ & 184824 < d \leq 0.9535 \ \textit{and} \ k_{\{12,12\}}^{\textit{sw}} - \{12,1\} \\ & < d \leq 0.87953 \ \textit{and} \ k_{\{12,12\}}^{\$ 0.032569 and 0 ≤ k < k, or<br>
0.032569 < d ≤ 0.184824 and 0 ≤ k < k,  $^{860}$ <br>
0.184824 < d ≤ 0.9535 and 0 ≤ k < k,  $^{860}$ <br>
1) 0.184824 < d ≤ 0.9535 and 0 ≤ k < k,  $^{860}$ <br>
1) when either<br>
0.184824 < d ≤  $\frac{2}{3}$  and  $k_{($ 0.032569 < d ≤ 0.052509 and 0 ≤ k < k, or<br>
0.032569 < d ≤ 0.184824 and 0 ≤ k < k  $^{80}$ <br>
1) uben either<br>
1) when either<br>
1) 0.184824 < d ≤  $\frac{2}{3}$  and  $k$ **a.3)** 0.184824 < e<br>  $g(12, 1)$  when either<br> **b.1)** 0.184824 < e<br> **b.2)**  $\frac{2}{3} < d \le 0.8$ <br> **b.3)** 0.87953 < d<br> **b.4)** 0.9535 < d<br> **e**<br>  $g(1, 2)$  when either<br> **c.1)** 0.032569 < e<br> **c.2)** 0.184824 < e<br>  $g(12, 0)$  when  $\frac{$ 0.184824 <  $d \leq \frac{2}{3}$ <br>
1  $\frac{2}{3} < d \leq 0.87953$ <br>
1  $0.87953 < d < 0.9$ <br>
1  $0.9535 < d < 1$  a<br>
1 when either<br>
1  $0.032569 < d \leq 0$ <br>
1  $0.184824 < d \leq \frac{2}{3}$ <br>
1  $0.032569 < d \leq 0.8$ <br>
1  $0.184824 < d \leq \frac{2}{3}$ <br>
1  $0.0184824 < d \leq \frac$  $k^{sw}_{(12)}$ <br> $\begin{array}{l} 1 \leq w \leq k \ \frac{1}{2} \leq k$  $k < k_{(12)}^{sw}$ <br>  $< k_{(12)}^{sw}$ <br>  $< k_{(12)}^{sw}$ <br>  $> k < k$ <br>  $>$ <br>  $k < k$ <br>  $>$ <br>  $\leq k < k$ <br>  $>$ <br>  $\leq k < k$ <br>  $>$ <br>  $\leq k <$ <br>  $\leq k$ <br>  $>$ <br>  $\leq k <$ <br>  $\leq k$ <br>  $>$ <br>  $\leq k <$ <br> 184824 <  $d \leq \frac{2}{3}$  and  $k_{(12)}^s$ <br>
<  $d \leq 0.87953$  and  $k_{(12)}^{sw}$ <br>
87953 <  $d < 0.9535$  and<br>
9535 <  $d < 1$  and  $0 \leq h$ <br>
en either<br>
184824 <  $d \leq \frac{2}{3}$  and  $k_{(12)}^{ss}$ <br>
184824 <  $d \leq \frac{2}{3}$  and  $k_{(12)}^{ss}$ <br>
hen  $k < k_{(12)}^{sw}$ <br>  $k_{(12)} \leq k$ <br>  $k_{(12)} \leq k$ <br>  $k_{(13)} \leq k$ <br>  $k_{(23)} \leq k$ <br>  $k_{(30)} \leq k$ <br>  $k_{(4)(a-c)^2}$ <br>  $k_{(4)(a-c)^2$ 0.164824 <  $a \leq \frac{2}{3}$  and  $k_{(12,12)}^{ev}$ <br>
3  $\frac{2}{3} < d \leq 0.87953$  and  $k_{(12,12)}^{ev}$ <br>
0.957953  $<$  d < 0.9535 and  $k_{(12,12)}^{ev}$ <br>
0.9535 < d < 0.9535 and  $k_{(12,12)}^{ev}$ <br>
0.9535 < d < 1 and  $0 \leq k < \overline{k}$ <br>
when either<br>  $k^{exp}_{(12,1)-(1)}$ <br>  $k^{sw}_{(12,1)-(12)}$ ,<br>  $k < \overline{k}$ , or<br>  $k < \overline{k}$ , or<br>  $k < \overline{k}$ ,<br>  $k < \overline{k}$ ,<br>  $g(1,0)$  and the highe<br>  $\frac{c)^2}{4} > 0$  si<br>  $\frac{d(2-3d)(48)}{4(4-\frac{2}{3})}$ . Then<br>
we will co<br>
2) have t<br>
analyzed b  $\frac{1}{3}$  < a ≤ 0.67953 and k<sub>(12,12)</sub>-(c)<br>
0.87953 < d < 0.9535 and k<sub>(12,12)</sub>-(c)<br>
0.9535 < d < 0.9535 and k<sub>(12,12)</sub>-(c)<br>
0.9535 < d < 1 and 0 ≤ k < k.<br>
when either<br>
0.032569 < d ≤ 0.184824 and k<br>
0.184824 < d ≤  $\$ **b.4)**  $0.9535 < d$ <br>  $g(1, 2)$  when either<br> **c.1)**  $0.032569 <$ <br> **c.2)**  $0.184824 <$ <br>  $g(12, 0)$  when  $\frac{2}{3} <$ <br> **oof:** We first show<br>
ted in social welfa:<br>
erence  $W(g(1, 1))$ <br>  $\frac{u-c)^2}{76} > \overline{k}$ , and the<br>  $g(1, 2)) - W(g(1, 1))$ 0.032569 < d ≤ 0.184824 and  $k_{(12)}^{sw}$ <br>
0.032569 < d ≤ 0.184824 and  $k_{(12,1)-(1,2)}^{sw}$ <br>
0.184824 < d ≤  $\frac{2}{3}$  and  $k_{(12,1)-(1,2)}^{sw}$ <br>
0) when  $\frac{2}{3}$  < d ≤ 0.87953 and  $k_{(12,1)}^{sw}$ <br>
We first show that the distributio  $k < \overline{k}$ , or<br>  $< \overline{k}$ .<br>
(1,0) and<br>
ion netwo<br>
positive<br>  $\frac{1}{2}$  highest<br>  $> 0$  sinc<br>
social we<br>  $\frac{2-3d}{4(4-d)^2}$ . Then,  $V$ <br>
will constant by will constant by Then,  $V$ <br>
have to<br>
nalyzed a 0.184824 <  $d \leq \frac{2}{3}$ <br>
0) when  $\frac{2}{3} < d \leq 0$ .<br>
We first show that 1<br>
social welfare terr<br>  $\frac{1}{2}$  W( $g(1,1)$ ) – W( $\frac{1}{2}$ <br>  $\frac{1}{2}$ , and therefore<br>  $\frac{1}{2}$  M( $g(1,1)$ ) =  $\frac{1}{2}$ <br>
distribution networl<br>
the d  $k^{sw}_{(12)}$  and string on  $k^{sw}_{(12)}$  and  $k^{sw}_{(12)}$  and  $k^{sw}_{(12)}$  and  $k^{sw}_{(12)}$  and  $k^{sw}_{(13)}$  and  $k^{sw}_{(13)}$  and  $k^{sw}_{(14)}$  and  $(-1,2)$   $\leq$ <br>  $k < \overline{k}$ .<br>
2,0)  $\leq$ <br>
works<br>
listribu<br>  $- 2k$ <br>
ttain t<br>
aximiz<br>
aximiz<br>
(12, 12<br>
(12, 12<br>
to be c.2) 0.18482<br>  $g(12, 0)$  when  $\frac{2}{3}$ <br>
pof: We first shed in social we<br>
erence  $W(g(1, 1-\frac{c)^2}{76}) > \overline{k}$ , and<br>  $g(1, 2)) - W(g($ <br>
the distributio<br>
Next, the diffesion<br>
independent o<br>  $0 < d < \frac{2}{3}$ ; and<br>  $d < \frac{2}{3}$  where<br>  $d \leq \frac{2}{3}$  and  $k_{(12,1)-(1)}^{sw}$ <br>  $d \leq 0.87953$  and  $k_{(12)}^{sw}$ <br>
ow that the distribution<br>
fare terms by some ot<br>  $d = \frac{17(a)}{2}$ <br>  $d = \frac{17(a$  $k < \overline{k}$ .<br>  $g(1, 0)$ <br>
ition r<br>
is posi<br>
he hig<br>  $\frac{y^2}{2} > 0$ <br>
i.e socia<br>  $\frac{d(2-3d)}{4}$ <br>  $\frac{2}{3}$ . Th<br>
we will<br>
2) have<br>
analy  $g(1, 0)$  and  $g(1, 1)$  are always dom-<br>tion network. First note that the<br>is positive for all  $k \in [0, \overline{k})$  since<br>he highest social welfare. Second,<br> $\frac{y^2}{4} > 0$  since  $1 > d > 0$ , and there-<br>e social welfare.<br> $\frac{d(2-3d)(4$  $W(g(1,1)) - W(g(1,0)) = \frac{1/(a-c)^2}{288}$ <br>  $\overline{k}$ , and therefore  $g(1,0)$  does not  $\overline{k}$ <br>  $-W(g(1,1)) = \frac{(184-160d-12d^2+20d^3-18(4-d)^2(2+d)^2)}{18(4-d)^2(2+d)}$ <br>
stribution network  $g(1,1)$  does not r<br>
che difference  $W(g(1,2)) - W(g(12,2))$ <br>
e − 2k is positive for all  $k \in [0, k)$  since<br>tain the highest social welfare. Second,<br> $\frac{6d^4}{(a-c)^2} > 0$  since  $1 > d > 0$ , and there-<br>aximize social welfare.<br>(1))) =  $\frac{d(2-3d)(48+10d-39d^2-2d^3-4d^4)(a-c)^2}{4(4-d)^2(2-d)^2(2+d)^2(1+d$ > k, and therefore  $g(1,0)$  does not attain the highest social welfare. Second,<br>
)) –  $W(g(1,1)) = \frac{(184 - 160d - 12d^2 + 20d^3 - 5d^4)(a - c)^2}{18(4 - d)^2(2 + d)^2} > 0$  since  $1 > d > 0$ , and there-<br>
distribution network  $g(1, 1)$  does not W(g(1,2)) – W(g(1,1)) =  $\frac{1.184-160d-12d^2+20d^2-5d^2+(20d^2-5d^2)(4-6)^2}{18(4-d)^2(2+d)^2}$ <br>fore the distribution network  $g(1, 1)$  does not maximize<br>Next, the difference  $W(g(1, 2)) - W(g(12, 0)) = \frac{d}{d}$ <br>also independent of k and  $18(4-a)(2+a)$ <br>2) does not m<br> $-W(g(12,0))$ <br>3 does not m > 0 since  $1 > d > 0$ , and there-<br>ocial welfare.<br> $\frac{-3d}{(48+10d-39d^2-2d^3-4d^4)(a-c)^2}$  is<br> $\frac{4(4-d)^2(2-d)^2(2+d)^2(1+d)^2}{4(bd-2d)(2d-2d)}$  is<br>Then,  $W(g(1,2)) > W(g(12,0))$ <br>will consider two cases: (a) for<br>nave to be considered, and (b)<br>a  $g(1, 1)$  does not maximize social welfare.<br>  $(1, 2)$ ) –  $W(g(12, 0)) = \frac{d(2-3d)(48+10d-36)}{4(4-d)^2(2-d)}$ <br>
sitive as long as  $0 < d < \frac{2}{3}$ . Then,  $W(g(10))$ <br>
osite otherwise. Thus, we will consider<br>
2),  $g(12, 1)$  and  $g(12, 12)$ Next, the difference  $W(g(1,2)) - W(g(12,0)) = \frac{d(2-3d)(48+10d-39d^2-2d^2-4d^2)(d-d)^2}{4(4-d)^2(2-d)^2(2+d)^2(1+d)^2}$  is<br>
b independent of k and positive as long as  $0 < d < \frac{2}{3}$ . Then,  $W(g(1,2)) > W(g(12,0))$ <br>  $0 < d < \frac{2}{3}$ ; and the opposite  $W(g(1, 2)) - W(g(12, 0)) = \frac{a_{(2-3d)(48+10d-39d}-2d-2d-4d)(4d-6)}{4(4-d)^2(2-d)^2(2+d)^2(1+d)^2}}$ d positive as long as  $0 < d < \frac{2}{3}$ . Then,  $W(g(1, 2)) > W(g(12, 0))$ posite otherwise. Thus, we will consider two cases: (a)  $g(1, 2)$ ,  $g(12, 1)$  also independent of k and positive as long as  $0 < d < \frac{2}{3}$ . Then, k and positive as long as  $0 < d < \frac{2}{3}$ <br>l the opposite otherwise. Thus, we<br>only  $g(1,2)$ ,  $g(12,1)$  and  $g(12,12)$ <br>e the distribution networks to be a<br>40 W( $g(1, 2)$ ) > W( $g(12, 0)$ )<br>nsider two cases: (a) for<br>be considered, and (b)<br>are  $g(12, 0)$ ,  $g(12, 1)$  and for  $0 < d < \frac{2}{3}$ ; and the opposite otherwise. Thus, we will consider two cases: (a) for  $d < \frac{2}{3}$ <br> $d < \frac{2}{3}$  w<br> $d < 1$ <br>12).  $0 < d < \frac{2}{3}$  where only  $\langle d \rangle < \frac{2}{3}$ <br>  $\langle \frac{2}{3} \rangle < d$ <br>  $\langle 12, 12 \rangle$  $g(1, 2)$ ,  $g(12, 1)$  and  $g(12, 12)$  have to be considered, and (b) distribution networks to be analyzed are  $g(12, 0)$ ,  $g(12, 1)$  and 40 for  $\frac{2}{3} < d < 1$  where the distribution networks to be analyzed are  $g(12,0), g(12,1)$  and  $g(12, 12)$ . % < d < 1 where the distribution networks to be analyzed are  $g(12,0)$ ,  $g(12,1)$  and  $g(12)$ .  $g(12, 12)$ .

(a)  $0 < d < \frac{2}{3}$ .

We first define the thresholds on  $k$  that indicate which one of the three distribution networks  $g(1,2)$ ,  $g(12,1)$  and  $g(12,12)$  is the one that achieves the greatest social welfare.

- The difference  $W(g(12,1)) - W(g(1,2))$  is positive if  $k < (69632 + 26624d +$  $25344d^{2} + 130048d^{3} - 29984d^{4} - 141504d^{5} - 30548d^{6} + 18460d^{7} + 2967d^{8} + 68d^{9} +$  $\frac{(d-\epsilon)^2}{144(4-d)^2(2+d)^2(1+d)^2(16-7d^2)^2}$ . Denote by  $k_{(12,1)-(1,2)}^{sw}$  the later expression, which estion of *d*, is always positive and intersects  $\overline{k}$  in the interval  $0 < d < \frac{2}{3}$  at 296709, being  $k_{(12,1)-(1,2)}^{sw} > \over$  $rac{2}{3}$  at  $0.0296709 < d < \frac{2}{3}$ 

 $< d < \frac{2}{3}$ <br>
We first<br>
defined  $\frac{2}{3}$ <br>
We first<br>
activorks<br>
welfare.<br>
The d<br>
25344 $d^2$ <br>
335 $d^{10}$  )<sub>1</sub><br>
is a func<br>  $d = 0.02$ <br>
329670<br>
The di<br>
3d<sup>4</sup> )<sub>9(4–</sub><br>
is always<br>  $k_{(12,12)-1}^{sw}$ <br>
The di<br>
(4352 –<br>
The di<br>
(435 k that indicate which one of the three distribution<br>  $\ell_1(2,12)$  is the one that achieves the greatest social<br>  $\ell_2(6(1,2))$  is positive if  $k < (69632 + 26624d +$ <br>  $141504d^5 - 30548d^6 + 18460d^7 + 2907d^8 + 68d^9 +$ <br>  $\overline{\tau}$ . g(1, 2), g(12, 1) and g(12, 12) is the one that achieves the greatest social<br>Verence  $W(g(1,1)) - W(g(1,2))$  is positive if  $k < (80832 + 20621d - 1300486^6 - 29084d^4 - 141694e^6 - 20084d^4 - 141694e^6 - 20084d^5 - 18046e^6 - 20084d^3 - 1$ W(g(12, 12)) − W(g(1,2)) is positive if  $k < (69632 + 26624d + 2674d^2 + 6864d + 2141304d^2 + 20804d + 141304d^3 + 2087d^2 + 68d^4 + 2087d^2 + 68d^4 + 2087d^2 + 68d^2 + 208d^2 + 208$ 2534 $H^2 = 180086t^2 - 2988t^2t^2 - 141418t^2 - 3164t^2 - 1414t^2 - 296t^2 + 88t^2 - 144t^2 - 296t^2 + 84t^2 - 296t^2 + 84t^2 - 296t^2 + 84t^2 - 296t^2 - 84t^2 - 84t^2$ 335d<sup>10</sup>)  $\frac{(u-c)}{144(4-d)^2(2+d)^2(1+d)^2}$ <br>
is a function of d, is alvearing the direct point of d, is alvearing to 0.0296709 < d <  $\frac{2}{3}$ .<br>
The difference  $W(g(123d^4)^{\frac{2}{2}(-d)^2(2-d)^2(2-d)^2(1+d)^2(1+d)^2(1+d)^2(2-d)^2(2-d)^2(2+d)^2(1+d)^$  $k_{(12)}^{\text{out}}$  sect d < osit and a sect d of  $k_{(12)}^{\text{out}}$  and  $k_{(1$ is a function of d, is always positive and intersects k in the interval  $0 < d < \frac{1}{2}$ <br>
d – 0.0296709. Leting  $k_{1,1,1}^{(n)} = 1.926709$  and the opposite<br>
of 0.0296709  $\alpha$  d  $\alpha \leq \frac{1}{2}$ .<br>
The difference IV ( $p(1,2,1)$ ) i d = 0.0296709, being  $k_3^{sw}$ <br>
0.0296709 < d <  $\frac{2}{3}$ .<br>
The difference  $W(g(12, 3d^4))\frac{(2-d+d^2)(a-c)^2}{9(4-d)^2(2-d)^2(2+d)^2(1+d)}$ <br>
is always positive and interview in the  $k_{[12,12)-(1,2)}^{su} > \overline{k}$  for  $0 < d$ <br>
- The difference  $W(g($  $d = 0.0296709$ , being  $k_{(12,1)-(1,2)}^{sw} > \overline{k}$  for  $0 < d < 0.0296709$  and the opposite for<br>
0.0296709 <  $d < \frac{2}{3}$ .<br>
- The difference  $W(g(12, 12)) - W(g(1,2))$  is positive if  $k < (34 + 3d - 30d^2 - d^3 + 3d^4)\frac{(2-d+d^2)(a-c)^2}{9(4-d)^2(2-d)^2$ > k for 0 < d < 0.0296709 and the opposite for  $(g(1,2))$  is positive if  $k < (34 + 3d - 30d^2 - d^3 +$ <br>
lenote by  $k_{12,12j-1,12j}^{89}$  (he later expression, which<br>
at  $d = 0.0325694$  in the interval  $0 < d < \frac{2}{3}$ , being<br>
6694 and 0.0296709 <d< 2 - The difference  $W(g(12, 12)) - W(g(1, 2))$  is positive if  $k < (34 + 3d - 30d^2 - d^3 + d^3)$ W(g(1,212)) – W(g(1,2)) is positive if  $k < (34 + 3d - 30d^2 - d^3 + 9d^2 - 2d^2)$ <br>  $W(g(1, 2, 2)) = W(g(1, 2) - 1, 2)$  the later expression, which  $\frac{(6\pi - d)^2}{2(1^2d^2(1+2)^2 - 1)(2^2)}$  be later expression, which<br>
for  $0 < d < 0.0325694$  an 3d<sup>4</sup>)  $\frac{u^{2}-a+{d}}{9(4-d)^{2}(2-d)^{2}(2+d)^{2}(2+d)^{2}(2+d)^{2}}$ <br>is always positive and<br> $k_{(12,12)-(1,2)}^{sw} > \overline{k}$  for 0<br> $k_{(12,12)-(1,2)}^{sw} > \overline{k}$  for 0<br>- The difference  $W(g($ <br>(4352 - 3072d - 5632d<br>denote by  $k_{(12,12)-(12,1)}^{sw}$  and in  $\frac{(2-a+a^2)(a-c)^2}{9(4-d)^2(2-d)^2(2+d)^2(1+d)^2}$ . We denote by  $k_{(12,12)-(1,2)}^{sw}$  the later expression, which ways positive and intersects  $\overline{k}$  at  $d = 0.0325694$  in the interval  $0 < d < \frac{2}{3}$ , being  $\frac{12}{12}-(1,2) > \overline{k}$  for  $k_{(12)}^{\text{sw}}$  the is p  $-$  4 w  $k_{(12,1)}^{\text{sw}}$  the is p  $-$  4 w  $k_{(12,1)}^{\text{sw}}$  or 0 while  $k_{(12)}^{\text{sw}}$  at (  $<$   $k_{(12)}^{\text{sw}}$  )  $>$  0  $\times$ is always positive and intersects k at  $d = 0.0325694$  in the interval  $0 < d < \frac{2}{3}$ , being k at d = 0.0325694 in the interval 0 <d< 2  $k_{(12)}^{\text{sw}}$  - T (43 den and and  $k_{(12)}^{\text{sw}}$  of  $k_{(12)}^{\text{sw}}$  dis  $k$  distribution of  $k_{(12)}^{\text{sw}}$  of  $k_{(12)}^{\text{sw}}$  or  $k_{(12)}^{\text{sw}}$  or  $k_{(12)}^{\text{sw}}$  or  $k_{(12)}^{\text{sw}}$  $(12,12)$   $(1,2)$ <br>The different<br> $4352 - 307$ <br>enote by k<br>nd intersec > $> k$  for  $0 < d < 0.0325694$  and the opposite for  $0.0325694 < \epsilon < \frac{d}{8}$ <br>
nee  $W(g(12,12)) - W(g(12,1))$  is positive if  $k < \frac{1}{14422-39^2(1+\delta)^2(624)}$ <br>  $2d - 5632d^2 + 3040d^3 + 1096d^4 - 476d^3 + 1081d^6 - 68d^2 - 335d^6)$ <br>  $2^8$ <br>  $2^8$  $k_{(12,12)-(1,2)}^{sw} > \overline{k}$  for  $0 < d < 0.0325694$  and the opposite for  $0.0325694 < d < \frac{2}{3}$ . - The difference W(g(12, 12)) – W(g(12, 1)) is positive if  $k < \frac{1}{144(2-\theta)^2(1+\theta)^2}$ <br>
5632 $d^2 + 3040d^3 + 1096d^4 - 476d^5 + 1081d^6 - 68d^7 - 335$ <br>  $>_{0-1(x,1)}$  the later expression, which is positive for  $0 < d <$ <br>
at  $d = 0.0359544$ , being  $k_{\{$ 144(2−d) (1+d) (10−7d)<br>
- 68d<sup>7</sup> - 335d<sup>8</sup>). We<br>
be for  $0 < d < 0.953505$ <br>
for  $0 < d < 0.0359544$ <br>
hat at  $d = 0$  and at (4352 – 3072d – 5632d<sup>2</sup> + 3040d<sup>2</sup> + 1096d<sup>4</sup> – 476d<sup>3</sup> – 1081d<sup>2</sup> – 68d<sup>7</sup> – 335d<sup>3</sup>). We<br>denote by  $k_{\text{eff,1,2}}^{R}$ , 102, 11<sup>1</sup> to kier expression, which is positive for  $0 < d < d$  0.03559544<br>and the opposite for 0.03595 denote by  $k_{(12,12)-(12,1)}^{sw}$  the later expression, which is positive for  $0 < d < 0.953503$  $k_{(12)}^{\text{sc}}$  and the contracts ppose of the contract of  $(12,1)$  and term  $(12,1)$  and term  $(13,1)$  and term  $(1569,1)$  and term  $(1569,1)$  and term  $(13,1)$  and term  $(13,1)$  and term  $(13,1)$  and term  $(13,1)$  and po  $\frac{sw}{(12,12)-(12,1)}$  the later expression, which is positive for 0<br>
ts  $\overline{k}$  at  $d = 0.0359544$ , being  $k_{(12,12)-(12,1)}^{sw} > \overline{k}$  for 0 <<br>
posite for 0.0359544 <  $d < \frac{2}{3}$ . Further note that at<br>
the three thresholds coi  $d < 0.9359544$ <br>  $d = 0$  and at<br>
y are ranked as<br>  $\langle \frac{2}{3}$  the ranking<br>
subcases can be<br>  $\langle k_{(12,12)-(12,1)}^{sw} \rangle$ <br>
are positive and<br>  $\langle k_{(12,12)-(1,2)}^{sw} \rangle$ <br>  $\langle k_{(12,12)-(1,2)}^{sw} \rangle$ <br>  $\langle k_{(12,12)-(1,2)}^{sw} \rangle$ <br>  $g(12,12))$  is th and intersects k at  $d = 0.0359544$ , being  $k_{(12)}^{sw}$ <br>ite for  $0.0359544 < d < \frac{2}{3}$ .<br>
three thresholds coincide; for<br>  $k_{(12,12)-(1,2)}^{sw} < k_{(12,12)-(1,2)}^{sw}$ , will  $\langle k_{(12,12)-(1,2)}^{sw} < k_{(12,12)-(1,2)}^{sw}$ , will<br>  $k_{(12,12)-(1,2)}^{sw} < k_{(12,1)-(1,2)}^{$ (12,12)−(12,1)<br>Further<br>or  $0 < d <$ <br>while for 0.<br>e<sub>2)</sub> Then, tl > k for  $0 < d < 0.0359544$ <br>
note that at  $d = 0$  and at<br>
1.184824 they are ranked as<br>
84824 <  $d < \frac{2}{3}$  the ranking<br>
e following subcases can be<br>  $k_{(12,12)-(1,2)}^{sw} < k_{(12,12)-(12,1)}^{sw}$ <br>
fined above are positive and<br>  $\sum_{j= (1$ and the opposite for  $0.0359544 < d < \frac{2}{3}$ . Further note that at  $d = 0$  and at  $d = 184824$  the three thresholds coincide; for  $0 < d < 0.184824$  they are ranked as (12,1)−(1,2) (1,2)<br>
is  $k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-}^{sw}$ <br>
distinguished:<br>
For  $0 < d < 0.0296709$ , it if  $\frac{2}{3}$  the ranking distinguished:

- $\begin{aligned} \n\frac{sw}{(12,12)-(12,1)} < k^{sw}_{(12,12)-(1,2)} < k^{sw}_{(12,1)-(1,2)}. \n\end{aligned}$  Then, the following subcases can be inguished:<br>  $0 < d < 0.0296709$ , it follows that  $\overline{k} < k^{sw}_{(12,1)-(1,2)} < k^{sw}_{(12,12)-(1,2)} < k^{sw}_{(12,12)-(12,1)}.$ <br>
E  $(a.i)$  For  $0$  $\begin{aligned} \sup_{(12,1)-(1,2)} < k_{(12,12)-(1,2)}^{sw} < k_{(12,12)-(1,1)}^{sw}. \end{aligned}$ <br>differences defined above are positive and<br> $\sup_{(12,1)-(1,2)} < \overline{k} < k_{(12,12)-(1,2)}^{sw} < \frac{1}{12} \end{aligned}$ Then, for all k belonging to  $[0,\overline{k}]$  the three differences defined above are positive and then  $q(12, 12)$  maximizes the social welfare.
- $\frac{S_{w}}{(12,12)-(12,1)}$ . Then for all k belonging to  $[0, \overline{k}]$  we have that both  $W(g(12, 12))$ <br>  $W(g(1,2))$  and  $W(g(12, 12)) W(g(12,1))$  are positive and then  $W(g(12, 12))$  is treatest. Items (a.i) and (a.ii) together prove part greatest. Items (a.i) and (a.ii) together prove part a.1) of the proposition.
- and the opposite for 0.0359544 <  $d < \frac{2}{3}$ <br>  $d = 184824$  the three thresholds coincide;  $t \frac{20}{3}$ <br>  $k_{(12,11)-(1,2)}^{8} \le k_{(12,12)-(1,2)}^{8} \le k_{(12,12)-(1,2)}^{8} \le k_{(12,12)-(1,2)}^{8} \le k_{(12,12)-(1,2)}^{8} \le k_{(12,12)-(1,2)}^{8} \le k_{(12,12)-(1$  $d = 0$  and at<br>  $d = 0$  and at<br>  $\frac{2}{3}$  the ranking<br>  $\frac{2}{3}$  the ranking<br>  $\frac{2}{3}$  the ranking<br>  $\frac{2}{3}$ <br>  $\frac{2}{3}$ <br> d = 184824 the three thresholds coincide; for  $0 < d < 0.184824$  they are ranked as  $k_{12,11}^{(8)}$ ,  $(1,2) < k_{12,121}^{(8)}$ ,  $(1,2) < k_{12,122}^{(8)}$ , while for  $0.184824 < d < \frac{3}{3}$  the ranking  $k_{12,122}^{(8)}$ ,  $(1,2) < k_{12,122}^{$  $k_{(12}^{\text{su}}$  is  $k$  dist For The the Fo $k_{(12}^{\text{su}}$   $W($  gree  $)$  Fe  $k_{(12}^{\text{su}}$   $0 < W($  a.  $2$  $< k_{(12)}^{sw}$ <br>  $(2,1)$   $<$ <br>  $(0.029)$ <br>  $1 k$  bel<br>  $(12)$  ma<br>  $(20)$  ma<br>  $(20)$  mand  $W$ <br>  $(12)$   $(12)$ <br>  $(12)$   $(12)$ <br>  $(W)$ <br>  $(W)$  $\langle k_{(1,2)}^{ss} \rangle$ <br>  $\langle k_{(1,2)}^{ss} \rangle$ <br>
bllows<br>
blows<br>
blows<br>
blows<br>
blows<br>  $\{0, \overline{k}\}$ <br>
he soc<br>  $\{2569, i \quad k \quad \text{bel}$ <br>  $k \quad \text{bel}$ <br>  $\{954, i \quad \leq k \quad \leq \{2, 12\} - (\text{and } V)$ <br>
prove  $\frac{sw}{(12,12)-(12,1)},$  while for 0.184824 <  $d < \frac{2}{3}$ <br>  $\langle k_{(12,1)-(1,2)}^{sw} \rangle$ . Then, the following sub<br>  $\sqrt{12,12-(1,2)}$  is that  $\overline{k} \langle k_{(12,1)-(1,2)}^{sw} \rangle \langle k_{(12,1)-(1,2)}^{sw} \rangle \langle k_{(12,1)-(1,2)}^{sw} \rangle$ <br>  $\overline{k}$  the three difference  $k_{(12)}^{\text{out}}$  and  $\text{in}$  an  $\langle k_{(12)}^{sw}\rangle$ <br>296709<br>elongi<br>aximi<br> $\langle d \rangle$  then for  $W(g(1))$ <br> $\langle d \rangle$ <br> $\langle d \rangle$  then, for  $\langle k \rangle$ <br> $W(g(1))$ <br>positic  $\langle k_{(12)}^{sw}$ <br>s that  $\vec{k}$  the  $\vec{k}$  the  $\vec{k}$ <br>ocial w, it foll elongit<br> $W(g(1 \text{ together} \text{it} \text{it} \text{right}))$ <br> $\langle k_{(12)}^{sw}$ <br> $W(g($ <br>red. B  $d < d < 0.0296709$ , it follows that  $k < k_{(12)}^{sw}$ <br>for all k belonging to  $[0, \overline{k}]$  the three differe.<br> $g(12, 12)$  maximizes the social welfare.<br> $(12, 12)$  maximizes the social welfare.<br> $(12, 12)$  Then for all k belonging t  $< k_{(12)}^{\text{sw}}$ <br>define<br> $\frac{w}{(12,1)-(12,1)}$  on ave the and<br> $\frac{w}{(12,1)-(1,2)}$ <br> $\frac{w}{(12,1)-(1,2)}$ <br> $\frac{w}{(12,1)-(1,2)}$  $< k_{(12)}^{sw}$ <br>are pc<br> $k_{(12)}^{sw}$ <br> $W(g(12, y))$ <br> $g(12, y)$ <br> $(12, 12)$ <br> $(12, 12)$ <br> $(12, 12)$ <br> $(12, 12)$ <br> $(12)$ <br> $(12)$ <br> $(12)$ k belonging to [0, k] the three differences defined above are positive and<br>
) maximizes the social welfare.<br>  $0.9 < d < 0.032569$ , it follows that  $0 < k_{(12,1)-(1,2)}^{sw} < \overline{k} < k_{(12,12)-(1,2)}^{sw} < 0$ .<br>
Then for all k belonging to  $g(12, 12)$  maximizes the social welfare.<br>
0.0296709 <  $d$  < 0.032569, it follows th<br>
2)-(12,1). Then for all  $k$  belonging to [1,2)] and  $W(g(12, 12)) - W(g(12, 1))$  is<br>
est. Items (a.i) and (a.ii) together pro<br>
0.032569 <  $d$ (a.ii) For 0.0296709 < d < 0.032569, it follows that  $0 < k_{(12)}^{sw}$ <br>  $k_{(12,12)-(12,1)}^{sw}$ . Then for all k belonging to  $[0, \overline{k}]$  we have  $W(g(1,2))$  and  $W(g(12,12)) - W(g(12,1))$  are positive greatest. Items (a.i) and (a.ii) toge  $k < k$  is  $\{k\}$ <br>both  $W(g(12))$ <br> $\text{in } W(g(12))$ <br> $\geq \text{propositi}$ <br> $\leq k_{(12,12)}^{sw}$ <br> $\geq k_{(12,12)}^{sw}$ <br>following is satisfied.<br> $k_{(12,12)-1}^{sw}$  $\frac{1}{\pi}$   $\frac{1}{\pi}$  de  $\frac{1}{\pi}$  or es rt  $\frac{1}{\pi}$  $k_{(12)}^{sw}$ <br> $W($  gree  $k_{(12)}^{sw}$ <br> $0 < W($ <br> $a.2$ k belonging to  $[0, k]$  we have that both  $W(g(12, 12)) -$ <br>  $)-W(g(12, 1))$  are positive and then  $W(g(12, 12))$  is the<br> **i.ii)** together prove part **a.1)** of the proposition.<br>
954, it follows that  $0 < k_{(12, 1) - (1, 2)}^{sw} < k_{(12, 12$ W(g(1,2)) and W(g(12, 12)) – W(g(12, 11)) are positive and then W(g(12, 12)) is the<br>greatest. Items (a.i) and (a.ii) together prove part a.1) of the proposition.<br>
) For 0.032569 < d < 0.035954, it follows that  $0 < k_{(12,1$ (a.iii) For  $0.032569 < d < 0.035954$ , it follows that  $0 < k_{(12)}^{sw}$ <br>  $k_{(12,12)-(12,1)}^{sw}$ . Then, for  $0 \le k < k_{(12,1)-(1,2)}^{sw} < k_{(12,12)}^{sw}$ <br>  $0 < k_{(12,1)-(1,2)}^{sw} < k < k_{(12,12)-(1,2)}^{sw} < k < k_{(12,12)-(12)}^{sw}$ <br>  $W(g(12,12)) > W(g(1,2))$  and  $W(g($  $\frac{1}{12,-1,-1,2}$  (12,12)−(1,2)<br>  $\frac{1}{12,-1,2}$  (12,12)−(1,2)<br>
(12,1), the following ineq<br>
(12,1)) are satisfied. The<br>  $\frac{1}{2,1,-1,2}$  ( $k_{(12,12)-1,2}^{sw}$ )  $\langle k_{(12)}^{sw} \rangle$ <br>  $\overline{k} \langle k \rangle$ <br>
follow<br>
e satis<br>  $k_{(12)}^{sw}$  $(k, k)$ , or<br>aalities<br>n part<br> $k < k <$  $k_{(12)}^{\text{out}}$ <br>0 <  $W($ <br>a.2  $\begin{aligned} \n\mathcal{L}_{(12,12)-(12,1)}^{sw} \text{ Then, for } & 0 \leq k < k_{(12,1)-(1,2)}^{sw} < k_{(12,12)-(1,2)}^{sw} < k < k_{(12,12)-(1,1)}^{sw}, \text{ or } \\ \n& k_{(12,1)-(1,2)}^{sw} < k < k_{(12,12)-(1,2)}^{sw} < k < k_{(12,12)-(1,2)}^{sw} < k^{sw}, \text{ the following inequalities} \\ \n\mathcal{W}(g(12,12)) > W(g(1,2)) \text{$  $k < k_{(12)}^{s,w}$ <br>2)-(1,2)<br>dd  $W(g)$ <br>oved. E  $< k_{(12)}^{sw}$ <br>sw<br>(12,12)  $> W($ <br> $>$   $\left\langle k_{0}^{k}\right\rangle$ <br> $>$   $< k_{0}^{k}$  $k < k_{(12)}$ <br>  $k \leq k_{(12)}$ <br>  $k \leq k_{(12,12)}$ <br>  $k \leq k_{(12,12)}$  $0 < k_{(12,1)-(1,2)}^{sw} < k < k_{(12,12)-(1,2)}^{sw} < k < k_{(12,12)-(1,1)}^{sw}$ , the following inequalities <br>  $W(g(12,12)) > W(g(1,2))$  and  $W(g(12,12)) > W(g(12,1))$  are satisfied. Then part <br> **a.2)** in the proposition is proved. But for  $0 < k_{(12,1)-(1,2)}$  $\langle k_{(12)}^{sgn} \rangle$ <br> $(g(12))$  in (12,12)−(1,2)<br>(12,12) >  $W(g(1,2))$  and  $W(g(n, 12))$  and  $W(g(n, 12))$  and  $W(g(n, 12))$  $\langle k \rangle \langle k_{(12)}^{sw}$ <br> $W(g(1,2))$  is position is  $\langle k \rangle \langle k_{(12)}^{sw}(12,12) \rangle$ <br>3ut for 0  $\langle k \rangle$ <br>41  $W(g(12, 12)) > W(g(1, 2))$  and  $W(g(12, 12)) > W(g(12, 1))$  are satisfied. Then part <br> **a.2)** in the proposition is proved. But for  $0 < k_{(12, 1)-(1, 2)}^{sw} < k_{(12, 12)-(1, 2)}^{sw} < k <$ <br>
41 a.2) in the proposition is proved. But for <sup>0</sup>  $\langle k_{(12)}^{sw}$ (12,1)−(1,2)  $\langle k_{(12)}^{sw} \rangle$ (12,12)−(1,2) <k<

- $\begin{align*}\n\text{(12,12)} (12,1), & \text{it follows that} \\
\text{(12,12)} (12,1), & \text{if follows that} \\
\text{(12,12)} (12,1), & \text{(12,13)} \leq k \leq 0.184824, \\
\text{(12,11)} \leq k. & \text{then, for } 0 \leq k \leq k \leq k. \n\end{align*}$  $\begin{aligned} \mathcal{H}^{(12,1)-1,2)} &\leq k^{sw}_{(12,12)- (1,2)} \ \mathcal{H}^{(12,12)-1,2)} &\leq k^{sw}_{(12,12)- (12,1)} \ \mathcal{H}^{(12,12)-1,2} &\leq k^{sw}_{(12,12)- (1,2)} \leq k^{sw}_{(12,12)- (12,1)} \leq k \ \mathcal{H}^{(12,12)-1,2} &\leq k \leq k, \ g(1) \end{aligned}$ (12,12)–(12,1)<br>
or  $0 < k^{sw}_{(12,12)-(1,2)} < k < k^{sw}_{(12,12)-(1,2)} < k^{sw}_{($ welfare is achieved by  $g(12,12)$ ; while for  $0 < k_{(12,12)-(12,1)}^{sw}$ <br>  $k_{(12,12)-(12,1)}^{sw} < \overline{k}$  or  $0 < k_{(12,1)-(1,2)}^{sw} < k_{(12,12)-(1,2)}^{sw} < k_{(12)}^{sw}$ <br>
gives the highest social welfare. Items (a.iii) and (a.iv<br>
proposition.  $\begin{align} (12,1)^\perp(1,2) \ \times \left( \frac{12,12}{12,12} \right) < k < k \ \text{and} \ (\mathbf{a}.\mathbf{iv}) \ \text{prove part c.1} \ \times \left( k^{sw} \right) < k^{sw} \end{align}$ (12,12)−(12,1)<br>
ives the highest social welfare. Items (**a.iii**) and (**a.iv**) prove<br>
roposition.<br>
For 0.184824 <  $d < \frac{2}{3}$ , we have  $0 < k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(1,2)}^{sw} < k_{(12,12)-(1,2)}^{sw}$ gives the highest social welfare. Items  $(a.iii)$  and  $(a.iv)$  prove part  $c.1$ ) of the proposition.
- $k < k$ ,  $k < k$ ,  $k$ ,  $k$ ,  $k$ ,  $k$ ,  $k$ ,  $k$ ,  $W(g(1, 2)$  For 0.0<br>  $k^{sw}_{(12,12)-1}$  or 0 <  $k^{sw}_{(12,12)-1}$  or 0 <  $k^{sw}_{(12,12)-1}$  gives the propositi For 0.18<br>
Then, for 0.18<br>
Then, for 0.18<br>
Then, for 0.18<br>
Then, for 0.18<br>
T  $W(g(12, 12)) < W(g(1, 2))$  and  $W(g(12, 1))   
the highest social welfare. \nfollows that  $0 < k_{11,11}^{(12)} - (k_{12}^{(12)}) < k_{12,12}^{(2)} - (k_{12}^{(2)})   
 (k_{12,11}^{(2)} - (k_{12}^{(2)})   
 (k_{12,12}^{(2)})   
 (k_{12,12}^{(2)})   
 (k_{12,12}^{(2)})   
 (k_{12,12}^{(2)})   
 (k_{12,12}^{(2)})   
 (k_{12,1$$ W(g(1,2)). Then W(g(1,2)) gives the highest social welfare.<br>
For 0.033954 < d < 0.154824, it follows that  $0 < k_{1011}^{\text{NS}}$ ,<br>  $\hat{b}_{112123}^{\text{NS}}$ ,  $\hat{b}_{12123}^{\text{NS}}$ ,  $\hat{b}_{12133}^{\text{NS}}$ ,  $\hat{b}_{12133}^{\text{NS}}$ ,  $\hat{b}_{1213$ (a.iv) For 0.035954  $\leq d \leq 0.184824$ , it follows that  $0 \leq k_{\text{FB},111}^{(2)} - (3.1)$ ,  $\leq k_{\text{FB},111}^{(2)} - (3.1$  $< k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw}$ <br>  $< k_{(12,12)}^{sw}$ <br>  $< k_{(12,12)}^{sw}$ <br>  $< k_{(12)}^{sw}$ <br>  $< k_{(12)}^{sw}$ <br>  $< k_{(12)}^{sw}$ <br>  $\geq 304d$ <br>
ote by<br>  $2,0), g$ <br>  $2304d$ <br>
ote by<br>  $2,0), g$ <br>  $2304d$ <br>  $\overline{k}$ , al  $<$  2) ne  $\overline{k}$ . al  $<$  1)  $<$  1) dd  $+$  2, (in at  $\forall$  e as al re  $\overline{k}$ . t  $<$  $k_{(12)}^{sw}$  well  $k_{(12)}^{sw}$  pro Fo The well  $k_{(12)}^{sw}$  and  $< k$ . Then, for 0 ≤ k < k<sup>sw</sup><sub>(12</sub>)-(1,2)<br>  $0$  = (1,2) < k < k'<sub>(12)</sub>-(1,2); while  $\leq k$  or 0 < k<sup>sw</sup><sub>(12)</sub>-(1,2); while  $\leq k$  or 0 < k<sup>sw</sup><sub>(12,1</sub>)-(1,2) < k hest social welfare. Items<br>  $1 < d < \frac{2}{4}$ , we have 0 < k<sup>sw</sup>  $< k_{(12,1)}^{sw}$ <br>  $< k_{(12,1)}^{sw}$ <br>  $k_{(12,1)}^{sw}$ <br>  $k_{(12,1)}^{sw}$ <br>
and (<br>  $\leq k_{(12,1)}^{sw}$ <br>  $k_{(12,1)}^{sw}$ <br>  $k_{(12,1)}^{sw}$ <br>  $k_{(12,1)}^{sw}$ <br>  $k_{(12,1)}^{sw}$ <br>  $k_{(12,1)}^{sw}$ <br>  $\leq k_{(12)}^{s}$ <br>
We which  $\leq k_{(12)}^{s}$ <br>  $\leq k_{(12)}^{s}$ <br>
sp  $< k_{(12,12)}^{sw}$ <br>  $< k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw}$ <br>  $\frac{k_{(12,12)}^{sw}}{k}$ <br>  $\frac{k}{k}$  the  $< k < k < 1$ <br>  $\frac{k}{k}$  the  $< k < 1$ <br>  $\frac{1}{14(2-i)}$ <br>  $\frac{1}{144(2-i)}$ <br>  $\frac{1}{144(2-i)}$ <br>  $\frac{1}{144(2-i)}$ <br>  $\frac{k_{(12,12)}^{sw}}{k}$ <br>  $\frac{k_{(12,12)}^{sw}}{k}$  $< k$ ,  $>$  cial<br> $k < k$ ,  $>$  cial<br> $k < 1, 2$ )<br> $\geq k$ .  $\geq 1$ <br> $\geq k$ .  $\geq 2, 1$ )<br> $\geq k$ .  $\geq 2, 1$ <br> $\geq k$ .  $\geq$  $\langle k_{12}^{sw} \rangle$ <br>  $\langle k_{12}^{sw} \rangle$  and  $k_{21}^{sw}$ <br>  $\langle k_{12}^{sw} \rangle$  and  $\langle k_{12}^{sw} \rangle$  are by  $k$ <br>  $\langle k_{12}^{sw} \rangle$  and  $\langle k_{12}^{sw} \rangle$ <k<ksw  $< k_{(12)}^{so}$ <br>  $< k_{(21)}^{so}$ <br>  $k_{(12)}^{so}$ <br>  $k_{(12)}^{so}$ <br>  $k_{(12)}^{so}$ <br>  $k_{(12)}^{so}$ <br>  $k_{(2-d)}^{so}$ <br>  $k_{(2-d)}^{so}$ <br>  $k_{(12,1)}^{so}$ <br>  $k_{(12,1)}^{$ < k the greatest social<br>  $0 \le k \le m$ <br>  $0 \le k \le 1, 2, -1, 2, 2, \le k \le 2) - (12, 1) \le k \le k$ <br>  $2) - (12, 1) \le k \le k$ ,  $g(1, 2)$ <br>
prove part **c.1**) of the<br>  $-(1, 2) \le k \le m$ <br>  $\sqrt{k}$  the greatest social<br>  $\sum_{i=1}^{k} k \le k \frac{2w}{(12, 12) - (1, 2)}$  $g(12, 12)$ ; while for  $0 < k_{(12, 12) - (1, 2)}^{sw}$ <br>  $\leq k_{(12, 1) - (1, 2)}^{sw}$   $\leq k_{(12, 12) - (1, 2)}^{sw}$ <br>  $\leq k_{(12, 12) - (1, 2)}^{sw}$ <br>  $\leq k_{(12, 12) - (12, 1)}^{sw}$ <br>  $\leq k_{(12, 12) - (12, 1)}^{sw}$ <br>  $\leq k_{(12, 12) - (12, 1)}^{sw}$   $\leq k_{(12,$  $\langle k_{(12)}^{\text{sw}} | 2 \rangle$ <br>  $\langle k_{(12)}^{\text{sw}} | 2 \rangle$ <br>  $\langle k_{(12)}^{\text{sw}} | 2 \rangle$ <br>  $\langle k_{(12,1)}^{\text{sw}} | 2 \rangle$ <br>  $\langle k_{(12,1)}^{\text{sw}} | 2 \rangle$ <br>  $\langle k_{(12,1)}^{\text{sw}} | 2 \rangle$ <br>  $\langle k_{(12)}^{\text{sw}} | 2 \rangle$ <br>  $\langle k_{(12)}^{\text{new}} | 2 \rangle$ <br>  $\langle k_{(12)}^{\text{sw}} | 2 \rangle$  $g(1,2)$ <br>of the<br> $g(1,2) < \overline{k}$ .<br>t social<br> $-(1,2) < g(12,1)$ <br> $((1,2) < g(12,1))$ <br> $((1,2) <$ <br>ts **b.1**)<br>1) and<br> $324d^2 +$ <br> $(1,1) - (12,0)$ <br> $(1,2) < \overline{k}$ .<br>ts  $\overline{k}$  in that<br> $(2)^2$ . We<br>ersects<br>follows<br>in case<br>nerval<br>ses are<br> $\frac{1}{k}$ . gr  $k_{(12)}^{\text{sw}}$  pro Fo The well  $k_{(12)}^{\text{sw}}$  we  $k_{(12)}^{\text{sw}}$  and  $\langle k \text{ or } 0 \rangle \langle k \rangle_{(12)}^{sw}$ <br>
hest social well<br>  $k \langle k \rangle_{(12,12)- (12)}^{sw}$ <br>  $k \langle k \rangle_{(12,12)- (12)}^{sw}$ <br>
ained with  $g(12)$ <br>  $\overline{k}$  or  $0 \langle k \rangle_{(12,12)}^{sw}$ <br>
hest social well<br>  $k \langle k \rangle$  the grea<br>
the proposition.<br>
Le the threshold<br>  $< k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw}$ <br>  $d^7)$   $\frac{1}{144}$ <br>  $\frac{1}{144}$  $< k_{(12)}^{sw}$ <br>  $(k_{(12,1)}^{sw}$ <br>  $(k_{(12,1)}^{sw})$ <br>  $(k_{(12,12)}^{sw})$ <br>  $< k$ <br>  $(k_{(12,12)}^{sw})$ <br>  $k_{(12,12)}^{sw}$ <br>  $(k_{(12,12)}^{sw})$ <br>  $(k \geq 2)$  $k \leq k < k$ ,  $g(1, 2)$ <br>
bart **c.1)** of the<br>  $k_{(12,1)-(1,2)}^{sw} < \overline{k}$ .<br>
e greatest social<br>  $k_{(12,12)-(1,2)}^{sw} < k_{(12,12)-(1,2)}^{sw} < k_{(12,12)-(1,2)}^{sw} < k_{(12,12)-(1,2)}^{sw} < k_{(12,12)-(1,2)}^{sw}$ <br>
roves parts **b.1)**<br>
(0),  $g(12,1)$ (a.v) For 0.184824 <  $d < \frac{2}{3}$ <br>
Then, for  $0 \le k < k_1^{sw}$ <br>
welfare is attained with<br>  $k_1^{sw}$  welfare is attained with<br>  $k_2^{sw}$ <br>
welfare is attained with<br>  $k_3^{sw}$ <br>  $(k_{(12,1)-(1,2)}^{sw} < \overline{k}$  or  $0 <$ <br>
gives the highest soci  $\langle k_{(12,12)}^{sw} \rangle$ <br>  $k_{(12,12)}^{sw}$  For ei<br>  $k_{(12,12)}^{sw}$  For ei<br>  $k_{(12,12)}^{sw}$  Finall with the gree  $(12,0)$ <br>  $(12,0)$ <br>  $(12,0)$ <br>  $(12,0)$ <br>  $(k_{(12,12)}^{sw} \leq k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw} \leq k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw}$ <br>  $(k_{12,12)-(1,2)} < k_{(12,12)-(1,2)} < k$  (12,17)−(1,2)<br>  $(k_{(12,12)-(1,2)}< k < k$  (12,12)−(1,2)<br>  $k < k_{(12,12)-(1,2)}< k < k_{(12,12)-(1,2)}< k, g$ <br>  $(k_{(12,12)-(1,2)}< k < k_{(12,1)-(1,2)}< k, g$ <br>  $(k_{(12,12)-(1,2)-(1,2)-(1,2)}< k, s$  $\langle k^{sw}_{(12,1)} \rangle$ <br>sw  $\langle k^{sw}_{(12,1)} \rangle$ <br> $\langle k^{sw}_{(12,12)} \rangle$ <br> $\langle k^{sw}_{(12,12)} \rangle$ <br> $\langle k^{sw}_{(12,12)} \rangle$ <br> $\langle k^{sw}_{(12,12)} \rangle$ <br>which al wel:<br> $\frac{c}{2}$ <br> $\frac{c}{3}$ <br> $\langle k^{sw}_{(12,12)} \rangle$ <br> $\langle k^{sw}_{(12,12)} \rangle$ <br> $\langle k^{sw}_{(12,12)} \rangle$ <br> $\langle k^{sw}_{(12,12)} \rangle$ <br> $\langle k^{sw$  $< k_{(12)}^{sw}$ <br>  $k < k_{(12)}^{sw}$ <br>  $k < k_{(22)}^{sw}$ <br>  $< k_{(32)}^{sw}$ <br>  $< k_{(42-d)}^{sw}$ <br>  $= 2304$  $< k$ .  $\frac{1}{2}$   $<$   $\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac{1}{2}$  and  $d^2 + \frac{1}{k}$ .  $\frac{1}{k}$  and  $d^2 + \frac{1}{k}$  in that We sects lows case erval s are  $\frac{1}{k}$ .  $\frac{1}{k}$  reat- $k < k$ Then, for  $0 \leq k < k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(1,2)}^{sw} < k_{(12,1)-(1,2)}^{sw} < \overline{k}$  the greatest social  $k < k_{(12)}^{sw}$ <br>
ined with  $\overline{k}$  or  $0 <$ <br>
est socing is the proportion of  $k < \overline{k}$  the proportion of  $k < \overline{k}$  the proportion of  $4-1990e$ <br>
est socing the  $W(g(14-1990e))$ <br>  $k = k$  the  $W(g(112)-(12,02e))$ <br>  $k = k$  or the  $k$ <br>  $k$ (12,12) (12,11) (12,12) (1,2)<br>
(12,11) (1,2)<br>
(12,12)-<br>  $\langle k_{(12)}^{sw}\rangle$ . For<br>  $(12,1)$   $\langle k_{(12)}^{sw}\rangle$ . Fin<br>
socia<br>  $\frac{1}{2}$   $k$  the  $\frac{1}{3}$ <br>  $\frac{1}{3}$   $\frac{1}{3}$ <br>  $\frac$  $< k_{0.12}^{sw}$ <br>  $< k_{(12)}^{sw}$ <br>  $< k_{(12)}^{sw}$ <br>  $\leq k_{(12)}^{sw}$ <br>  $\leq 0 < R$ <br>
is  $W((a-c)^2)$ <br>
is  $W((a-c)^2)$ <br>  $\leq (a-c)^2$ <br>  $\leq (1+d)$ <br>  $\leq 3$ <br>  $\leq 795$ ;<br>
salwa<br>
for  $\frac{2}{3}$ <br>  $\leq 795$ ;<br>
positive<br>
then, i<br>  $d > 0$ <br>  $-W((a-c)^2)$ <br>  $\leq k$ <br>  $\langle k \rangle \langle k \rangle \langle k \rangle \langle k \rangle \langle k \rangle$ <br>  $\langle k \rangle \langle k \rangle \langle k \rangle \langle 12,12 \rangle - (1,2) \langle k \rangle \langle k \rangle \langle 12,12 \rangle - (1,2) \langle k \rangle \langle k \rangle \langle k \rangle \langle 12,12 \rangle - (1,$ welfare is attained with  $g(12, 12)$ . For either  $0 < k_{(12, 12)-(12, 1)}^{sw} < k < k_{(12, 12)-(1, 2)}^{sw} <$  $(12,12)-(12,1)$  (12,12)−(1,2)<br>  $0 < k < k_{(12,12)-(12,1)}^{sw} < k, g(12,12)$ <br>  $0 < k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(1,2)}^{sw}$ <br>  $V(g(1,2))$ . This proves parts **b**. gives the highest social welfare. Finally, for  $0 < k_{(12,12)-(12,1)}^{(12,12)-(12,1)} < k$ <br>  $k_{(12,1)-(1,2)}^{sw} < k < \overline{k}$  the greatest social welfare is  $W(g(1,2))$ . This provand **c.2)** of the proposition. (12,12)−(12,1) (12,12)−(1,2)<br>(1,2)). This proves parts **b**<br>ch one of  $g(12,0)$ ,  $g(12,1)$ and c.2) of the proposition.
- (b)  $\frac{2}{3} < d < 1$ .

(12,1)−(1,2)<br>
nd **c.2)** o:<br>  $d < 1$ .<br>
We now de 3 We now define the thresholds on

 $g(12, 12)$ . For either  $0 < k_{(12, 12)-(12, 1)}^{sw}$ <br>  $s_w^{sw}$ <br>  $s_w^{sw}$ <br>  $s_w^{sw}$ <br>  $s_w^{sw}$   $(12, 12)-(12, 1)$   $\leq k_{(12, 12)-(1, 2)}^{sw}$ <br>  $s_w^{sw}$ <br>  $s_w^{sw}$   $s_w^{sw}$   $s_w^{sw}$   $s_w^{sw}$   $s_w^{sw}$ <br>  $s_w^{sw}$   $s_w^{sw}$ <br>  $s_w^{sw}$   $-267d^6 + 335d^7$ <br>  $k < k$ sw<br>  $k(12, 1) - (1, 2)$ <br>  $k(12, 12) - (1, 2)$ <br>  $k(12, 12) - (1, 2)$ <br>  $k(12, 12) - (1, 2)$ (1)  $<$  1) and  $+$  2, 0 at  $\frac{1}{k}$  at  $k_{(12)}^{sw}$  and  $\lt \leq$  We  $g(1 - T)$  den a  $k_{(12)}^{sw}$  and  $\lt \leq$  We  $g(1 - T)$  den a  $k_{(12)}^{sw}$  and  $k_{(12)}^{sw}$  and  $\leq$  and  $k_{(12)}^{sw}$  and  $\leq$  and  $k < k$  or  $0 < k_{(12)}^{sw}$ <br>
ighest social words in the special words  $k < \overline{k}$  the gree the proposition<br>
ine the threshorolic means the one that acroite  $W(g(12, 1))$ <br>  $W(g(12, 1))$ <br>  $W(g(12, 1))$ <br>  $\frac{2}{3} < d < 1$  at  $\frac{2$  $< k_{(12)}^{sw}$ <br>  $\leq k_{(12,12)}^{sw}$ <br>  $\leq k_{(12,$  $k < k < k_{(12)}^{sw}$ <br>  $k_{(12,12)-(12)}^{sw}$ <br>  $(g(1,2))$ . T<br>  $(g(1,2))$ . T<br>  $(g(1,2))$ . T<br>  $\frac{1}{3}k < (438)$ <br>  $\frac{2}{3} < d < 0$ .<br>
3.<br>
We if  $k < 1$ <br>
s always p<br>
for  $\frac{2}{3} < d$ <br>
9.956154.<br>  $(g(12,1))$  i<br>
resholds a<br>  $g,0$ . The fo<br>  $\langle k, g(12, 1)$ <br>  $\frac{w}{12,12)-(1,2)}$   $\leq$ <br>  $\frac{w}{12,12)-(1,2)}$   $\leq$ <br>  $\geq$  parts **b.1)**<br>  $g(12, 1)$  and<br>  $4d-5824d^2 +$ <br>  $\frac{w}{(12,1)-(12,0)}$ <br>  $\frac{d^2}{(1+d)^2}$ . We<br>
md intersects  $\overline{k}$  in follows<br>
me as in case<br>  $\frac{13-(12,0)}{k$  $< k_{012}^{\text{sw}}$ <br>  $W(g(1$ <br>
which well  $W(g(1$ <br>  $W(g(1$ <br>  $\frac{-c)^2}{1+d})(16-$ <br>  $\frac{-c)^2}{1+d}$ <br>  $\frac{1}{3}$ <br>  $\frac{1}{$  $< k_{(12)}^{sw}$ <br>2, 0), 2<br>2304 $a$ <br>note by<br>md int<br>53 it 1<br> $\frac{2d+12d^2}{(2-d)^2}$ <br>ive an esammies in  $\frac{2d+12d^2}{(12,d)^2}$ <br>ive an int<br>is in  $\frac{2d+12d^2}{(12,d)^2}$ <br> $k_{(12,1)}^{sw}$ (1) and  $+$   $\frac{1}{2}$ ,  $\frac{1}{2}$  at  $\frac{1}{2}$   $\frac{1}{2$  $k_{(12)}^{\text{out}}$  and  $\leq$  We  $g(1 - T)$  368 the  $k_{(12)}^{\text{out}}$  and  $k_{(12)}^{\text{out}}$  and  $k_{(12)}^{\text{out}}$  and  $\frac{2}{3}$   $\leq$  and  $F$  or The est  $\langle k \rangle \langle k \rangle$  the greatest social welfare is  $W(g(1, 2))$ . This proves parts b.1)<br>the proposition.<br>Since the densities on  $k$  that indicate which one of  $g(12, 0)$ ,  $g(12, 1)$  and<br>the one that schieves the greatest social we  $d < 1$ .<br>We now  $d$ <br> $d(12, 12)$ <br>The diffication  $d^3+2$ <br> $d^3+2$ <br> $e$  later<br>the interview  $d^3+2$ <br> $e^{3w}$ <br> $e^{2w}$ k that indicate which one of  $g(12, 0)$ ,  $g(12, 1)$  and<br>the greatest social welfare.<br> $g(12, 0)$ ) is positive if  $k < (4352 - 2304d - 5824d^2 + 335d^7)\frac{(a - c)^2}{144(2 - d)^2(1 + d)(16 - 7d^2)^2}$ . Denote by  $k_{(12, 1)}^{\text{exp}} - (12, 1)$ <br>meti  $g(12, 12)$  is the one that achieves the greatest social welfare.<br>
The difference  $W(g(12, 1)) - W(g(12, 0))$  is positive if  $k < ($ <br>  $3680d^3+2270d^4-1990d^5-267d^6+335d^7) \frac{(a-c)^2}{144(2-d)^2(1+d)(16-d^2)^2}$ <br>
the later expression, whi - The difference  $W(g(12, 1)) - W(g(12, 0))$  is positive if  $k < (4352 - 2304d - 5824d^2 +$ W(g(12, 1)) – W(g(12, 0)) is positive if  $k < (4352 - 2304d - 5824d^2 + 1990d^5 - 267d^6 + 335d^7)\frac{(a \cdot c)^2}{144(2 - d)^2(1+d)(16 - 7d^2)^2}$ . Denote by  $k_{(12,1)}^{exp} - (12, 12)$  is a function of d, is always positive and intersects k in  $\frac{(a-c)^2}{144(2-d)^2(1+d)(16-7d^2)^2}$ . Denote by  $k_{(12,1)-(12,0)}^{sw}$ <br>
1 of *d*, is always positive and intersects  $\overline{k}$  in<br>
Then, for  $\frac{2}{3} < d < 0.87953$  it follows that<br>
r *d* > 0.87953.<br>
, 0)) is positive if  $k < \frac{(17-2d+1$ the interval  $\frac{2}{3} < d < 1$  at  $\frac{d}{b} = 0.87953$ . Then, for  $\frac{2}{3} < d < 0.87953$  it follows that  $\sim$   $\sim$   $\sim$   $\sim$   $\sim$   $\sim$   $\sim$   $\sim$ 

3680d<sup>3</sup>+2270d<sup>4</sup>-1990d<sup>5</sup>-267d<sup>6</sup>+335d')  $\frac{u-c_0}{144(2-d)^2(1+d)}$ <br>the later expression, which is a function of d, is alwe<br>the interval  $\frac{2}{3} < d < 1$  at  $\frac{d}{b} = 0.87953$ . Then, for  $\frac{2}{3}$ <br> $k_{(12,1)}^{so}$ -(12,0)  $$\overline{k}$$  $k_{(12)}^{\text{out}}$ <br>  $k_{(12)}^{\text{out}}$ <br>  $\text{in } (a-1)$ <br>  $\text{in } (a+1)$ <br>  $\text{in } (12)$ <br>  $\text{in } (12)$ <br>  $\text{in } (12)$ <br>  $\text{in } (12)$ the later expression, which is a function of d, is always positive and intersects k in<br>
the interval  $\frac{2}{3} < d < 1$  at  $\frac{d}{s} = 0.87953$ . Then, for  $\frac{2}{3} < d < 0.87953$  it follows that<br>  $k_{1121-12,00}^{(m)} < \overline{k}$ , the opp  $d < d < 1$  at  $\frac{d}{b}$ <br>  $\overline{k}$ , the opposition  $\overline{k}$ , the opposition  $W(g(12, 12))$ <br>  $(12)-(12,0)$  the lift  $(12, 0)$  is  $\overline{k}$ , the conduct of the difference of  $\overline{k}$ , the following  $\overline{k}$ ,  $(12, 1)$  is  $(12, 12)$ <br>  $(1$  $\frac{d}{b} = 0.87953$ . Then, for  $\frac{2}{3}$ <br>ite follows for  $d > 0.87955$ <br>
(b))  $- W(g(12,0))$  is positival in the sphere of  $l > 0$ <br>
(b) is positic head in the sphere of  $d > 0$ <br>
erence  $W(g(12,12)) - W(\text{lowing ranking for the th})$ <br>  $k_{(12,12)-(12,0)}^{sw} <$  $d < d < 0.87953$  it follows that<br>  $e$  if  $k < \frac{(17 - 2d + 12d^2)(a - c)^2}{144(2 - d)^2(1 + d)^2}$ . We<br>
always positive and intersects<br>
or  $\frac{2}{3} < d < 0.956154$  it follows<br>
956154.<br>  $n(12, 1)$  is the same as in case<br>
esholds applies in t  $k_{(12)}^{\text{out}}$  - T den a that  $\frac{1}{k}$  a that  $\frac{2}{3}$  < ana For The est (12,1)−(12,0)<br>
The difference by k<br>
at  $d = 0.9$ <br>
hat  $k_{(12,12)}^{sw}$  $\langle k, \text{ the opposite follows for } d > 0.87953.$ <br>
nce  $W(g(12, 12)) - W(g(12, 0))$  is positive<br>  $\sum_{12,12}^{w}$  $(12,0)$  the later expression, which is<br>
56154 in the interval  $\frac{2}{3} < d < 1$ . Then, for<br>  $\langle (12,0) \rangle \langle k, \text{ the opposite follows for } d > 0.9$ <br>
cold for the diffe - The difference  $W(g(12, 12)) - W(g(12, 0))$  is positive if  $k < \frac{(1\ell - 2d + 12d^2)(d - c)^2}{144(2-d)^2(1+d)^2}$ . We denote by  $k_{(12, 12)-(12, 0)}^{sw}$  the later expression, which is always positive and intersects  $\overline{k}$  at  $d = 0.956154$  denote by  $k_{(12,12)-(12,0)}^{sw}$  the later expression, which is always positive and intersects <sup>sw</sup><sup>(12,12)–(12,0) the later expression, which is always positive and intersects<br>
956154 in the interval  $\frac{2}{3} < d < 1$ . Then, for  $\frac{2}{3} < d < 0.956154$  it follows<br>
<sub>−(12,0)</sub> <  $\overline{k}$ , the opposite follows for  $d > 0.95$ that  $k_{(12, 12)-(12, 0)}^{sw} < \overline{k}$ , the opposite follows for  $d > 0.956$ 

 $W(g(12, 12)) - W(g(12, 0))$  is positive if  $k < \frac{111-2a+12a-10a-1}{144(2-d)^2(1+d)^2}$ )-(12,0) the later expression, which is always positive and interial is 4 in the interval  $\frac{2}{3} < d < 1$ . Then, for  $\frac{2}{3} < d < 0.956154$  it for  $k^{30}$ <br>  $k^{31}$ k at  $d = 0.956154$  in the interval  $\frac{2}{3}$ <br>that  $k_{(12,12)-(12,0)}^{sw} < \overline{k}$ , the opposite-<br>The threshold for the difference  $V$ <br>(a),  $k_{(12,12)-(12,1)}^{sw}$ . The following r<br> $\frac{2}{3} < d < 1$ :  $k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-}^{sw}$ <br>a  $d < d < 1$ . Then, for  $\frac{2}{3}$ <br>follows for  $d > 0.956$ <br> $(g(12, 12)) - W(g(12))$ <br>mking for the threshot<br> $(12,0) \leq k_{(12,1)-(12,0)}^{sw}$ .<br> $(k_{(12,12)-(12,1)}^{sw} \leq k_{(12,12)-(12,0)}^{sw} \leq k_{(12,12)-(12,0)}^{sw}$ .<br> $k_{(12,12)-(12,0)}^{sw} \leq k_{(12,12)-(12,0)}^{$  $< d < 0.956154$  it follows<br>54.<br>1)) is the same as in case<br>lds applies in the interval<br>'he following subcases are<br> $\frac{1}{(12,0)} < k^{sw}_{(12,1)- (12,0)} < \overline{k}$ .<br> $\frac{1}{(12,1)- (12,0)} < k^{sw}_{(12,12)- (12,1)} < k <$  $k^{sw}_{(12)}$  the  $k^{sw}_{(12)}$  denotes  $k^{sw}_{(12)}$  denotes  $k^{sw}_{(12)}$  denotes  $k^{sw}_{(12)}$  denotes  $\frac{2}{3}$  denotes  $\frac{$ (12,12)−(12,0)<br>
threshold fo<br>  $\frac{sw}{(12,12)-(12,1)}$ <br>
< 1 :  $k_{(12,12)}^{sw}$ <br>
ied:  $\langle k, \text{ the opposite follows for } d > 0.956154.$ <br>
r the difference  $W(g(12, 12)) - W(g(12, 1))$ <br>
The following ranking for the thresholds<br>  $\langle k^{sw}_{(12,12)-(12,0)} \rangle \langle k^{sw}_{(12,1)-(12,0)} \rangle$ . The<br>
53, we have  $0 \langle k^{sw}_{(12,12)-(12,1)} \rangle \langle k^{sw}_{(12,12)-(12,0)} \rangle$ . The<br> - The threshold for the difference  $W(g(12, 12)) - W(g(12, 1))$  is the same as in case  $W(g(12, 12)) - W(g(12, 1))$  is the same as in case<br>ranking for the thresholds applies in the interval<br> $- (12,0) < k_{(12,1)}^{sw} - (12,0)$ . The following subcases are<br> $k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(12,0)}^{sw} < k_{(12,12)-(12,0)}^{sw} < k_{(12,12)-(12$ (a), <sup>sw</sup> (12,12)−(12,1). The following ranking for the thresholds applies in the interval<br>  $\langle 1 : k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(12,0)}^{sw} < k_{(12,1)-(12,0)}^{sw}$ . The following subcases are<br>
ced:<br>  $\langle d < 0.87953$ , we have  $0 < k_{(12,12)-(12$  $\frac{2}{3}$ analyzed:

 $k_{(12)}^{\text{out}}$ <br> $d <$ <br> $72e$ c<br> $\frac{2}{3}$ <br> $\leq$ <br> $\frac{2}{3}$ <br> $\leq$ <br> $\frac{2}{3}$ <br> $\leq$  $d < 1$ :  $k_{(12)}^{sw}$ <br>
alyzed:<br>
or  $\frac{2}{3} < d < 0$ .<br>
hen, for  $0 \le$ <br>
t social welfa  $\begin{aligned} \n\mathcal{L}_{(12,12)-(12,1)} &< k_{(12,12)-(12,0)}^{sw} < k_{(12,1)-(12,0)}^{sw} \text{ The following subcases are} \ \n\mathcal{L}_{(12,12)-(12,0)} < k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(12,0)}^{sw} < k_{(12,12)-(12,1)}^{sw} \leq k < k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(12,0)}^{sw} < k < k_{(12,12)-(12,$  $\langle k_{(12)}^{s,w}\rangle$ <br>have<br> $(12)-(1)$  ined  $\langle k_{(12)}^{sw} \rangle$ <br>  $\langle k_{(12)}^{w} \rangle$ <br>  $\langle k_{(2,12)}^{w} \rangle$ <br>  $\langle k_{(12)}^{w} \rangle$ (b.i) For  $\frac{2}{3} < d < 0.87953$ , we have  $0 < k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(12,0)}^{sw} < k_{(12,1)-(12,0)}^{sw} < \overline{k}$ . Then, for  $0 \le k < k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(12,0)}^{sw} < k_{(12,1)-(12,0)}^{sw} < k$  the g<br>est social welfare is obtained with  $g(12,12)$ . For either  $0 < k_{(12,12)-(12,1)}^{sw} <$ <br>42  $d < d < 0.87953$ , we have  $0 < k_{(12)}^{sw}$ <br>for  $0 \le k < k_{(12,12)-(12,1)}^{sw}$ <br>cial welfare is obtained with  $g(1)$  $\langle k_{(12)}^{s,w}\rangle$ <br>  $\langle k_{(12)}^{(12)}\rangle$  $< k^{sw}_{(12)}$ <br>  $k^{0.0}_{(12,12)}$  $\langle k.$ <br>reat-<br> $k<$  $k < k_{(12)}^{sw}$ <br>e is obta (12,12)−(12,1) (12,12)−(12,0) (12,1)−(12,0) est social welfare is obtained with  $g(12, 12)$ . For either  $0 < k_{(12)}^{sw}$ <br>42  $\langle k_{(12)}^{30}$ <br>g(12, 42  $\langle k_{(12)}^{s,w}\rangle$ <br>ither (  $\langle k \rangle$  the great-<br>2)-(12,1)  $\langle k \rangle$ g(12, 12). For either  $0 < k_{(12)}^{sw}$ <br>42 (12,12)−(12,1) <k<

(12,12) (12,1) (12,0)<br>  $\frac{sw}{(12,1)-(12,0)} < \overline{k}$ ,  $g(12,1)$  gives the highest social welfare, and part **b.2)** of the<br>
ition is proved. Finally, for  $0 < k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(12,0)}^{sw} < k_{(12,1)-(12)}^{sw}$ <br>
the greatest socia (12,1)−(12,0)<br>
(16) is provided in the greate<br>
For 0.8795 sition is proved. Finally, for  $0 < k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(12,0)}^{sw} < k_{(12,1)-(12,0)}^{sw} < k < \overline{k}$  the greatest social welfare is  $W(g(12,0))$ , and part **d**) of the proposition is proved.

- $k_{(12)}^{sw}$  is the contract of  $k_{(12)}^{sw}$  is the contract of  $k_{(12)}^{sw}$  and  $k < k_{(12)}^{sw}$  pro  $\overline{k}$   $k_{(12)}^{sw}$  g(1  $\overline{e}$  are  $k_{(12)}^{sw}$  of  $k_{(12)}^{sw}$  soc  $\overline{e}$  ). For  $k_{(12)}^{sw}$  soc  $\overline{e}$  and  $\overline{e}$  and  $\langle k_{01}^{\text{max}}\rangle$ <br>  $\langle k_{01}^{\text{max}}\rangle$ <br>  $\overline{k}$ ,  $g(1)$ <br>
d. Fin<br>
social<br>  $\langle d \rangle$ <br>
Then,<br>
ocial wirth a.3<br>  $\langle k_{12,1}^{\text{max}}\rangle$ <br>  $\langle k_{12,1}^{\text{max}}\rangle$ <br>  $\langle k_{(12,1)}^{\text{max}}\rangle$ <br>  $\langle k_{(12,1)}^{\text{max}}\rangle$ <br>  $\langle k_{(12,1)}^{\text{max}}\rangle$ <br>  $\langle k_{(12,$  $\langle k \text{ or } 0 \rangle \langle k_{(12,12)-(12,1)}^{sw} \rangle$ <br>the highest social<br> $\langle k_{(12,12)-(12,1)}^{sw} \rangle$ <br> $W(g(12,0)),$  and  $1$ <br>follows that  $0 \langle k_{(12,12)-(12,1)}^{sw} \rangle$ <br>tained with  $g(12, 1)$ <br>otained with  $g(12, 1)$ <br>otained with  $g(12, 1)$ <br>roposition is  $< k_{(12)}^{\text{sw}}$ <br>  $< k_{(12)}^{\text{sw}}$ <br>  $k_{(2,0)}^{\text{sw}}$ <br>  $k_{(12,0)}^{\text{sw}}$ <br>  $k_{(12,12)}^{\text{sw}}$ <br>  $k_{(12,12)}^{\text{sw}}$ <br>  $k_{(12,11)}^{\text{sw}}$ <br>  $k_{(12,11)}^{\text{sw}}$ <br>  $k_{(12,11)}^{\text{sw}}$ <br>  $k_{(12,11)}^{\text{sw}}$ <br>  $k_{(12,11)}^{\text{sw}}$ <br>  $k_{(12,11)}^{\text{sw}}$ <br>  $\begin{aligned} \text{Prop} &> p &> \text{prop} \text{or} \end{aligned}$   $\begin{aligned} &\geq k < \text{proved}. \ &\leq k < \text{1}) - (12,0) \text{, (b.i)} \ &\leq (12,1) < k < \overline{k} < \text{1}) \text{ of the} \ &\leq (12,0) < \overline{k} < \text{greatest} \ &\leq (12,0) < \text{1}) \end{aligned}$  $k_{(12)}^{sw}$  sitile  $k_{(12)}^{sw}$  sitile and  $k < k_{(12)}^{sw}$  the and  $k < k_{(12)}^{sw}$  proper  $\overline{k}$  ,  $k_{(12)}^{sw}$  sociallecty  $g(1$  are  $k$  sociallect  $g$  and  $k$  substantiallect  $\overline{s}$  conditionallect  $\overline{s}$  conditionallect  $k$ , g(12, 1) gives the highest social welfare, and part b.2) of the proposition<br>
ond Finally, for it b  $k$ (Eag)-(14), and part d) (dependential points of<br>  $k$  to social welfare is  $W(g(12,01))$ , and part d) of the proposit  $\langle k \rangle_{(12)}^{88}$ <br>  $\langle k \$  $g(12,12)-(12,1)$  (12,12)−(12,0) (12,1)−(12,0)<br>  $g(12,0)$ ), and part **d)** of the proposition is power that  $0 < k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(12,1)}^{sw}$ <br>  $k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(12,0)}^{sw} < \overline{k} < k_{(12,12)-(12,1)}^{sw}$  $< k_{(12)}^{sw}$ <br>  $< k_{(12)}^{sw}$ <br>  $< k_{(12)}^{sw}$ <br>  $< k_{(12)}^{sw}$ <br>  $< k_{(12,12)}^{sw}$ <br>  $< k_{(12,12)}^{sw}$ <br>  $< k_{(12,12)}^{sw}$ <br>  $< k$ <br>  $< 2) - (12$ <br>  $> 2) - (12$ <br>  $> 38$ <br>  $> 2)$ : 38<br>  $> 38$ <br>  $> 2)$ : 38<br>  $> 38$ <br>  $> 2)$ : 38<br>  $> 38$ <br>  $> 2)$ :  $> 38$  $\langle k_{(12)}^{sw} \rangle$ <br>  $\langle k_{(12)}^{sw} \rangle$ <br>  $\langle k_{(12)}^{sw} \rangle$ <br>  $\langle k_{(12)}^{sw} \rangle$ <br>  $\langle k_{(12,1)}^{sw} \rangle$ <br>  $\sqrt{k}$ <br>  $\lt \overline{k}$ <br>  $\lt$   $-$  (12,0)<br>  $\sqrt{(b \cdot i)}$ <br>  $\lt \sqrt{k}$ <br>  $\lt \sqrt{k}$ <br>
of the<br>  $\sqrt{2,0}$ <br>  $\lt \overline{k}$ <br>  $\lt$  eatest<br>  $\sqrt{2,0}$ <br>  $\lt \sqrt{k}$ <br>  $\approx 2,0$ <br>  $\sqrt{2}$ <br>  $\approx 0$ k the greatest social welfare is W(g(12, 0)), and part d) of the proposition is proved.<br>
For Bre Willak social welfare is W(g(12, 0)), and part d) of the proposition is proved.<br>
For Bre Willace is obtained with  $g(12, 12)$ (b.ii) For 0.87953 < d < 0.9535, it follows that 0 < k $k_{\text{eff}}^{\text{max}}$ <br>  $k_{\text{eff},21}^{\text{max}}$  = (rag.) Then, for 0  $\leq k \leq k_{\text{eff},21}^{\text{max}}$  of the presentes social welfare is obtained with  $g(12,12)$ <br>
the greatest social wel  $\begin{align*} (12,12)^{-1}(12,1) \times (12,12) \$  $< k_{0.0}^{\text{sw}}$ <br>  $< k_{(12)}^{\text{sw}}$ <br>  $\text{inj}$  it are  $0 < \frac{sw}{(12,12)-1}$ <br>  $< \infty$ <br>  $< k_{(12)}^{\text{sw}}$ <br>  $< k_{(12)}^{\text{sw}}$ <br>  $< k \leq 1$ <br>  $\text{inj}$  $\begin{array}{l} - (12,0) \ \hline (12,1) \end{array}$ ,  $(\mathbf{b}.\mathbf{i})$ <br>  $\begin{array}{l} 12,1) \end{array}$  <  $\begin{array}{l} < \overline{k} \end{array}$ <br>  $\begin{array}{l} < \overline{k} \end{array}$  of the<br>  $\begin{array}{l} 12,0) \end{array}$  <  $\begin{array}{l} < \overline{k} \end{array}$ <br>  $\begin{array}{l} \hline 12,0) \end{array}$  < decreases and the distributi  $k_{(12)}^{sw}$  the and  $k < k_{(12)}^{sw}$  the and  $k < k_{(12)}^{sw}$  pro  $\big)$  F  $< k_{(12)}^{sw}$  soc  $\big)$  F  $k_{(12)}^{sw}$  soc  $k_{(12)}^{sw}$  soc  $k_{(12)}^{sw}$  soc  $s$  and  $s$  and  $s$  and  $s$  and  $s$  conducts  $s$  and  $s$  and  $s$  conducts  $s$  an <sup>sw</sup><sub>(12,1)−(12,0)</sub>. Then, for  $0 \le k < k_{(12,12)-(12,1)}^{sw} < k_{(12,12)-(12,0)}^{sw} < k < k_{(12,1)-(12,0)}^{sw}$ <br>he greatest social welfare is obtained with  $g(12,12)$ . Considering items (a.v), (b.i)<br>nd (b.ii), part **a.3**) of the proposi  $k < k_{(12)}^{sw}$ <br>botained<br>proposit<br>)-(12,0) c<br>e highes<br>e highes<br>4, it folle<br>either  $k$ <br> $< 0 < k_{(12)}^{sw}$ <br>h  $g(12, 1)$ <br>follows t<br> $\leq k < 1$ <br>collows t<br> $\leq k$ <br> $\leq k$ <br>nston (1<br>1994). "Journal of Economic Reformal of Economic Reform  $< k_{(12)}^{sw}$ <br>  $(12).$  (red. F<br>  $(12).12)$  (red. F<br>  $(12).12)$  (led. F<br>  $(12.12)$   $< 0$ <br>  $< k <$ <br>  $> 0$  (12.12)<br>  $> 0$ <br>  $< k <$ <br>  $> 0$  (12.12)<br>  $> 0$ <br>  $< k <$ <br>  $> 0$ <br>  $\geq 0$ <br>  $\geq k <$ <br>  $> 0$ <br>  $\geq 0$ <br>  $\geq k <$ <br>  $> 0$ <br>  $\geq 0$ <br>  $\geq k$  $k < k$  ( $k_{(12)}$ <br>
g items (**a.**  $0 < k_{(12,12)}^{sw}$ <br>  $k_{(12,12)}^{sw}$ <br>  $k_{(12,1$ the greatest social welfare is obtained with  $g(12, 12)$ . Considering items  $(a.v)$ ,  $(b.i)$ g(12, 12). Considering items (a.v), (b.i)<br>proved. For either  $0 < k_{12,12}^{sm}(-12,1) < k_{(12,12)-(12,1)}^{sm} < k_{(12,12)-(12,1)}^{sm} < k_{(12,12)-(12,1)}^{sm} < k_{(12,12)-(12,1)}^{sm} < k_{(12,12)-(12,1)}^{sm} < k < \bar{k} <$ al welfare. This proves part b.3) of the<br> and (b.ii), part a.3) of the proposition is proved. For either  $0 < k_{(12,12)-(12,1)}^{sw} <$  $(12,12)$  (12,1)<br>  $(2,0) < k < k$ <br>
(12,1)<br>  $(12,12)$ <br>  $(12,12)$ <sup>sw</sup><sub>(12,12)−(12,0)</sub> <  $k < k_{(12,12)-(12,0)}^{sw}$  or 0 <  $k_{(12,12)-(12,1)}^{sw}$  <  $k_{(12,12)-(12,0)}^{sw}$ <br>  $(12,1)$  gives the highest social welfare. This proves part<br>
sition.<br>
0.9535 <  $d$  < 0.956154, it follows that  $k_{(12,12)-($ proposition.
- $\langle k_{(12)}^{sw} \rangle$  = (12,0)<br>  $\langle k_{(12)}^{sw} \rangle$  is obtained in the munits are formulated in the munission of the munission of the munission of the mu  $\leq$   $\leq$  $k < k_{(12)}^{sw}$ <br>  $k_{(12,1)-}^{sw}$ <br>  $k_{(12,1)-}^{sw}$ <br>  $k < k_{(12,1)-}^{sw}$ <br>  $k_{(12,1)-}^{sw}$ <br>  $k_{(12,1)-}^{sw}$ <br>  $k_{(12,1)-}^{sw}$ <br>  $g(12,1)$ <br>  $g(12,1)$ <br>  $\textbf{percent}$ <br>  $\textbf{element}$ <br>  $\textbf{element$  $k < k$  ( $k_{12}$ )<br>  $k$  ( $k_{21}$ )<br>  $k$ ) gives t<br>  $k$  ( $k$ )<br>  $k$ ) gives t<br>  $k$  ( $k$ )<br>  $k$ ). Then, for all<br>  $k$  ( $k$ ) and F. Blootting and Ect<br>  $k$  ( $k$ ). The  $k$  ( $k$ ). Herry<br>  $k$ ). The  $k$  ( $k$ ). The  $k$  ( $k$ ). The  $k$  ( $k$ )  $< k_{(12)}^{sw}$ <br>  $< k_{(12)}^{sw}$ <br>  $= (12,0)$ <br>  $k_{(12,12)}^{sw}$ <br>  $=$   $k$  $< k_{(12)}^{sw}$ <br>  $< k_{(12)}^{sw}$ <br>  $< k < k_{(12)}^{sw}$ <br>  $0 < k_{(12)}^{sw}$ <br>  $0 < 1$  welf<br>
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2)-(12,0)  $\langle k \rangle$ <br>
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20 for Echail  $k_{(12)}^{\text{sw}}$  pro  $\bigr)$  F  $\leq k$   $k_{(12)}^{\text{sw}}$  soc  $\bigr)$  F  $k_{(12)}^{\text{sw}}$  soc  $\bigr)$  F  $k_{(12)}^{\text{sw}}$  soc  $\bigr)$  F  $k_{(12)}^{\text{sw}}$  soc  $\bigr)$  F  $k_{(12)}^{\text{sw}}$  soc  $\bigr)$  E  $k_{(12)}^{\text{sw}}$  soc  $\bigr)$  and  $\bigr)$  and  $\bigr)$  $\begin{aligned} &\overset{sw}{(12,1)-(12,0)}, \ &\text{toposition.} \ &\text{For } 0.9535 \ &\lt k^{sw}_{(12,1)-1} \end{aligned}$ g(12, 1) gives the highest social welfare. This proves part b.3) of the<br>  $< d < 0.956154$ , it follows that  $k_{(12,12)-(12,1)}^{exp} < 0 < k_{(12,12)-(12,0)}^{exp} < 0$ <br>  $\alpha_{120}$ . Then, for either  $k_{(12,12)-(12,1)}^{exp} < 0 \le k < k_{(12,12)-(12,0)}^{exp} <$ (b.iii) For 0.9535 < d < 0.956154, it follows that  $k_{112}^{m}$ <br>  $\overline{k}$  <  $k_{12,11}^{m}$  -(12,0). Then, for either  $k_{12,121}^{m}$  -(12,1)<br>  $k_{12,11}^{m}$  -(12,0) or  $k_{12,121}^{m}$  -(12,1) < 0 <  $k_{12,121}^{m}$ -(12,1)<br>
(b.iv)  $\begin{align*} (12,12) \quad (12,12) \quad (12,12) \ \frac{1}{2} < 0 \leq k < k^{sw} \ \leq k < \overline{k} < k^{sw} \ \leq k < k^{sw} \ \end{align*}$  $< 0 < k_{(12)}^{sw}$ <br>  $< k_{(12,12)-(12,0)}^{sw}$ <br>  $< \overline{k} < k_{(12,12)-(12,0)}^{sw}$ <br>  $< \overline{k} < k_{(12)}^{sw}$ <br>  $< \overline{k} < k_{(12)}^{sw}$ <br>  $<$   $\overline{k} < k_{(12)}^{sw}$ <br>  $<$   $\overline{k}$   $< k_{(12)}^{sw}$ <br>  $<$   $\overline{k}$   $< k_{(12)}^{sw}$ <br>  $<$   $\overline{k}$   $< k_{(12)}^{sw}$ <br>  $<$   $\overline{k}$ (vertical vertical verti (12,1)−(12,0). Then, for either  $k_{(12,12)-(12,1)}^{sw} < 0 \le k < k_{(12,12)-(12,0)}^{sw} < k < k_{(12,12)-(12,0)}^{sw} < k$  (12,12)−(12,0) or  $k_{(12,12)-(12,1)}^{sw} < 0 < k_{(12,12)-(12,0)}^{sw} < k < \overline{k} < k_{(12,1)-(12,0)}^{sw}$  the greatest ocial welfare is obtained social welfare is obtained with  $q(12, 1)$ .
- $k < k_{(12)}^{sw}$ <br>  $k_{(12,1)- (12)}^{sw}$ <br>  $k_{(12,1)- (12)}^{sw}$ <br>  $k_{(12,1)- (12)}^{sw}$ <br>  $s^{sw}$  $k_{(12,1)}^{sw}$ <br> $k_{(12,1)}^{sw}$ <br> $(k_{(12,1)}^{sw})$ . "<br> $k$ <br> $k$   $\overline{k}$ <br>  $< 0 \leq k < k_{(12)}^{sw}$ <br>  $k < \overline{k} < k_{(12,1)-1}^{sw}$ <br>  $(12,1) < 0 < \overline{k} <$ <br>
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s-European Com  $\frac{1}{2,0}$   $\leq$ <br>d with<br>bllusive<br>bllusive<br>bllusive<br>conomic<br>DG for  $k_{(12)}^{\text{sw}}$  soc.) First  $k_{(12)}^{\text{sw}}$  soc. ) First  $g(1$  and  $g(1$  are  $\ell$  belief variation.  $k_{(12)}^{\text{out}}$  is obviously and and any  $n$  and  $n$  and  $4-1$  of  $k$  is  $k$  is  $k$  is  $k$  in  $k$  $< 0 < k_{012}$ <br>th  $g(12, 1)$ <br>follows th<br> $0 \le k < \overline{k}$ <br> $\ge k < \overline{k}$ <br> $\ge 0$ <br> $\ge k < \overline{k}$ <br> $\ge 0$ <br> $\ge k < \overline{k}$ <br> $\ge 0$ <br> $\ge 0$ <br>*nomic Re*<br>inston (1994). "Economic ey (1996)<br>ts," Economic security Economic  $k < k < k$ <br>  $k < k < k$ <br>  $k = (12,1) < 0 < k$ <br>
test social well<br>
sharing Agree<br>  $k = 387-411$ .<br>
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Ilows that<br>  $\leq k < \overline{k}$ <br>  $(2004)$ .<br>  $mic Rev$ <br>
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994). "Expansion (19<br>
994). "Expansion (1996).<br>
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4 (b.iv) For 0.956154 < *d* < 1, it follows that  $k_{(12)}^{sw}$ <br>  $k_{(12,1)- (12,0)}^{sw}$ . Then, for all  $0 \le k \le \overline{k}$  the g<br>  $g(12,1)$ . ■<br> **References**<br>
[1] Belleflanme, P. and F. Bloch (2004). "Mark<br>
Networks," *International Ec*  $\sum_{(12,12)-(12,0)}^{sw}$  (12,12)−(12,0). Then, for all  $0 \le k < \overline{k}$  the greatest social welfare is obtained w<br>(12,1).  $\blacksquare$

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- $<$  0  $<$   $k$   $<$   $k$ th<br>th<br>we<br>all<br>all<br>d-<br>ic  $k_{(12)}^{\text{sw}}$ <br> $g(1$ <br> $g(1$ <br> $B$ elletv $3$ err $\mathbb{E}$ cor $\mathbb{E}$ err $\mathbb{E}$ cor $\mathbb{E}$ aba  $k < k$  the greatest social welfare is obtained with<br>  $k < k$  the greatest social welfare is and Collusive<br>  $k$ ic  $Review 45(2): 387-411$ .<br>
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