Endogenous growth cycles

Klaus Wälde[1], European Commission and University of Dresden
DG Economic and Financial Affairs, BU1 5 - 179, 1160 Brussels, Belgium
CESifo and IRES
www.waelde.com, Klaus.Waelde@cec.eu.int
Phone + 32.2.29 - 52800, Fax + 32.2.29 - 67291
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Current explanations why a growing economy necessarily goes through periods of high and low growth predict countercyclical R&D investment. As this is very controversial from an empirical perspective, a stochastic Poisson model of endogenous growth cycles is presented where the determinants of the cyclical behaviour of R&D investment are analytically studied. Providing an explicit expression for the expected length of a cycle shows that high frequency fluctuations can indeed be understood by this approach. It is also shown how small technological improvements translate into large aggregate fluctuations.

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1 Introduction

Why do growing economies experience periods of high and low growth? Traditional answers to this question mainly stress exogenous shocks or non-linearities whose effects are analyzed in stationary economies. Recently, however, several authors have proposed mechanisms that allow to understand both endogenous short-run fluctuations and endogenous long-run growth in a unified setup. Francois and Lloyd-Ellis (2003) show how bunching of innovation can occur in a quality-ladder growth model. Deterministic cyclical growth results. Freeman, Hong and Peled (1999) show how the introduction of new technologies that require prior accumulation of research experience leads to deterministic cyclical growth as well. The "portfolio approach" by Bental and Peled (1996), Matsuyama (1999, 2001) and Wälde (2002) stresses how the choice of investors between financing capital accumulation and R&D and implied endogenous jumps in productivity leads to cyclical deterministic growth in Matsuyama's work or cyclical stochastic growth in models of Bental and Peled and Wälde.[2]

A common prediction of almost all[3] of these models is a countercyclical allocation of resources to R&D. In periods of high growth of GDP, few resources are allocated to R&D. With low growth, resource allocation to R&D is high. Empirically, this prediction is not easy to defend. While Francois and Lloyd-Ellis (2003) cite evidence for countercyclical *non*-R&D activites like e.g. reorganization and training activities, Saint-Paul (1993) remains inconclusive concerning R&D expenditure. The currently most comprehensive study on cyclical properties of R&D expenditure (Wälde and Woitek, 2004), however, finds that R&D expenditure in G7 countries is procyclical in the majority of cases. This is in line with e.g. Fatás (2000) who also finds that R&D expenditure in the US is procyclical.

Given this empirically disputed evidence and the seemingly counter-factual prediction of existing models, the first objective of the present paper is to clarify the determinants of the cyclical behaviour of R&D investment and to understand conditions under which R&D expenditure is procyclical. As the paper is in the tradition of the portfolio approach, it builds its explanation of endogenous fluctuations on investment decisions of individuals who can use their savings to finance capital accumulation or R&D. As a consequence, determinants of the individual portfolio choice are, upon aggregation, determinants of aggregate cyclical behaviour of R&D. Shadow prices of capital, the riskiness of R&D and individual dividend payments in case of successful R&D all play a role.

On the equilibrium path we analyze, relative shadow prices and the riskiness of R&D is constant. As in our setup dividend payments to successful R&D increase over the cycle, individuals are induced to shift more and more resources to R&D as the economy grows. R&D investment that grows as the economy grows implies procyclical R&D investment.

Apart from a different prediction concerning cyclical behaviour of R&D investment, the present paper also stresses a "less drastic" mechanism why growing economies fluctuate.

[2]An alternative explanation of endogenous fluctuations and growth is presented by Redding (2002) which will be discussed below. The term 'endogenous' as used here should not be confused with endogenous in the sense of the sunspot literature (e.g. Aloi, Lloyd-Braga and Whitta-Jacobsen, 2003) where some exogenous source of uncertainty is still required. Endogenous here and in the cited literature means that the intensity with which an economy fluctuates depends on decisions made by agents populating this economy.

[3]Francois and Lloyd-Ellis have reorganisation and other non-R&D activities in mind when modelling investment. In this sense, their model does not make any statement about cyclical properties of R&D.

While Matsuyama builds his explanation on a one-period patent protection for new varieties, Bental and Peled assume new technologies to be common knowledge after one period. Here, capital accumulation and R&D take place in equilibrium as well. As long as risky R&D is not successful, the economy accumulates capital at a decreasing growth rate resulting from decreasing returns to capital accumulation. When research is successful, a better capital good is available and total factor productivity increases, i.e. a "technology jump" occurs.[4] Returns to capital accumulation go up and a boom results. As in other work, successful research increases labour productivity. This increase by itself, however, is enough for periods of high and low growth and no limited patent protection, sudden common knowledge of a technology or a reduction in the physical capital stock as in Wälde (2002) is required.

The third objective of the present paper is to understand whether large jumps in technology are required to understand realistic aggregate fluctuations. By presenting a continuous-time model, a closed form solution is available for the entire transition path towards (a temporary) long-run steady state for certain parameter values and despite aggregate uncertainty. This allows to analytically analyze the expected growth rate and the expected length and amplitude of cycles. It turns out that small jumps can cause realistically large aggregate fluctuations.

A further contribution lies in clarifying what type of fluctuations can be understood by the portfolio approach to economic fluctuations. It is sometimes argued that this approach is useful for understanding fluctuations of low frequency but not high frequency fluctuations of say 4 to 5 years. With an analytical expression for the expected length it can be shown that high frequency fluctuations can well be understood by the portfolio approach to economic fluctuations.

The papers most closely related to the analysis presented here are Redding (2003), de Hek (1999), de Hek and Roy (2001) and Bental and Peled (1996). Redding distinguishes between fundamental and secondary developments and shows that path-dependence and technological lock-in are natural features in this setup. Growth rates in his model fluctuate. As both the development of fundamentally new technologies and successive secondary development can be considered as R&D, his model predicts continuous R&D expenditure over the cycle. While not analysed, R&D expenditure in his setup could indeed be procyclical. The present paper also has a certain and an uncertain investment possibility and fluctuating growth rates. It does not focus on technological lock-in, however, but allows for risk averse agents, capital accumulation and an infinite planning horizon, features not taken into consideration by Redding. The present paper shares with de Hek (1999) and de Hek and Roy (2001) the belief that explicit analytical solutions for expected growth rates are useful for understanding economic mechanisms. By allowing for rare shocks here (modelled by a Poisson process which differs from other processes where shocks occur in every period) and thereby stating that large changes in technology take place not continuously and gradually but at discrete points and in discrete steps, the propagation mechanism of a shock is highlighted and more emphasis can be put here on cyclical properties of growing economies. Bental and Peled's (1996) analysis is the starting point of the literature on endogenous fluctuations and growth. Allowing for risk-averse households and undertaking an analysis in continuous time delivers

[4] The term technology *jump* will be used to distinguish endogenous discrete changes in (total factor or labour) productivity from exogenous *shocks* to productivity.

additional results to the ones of Bental and Peled.

Technically, the paper extends the literature on stochastic continuous time models. The majority of the contributions to this literature uses Brownian motion as their source of uncertainty (e.g. Grinols and Turnovsky, 1998). The present paper uses Poisson uncertainty as occasional jumps are more appropriate for modeling cyclical growth. Poisson uncertainty has been used in the economics literature by e.g. Farzin et al. (1998), Hassett and Metcalf (1999) and Venegas-Martínez (2001).

2 The model

2.1 Technologies

Technological progress is labour augmenting and embodied in capital. A capital good K_j of vintage j allows workers to produce with a labour productivity of A^j, where $A > 1$ is a constant productivity parameter. Hence, a more modern vintage $j+1$ implies a labour productivity that is A times higher than labour productivity of vintage j. The production function corresponding to this capital good reads

$$Y_j = K_j^{\alpha} \left(A^j L_j\right)^{1-\alpha}. \tag{1}$$

The amount of labour allocated to this capital good is denoted by L_j, $0 < \alpha < 1$ is the output elasticity of capital. The sum of labour employment L_j per vintage equals aggregate constant labour supply, $\Sigma_{j=0}^{q} L_j = L$, where q is the most advanced vintage currently available.

Independently of which vintage is used, the same type of output is produced. Aggregate output is used for producing consumption goods C, investment goods I and it is used as an input R for doing R&D,

$$C + I + R = Y = \Sigma_{j=0}^{q} Y_j. \tag{2}$$

All activities in this economy take place under perfect competition. Good Y will be chosen as numeraire. Its price and the price of the consumption, investment and research good will therefore be identical,

$$p_Y = p_c = p_I = p_R, \tag{3}$$

and constant throughout the paper; we will nevertheless use price variables at various places (and not normalize to unity) as this makes some relationships more transparent.

The objective of R&D is to develop capital goods that yield a higher labour productivity than existing capital goods. R&D is an uncertain activity which is modeled by the Poisson process q (as in Aghion and Howitt, 1992 or Redding, 2002). The probability per unit of time dt of successful R&D is given by λdt, where λ is the arrival rate of the process q. This economy-wide arrival rate is an increasing function of the amount of resources R used for R&D,

$$\lambda = \frac{R}{D} h\left(\frac{R}{D}\right) = \left(\frac{R}{D}\right)^{1-\gamma}, \quad 0 < \gamma < 1. \tag{4}$$

At the level of an individual firm f, there are constant returns to scale, $\lambda_f = D^{-1} h\left(R/D\right) R_f$ and a firm chooses resources R_f, taking the "difficulty" function D and the externality $h\left(.\right)$

as given. As firm-level Poisson processes q_f can be added up, we obtain (4) at the sectoral level where $h(.)$ implies decreasing returns to scale. The parameter γ can be thought of as close to but different from zero.[5]

The exogenous function D captures the "difficulty" to make an invention, as in Segerstrom (1998). Given a certain amount of resources R, the probability to find a better capital good is lower, the higher the difficulty D. The primary objective is to remove the well-known scale effect (e.g. Jones, 1995) in the present model. We will therefore assume that the difficulty increases in the value K^c of the observed capital stock, i.e. the capital stock measured in units of the consumption good,

$$D = D_0 K^c, \quad D_0 > 0. \tag{5}$$

As we will see later, growth of the capital stock K^c can be split into an (unbounded) trend component driven by better technologies, i.e. increases in q, and into a (bounded) cyclical component. Including the trend component in the difficulty function captures the fact that more resources are required to find better technologies at a constant arrival rate, the more discoveries have been made in the past. A lot of empirically support is available for this specification both on the micro- and macro level (Segerstrom, 2002; Jones 1995). The intuition is simply that with a given pool of innovative ideas, it is more difficult to find a new one when many ideas are already in use. The capital stock K^c and therefore the difficulty D also increases (up to an upper bound) due to the cyclical component of K^c. This latter assumption is made primarily for analytical convenience. It will particularly allow us to compute explicit expressions for the expected arrival rate of new technologies and thereby the expected length of a cycle and the expected growth rate. This in turn allows us to explicitly study determinants of these quantities. Section 6 will show that an analysis using closed form solutions can also be undertaken for a difficulty function where only the number of past inventions increases the difficulty to invent,

$$D = D_0 A^q, \quad D_0 > 0. \tag{6}$$

When R&D is successful, q increases to $q + 1$ and a first prototype κ of a production unit that yields a labour productivity of A^{q+1} becomes available. In more conventional quality ladder models, output of successful research is modeled as an intangible good, a blueprint. Owners of the blueprint sell goods constructed accordingly and obtain profits due to some market power. These profits are used to cover R&D costs. Here, engineers actually construct a first machine that implies this higher labour productivity. Instead of thinking about how a new good or variety can be produced, researchers build one.[6] This allows us to understand R&D in a decentralized economy as a perfectly competitive process: Those who have financed R&D obtain a tangible good, a production unit, whose (expected discounted) capital rewards obtained under perfect competition cover R&D costs. Hence, no monopoly profits are required.[7]

[5] This discussion of figure 3 shows why attention is restricted to $0 < \gamma < 1$.

[6] One can think of this prototype as a pilot plant in the sense of Rosenberg (1994).

[7] This is in the spirit of Hellwig and Irmen (2001) or Boldrin and Levine (2001, 2002). The reason why R&D can be reconciled with perfect competition is different here, however.

Concerning the size of the first machine resulting from successful research, we assume it equals a constant share of the current capital stock,

$$\kappa = \kappa_0 K^c, \quad 0 < \kappa_0 \ll 1. \tag{7}$$

This specification can be motivated in two ways: first, the longer the research process lasts, the more time researchers have spent both thinking about the design of a new machine and actually building it. Hence, the longer research lasts, even though the crucial breakthrough has not yet occurred, the larger the new machine. As K^c increases over time, this relationship is captured by (7). Second, when Intel or AMD develop a new processor or when Nokia develops a new cell-phone, total dividend payments of successful research should be relatively constant compared to total wealth of the economy, no matter how many innovations or how much capital accumulation took place before. Specifying κ as a small but constant percentage κ_0 of the capital stock K^c captures this effect.[8]

Each vintage of capital is subject to depreciation at the constant rate δ. If investment in vintage j exceeds depreciation, the capital stock of this vintage increases in a deterministic way,

$$dK_j = (I_j - \delta K_j)\, dt, \quad j = 0...q. \tag{8}$$

When research is successful, the capital stock of the next vintage $q+1$ increases discretely by the size κ of the first new machine of vintage $q+1$,

$$dK_{q+1} = \kappa dq. \tag{9}$$

Afterwards, (8) would apply to the vintages $j = 0...q+1$.[9]

Before describing households, we derive some straightforward equilibrium properties that both simplify the presentation of the production side and, more importantly, the derivation of the budget constraint of households in the next section.

Allowing labour to be mobile across vintages $j = 0...q$ such that wage rates equalize, total output of the economy can be represented by a simple Cobb-Douglas production function (cf. appendix 9.1)

$$Y = K^\alpha L^{1-\alpha}. \tag{10}$$

Vintage specific capital stocks have been aggregated to an aggregate capital index K,

$$K = K_0 + BK_1 + ... + B^q K_q = \Sigma_{j=0}^q B^j K_j, \tag{11}$$

$$\text{where} \quad B \equiv A^{\frac{1-\alpha}{\alpha}}. \tag{12}$$

[8] An alternative specification could keep dividend payments at a constant level, $\kappa = \kappa_0$, i.e. a new technology always has the same size. In a growing economy, total dividend payments from new technologies relative to the capital stock would permanently fall which excludes balanced growth. Yet another specification where $\kappa = \kappa_0 A^q$, similar to (6), would make dividend payments acyclical which contradicts empirical evidence (e.g. Campbell, 1999). Given these alternatives, the specification in (7) is more convincing. The advantage of (7) over $\kappa = \kappa_0 A^q$ is also analytical as $\kappa = \kappa_0 A^q$ would require numerical solution methods. See footnote 19 on how to endogenize κ.

[9] Formally, this equation is a stochastic differential equation driven by the Poisson process q whose arrival rate λ is given in (4). The increment dq of this process can either be 0 or 1. Successful R&D means $dq = 1$.

If K_0 is thought of as the "number of machines" of vintage 0, K gives the number of machines of vintage 0 that would be required to produce the same output Y as with the current mix of vintages.

Given that the price of an investment good does not depend on where this investment good is used, that depreciation is the same for all investment goods and given that value marginal productivities,

$$w_j^K = p_c \frac{\partial Y}{\partial K} B^j, \tag{13}$$

are highest for the most advanced vintage, investment takes place only in the currently most advanced vintage q, $I_j = 0 \; \forall j < q$, $I_q = I$. Hence, the evolution of the aggregate capital index K follows from (8) and (9) by applying Ito's Lemma to (11),

$$dK = (B^q I - \delta K)\, dt + B^{q+1} \kappa dq. \tag{14}$$

The capital stock increases continuously as a function of effective investment $B^q I$ minus depreciation.[10] As the prototype increases the capital stock of vintage $q+1$ in case of successful research by κ, it increases the capital index (11) by $B^{q+1}\kappa$.

As long as investment is positive, the price v_q of an installed unit of the most recent vintage of capital equals the price of an investment good, $v_q = p_I$. As different vintages are perfect substitutes in production (11), prices of different vintages are linked to each other by

$$p_I = v_q = B^{q-j} v_j, \quad \forall j = 0...q. \tag{15}$$

Further, the price p_K of one efficiency unit of capital (which corresponds to one unit of capital of vintage 0) is a decreasing function of the most advanced vintage q,

$$p_K = B^{-q} p_I. \tag{16}$$

This also reflects the term B^q in the capital accumulation equation (14) and provides a link between the capital index K and the capital stock K^c as observed in the data. Multiplying the capital index by the price of one efficiency unit of capital and dividing by the price of the consumption good (which equals the price of the investment good) gives the value of the capital stock in terms of the consumption good,

$$K^c \equiv \frac{p_K}{p_I} K = B^{-q} K. \tag{17}$$

This quantity will play an important role when looking later at the empirical predictions of the model.

2.2 Households

There is a discrete finite number of households in this economy. Each household is sufficiently small to neglect the effects of own behaviour on aggregate variables. Households

[10]This is similar to Solow-type vintage models of e.g. Greenwood, Hercowitz and Krusell (1997).

maximize expected utility $U(t)$ given by the sum of instantaneous utility $u(.)$ resulting from consumption flows $c(\tau)$, discounted at the time preference rate ρ,

$$U(t) = E_t \int_t^{\infty} e^{-\rho[\tau - t]} u(c(\tau)) d\tau, \tag{18}$$

where instantaneous utility $u(.)$ is characterized by constant relative risk aversion,

$$u(c(\tau)) = \frac{c(\tau)^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma > 0. \tag{19}$$

For saving purposes, a household can buy capital and finance R&D. When she buys capital, her wealth a in terms of the consumption good increases in a deterministic and continuous way. This increase depends on the difference between real capital and labour income $ra + w$ minus real R&D investment i and real expenditure c for consumption. This is the "dt-term" on the right hand side of her budget constraint (which is derived in appendix 9.2),

$$da = (ra + w - i - c)\, dt + \left(\kappa \frac{i}{R} - sa \right) dq, \tag{20}$$

where the interest rate is given by

$$r = B^q \frac{\partial Y}{\partial K} - \delta. \tag{21}$$

When financing R&D, i.e. when i is positive, successful research changes her wealth in a discrete way, as shown by the "dq-term" in (20). *Total* dividend payments after a successful research project depend on the price and the size κ of the prototype. As κ, once developed, is the most modern vintage, its price equals by (15) the price of the investment good. Hence by (3), total dividend payments in terms of the consumption good are given by κ. These payments need to be divided among investors in the successful project. We assume a simple "division rule": A household receives the same share of total dividend payments of the successful research project that she has contributed to financing this project. As R&D is undertaken under perfect competition, the sum of individual real R&D investment i equals resources R from (2) allocated to the R&D sector. The household therefore receives the share i/R.[11]

A negative effect of successful research stems from the devaluation of capital. When a new vintage is found, i.e. when q increases by one, the price of older vintages relative to the consumption good fall as by (15) and (3) $v_j/p_c = B^{-(q-j)}$. Capital owners therefore experience a certain reduction in their real wealth. The share of assets that is "lost" due to this devaluation is denoted by s and given by[12]

$$s = \frac{B - 1}{B}. \tag{22}$$

[11] If we had not a finite and discrete number of households as stated at the beginning of this subsection, this ratio would be zero and maximization would not give reasonable results.

[12] Greenwood, Hercowitz and Krusell (1997, p. 361), analyzing the long-run effects of technological change limited to investment goods, distinguish between economic depreciation (which would be s here) and physical depreciation (corresponding to δ).

3 Solving the model

A household's choice variables are the consumption flow c and real R&D investment i. By choosing consumption, the household solves her consumption-savings problem. By choosing R&D investment, she determines the amount of savings going to capital accumulation, i.e. she solves the portfolio problem. One optimality condition describes the evolution of consumption by a Keynes-Ramsey rule. The second one is an arbitrage condition describing the optimal allocation of savings to capital accumulation and R&D. These two optimality conditions, aggregated over households in an appropriate way, together with the expression for the arrival rate (4) and an equation describing capital accumulation similar to (14) with (2) describe equilibrium of this economy (given initial conditions for the capital stock and consumption.) The next subsection presents four such equations.

3.1 The cyclical components

We first focus on understanding the cyclical components of our growth paths. One can split trajectories K and C of the capital index and aggregate consumption into trend components $A^{q/\alpha}$ and A^q and cyclical components $\hat{K}$ and $\hat{C}$ according to

$$K \equiv \hat{K} A^{q/\alpha}, \quad C \equiv \hat{C} A^q. \tag{23}$$

Trend components are not identical due to the vintage structure of capital. We will nevertheless eventually analyze a balanced cyclical growth path where K^c from (17) (and not K) and C grow at the same expected rate. With this specification, cyclical components are without trend.

Expressed in our cyclical components (23), the Keynes-Ramsey rule is (cf. appendix 9.3)

$$-\frac{u''\left(\hat{C}\right)}{u'\left(\hat{C}\right)} d\hat{C} = \left\{ r - \rho - \lambda \left[1 - (1-s) \frac{u'\left(A\tilde{\hat{C}}\right)}{u'\left(\hat{C}\right)} \right] \right\} dt - \frac{u''\left(\hat{C}\right)}{u'\left(\hat{C}\right)} \left\{ \tilde{\hat{C}} - \hat{C} \right\} dq. \tag{24}$$

Consumption rises in a continuous fashion (the dt-term) when the interest rate exceeds the time preference rate and the arrival rate times the expression in squared brackets.[13] With an arrival rate of zero, this is the well-known relationship from deterministic models. With a positive arrival rate, if all wealth was lost in case of successful research, i.e. assuming $s = 1$ for interpretational purposes, the interest rate would have to exceed the sum of the time preference rate and the arrival rate in order for consumption to grow. This reflects the fact that consumption is only postponed if returns r compensate for the risk of losing all wealth. When the economic devaluation s is small, wealth is not entirely lost and consumption is postponed also for lower returns r. The extent to which a change in s influences the level of returns required for consumption growth depends on the ratio $u'\left(A\tilde{\hat{C}}\right)/u'\left(\hat{C}\right)$ of marginal utility from consumption after and before successful research.[14] This ratio equals

[13] As introduced in (23), the cyclical component of a variable X is denoted by $\hat{X}$. Where no ambiguity arises, we will nevertheless talk about e.g. consumption $\hat{C}$ rather than (correctly) the cyclical component $\hat{C}$ of consumption in order to avoid too much repetition.

[14] A tilde ($\tilde{}$) denotes the value of a quantity immediately after successful research.

by the first order condition for consumption the ratio of shadow prices of capital after and before successful research. With a high shadow price of capital after R&D, the growth rate of consumption rises simply because successful R&D is desirable.[15] The dq-term gives discrete changes in case of successful R&D. It is tautological, however: When q jumps and $dq = 1$ and $dt = 0$ for this small instant of the jump, (24) says $d\hat{C} = \tilde{\hat{C}} - \hat{C}$. Hence, (the cyclical component of) consumption after successful R&D, $\tilde{\hat{C}}$, needs to be determined in an alternative way.

The first order condition for R&D is satisfied if the certain return from capital accumulation equals the expected return from R&D,

$$u'\left(\hat{C}\right) = \lambda u'\left(A\tilde{\hat{C}}\right)\kappa/R = u'\left(A\tilde{\hat{C}}\right)\lambda^{-\gamma/(1-\gamma)}\kappa_0/D_0. \tag{25}$$

The certain return is given by the shadow price of wealth a on the LHS (which by the first order condition for consumption equals marginal utility from *current* consumption). The expected gain from a marginal unit of savings into R&D on the RHS is given by the arrival rate times the shadow price of wealth after successful research (which equals marginal utility from consumption *after* successful research) times "marginal dividend payment" κ/R. The second equality uses (4), (5) and (7).

Equation (4), rewritten in order to obtain the amount of resources required for R&D as a function of the arrival rate λ, with (5), (17) and (23) gives the cyclical component $\hat{R}$ of R&D resources,

$$\hat{R} \equiv A^{-q}R = \lambda^{1/(1-\gamma)}D_0\hat{K}. \tag{26}$$

The final equation combines (14), describing the evolution of the capital index, with the goods market clearing condition (2) and uses (23). Letting $\hat{Y} = \hat{K}^a L^{1-\alpha}$ describe the cyclical component of GDP, it reads

$$d\hat{K} = \left\{\hat{Y} - \hat{R} - \hat{C} - \delta\hat{K}\right\}dt + \left\{A^{-1/\alpha} + A^{-1}\kappa_0 - 1\right\}\hat{K}dq. \tag{27}$$

The deterministic dt-term is self-explanatory. The stochastic dq-term shows that the change in the capital stock is given by the difference between the new capital stock $\left(A^{-1/\alpha} + A^{-1}\kappa_0\right)\hat{K}$ and the old capital stock $\hat{K}$. The new capital stock equals the old capital stock times $A^{-1/\alpha}$, which is a consequence of the detrending rule (23), plus the size of the new machine. As the new machine by (7) is proportional to the observed capital stock before successful R&D, its size after successful R&D is reduced by the factor A.

Equations (24)-(27), given initial conditions, describe the equilibrium of our economy. Given this system, we now have to understand whether a unique solution exists and what its properties are. A formal proof in this generality is beyond the scope of this paper. It would have to follow the literature on functional differential equations (e.g. Hale and Verduyn, 1993) due to the retarded term $\tilde{\hat{C}}$ in (24). Intuitively, it is easy to understand, however, that a solution to (24)-(27) exists indeed and is unique.

If we replace the arrival rate in (24) by the expression resulting from (25), we have a differential equation which gives the change of consumption as a function of the capital index,

[15] When returns to R&D are very high, the expression in squared brackets can even be negative and the presence of an R&D process has a positive effect on consumption growth.

consumption itself, exogenous quantities and $\tilde{\hat{C}}(t)$, the cyclical component of consumption after successful R&D. Equation (27), after having inserted (26) with the arrival rate again replaced by the expression from (25) gives us the change of the capital index as a function of the capital index, consumption, exogenous quantities and $\tilde{\hat{C}}(t)$. If we knew $\tilde{\hat{C}}(t)$, we would have a two-dimensional differential equation system in $\hat{K}(t)$ and $\hat{C}(t)$ which, provided initial conditions $\hat{K}_0$ and $\hat{C}_0$, gives a unique path $\left\{\hat{K}(t), \hat{C}(t)\right\}$.

The crucial step in understanding existence and uniqueness of such a solution is that on the optimal path (in an analogy, think of the saddle path in an optimal growth model), i.e. on the path where the initial consumption level $\hat{C}_0$ is optimally chosen, consumption is a function of the current capital index only (and not of q),

$$\hat{C} = \hat{C}\left(\hat{K}(t)\right). \tag{28}$$

As a consequence, the consumption level $\tilde{\hat{C}}(t)$ after successful research obeys the same functional relationship (28) as any other consumption level. It is determined by $\tilde{\hat{C}}(t) = \hat{C}\left(\tilde{\hat{K}}(t)\right)$, i.e. the consumption level corresponding to the capital stock $\tilde{\hat{K}}(t)$ after successful research. As this capital stock can be computed from (27) by setting $dt = 0$ and $dq = 1$, one just needs to insert $\tilde{\hat{K}}(t)$ into (28) to obtain $\tilde{\hat{C}}(t)$. The jump in consumption is therefore such that the system jumps from $\left(\hat{K}, \hat{C}\right)$ to $\left(\tilde{\hat{K}}, \tilde{\hat{C}}\right)$ where both capital-consumption pairs are on the optimal path $\hat{C}\left(\hat{K}(t)\right)$. This completes the illustration of existence and uniqueness of a solution of the above system. The next section and section 6 prove formally (for certain parameter sets) that such a unique path actually exists.

3.2 A linear policy rule

One can prove the existence of a unique solution as just informally described and derive its properties for a certain set of parameters. By focusing on this solution, we can derive many interesting predictions. We argue later and it will become clear that many findings hold more generally.

Theorem 1 *If the share of capital in GDP equals the inverse of the intertemporal elasticity of substitution, i.e. if $\alpha = \sigma$, the arrival rate λ is constant and given by*

$$\lambda = \left(\xi^{-\sigma}\kappa_0 D_0^{-1}\right)^{(1-\gamma)/\gamma}, \tag{29}$$

where

$$\xi = \kappa_0 + B^{-1}. \tag{30}$$

Further, the cyclical component of consumption is a linear function of the cyclical component of the capital index,

$$\hat{C} = \Psi\hat{K}, \tag{31}$$

where Ψ is a constant as well,

$$\Psi = \frac{\rho + \lambda \left[1 - (1-s)\,\xi^{-\sigma}\right] + (1-\sigma)\,\delta}{\sigma} - \lambda^{1/(1-\gamma)} D_0. \tag{32}$$

Finally, the jump in capital and consumption is given by

$$\frac{\tilde{\hat{C}}}{\hat{C}} = \frac{\tilde{\hat{K}}}{\hat{K}} = A^{-1}\xi. \tag{33}$$

Proof. cf. appendix 9.4. ■

Clearly, the results to be presented hold exactly only for $\alpha = \sigma$. How reasonable is such an assumption (made e.g. also in deterministic models of Xie, 1991, 1994)? When the capital share is understood in a narrow sense, i.e. when $\alpha = 1/3$, the intertemporal elasticity of substitution ε equals $\varepsilon = 1/\sigma = 3$. Compared to usual *average* estimates of ε lying between 0 and 1 (e.g. Vissing-Jørgensen, 2002), this appears high. Taking α to capture the output elasticity of capital in a broad sense (including human capital), i.e. $2/3 < \alpha < 1$, the intertemporal elasticity of substitution lies between 1 and 1.5. Allowing for household *heterogeneity* and estimating ε for households that hold assets (in contrast to those that do not), Vissing-Jørgensen found values of ε in this range. Hence, with $2/3 < \alpha < 1$, the implied value for ε appears reasonable.

Independently of what the appropriate value for α is exactly, however, assuming $\alpha = \sigma$ is required only for obtaining analytical results. Many findings for this case should hold for parameter values $\alpha \neq \sigma$ as well. In fact, this is analytically shown for another parameter restriction that implies a closed form solution in section 6.1. Further confirmation would require numerical approaches for which analytical results could serve as benchmark cases.

3.3 Equilibrium

For $\alpha = \sigma$, equilibrium of our economy can be described by the consumption rule (31) and the differential equation (27) for the cyclical component of capital, which, with (26) and (30), can be written as

$$d\hat{K} = \left\{\hat{Y} - \hat{C} - \hat{\delta}\hat{K}\right\} dt + \left\{A^{-1}\xi - 1\right\} \hat{K} dq, \tag{34}$$

where $\hat{\delta} = \delta + \lambda^{1/(1-\gamma)} D_0$. Figure 1 illustrates the evolution of capital and consumption. It plots $\hat{K}$ on the horizontal and $\hat{C}$ on the vertical axis. Zero-motion lines $d\hat{K} = 0$ and $d\hat{C} = 0$ follow for $dq = 0$ from (34) and from (24) with (25) and (26). In both cases, properties of the equilibrium path (29) and (33) have to be used. Using equilibrium properties for plotting zero motion lines might at first appear confusing, it is typical, however, of systems with retarded arguments as $\tilde{\hat{C}}$. Zero motion lines as e.g. the dt term for consumption in (24), depend on $\tilde{\hat{C}}$, i.e. consumption after a jump. As this value is known only when the current trajectory (28) of the economy is known (as the jump is from the current point on this trajectory to some other point on this same trajectory), the zero motion line depends on the current trajectory. As we want to analyze zero motion lines for equilibrium, the equilibrium trajectory needs to be known. In the solution of deterministic control problems, zero motion lines can be computed *before* the equilibrium trajectory is known. Here, equilibrium has to be found first and zero-motion lines have an illustrative purpose only. Nevertheless, zero-motion lines have the usual shape and laws of motion indicated by arrows are identical to standard Ramsey growth models. This allows us to describe a typical cycle of our growing economy as follows:

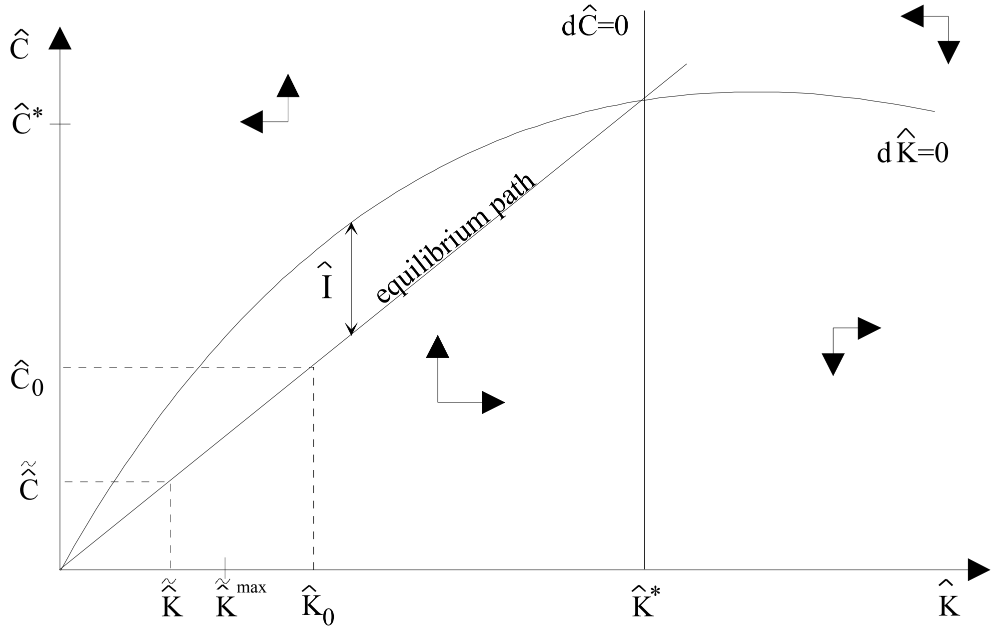

Figure 1: The equilibrium path in a phase diagram

Let the economy start with some historically given capital index $\hat{K}_0$. With consumption given by $\hat{C}_0$, the economy is on the equilibrium path and approaches the steady state as long as research is not successful, i.e. $dq = 0$. The capital stock and GDP grow, research is being undertaken. At some point, research is successful, a better vintage is available and q increases by one. The new level of the cyclical component $\hat{K}$ of the capital stock by (27) amounts to $\tilde{\hat{K}} = \left(A^{-1}\kappa_0 + A^{-1/\alpha}\right)\hat{K}$ and changes due to two factors: First, it decreases because of the factor $A^{-1/\alpha} < 1$, originating from the detrending rule (23). Second, it increases by the size of the new machine, i.e. by $A^{-1}\kappa_0$. Overall, the capital index drops if the relative size κ_0 of the new machine is small enough. As this is the only empirically reasonable assumption, we set

$$A^{-1}\xi = A^{-1}\kappa_0 + A^{-1/\alpha} < 1 \tag{35}$$

and the economy finds itself at a point $\left(\tilde{\hat{K}}, \tilde{\hat{C}}\right)$ after successful research, where $\tilde{\hat{K}} < \hat{K}$. There, it starts growing again through accumulating capital of the new vintage and it approaches the steady state until the next jump occurs.

Result 1 *Equilibrium cyclical growth takes place on a path where capital is accumulated and R&D is undertaken and where better vintages causing fluctuations and growth come at* *random points in time. The expected length between two vintages and whether new vintages are developed at all is endogenously determined by the households' investment decisions.*

Returning to the informal proof of a unique solution to the system (24) - (27), this phase diagram illustrates (and the theorem has proven) that on the optimal path there is a functional relationship as in (28) indeed. Further, $\tilde{\hat{C}}(t) = \hat{C}\left(\tilde{\hat{K}}(t)\right)$, i.e. a jump in the capital stock implies a jump in consumption such that the economy jumps to some other point on the path on which it found itself before the jump. The level of the capital index $\tilde{\hat{K}}$ after a jump is bounded,

$$0 < \tilde{\hat{K}} < A^{-1}\xi\hat{K}^{*} \equiv \tilde{\hat{K}}^{\max}. \tag{36}$$

By (33), it is strictly positive as the capital index $\hat{K}$ before the jump is positive. It is also strictly smaller than the level where it would end up when jumping back from the steady state $\hat{K}^{*}$ (as the steady state is never reached).

4 Plausibility of equilibrium paths

4.1 Short-run fluctuations

Let us look at the evolution of variables as they are actually "observed" by re-transforming cyclical components into observed variables. One realization of actual variables is depicted in figure 2.

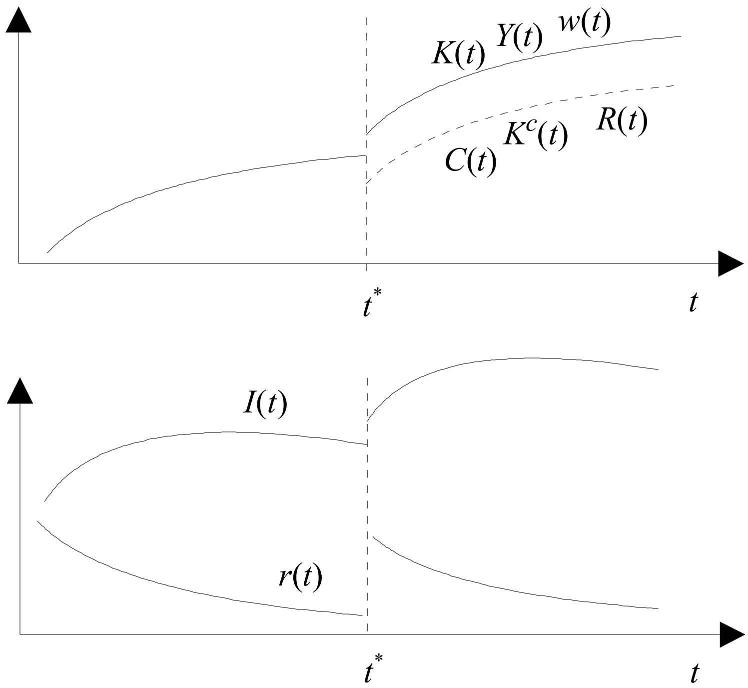

Figure 2: Qualitative properties of cycles

The capital index K increases smoothly as long as no jump occurs, as by (23) and with $dq = 0$ it is proportional to $\hat{K}$. As $\hat{K}$ approaches the steady state with an ever decreasing growth rate, it has an upper bound which it never reaches. The same therefore holds for K. When a jump occurs, the capital index unambiguously increases according to (14) by $\tilde{K} - K = B^{q+1}\kappa$. With (7) and (17), we obtain

$$\frac{\tilde{K}}{K} = B\kappa_0 + 1 = B\xi > 1. \tag{37}$$

An immediate implication of the time path of K for GDP, following from (10), is that GDP increases smoothly as well when research is not successful and jumps as a result of successful research by

$$\frac{\tilde{Y}}{Y} = \left(\frac{\tilde{K}}{K}\right)^{\alpha} = (B\xi)^{\alpha} > 1. \tag{38}$$

The same holds true for real wages as by the Cobb-Douglas structure in (10) they are a constant share of GDP, $w/p_c = (1-\alpha)\,Y/L$. Figure 2 illustrates that K, Y and w qualitatively behave in the same way. The jumps are not the same as (37) and (38) show.

Quantitatively, the predicted jumps in K, Y and w are small as they stem from the discovery of the new prototype in research. Compared to the existing aggregate capital index, this is small. In terms of the model and (37), the increase of e.g. K is by $B\kappa_0$ %. With κ_0 close to zero in order to ensure a realistic size of dividend payments relative to the capital stock as specified in (7), this jump is close to zero as well and would not be visible in real world data.

Consumption C, the observed capital stock K^c from (17) and R&D investment R are all proportional to $\hat{K}$ (cf. appendix 9.5). In periods without jumps, they behave qualitatively identical to, say, GDP. Computing the jump of these variables, however, gives

$$\frac{\tilde{R}}{R} = \frac{\tilde{C}}{C} = \frac{\tilde{K}^c}{K^c} = \xi = B^{-1}\frac{\tilde{K}}{K}. \tag{39}$$

The jump is lower than the jump of the capital index and can be a drop (the dotted line in figure 2), as ξ can be smaller than unity. With plausible parameter values, i.e. with κ_0 very small and B within a reasonable range (to be discussed later), ξ is smaller than unity indeed. Investment in R&D, consumption and the observed capital stock drop after an innovation. Individuals postpone consumption as a new technology promises higher returns to capital accumulation.[16]

The cyclical component of investment $\hat{I} = \hat{Y} - \hat{C} - \hat{\delta}\hat{K}$ is given by the distance between the equilibrium path and the zero-motion line as depicted in figure 1. As observed investment I by applying (23) is proportional to $\hat{I}$, its behaviour over the cycle is identical to $\hat{I}$. Note that investment I can be non-monotonic even without jumps: Let the upper bound $\tilde{\hat{K}}^{\max}$ for the capital index after the jump in (36) be lower than the capital stock where $\hat{I}$ is at its maximum (as depicted in figure 1). Then any jump from a capital stock sufficiently close to the steady state (say from the "th" of path in figure 1) implies non-monotonic investment

[16] Consumption drops also as the $\sigma = \alpha$ assumption implies an intertemporal elasticity of substitution that exceeds 1. Consumption would not drop for lower elasticities (cf. section 6.1 and footnote 20).

between jumps (as depicted in figure 2). When a jump occurs, investment unambiguously increases (cf. appendix 9.5).

Finally, the interest rate (21) with (10) and (23) is $r = \alpha \left(L/\hat{K} \right)^{1-\alpha} - \delta$. It jumps when research is successful and $\hat{K}$ falls. This induces a boom, i.e. a phase of growth rates above average. The interest rate falls smoothly as $\hat{K}$ increases and the economy eventually has growth rates below average. Overall, the interest rate is without trend.

4.2 Long-run growth

We measure the growth rate between today in t and some future point T by the difference in logarithms $g_{T,t} \equiv \ln Y(T) - \ln Y(t)$. Inserting the production function (10) and using the martingale property of $q(t) - \lambda t$, gives an expected growth rate per unit of time of (cf. appendix 9.6)

$$Eg_t \equiv \frac{E_t g_{T,t}}{T-t} = \lambda \ln A + \alpha \frac{E_t \ln \hat{K}(T) - \ln \hat{K}(t)}{T-t}. \tag{40}$$

The first term is the expected growth rate of the stochastic trend, the second term describes the contribution of transitory growth towards the expected capital stock in T. As the numerator of the second term is bounded and the denominator goes to infinity when the future point T is sufficiently far in the future, we focus on the first term $\lambda \ln A$ as the central determinant of expected growth.

Result 2 *From (29) and (30), the arrival rate is given by*

$$\lambda = \left(\frac{\kappa_0}{\left[\kappa_0 + B^{-1}\right]^\sigma D_0} \right)^{(1-\gamma)/\gamma}. \tag{41}$$

For decreasing returns to scale in the R&D sector ($0 < \gamma < 1$), the arrival rate increases in A and falls in D_0, i.e. it increases when innovations are more important and become less difficult. If in addition $(1-\sigma)\kappa_0 + B^{-1} > 0$ (cf. appendix 9.7), which holds on our equilibrium path where $\sigma = \alpha$, the arrival rate increases when dividend payments increase,

$$\frac{\partial \lambda}{\partial A} > 0, \quad \frac{\partial \lambda}{\partial D_0} < 0, \quad \frac{\partial \lambda}{\partial \kappa_0} > 0.$$

There are scale effects neither in the arrival rate nor in the expected growth rate which is also due to the difficulty function (5). Other parameters that sometimes appear in growth rates (e.g. the time preference rate or the depreciation rate) do not have a growth effect. They have a level effect though, as they affect the behavior of cyclical components via (32).

In order to fully understand why these results hold only for decreasing returns to scale and why the effect of dividend payment κ_0 is ambiguous, consider again the household's first order conditions for R&D investment (25), $u'\left(\hat{C}\right) = \lambda u'\left(A\tilde{\hat{C}}\right)\kappa/R$. For a given capital stock $\hat{K}$ and with (31), certain returns from capital accumulation, $u'\left(\hat{C}\right) = \left(\Psi\hat{K}\right)^{-\sigma}$ are independent of R&D investment R and can therefore be depicted as a horizontal line in the following figure.[17] This independence is intuitively clear as certain returns from capital accumulation are a direct consequence of the marginal productivity of capital which, given the aggregate technology (10), is independent of how output is used.

Expected returns can be expressed with (4), (5), (17), (23) and (26) as the product of the "riskiness of R&D" λ, the shadow price of capital $u'\left(A\tilde{\hat{C}}\right)$ and individual dividend payments κ/R in case of successful R&D, i.e. $\lambda u'\left(A\tilde{\hat{C}}\right)\kappa/R = \left(\hat{R}/\left(D_0\hat{K}\right)\right)^{1-\gamma}\left(\Psi\left[\kappa_0 + B^{-1}\right]\hat{K}\right)^{-\sigma}$ $\kappa_0\hat{K}/\hat{R}$. This shows that expected returns fall in $\hat{R}$ for decreasing returns to scale in the R&D sector, i.e. for $0 < \gamma < 1$. Decreasing returns therefore guarantee stability of optimal R&D investment $\hat{R}^*$: if R&D investment lay slightly to the left (right) of $\hat{R}^*$, expected returns would exceed (fall short of) certain returns and R&D investment would increase (decrease) until it reaches $\hat{R}^*$. For increasing returns ($\gamma < 0$), expected returns to R&D would increase in R (as the dashed line shows) and the equilibrium point at $\hat{R}^*_{\gamma<0}$ would be unstable. For $\gamma = 0$, expected returns would be horizontal as well and agents would find it optimal to invest all savings either into R&D *or* into capital accumulation. R&D investment would be countercyclical (Wälde, 2002).

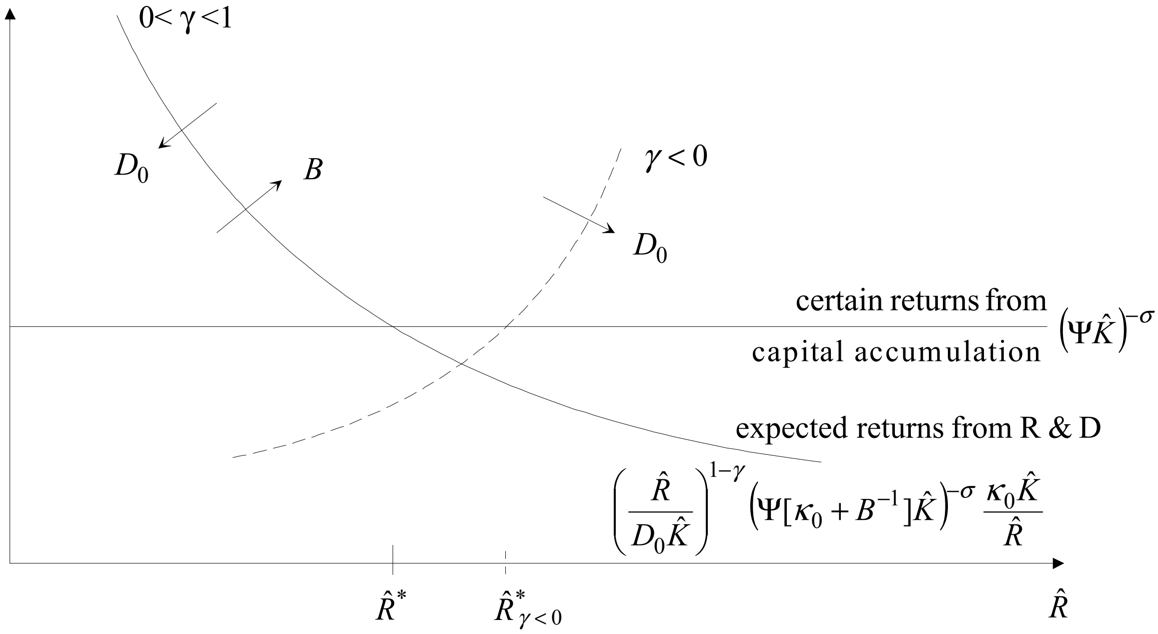

Figure 3: Equilibrium R&D investment

An increase in difficulty D_0 decreases expected returns, as the expression in the figure immediately shows, and R&D investment falls.[18] If expected returns increased in $\hat{R}$ for $\gamma < 0$, this result would reverse. As increasing returns to R&D must be ruled out because

[17] I am grateful to Sjak Smulders for having suggested the presentation of such a figure. A corresponding figure in deterministic models would show returns from capital accumulation as expressed in (21) and *certain* returns from R&D.

[18] The effect of any parameter change on Ψ can be neglected as it has the same effect on certain returns and on expected returns. It therefore cancels out.

of the stability aspect just described, we will limit all subsequent discussion to the case of decreasing returns. A larger κ_0 increases expected returns as it increases dividend payments $\kappa_0 \hat{K}$. It decreases expected returns as the shadow price $\left(\Psi\left[\kappa_0 + B^{-1}\right]\hat{K}\right)^{-\sigma}$ of capital falls. Which effect is stronger depends on the elasticity of substitution parameter σ.

5 The nature of cycles

5.1 Cyclical behaviour of R&D investment

We analyze cyclical properties of endogenous variables by using an inequality attributed to Čebyšev (e.g. Mitrinović, 1970, ch. 2.5, th. 10): Let two functions obey $f'(x)\,g'(x) \gtreqless 0$ on an interval $]a,b[$. Then $\int_a^b p(x)\,dx \int_a^b p(x)\,f(x)\,g(x)\,dx \gtreqless \int_a^b p(x)\,f(x)\,dx \int_a^b p(x)\,g(x)\,dx$ for an integrable function $p(x) > 0$ on $]a,b[$. Applying this to our question, let X be a random variable with density $p(X)$ and support $[a,b]$ and $f(X)$ and $g(X)$ two transformations for which $f'(x)\,g'(x) \gtreqless 0$ for all realizations x of X. Then $\int_a^b p(x)\,dx = 1$ and the inequality says $Ef(X)\,g(X) \gtreqless Ef(X)\,Eg(X)$ which is identical to saying that the covariance of these transformed random variables is given by $\text{cov}(f(X), g(X)) \gtreqless 0$. Simply speaking, when two variables "move in the same direction" ($f'(x)$ and $g'(x)$ are both either positive or negative), their covariance and correlation coefficient are positive. In terms of business cycle analysis, when $g(X)$ represents output, $f(X)$ would be procyclical.

As a first application of Čebyšev's inequality, consider the correlation of the interest rate with output. As before and as is custom in empirical work, we consider cyclical components only. We therefore detrend output by applying (23) to (10) and removing the resulting trend term A^q. For the trendless interest rate, only (21) is applied in order to get an expression in terms of $\hat{K}$. Taking $\hat{K}$ as our random variable X and the cyclical component of GDP and of the interest rate as transformations, $f\left(\hat{K}\right) = \hat{K}^{\alpha} L^{1-\alpha}$ and $g\left(\hat{K}\right) = \alpha\left(L/\hat{K}\right)^{1-\alpha} - \delta$, a negative correlation is found. The intuition, given Čebyšev's inequality is simple: The interest rate falls in the capital stock, GDP rises. As they move in opposite directions, they are negatively correlated, i.e. the interest rate is countercyclical.

While the interest rate falls in the capital stock in all models with standard neoclassical production functions, models with shocks to total factor productivity often imply (e.g. King and Rebelo, 1999, p.939) that GDP and the interest rate are positively correlated, in contrast to what is empirically observed. The present model departs in one important way from other setups, causing this result: Jumps of labour productivity cause long-run growth and do not play any role in determining the cyclical component. In fact, the cyclical behaviour of the interest rate and the cyclical component of GDP is entirely determined by $\hat{K}$. Hence, this unambiguous countercyclical behaviour of the interest rate. In more traditional models, shocks to total factor productivity are central to understanding cyclical behaviour. As both GDP and the interest rate increase in TFP, a procyclical relationship is usually found.

Now use Čebyšev's inequality to understand why R&D investment and GDP are positively correlated here. If we express the cyclical components of R&D investment R and GDP Y as a function of the random variable $\hat{K}$, we can deduce the sign of their correlation coefficient by checking the sign of $\hat{R}'\left(\hat{K}\right)\hat{Y}'\left(\hat{K}\right)$. As $\hat{Y}'\left(\hat{K}\right) > 0$ and on our equilibrium path,

$\hat{R}'\left(\hat{K}\right) > 0$ by (27) and (29), R&D is procyclical.

To understand why $\hat{R}$ increases as $\hat{K}$ increases, look again at figure 3. An increase in $\hat{K}$ decreases the shadow price of capital before and after the jump, $\left(\Psi\hat{K}\right)^{-\sigma}$ and $\left(\Psi\left[\kappa_0 + B^{-1}\right]\hat{K}\right)^{-\sigma}$, in the same way. Changes in shadow prices are therefore neutral and do not affect R&D investment. An increase in $\hat{K}$ increases expected returns, i.e. the expected returns curve shifts outward, as dividend payments $\kappa_0\hat{K}$ rise. At the same time, it decreases expected returns through the increase in difficulty $D_0\hat{K}$ that decreases the arrival rate. Due to decreasing returns in the R&D sector, the dividend payment effect is stronger than the difficulty effect. The expected returns curve shifts outward and R&D investment rises.

Result 3 *Dividend payments and the difficulty to invent increase as capital is accumulated. The investment encouraging effect of higher dividend payments overcompensates the discouraging effect of higher difficulty due to decreasing returns in the R&D sector. R&D investment is procyclical.*[19]

Note that the relative shadow price, which for $\alpha = \sigma$ is given by $\left(\Psi\hat{K}\right)^{-\sigma} / \left(\Psi\left[\kappa_0 + B^{-1}\right]\hat{K}\right)^{-\sigma}$ and is therefore independent of $\hat{K}$, will be a function of $\hat{K}$ for other parameter sets. If it is a decreasing function of $\hat{K}$, procyclical R&D investment will be preserved. If, however, it is *strongly* increasing in $\hat{K}$, R&D investment will become countercyclical. If through numerical analysis parameter sets can be identified for which R&D investment becomes countercyclical, empirically not always clear-cut findings for R&D could potentially be given a better interpretation.

5.2 Jumps and aggregate fluctuations

The aggregate impact of jumps can be measured by the length and the amplitude of fluctuations. As on our equilibrium path the arrival rate λ in (41) is constant, the expected length of a cycle is simply its inverse λ^{-1},

$$E\text{Length} \; = \lambda^{-1} = \left(\frac{\left[\kappa_0 + B^{-1}\right]^{\sigma} D_0}{\kappa_0}\right)^{(1-\gamma)/\gamma}. \tag{42}$$

The amplitude of a cycle can be measured by the distance between the maximum and the minimum of the *log* of the cyclical component of GDP, $\hat{Y} = \hat{K}^{\alpha}L^{1-\alpha}$. As the GDP ratio is given by $\hat{Y}/\tilde{\hat{Y}} = \left(\hat{K}/\tilde{\hat{K}}\right)^{\alpha}$, the distance is with (35)

$$\text{Amplitude} = \ln\left(\hat{Y}/\tilde{\hat{Y}}\right) = \alpha \ln\left(A^{-1}\kappa_0 + A^{-1/\alpha}\right)^{-1} \geq \ln A, \tag{43}$$

[19] At present, the mechanism that dividend payments increase when the capital stock increases is exogenously put into (7). Dividend payments that increase in the capital stock could naturally be endogenized, however: Imagine a differentiated capital structure like $\left(\Sigma_{j=1}^{q} K_j^{\theta}\right)^{1/\theta}$ where new capital goods are not perfect substitutes to old vintages as in (11). The marginal productivity of an additional vintage $q+1$ is then higher, the higher the capital stock of old vintages, i.e. the higher the K_js. Dividend payments would therefore grow when K_q is accumulated, just as in the present "short-cut" specification (7).

where the approximation used that κ_0 is close to zero.

When the increase in labour productivity from one vintage to the next one is measured in the same way, one finds $\ln(A^{q+1}/A^q) = \ln A$. Comparing this with the aggregate effect (43), one obtains the following

Result 4 *The direct effects of successful R&D are small as (i) labour productivity increases only for the new vintage $q+1$ and remains unaffected for old vintages $0...q$ and (ii) the size κ_0 of the new machine relative to the aggregate capital stock is small. Nevertheless, given the subsequent accumulation of this new vintage, the aggregate effects as measured by the amplitude in (43) are of the same order of magnitude as the vintage-specific jump of labour productuvity of* $\ln A$.

The expected length of a cycle (42), given decreasing returns in R&D $(0 < \gamma < 1)$, goes to infinity as the size of the machine goes to zero, $\lim_{\kappa_0 \to 0} E\text{Length} = \infty$, i.e. small jumps can cause long-lasting aggregate fluctuations. This result is fairly straightforward when recalling the discussion of the arrival rate (41). A smaller size κ_0 means smaller dividend payments. Less resources are allocated to R&D and the arrival rate falls. The expected length of a cycle, being its inverse, increases. The interesting effect is the strong nonlinearity of (42) in κ_0. When κ_0 goes linearly to 0, the expected length quickly increases.

5.3 Which frequencies can we understand?

Looking at (42), the model seems flexible to capture both high and low frequencies. In order to be reasonable, the model should also on average predict realistic growth rates (40). Taking for illustration purposes the average length of post-World War II business cycles to be 5 years in OECD countries and the average growth rate to be 2%, we obtain two conditions,

$$E\text{Length} = \lambda^{-1} = 5 \text{ years}, \tag{44}$$

$$Eg_t = \lambda \ln A = 2\%. \tag{45}$$

They immediately imply $\ln A = .1 \Leftrightarrow A \approx 1.1$ from inserting $\lambda = .2$ from the first into the second condition. Given the expression for the expected length (42), one parameter of the remaining κ_0, D_0 and γ is therefore fixed by (44). (The parameter σ is pinned down on our equilibrium path by $\sigma = \alpha$.) As γ is limited to lie between 0 and 1 and κ_0 should be small following the discussion of (7), D_0 would have to (and could) adjust in order to satisfy (44). Hence,

Result 5 *The model can be used to jointly analyze endogenous high-frequency fluctuations and growth.*

6 Extensions

This section shows that qualitative results presented so far are robust to changes in (some) fundamentals of the economy: We relax the assumption of theorem 1 and study implications of the alternative difficulty function (5). We also show that the model presented so far can be considered as a shortcut for a multi-sector version that has qualitatively very similar properties.

6.1 An optimal constant saving rate

Clearly, closed form solutions can not be found for all parameter sets where $\alpha \neq \sigma$. There exists, however, a closed form solution for consumption and R&D expenditure for a particular parameter set where α differs from σ. This parameter set implies that households optimally choose a constant saving rate. This result is known from deterministic models (e.g. Kurz, 1968; Barro, Mankiw and Sala-i-Martin, 1995) and, as the following theorem shows, can be extended to the stochastic model presented here as well.

Theorem 2 *If*

$$\rho - (\alpha\sigma - 1)\,\delta = \sigma\lambda^{1/(1-\gamma)} D_0 - \lambda \left[1 - (1-s)\left(A^{1-\alpha}\xi^{\alpha}\right)^{-\sigma}\right], \tag{46}$$

the cyclical component of consumption is

$$\hat{C} = \Psi \hat{K}^{\alpha} L^{1-\alpha}, \tag{47}$$

where the consumption rate is $\Psi = 1 - 1/\sigma$. Further, the arrival rate λ is constant and given by

$$\lambda = \left(\left(A^{1-\alpha}\xi^{\alpha}\right)^{-\sigma} \kappa_0 / D_0\right)^{(1-\gamma)/\gamma}, \tag{48}$$

where ξ is defined as in (30).

Proof. cf. appendix 9.8. ■

While the condition (46) might not look intuitive at first sight, the left-hand side is the expression known from deterministic models. When $\rho = (\alpha\sigma - 1)\,\delta$ in an optimal deterministic continuous time growth model, optimal saving behaviour implies a constant saving rate equal to the intertemporal elasticity of substitution, $s = \sigma^{-1}$. This clearly requires $\sigma^{-1} < 1$ to make sense and it nicely presents an alternative to $\alpha = \sigma$ where $\sigma^{-1} > 1$. In our stochastic setup, additional terms on the right-hand side appear. The deterministic case is clearly a special case for $\lambda = 0$. After inserting the arrival rate (48) into this expression it can be shown (cf. appendix 9.8) that it is always positive but close to zero. Hence, condition (46) can be understood as in the deterministic case to represent preferences of households where the intertemporal elasticity of substitution is below unity.

This parameter restriction implies a constant saving rate σ^{-1} which gives the cyclical component of consumption (47) as a constant fraction of the cyclical component of GDP. The equilibrium path in figure 1 would therefore be a curve just below the zero motion line for capital, starting in the origin and going through the steady state. The arrival rate (48) is constant as well. The difference to (29) in theorem 1 is minor and all comparative static properties are preserved. Further, resource allocation to R&D is still procyclical which follows directly from (26) due to the constant arrival rate. Economically, the reasoning is identical to the one discussed after figure 3.[20]

[20] Exploring other equilibrium properties on this path would be very interesting. Using the jump of cyclical capital from (27), the consumption rule (47) and the transformation (23) gives $\tilde{C}/C = (B\kappa_0 + 1)^{\alpha}$. The jump in consumption, in contrast to the $\alpha = \sigma$ case, is therefore always positive.

6.2 An alternative difficulty function

Let the difficulty function now be given by (6) instead of (5). As briefly mentioned in the model section, if the difficulty function captures the idea that inventions become more difficult as the number of previous inventions rises, e.g. due to a finite pool of ideas, the difficulty function should be a function of q only and not also of $\hat{K}$. In that case, the following closed form solution is available.

Theorem 3 *If, as in theorem 1, $\alpha = \sigma$ and if the condition*

$$\left(B^{-1} + \kappa_0\right)^{\sigma} = B^{-1} + \sigma\kappa_0 \tag{49}$$

is met, the arrival rate

$$\lambda = \left(\xi^{-\sigma}\kappa_0 D_0^{-1}\hat{K}\right)^{(1-\gamma)/\gamma} \tag{50}$$

is an increasing function of the cyclical component $\hat{K}$ of capital, where ξ is defined as in (30). Further, the cyclical component of consumption is a linear function of the cyclical component of the capital index, $\hat{C} = \Psi\hat{K}$, where $\Psi = \left(\left(1-\sigma\right)\delta + \rho\right)/\sigma$.

Proof. cf. appendix 9.9. ■

This theorem uses two conditions, the well-known $\alpha = \sigma$ condition and the additional one in (49). This condition pins down κ_0 which can be shown to lie in the interval between 0 and 1 as required by (7).

Expected returns are now$\lambda u'\left(A\tilde{\hat{C}}\right)\kappa/R = \left(\frac{\hat{R}}{D_0}\right)^{1-\gamma}\left(\Psi\left[\kappa_0 + B^{-1}\right]\hat{K}\right)^{-\sigma}\kappa_0\hat{K}/D_0$ and certain returns are $\left(\Psi\hat{K}\right)^{-\sigma}$. Undertaking a similar reasoning to the one illustrated in figure 3 shows that an increase of the cyclical component of capital affects the allocation of resources to R&D only via the channel of higher dividend payments. The discouraging effect due to a higher difficulty resulting from an increase in $\hat{K}$ is now no longer present in the first term $\left(\hat{R}/D_0\right)^{1-\gamma}$ representing the arrival rate. In fact, whereas before in (26) resource allocation was a linear function of the capital stock, $\hat{R} = \lambda^{1/(1-\gamma)}D_0\hat{K}$, it now increases strongly over-proportionally in $\hat{K}$, $\hat{R} = \left(\xi^{-\sigma}\kappa_0 D_0^{-1}\hat{K}\right)^{1/\gamma}D_0$. The procyclical nature of R&D expenditure is therefore preserved also here.

Comparing the new expression of the arrival rate (50) with the original one in (29) shows that the difference consists in the capital term $\hat{K}^{(1-\gamma)/\gamma}$ being added. This means that comparative static properties for the arrival rate for each point on the cycle, i.e. for a given $\hat{K}$, remain valid as well. The arrival rate (50) also stresses, probably more forcefully than the constant expression in (29), the endogeneity of technological jumps. Towards the beginning of the cycle when dividend payments to successful R&D are low, few resources are allocated to R&D and the arrival rate is low. As dividend payments rise, agents find it more profitable to invest in R&D and the probability of a technological jump increases.

6.3 A multi-sector version

Consider an economy with N sectors. Output in sector i is given by $Y_i = \Gamma_i K_i^{\alpha_i}\left(A^{q_i}L_i\right)^{1-\alpha_i}$, where Γ_i is a constant and sector specific total factor productivity parameter, α_i is the output elasticity of capital, K_i and L_i are capital and labour allocated to this sector and A^{q_i} is the current labour productivity. All sectors produce intermediate goods that are assembled to give a final good Y,

$$Y = \Pi_{i=1}^{N} Y_i^{\gamma_i} = \Pi_{i=1}^{N} \left(\Gamma_i K_i^{\alpha_i} (A^{q_i} L_i)^{1-\alpha_i}\right)^{\gamma_i}. \tag{51}$$

Intermediate goods differ in their "importance" for the final good, captured by differences in γ_i. Allowing this assembly process to take place under perfect competition requires constant returns to scale,

$$\Sigma_{i=1}^{N} \gamma_i = 1. \tag{52}$$

Labour is instantaneously mobile across sectors. Its marginal productivity in any two sectors therefore needs to equalise. Making a similar argument for capital, taking factor market clearing conditions into account, i.e. $\Sigma_{i=1}^{N} L_i = L$ and $\Sigma_{i=1}^{N} K_i = K$, and putting all constants into Y_0, allows to rewrite the aggregate technology (51) as (cf. appendix 9.10)

$$Y = Y_0 A^{\Sigma_{i=1}^{N} q_i (1-\alpha_i)\gamma_i} K^{\Sigma_{i=1}^{N} \alpha_i \gamma_i} L^{\Sigma_{i=1}^{N} (1-\alpha_i)\gamma_i}. \tag{53}$$

This expression shows that all investment concerning improvement of technologies will, at identical R&D cost, be channelled into the sector that has the highest marginal contribution to output. This contribution consists in the contribution to labour productivity, $1-\alpha_i$, and in the importance of the sector in aggregate output, γ_i. The literature usually makes the assumption that sectors are completely symmetric, i.e. $\gamma_i = \gamma$, and, as often labour is the only factor of production, $\alpha_i = 0$. We can slightly relax this by assuming $(1-\alpha_i)\gamma_i \equiv \phi$. The technology then becomes with (52)

$$Y = Y_0 A^{\phi \Sigma_{i=1}^{N} q_i} K^{\Sigma_{i=1}^{N} \alpha_i \gamma_i} L^{\Sigma_{i=1}^{N} (1-\alpha_i)\gamma_i} \equiv Y_0 A^{\phi q} K^{\beta} L^{1-\beta} \tag{54}$$

and, as elsewhere in the literature, investors into R&D are indifferent between sectors. One can therefore assume that investment is equally spread and each sector has the same probability of improving its technology. On the aggregate level, this is equivalent to having just one R&D sector with one Poisson process $q \equiv \Sigma_{i=1}^{N} q_i$ whose arrival rate depends on the aggregate amount of resources invested into R&D, $\lambda = \lambda(R)$ as in (4).

Do sectoral improvements in technology wash out and result in constant growth rates on the aggregate level? Given the aggregate technology (54) clearly shows that sectoral jumps result in aggregate fluctuations. When total factor productivity in sector i increases by a factor A, this increases aggregate total factor productivity by the factor $A^{\phi} = A^{(1-\alpha_i)\gamma_i}$. Each sectoral technology jump therefore induces a discrete increase not only of sectoral total factor productivity but also, due to general equilibrium effects caused by factors of production moving into the technologically just advanced sector, a discrete increase in returns to capital accumulation in all sectors.[21] A sectoral jump therefore implies a boom in all sectors at the same time: there is co-movement of sectors.

Can such mechanism convince quantitatively? If on average the economy grows at 2% per year and there is a jump every 5 years as in (44) and (45), there would have to be an increase of total factor productivity every five years of 10% as well, i.e. $A^{\phi} = 1.1$. This would

[21] Other multi-sector models where sectoral technology jumps imply aggregate smooth growth assume a continuum of sectors and not a discrete and countable number of sectors as here.

require a sectoral increase, with a labour share of roughly 2/3 and, for simplicity, $\gamma_i = N^{-1}$, of $A = 1.1/\left[(1-\alpha_i)\,\gamma_i\right] \approx N * 2.2/3$. Whether such an increase is plausible depends on the view one takes on the number of *technologically independent* sectors in an economy. When there are many (say $N \gg 10$), technological increases in a sector will be relatively large per technological jump and occur rarely. This might not be very convincing.

Many sectors are not technologically independent, however. Using an argument made by Horvath (2000), an input output matrix is usually characterised by a few full rows and columns with many zeros. This means that they are are 'key sectors' whose output is used in many other sectors. These sectors are then technologically dependent. A technology jump in one of these key sectors therefore translates into a sufficiently large aggregate effect. Representing a multi-sector model as in (51) is therefore useful to demonstrate the basic transmission between sectoral jumps and aggregate effects, when it comes to the question of quantitative importance, however, properties of an input-output matrix need to be taken into account. When this is done, it is clear that due to these key sectors, a specification as in (51) is most convincing with few sectors, $N < 10$. Some law of large numbers is therefore not excluded on theoretical grounds but on empirical properties of real world economies.

Strong empirical support for this "mushroom view" of technological progress comes from Harberger (1998) and the literature cited therein. Technological progress is concentrated in a few industries at a given point in time, while technological progress at some other point in time is concentrated in a different set of industries. Technological progress is therefore not uniform across sectors (the yeast view) but TFP increases sometimes here, sometimes there, similar to the growth of mushrooms.

7 Conclusion

The starting point of this paper was the belief that economic fluctuations can originate endogenously from within an economy. R&D and the development of more efficient production units was presented as one mechanism causing an economy to grow by going through periods of high and low growth. The mechanism causing fluctuations is endogenous in the sense that the economy could grow (at least up to an upper bound) also without investment in R&D. It is the intentional choice of investors to finance the development of new technologies which causes fluctuations. If no investment took place, no fluctuations would be observed.

The first objective was to clarify the determinants of cyclical behaviour of R&D investment. In empirical work, R&D investment tends to be procyclical, while only scarce evidence for some countercyclical behaviour of R&D can be found. The present paper has shown that the cyclical behaviour can be understood by analyzing a portfolio decision problem. Understanding the determinants of this decision problem means understanding the cyclical behaviour of R&D. The driving force of procyclical R&D expenditure are procyclical dividend payments.

This relationship is not visible in previous work as in deterministic setups or stochastic setups with risk neutral agents a portfolio decision problem usually has a bang-bang property where either investment in capital accumulation or R&D takes place. As agents are risk averse here and there is an internal solution to the portfolio decision problem (due to the small externality γ), procyclical dividend payments can play this role and cause procyclical

R&D expenditure.

The average rate at which new technologies arrive and thereby the expected growth rate of the economy depends, among others, on total dividend payments, the increase in labour productivity due to new technologies and the returns to scale in the R&D sector. These determinants also pin down the average length of a cycle. The expression for the expected length of the cycle has shown that small technology jumps in an economy can easily have large effects on the aggregate level. By fixing some parameter values, it has also been shown that the model can be used to jointly study both high-frequency fluctuations and long-run growth.

Clearly, there are shortcomings that need to be addressed in future work. First, the present paper does not analyse recessions as growth rates of GDP are always positive. None of the existing papers on endogenous fluctuations and growth takes unemployment into account. Employment effects of fluctuations, however, are central in policy discussions about growth and business cycles. Combining unemployment with endogenous growth cycles should lead to a better understanding of recessions and therefore business cycles. Second, the model should numerically be solved for a broader class of parameter values. This would expand our understanding of the determinants of endogenous fluctuations. Finally, while the length of a cycle is stochastic, the amplitude is not, as each new technology increases labour productivity by the same factor A. Introducing either a stochastic or endogenous increase would allow to study the determinants and the effects of large and small technology jumps in an economy.

8 Appendix

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