Optimal asset allocation for pension funds under mortality risk during the accumulation and decumulation phases*

Paolo Battocchio†, Francesco Menoncin‡, Olivier Scaillet§

Abstract

In a financial market with one riskless asset and n risky assets following geometric Brownian motions, we solve the problem of a pension fund maximizing the expected CRRA utility of its terminal wealth. By considering a stochastic death time for a subscriber, we solve a unique problem for both accumulation and decumulation phases. We show that the optimal asset allocation during these two phases must be different. In particular, during the first phase the investment in the risky assets should decrease through time to meet future contractual pension payments while, during the second phase, the risky investment should increase through time because of closeness of death time. Our findings also suggest that it is not optimal to manage the two phases separately.

JEL: G23, G11.
Key words: pension fund, mortality risk, asset allocaiton.

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*The authors thank V. Young for helpful comments. The two first authors are grateful for the financial support from the Belgian French Community's program "Action de Recherches Concertée 99/04-235". The third author gratefully acknowledges financial support from the Swiss National Science Foundation through the National Center of Competence: Financial Valuation and Risk Management. Part of this research was done when he was visiting THEMA and IRES.

†IRES, Université catholique de Louvain, Place Montesquieu, 3, 1348 - Louvain-La-Neuve, Belgium. Tel: 0032-10-474484; fax: 0032-10-473945; e-mail: battocchio@ires.ucl.ac.be.

‡IRES, Université catholique de Louvain, Place Montesquieu, 3, 1348 - Louvain-La-Neuve, Belgium. Tel: 0032-10-474035; fax: 0032-10-473945; e-mail: menoncin@ires.ucl.ac.be.

§HEC Geneve and FAME, UNI MAIL, Faculte des SES, Bd Carl Vogt 102, CH - 1211 Geneve 4 Suisse. Tel: 0041-22-7058816; fax: 0041-22-7058104; e-mail: olivier.scaillet@hec.unige.ch.

1 Introduction

In this work we analyse optimal asset allocation by a pension fund which maximizes the expected utility of its final wealth. Unlike the analyses studying the problem of a non-actuarial institutional investor (a general framework can be found in Menoncin, 2002), the case of a pension fund requires the introduction of two new characteristics: (i) the different behaviour of the fund wealth during the accumulation and the decumulation phases (hereafter, APh and DPh, respectively), and (ii) the mortality risk. We want to develop a set up aimed at finding out how and how much this mortality risk affects the optimal asset allocation.

The existing literature dealing with the asset allocation problem for a pension fund, completely neglects the mortality risk and partially takes into account the problem of distinguishing the accumulation and the decumulation phases. In particular, Boulier, Huang, and Taillard (2001), and Battocchio and Menoncin (2002) just deal with the investment problem during the APh while Blake, Cairns, and Dowd (2000) take into account only the distribution phase. Instead, the only literature explicitly taking into account the mortality risk problem is the actuarial literature (see, e.g., Young and Zariphopoulou, 2002a,b for optimal asset allocation under an exponentially distributed investment horizon).

The only work, at least at our knowledge, which considers both the mortality risk and the difference between the APh and the DPh is the paper by Charupat and Milevsky (2002). They analyse the interaction between financial risk, mortality risk, and consumption towards the end of the life cycle. Their main result is that for constant relative risk aversion (CRRA) preferences and geometric Brownian motion dynamics, the optimal asset allocation during the DPh is identical to the APh, which is the classical Merton's (1971) solution. Nevertheless, they solve two different problems: (i) they maximize, for the fund manager, the expected utility of fund terminal wealth during the APh, and (ii) they find, for the consumer-investor, the optimal consumption-portfolio during the DPh. In their setting it is up to the consumer to choose how to allocate his wealth after the accumulation phase.

In this paper, instead, we want to present the case of a pension fund which manages the investor's wealth during both phases. Thus, during the APh, the fund wealth increases because of the contributions paid by the subscriber while, during the DPh, it decreases because of the pension paid by the fund. Thus, we suppose there is no choice at the retirement date but to receive a pension until the death time (τ). Here, we suppose τ to be stochastic and, in particular, we find a closed form solution to the asset allocation problem when it is exponentially distributed while we show an approximated solution when it is distributed according to a Weibull random variable.

Even if we take into account the simple framework after Charupat and Milevsky (2002) with geometric Brownian motion and a CRRA utility function, we show that their result is not robust. In fact, after solving a unique problem for the optimal asset allocation during the whole life of the fund, we find two different portfolio compositions during the APh and the DPh. More precisely, we find that during the APh the amount of wealth invested in the risky assets must decrease through time while, after the retirement date, it must (rapidly) increase.

As we have already highlighted, the risk aversion of the pension fund we take into account is described by a CRRA utility function. Nevertheless, in order to take into account the engagement of the fund to provide the subscriber with a (constant) pension rate, we use the so-called "state-dependent" preferences (see, e.g. Merton, 1990, Section 6.4). In particular, we suppose that during the APh the fund can obtain some utility only from the "new" wealth it is able to create thanks to its investment strategy, without obtaining any utility from the contributions paid by the subscriber. In fact, these contributions will have to be paid back to the subscriber as pensions.

For the sake of simplicity, in our model we keep constant the contribution and the pension rates and we compute a feasibility (equilibrium) condition on them for making it convenient to subscribe the contract both for a pension fund and for a worker. This equilibrium condition has already been used in the literature about the pension funds (see, e.g. Josa-Fombellida and Rincón-Zapatero, 2001).

Through this work we consider agents trading continuously in a frictionless, arbitrage-free market. Furthermore, we do not need the hypothesis of completeness for the financial market.

The paper is structured as follows. The framework is outlined in Section 2. First we describe the financial market. Then we compute the feasibility condition on the contribution and pension rates when the death time follows a Weibull distribution. Eventually we present the state-dependent utility underlying the financial decision problem. In Section 3 we compute the optimal portfolio and discuss the main practical implications of our results for the management of a pension fund. Section 4 concludes.

2 The model

We consider a financial market where there exist n risky assets and one riskless asset paying a constant interest rate r, whose dynamics are described by:

$$\underset{n\times 1}{dS} = \underset{n\times n}{I_S}\left(\underset{n\times 1}{\mu dt} + \underset{n\times k}{\Sigma'}\underset{k\times 1}{dW}\right), \quad dG = Grdt,$$

where I_S is a square diagonal matrix containing the elements of vector S and W is a $k-$dimensional Wiener process. Both μ and Σ are supposed to be constant. The fund wealth process R is then equal to

$$R = \theta' S + \theta_0 G,$$

where θ and θ_0 are the number of risky asset and the number of riskless asset held, respectively. Its associated SDE is simply:

$$dR = \theta' dS + \theta_0 dG + d\theta' (S + dS) + G d\theta_0.$$

The self-financing condition implies that the two last terms must be equated to zero or, when consumption is considered, must finance the consumption rate. In the case of a pension fund, the self-financing condition must ensure that the changes in portfolio composition (the two last terms) must: (i) be financed by the subscribers' contributions rate $u(t)$ during the accumulation phase, and (ii) finance the pension rate $v(t)$ paid to the subscribers during the decumulation phase. For the sake of simplicity, in what follows we suppose both u and v to be constant.

Let T indicate the (deterministic) date at which the subscriber retires, and let

$$\phi(t) = \begin{cases} 1, & \text{if } t \leq T, \\ 0, & \text{if } t > T. \end{cases}$$

Accordingly, the dynamic budget constraint can be written as

$$dR = (Rr + w'M + k)\, dt + w'\Sigma' dW, \tag{1}$$

where[1]

$$\begin{aligned} M &\equiv (\mu - r\mathbf{1}), \quad w \equiv I_S \theta, \\ k &= u\phi - v(1-\phi), \end{aligned} \tag{2}$$

and $\mathbf{1}$ is a vector of 1s.

In Charupat and Milevsky (2002) each dollar of new income flowing into the fund (u) is allocated separately and treated as a new problem. Thus, they completely neglect the role of u during the APh and they solve for $u = 0$. In our approach, instead, we treat u as a planned flow which the fund manager can rely on. Furthermore, as Merton (1990, Section 5.7) underlines, it is not necessary to treat the new financial flows (u) as they could be borrowed against, since the investor behaves "as if" this would be true.

2.1 The feasibility condition

The constant level of the contribution and the pension rates (u and v respectively) cannot be both freely chosen by the fund. Here, we take into account the case of a pension fund letting its subscribers choose the (constant) contribution rate (u) they prefer. The (constant) pension rate (v) is accordingly chosen. In particular, we know that, at time $t = 0$, from the point of view of the subscriber (pension fund), the expected present value of all pensions cannot be lower (higher) than the expected present value of all payments. Thus, we just equate the expected present value of pensions and payments by putting

$$\mathbb{E}_0^\tau \left[\int_0^\tau k(s)\, e^{-rs} ds \right] = 0.$$

[1] We underline that $w \in \mathbb{R}^{n \times 1}$ contains the amount of money invested in each risky asset.

This condition can be transformed into a condition on the ratio v/u by substituting the expression for k in (2):

$$\frac{u}{v} = \frac{\mathbb{E}_0^\tau \left[\int_0^\tau e^{-rs} ds\right]}{\mathbb{E}_0^\tau \left[\int_0^\tau \phi(s) e^{-rs} ds\right]} - 1.$$

Now, since we can write

$$\int_0^\tau e^{-rs} ds = \frac{1 - e^{-r\tau}}{r},$$
$$\int_0^\tau \phi(s) e^{-rs} ds = \begin{cases} \int_0^\tau \phi(s) e^{-rs} ds = \int_0^\tau e^{-rs} ds = \frac{1-e^{-r\tau}}{r}, & \tau < T, \\ \int_0^\tau \phi(s) e^{-rs} ds = \int_0^T e^{-rs} ds = \frac{1-e^{-rT}}{r}, & \tau \geq T, \end{cases}$$

we have

$$\mathbb{E}_0^\tau \left[\int_0^\tau e^{-rs} ds\right] = \frac{1}{r} - \frac{1}{r}\mathbb{E}_0^\tau \left[e^{-r\tau}\right],$$
$$\begin{aligned} \mathbb{E}_0^\tau \left[\int_0^\tau \phi(s) e^{-rs} ds\right] &= \int_0^T \frac{1 - e^{-r\tau}}{r} f(\tau)\, d\tau + \int_T^\infty \frac{1 - e^{-rT}}{r} f(\tau)\, d\tau \\ &= \frac{1}{r}\mathbb{P}(\tau < T) - \frac{1}{r}\int_0^T e^{-r\tau} f(\tau)\, d\tau + \frac{1 - e^{-rT}}{r}\mathbb{P}(\tau \geq T) \\ &= \frac{1}{r} - \frac{1}{r}\int_0^T e^{-r\tau} f(\tau)\, d\tau - \frac{1}{r} e^{-rT}\mathbb{P}(\tau \geq T), \end{aligned}$$

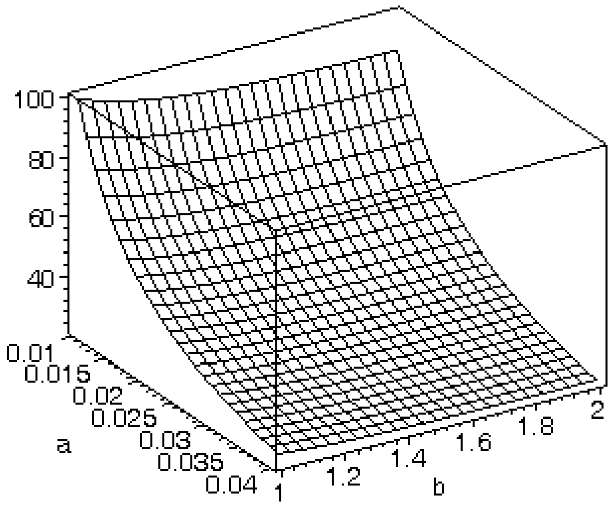
and a feasible ratio u/v can finally be written under the following form where $\mathbb{I}_A$ is the indicator function for the event A.

Definition 1 *A pair of contribution and pension rates* (u, v) *is said to be feasible if*

$$\frac{u}{v} = \frac{1 - \mathbb{E}_0^\tau \left[e^{-r\tau}\right]}{1 - \mathbb{E}_0^\tau \left[e^{-r\tau}\mathbb{I}_{\tau<T}\right] - e^{-rT}\mathbb{P}(\tau \geq T)} - 1, \qquad u, v > 0. \tag{3}$$

Let us remark that the event "death", happening in τ, can sometimes be affected by a series of explanatory variables. In particular, we are referring to the so-called "proportional hazard rate model" used in statistical analysis of transition data. Fortunately the form of the feasible ratio u/v remains unchanged, and we only need to compute the probability and expected values conditionally to the realization of the explanatory variables in (3) to accommodate this situation.

Figure 1: Expected time of death for the Weibull distribution



2.1.1 The Weibull distribution

Here, we explicitly compute the feasibility condition (3) by supposing that the death time τ follows the Weibull distribution, whose probability density function is given by

$$f(\tau) = \alpha\beta (\alpha\tau)^{\beta-1} e^{-(\alpha\tau)^{\beta}},$$

where $\alpha > 0$, $\beta > 0$. The case of the exponential distribution turns out to be a particular case of the Weibull distribution when $\beta = 1$. The Weibull distribution represents one of the most widely used model in survival analysis. The expected time of death has the following form:[2]

$$\int_0^{\infty} \tau f(\tau)\, d\tau = \frac{1}{\alpha}\Gamma\left(\frac{1+\beta}{\beta}\right),$$

whose behaviour is shown in Fig. 1. We see that if parameter α belongs to $[0.01, 0.04]$ then the expected death time goes from a value close to 20 to a value close to 100 years. For the numerical simulations that follow we will always consider values of β belonging to $[1, 2]$.

[2] We indicate with $\Gamma(t)$ the Gamma function having the following form:

$$\Gamma(t) = \int_0^{\infty} e^{-x} x^{t-1} dx.$$

The probability that τ is greater than T is easy to compute:

$$\mathbb{P}(\tau \geq T) = \int_T^{\infty} \alpha\beta(\alpha\tau)^{\beta-1} e^{-(\alpha\tau)^\beta} d\tau = e^{-(\alpha T)^\beta},$$

while the expected value in the numerator of (3) is as follows:

$$\mathbb{E}_0^\tau\left[e^{-r\tau}\right] = \int_0^{\infty} e^{-r\tau}\alpha\beta(\alpha\tau)^{\beta-1} e^{-(\alpha\tau)^\beta} d\tau = \int_0^1 e^{-\frac{r}{\alpha}(-\ln y)^{\frac{1}{\beta}}} dy,$$

where we have used the change of variable $y = e^{-(\alpha\tau)^\beta}$. Since this integral does not admit an algebraic solution, we may propose an approximation. Indeed we know the exact solution for the exponential case, i.e. when $\beta = 1$, and we may then think of approximating the integral via a Taylor expansion around $\beta = 1$:

$$\mathbb{E}_0^\tau\left[e^{-r\tau}\right] \cong \beta\frac{\alpha}{r+\alpha} + (\beta-1)\frac{\alpha}{(r+\alpha)^2}\left(r\ln\frac{\alpha}{r+\alpha} - r\gamma - \alpha\right),$$

where γ is the Euler constant.[3] The first term of the above expression obviously coincides with the explicit solution given by the exponential case. Slightly more tedious computation are required for $\mathbb{E}_0^\tau\left[e^{-r\tau}\mathbb{I}_{\tau<T}\right]$:

$$\begin{aligned}\int_0^T e^{-r\tau} f(\tau)\, d\tau \cong\; & \beta\alpha\frac{1-e^{-T(r+\alpha)}}{r+\alpha} + (\beta-1)\frac{\alpha}{(r+\alpha)^2}\left(r\ln\frac{\alpha}{r+\alpha} - r\gamma - \alpha\right) \\ & + (\beta-1)\frac{\alpha}{(r+\alpha)^2} e^{-(r+\alpha)T}\left(\ln(\alpha T)\left(-r + r\alpha T + \alpha^2 T\right) + \alpha\right) \\ & + (\beta-1)\frac{\alpha}{(r+\alpha)^2} r \int_{-\infty}^{-T(r+\alpha)} \frac{1}{x} e^x dx.\end{aligned}$$

Note that the integral of the last term can be neglected for sufficiently high values of T. Hence, after plugging these closed-form approximations into the feasible condition (3), we obtain the results presented in Table 1 for several values of α, β, T, and r.

Even if the approximation has been computed for β tending to 1, from Table 1 we can see that it remains good while β is far from 1. Furthermore, the approximated values seem to behave quite well even for closer time horizons ($T = 20$). Accordingly, we can easily show how the ratio u/v behaves with respect to the actuarial parameters α and β just by plotting the graphs of the approximated ratio. These graphs are shown in Fig. 2, where three different values of T and r are chosen. The first column of Fig. 2 shows the behaviour of u/v for $T \in \{20, 30, 50\}$, while the second column analyses how u/v changes for $r \in \{0.01, 0.03, 0.05\}$. The values of α and β belong to $[0.012, 0.016]$ and $[1, 2]$, respectively.

In particular, we highlight the following results:

[3] We recall

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \ln n\right).$$

Figure 2: Feasible ratio $u/v|_{\beta\to 1}$

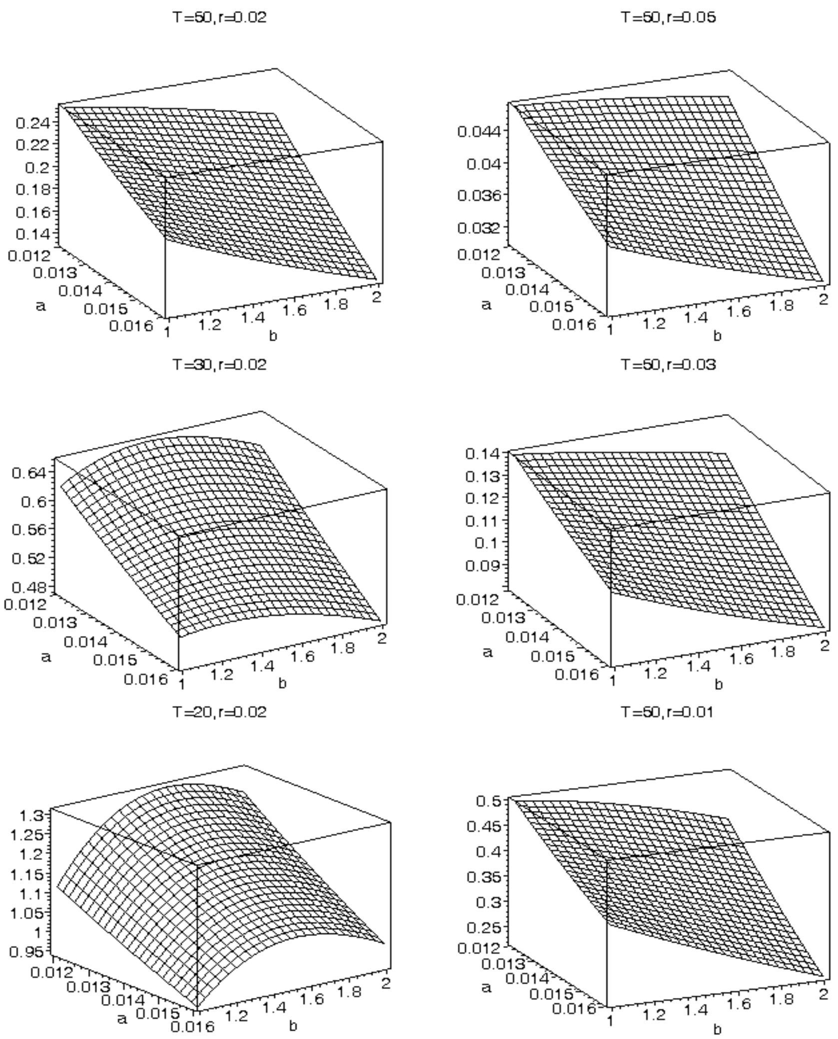


Table 1: Approximation of the feasible ratio

r	α	β	T	u/v	$u/v\|_{\beta\to1}$
0.02	0.01	1.5	50	0.2747	0.2826
0.02	0.01	1.5	30	0.6971	0.7376
0.02	0.01	1.5	20	1.2677	1.4399
0.02	0.01	1.3	50	0.2775	0.2863
0.02	0.01	1.7	50	0.2738	0.2769
0.02	0.01	1.9	50	0.2741	0.2693
0.02	0.005	1.5	50	0.4289	0.4236
0.02	0.008	1.5	50	0.3302	0.3343
0.02	0.02	1.5	50	0.1039	0.1051
0.01	0.01	1.5	50	0.5125	0.5801
0.03	0.01	1.5	50	0.1559	0.1555
0.04	0.01	1.5	50	0.0913	0.0898

1. when the time horizon T is sufficiently far away, the ratio u/v is decreasing with respect to both α and β. What changes is just the level of u/v which inversely depends on the interest rate r. In fact, when the riskless interest rate increases (decreases) it is easier (more difficult) to meet future payments and the pension fund can ask for a lower (higher) contribution rate;

2. when the pension horizon T is small, the ratio u/v is still decreasing with respect to α and presents a maximum for a given value of β. For better understanding this result, we recall that the hazard function for a Weibull distribution is given by $\alpha\beta\left(\alpha\tau\right)^{\beta-1}$. So, when the hazard rate increases (i.e. a near death is more likely) the contribution rate can decrease and vice-versa. Furthermore, while the hazard function is always increasing in α,[4] it is increasing in β for $\beta < -\left(\ln\left(\alpha\tau\right)\right)^{-1}$;[5]

3. the longer the pension horizon T the lower the ratio u/v. In fact, the pension fund can ask for lower (higher) contribution rates when these contributions are paid for a long (short) period of time;

4. the shape of u/v is not affected by the changes in r. The interest rate only affects the level of u/v without altering its behaviour with respect to the other parameters.

[4] In fact, we have

$$\frac{\partial}{\partial\alpha}\left(\alpha\beta\left(\alpha\tau\right)^{\beta-1}\right) = \beta^2\left(\alpha\tau\right)^{\beta-1}.$$

[5] In fact, the derivative

$$\frac{\partial}{\partial\beta}\left(\alpha\beta\left(\alpha\tau\right)^{\beta-1}\right) = \alpha\left(\alpha\tau\right)^{\beta-1}\left(1+\beta\ln\left(\alpha\tau\right)\right)$$

is positive when $\beta\ln\left(\alpha\tau\right) > -1$. Now, since $\alpha\tau$ is generally lower then 1, the inequality becomes $\beta < -\left(\ln\left(\alpha\tau\right)\right)^{-1}$.

2.2 The objective function

Since a pension fund does not consider any consumption problem, then it is just supposed to maximize the expected utility of its final wealth. Thus, the optimization problem can be written as

$$\max_{w} \mathbb{E}_0^{\tau} \left[U \left(R \left(\tau \right), \tau \right) \right],$$

subject to the dynamic constraint (1) and where $U(\bullet)$ is an increasing and concave function. Since the mortality risk is assumed to be independent of the financial risk, we can write the maximization problem as follows:

$$\max_{w} \mathbb{E}_0 \left[\mathbb{E}_0^{\tau} \left[U \left(R \left(\tau \right), \tau \right) \right] \right] = \max_{w} \mathbb{E}_0 \left[\int_0^{\infty} f(t) U \left(R(t), t \right) dt \right], \tag{4}$$

under the same dynamic constraint (1).

Now, we need to define the utility function $U(\bullet)$. The most widely used utility function in the literature is the CRRA function of the form $U(R) = \frac{1}{\delta} R^{\delta}$. Here, we use such a function with a little modification due to the specific nature of the pension fund problem. When the pension fund receives the contributions, it cannot obtain any utility from them since it will have to pay them back as pensions. Thus, the argument of the utility function we consider here is the wealth R diminished by the received contributions (during the accumulation phase) and augmented by the paid pensions (during the decumulation phase). In fact, when the pensions are paid, the corresponding amounts of money are freed and the pension can obtain some utility from them.

Accordingly, we define the utility function as follows:

$$U(R,t) = \frac{1}{\delta} \left(R(t) - \int_0^t k(s) e^{-r(s-t)} ds \right)^{\delta},$$

where the function $k(s)$ is as in (2). This approach is widely used in the literature (see Merton, 1990, Section 6.4) and the utility function we have supposed is known as "state-dependent" utility. In order to have an increasing and concave utility function the parameter δ must be less than one.

3 The optimal portfolio

After what we have presented in the previous section, the asset allocation problem for a pension fund can be written as

$$\begin{cases} \max_w \mathbb{E}_0 \left[\int_0^{\infty} f(t) \frac{1}{\delta} \left(R(t) - \int_0^t k(s) e^{-r(s-t)} ds \right)^{\delta} dt \right], \\ \text{with} \quad dR = (Rr + w'M + k) dt + w'\Sigma' dW, \\ \text{and} \quad R(0) = R_0. \end{cases} \tag{5}$$

The Hamiltonian for this problem is

$$\mathcal{H} = f(t)\frac{1}{\delta}\left(R(t) - \int_0^t k(s)\, e^{-r(s-t)} ds\right)^{\delta} + J_R\left(Rr + w'M + k\right) + \frac{1}{2}J_{RR}w'\Sigma'\Sigma w,$$

from which we have the set of first order conditions[6]

$$\frac{\partial\mathcal{H}}{\partial w} = J_R M + J_{RR}\Sigma'\Sigma w = 0 \Rightarrow w^* = -\frac{J_R}{J_{RR}}\left(\Sigma'\Sigma\right)^{-1} M,$$

where $J(R,t)$ is the value function solving the maximization problem and the subscripts indicate the partial derivatives of J. The HJB equation is

$$0 = J_t + f(t)\frac{1}{\delta}\left(R(t) - \int_0^t k(s)\, e^{-r(s-t)} ds\right)^{\delta} + J_R\left(Rr + k\right) - \frac{1}{2}\frac{J_R^2}{J_{RR}}\xi'\xi,$$

where $\xi \equiv \Sigma\left(\Sigma'\Sigma\right)^{-1} M$. For the value function, we try the form $J(R,t) = g(t)\, f(t)\, U(R,t)$ where $g(t)$ must be determined. So, after substituting this form into the HJB equation and carrying out some simplifications, we obtain that $g(t)$ must satisfy

$$\frac{\partial g}{\partial t} + \frac{\partial f(t)}{\partial t}\frac{1}{f(t)}g(t) + 1 + r\delta g(t) + \frac{1}{2}g(t)\frac{\delta}{1-\delta}\xi'\xi = 0,$$

whose boundary condition must guarantee the convergence of $J(R,t)$ when t tends to infinity. The precise form of function $g(t)$ is not important for computing the optimal portfolio composition. The inverse of the Arrow-Pratt risk aversion index computed on $J(R,t)$, in fact, does not depend on $g(t)$. So, we can finally write what follows.

Proposition 2 *The optimal portfolio composition solving Problem (5) is given by*

$$w^* = w_R^* + w_u^* + w_v^*, \tag{6}$$

where

$$\begin{aligned}
w_R^* &\equiv \frac{1}{1-\delta}R\left(\Sigma'\Sigma\right)^{-1} M,\\
w_u^* &\equiv -\frac{1}{1-\delta}u\left(\int_0^t \phi(s)\, e^{-r(s-t)} ds\right)\left(\Sigma'\Sigma\right)^{-1} M,\\
w_v^* &\equiv \frac{1}{1-\delta}v\left(\int_0^t \left(1-\phi(s)\right) e^{-r(s-t)} ds\right)\left(\Sigma'\Sigma\right)^{-1} M,
\end{aligned}$$

and u and v must verify (3).

[6] The first order conditions are necessary and sufficient because the objective function is strictly concave in R.

The first component w_R^* depends on the wealth level but not (explicitly) on time, w_u^* depends on the contribution rate and w_v^* depends on the pension rate. We underline that the component we have called w_R^* coincides with Merton's portfolio.

It is interesting to stress that the actuarial risk enters the optimal portfolio via the link that exists between u and v in the feasible condition (3). When this link is not considered, as in Charupat and Milevsky (2002), the portfolio composition is independent of the mortality risk.

Furthermore, it is important to stress that the optimal portfolio allocation in (6) does depend on the wealth level $R(t)$. Thus, it is not optimal to manage the accumulation and the decumulation phases separately and our model suggests to commit the management of the whole investment period to the same institutional investor.

The function $\phi(t)$ can be eliminated from (6) by considering separately the two following cases (in both cases w_R^* is the same):

1. $t \leq T$, we are in the APh and the components of the optimal portfolio are

$$w_u^* \equiv -\frac{1}{1-\delta}\frac{u}{r}\left(e^{rt}-1\right)\left(\Sigma'\Sigma\right)^{-1}M, \tag{7}$$

$$w_v^* \equiv 0, \tag{8}$$

2. $t > T$, we are in the DPh and we have

$$w_u^* \equiv -\frac{1}{1-\delta}\frac{u}{r}e^{rt}\left(1-e^{-rT}\right)\left(\Sigma'\Sigma\right)^{-1}M, \tag{9}$$

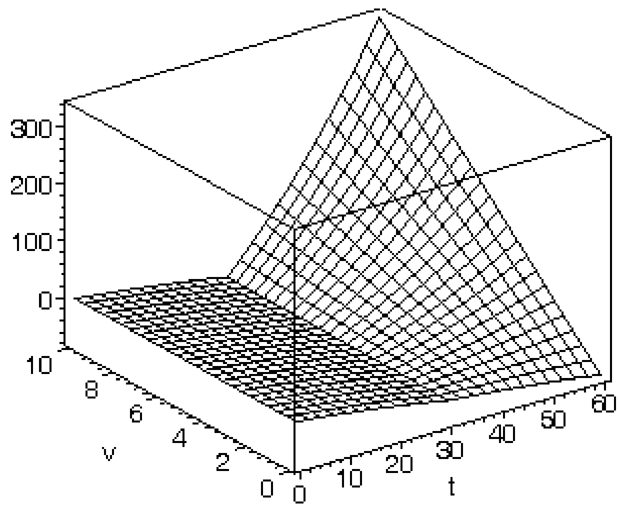
$$w_v^* \equiv \frac{1}{1-\delta}\frac{v}{r}\left(e^{r(t-T)}-1\right)\left(\Sigma'\Sigma\right)^{-1}M. \tag{10}$$

All stated results can be easily traced back to Merton's model by putting $u = v = 0$. In this case $w_u^* = w_v^* = 0$. During the accumulation phase ($t \leq T$), it is easy to check that w_u^* in (7) contains only negative numbers. Indeed $\delta < 1$, $\Sigma'\Sigma > 0$ by construction, $M > 0$ to preclude arbitrage,[7] and $e^{rt} > 1$. Thus, the optimal portfolio during the accumulation phase contains less risky assets than the optimal portfolio in the Merton's case. Furthermore, we can see that the vector w_v^* contains only positive elements (we recall that during the decumulation phase $t > T$). So, the behaviour of the optimal portfolio can be summarized as in the following corollary.

Corollary 3 *During the accumulation phase ($t < T$) the amount of wealth invested in the risky assets decreases through time, while during the decumulation phase ($t > T$) it increases.*

[7] The returns of the risky assets must be greater than the riskless rate. If this was not true, all investors would buy the riskless asset.

Figure 3: Behaviour of the function $\chi(t)$



The behaviour described in this corollary can be seen in Fig. 3 where we have plotted the following function which appears in (7)-(8) and (9)-(10):

$$\chi(t) = \begin{cases} -\frac{u}{r}\left(e^{rt}-1\right), & \text{if} \quad t \leq T \\ -\frac{u}{r}e^{rt}\left(1-e^{-rT}\right)+\frac{v}{r}\left(e^{r(t-T)}-1\right), & \text{if} \quad t > T \end{cases}$$

and where we have put $T = 30$, $r = 0.02$, and $u = 1$. While t is lower than the pension time T, the amount of money invested in the risky assets decreases. It begins increasing when t becomes higher than T. Furthermore, the higher the pension rate v, the sharper the increase in the risky profile of the optimal portfolio. The behaviour during the accumulation phase confirms the results after Boulier, Huang, and Taillard (2001) and Battocchio and Menoncin (2002).

In the deterministic case, that is to say when the subscriber of the fund never dies (i.e. $\alpha \to 0$), then the feasibility condition becomes $v = u\left(e^{rT}-1\right)$ and the optimal portfolio can be written as follows:

$$w^* = \frac{1}{1-\delta}\left(R - \frac{u}{r}\left(e^{r\min(t,T)}-1\right)\right)\left(\Sigma'\Sigma\right)^{-1}M.$$

In this case, during the accumulation phase ($t < T$) the optimal portfolio has the same behaviour as in the case of a "mortal" subscriber. Instead, during the decumulation phase ($t > T$) the component $w_u^* + w_v^*$ of the optimal portfolio becomes constant through time and remains negative. This leads to a different behaviour than the one plotted in Fig. 3. In particular, since the subscriber never dies, we cannot increase the riskiness of the optimal portfolio after the date T.

4 Conclusion

In this paper we have solved the asset allocation problem for a pension fund. The structure of the financial market is as follows: (i) there are n risky assets, following geometric Brownian motions, (ii) there exists a riskless asset paying a constant interest rate, and (iii) the market is not necessarily complete. Furthermore, the fund is supposed to have a state-dependent CRRA utility function.

We analyse the portfolio problem during both the accumulation and the decumulation phases when the death time of the subscriber is a stochastic variable (following a Weibull distribution). The contribution and the pension rates are supposed to be constant.

We show that the optimal asset allocation during the accumulation phase (APh) is different from the one during the decumulation phase (DPh). In particular, during the APh the investment in the risky assets should decrease through time for allowing the fund to guarantee the payment of the (constant) pension rate during the DPh. Instead, during the second phase when the pension is paid, the risky investment should increase through time. In fact, since the death of the subscriber becomes more and more likely, the remaining wealth can be invested in riskier and riskier portfolio allocation.

Finally, since the optimal asset allocation depends on the level of fund wealth, our model suggests that it is not optimal to manage the APh and the DPh separately. This is in agreement with conventional industry practice.

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