International Trade, Hedging and the Demand for Forward Contracts*

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Abstract

There is a huge literature on the effects of uncertainty on trade levels. One very strong result of that literature is that uncertainty should not matter, as long as well developed forward markets exist. The empirical implications of this result, however, are hard to find in the data. We model terms of trade uncertainty in a small open economy with uncertainty stemming from abroad and derive the equilibrium demand for forward contracts. It turns out that risk averse agents will not buy forwards at an actuarially fair price, thus rendering both the full-hedge theorem and the separation theorem of the aforementioned literature obsolete. Using real world data for Germany we calibrate our model. We find that in equilibrium risk averse agents will buy forward cover only for investment reasons. The amount of forwards purchased is around 20% of equilibrium imports. This is broadly in accordance with empirical observed ratios.

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1 Introduction

International trade in goods is characterized by uncertainty. Common sense and economic theory suggest that exporters, importers and households should try to hedge against this uncertainty. Natural candidates for hedging instruments are future and forward contracts. In fact, Ethier (1973) introduced the separation theorem and the full hedge theorem under exchange rate uncertainty, showing that demand for forward contracts perfectly compensates uncertainty. Benninga, Eldor and Zilcha (1985) and Kawai and Zilcha (1986) additionally discussed price level uncertainty, obtaining the same results. Recently this strong result has been subject to some qualifications. Viaene and Zilcha (1998) for example, consider additionally output and cost uncertainty and find that under this setup full-double hedge and separation fail to hold. Adam-Müller (2000) introduces inflation risk which cannot be hedged away and finds that full-hedge and separation break down if the two sources of risk in the model are not statistically independent. Market structure issues have been addressed as well, examples are Eldor and Zilcha (1987) and Broll and Zilcha (1992).

The empirical literature, though spares, does not support the strong theoretical predictions of the early literature. As Carse, Williamson and Wood (1980) and others have shown, only roughly one-third of the value of international trade is covered by forward contracts. Even equity flows are only poorly hedged. According to Hau and Rey (2003) only 8% of US equity holdings abroad are hedged against exchange rate risks. Furthermore, there exists a lively debate in the empirical literature as to whether exchange rate volatility depresses trade levels or not. This debate is related to the issue of demand for forwards in that often the argument is made that as long as agents have access to well developed forward markets, the uncertainty should not matter. Strikingly, the evidence is rather mixed and seems to be independent of the existence of well developed forward markets (see Coté (1994) for a survey on the empirical evidence and Wei (1998) for a discussion of the underlying causes).

This paper reconciles empirical findings with theoretical considerations. We build an infinite horizon small open economy model where one good is domestically produced with capital and labour, another good is imported. Both goods are consumed. Capital is accumulated and risk averse households hedge optimally against terms of trade uncertainty.[1] One forward contract

[1] In contrast to the majority of the literature on that topic households demand forwards, not firms. This, however, simply follows from the general equlibrium setup we use. Firms are owned by the households, who look "through" them. A similar argument is made in Bacchetta and Wincoop (1998, pp. 18).

allows (and obliges) them to buy one import good in the next period at a fixed price $\bar{p}_Y$.

We first study the determinants of demand for forwards. We show that the exogenous internationally given forward price $\bar{p}_Y$ is the crucial determinant of demand for forwards. If this price equals the expected price of the import good, households do not want to buy any forwards. (They would actually want to sell forwards.) If this price equals the price at which risk neutral households would be indifferent, risk averse households demand a positive amount of forward contract.

The reason for the fact that risk averse households want to sell forwards at actuarially fair prices lies in the concavity of their utility function in consumption levels. With consumption levels optimally chosen ex-post, indirect utility functions of individuals exhibit convexity in prices, though still concavity in expenditure. As expenditure is a function of prices as well, overall the indirect utility function exhibits convexity in prices and households are actually (price-) risk lovers. Positive demand therefore requires a price that is sufficiently low, e.g. the price offered by risk neutral households. Intuitively we could think of the risk averse households as not willing to commit themselves to a consumption decision, when faced with price uncertainty. They do not want to give away the option to adjust their consumption bundles.

We then calibrate the model by using realistic and reasonable parameter values. We find that between 10 and 20% of international trade is covered by forward contracts. The low ratios cited in the empirical literature are therefore not surprising and may reflect the curvature of utility functions of utility maximizing households. Partial equilibrium setups or setups focusing on risk neutral firms should therefore be extended to take this aspect into consideration.

We are not the first that find that full-hedge theorem and separation theorem does not hold. As argued above there is a substantial literature that finds that these two theorems will not hold as soon as certain conditions, i.e. independence of the underlying sources of uncertainty, are violated. Our result, however, is derived in a completely different manner. The crucial point is the decision structure of our agents. The standard approach assumes that *all* decisions are made *before* the resolution of uncertainty. In contrast we employ an alternative decision rule. In the first period, still before resolution of uncertainty, the agents decide upon their level of hedging and in the second, *after* the uncertainty is resolved, the agents actually make their consumption decision. Following this approach, agents will never be able to eliminate uncertainty from their budgets and hence are faced with a trade-off. Using this setup and considering normal conditions, i.e. actuarially fair insurance, risk averse agents will never buy forward cover.

The paper is structured as follows: Section 2 introduces the model, section 3 presents the solutions of the model and the following section discusses the properties of the equilibrium and makes some qualitative statements of the comparative static behavior of the system using a numerical calibration of the model. A brief discourse to options as a mean of comparison ends the theoretical discussion of the model. Section 5 concludes the paper.

2 The model

2.1 Technologies

We study a small open economy that produces one good X that is internationally traded. It imports a foreign consumption good Y which is not domestically produced. Domestic production requires capital K and labour L, which are non-tradable,

$$X_t = X(K_t, L_t). \tag{1}$$

Time is discrete and variables are indexed by t. The production function $X(.)$ has the standard neoclassical properties. Firms produce under perfect competition and factor rewards w_t^L and w_t^K for labour and capital are given by their value marginal productivities,

$$w_t^L = p_t^X \partial X_t / \partial L_t, \quad w_t^K = p_t^X \partial X_t / \partial K_t. \tag{2}$$

The number of units of the import good to be exchanged for one unit of the export good, i.e. international terms of trade p_t^X / p_t^Y at a point in time t are exogenously given to the economy and unknown in $t-1$. The probability distribution $f\left(p_t^X / p_t^Y\right)$ of p_t^X / p_t^Y is common knowledge. In what follows, we choose X as numeraire and denote its price by p^X,

$$p_{t+1}^X = p_t^X \equiv p^X.$$

One can therefore think of the price of the domestic good as a deterministic price and of the price of the foreign good as stochastic.

Domestic output X from the production process (1) is used for domestic consumption C_t^X, exports X_t^E and gross investment I_t,

$$X_t = C_t^X + X_t^E + I_t. \tag{3}$$

Capital grows according to

$$K_{t+1} = (1-\delta) K_t + I_t \tag{4}$$

where δ captures depreciation.

In addition to producing the good Y, foreign agents offer forward contracts.[2] At a cost of χ per unit to be paid in t, foreign agents agree in t to sell in $t+1$ one unit of the foreign good at the price $\overline{p}^Y$. This is equivalent to fixing in t next periods terms of trade at $p^X/\overline{p}^Y$. When forward contracts of total volume D_t are signed, foreign agents agree to sell D_t units of good Y at $\overline{p}^Y$ in $t+1$. Domestic buyers commit to buy in $t+1$ at this price, irrespective of the realization of p_{t+1}^Y.[3]

2.2 Households

The horizon of the economy is infinite. Agents in this economy live for two periods. They work in the first period and consume in the second period of their life. Consumption in the second period comprises both the domestically produced good and the foreign good.

2.2.1 Preferences and budget constraints

Let their utility function be given by

$$v = v\left(u\left(C_X, C_Y\right)\right),$$

where $u\left(C_X, C_Y\right)$ is some homothetic utility function and $v\left(.\right)$ determines the degree of risk aversion. For illustrating purposes, we will later use

$$u\left(C_X, C_Y\right) = C_X^{\alpha} C_Y^{1-\alpha}, \quad 0 < \alpha < 1 \tag{5}$$

$$v\left(x\right) = \frac{x^{\sigma}}{\sigma}, \quad \sigma > 0. \tag{6}$$

Note that the utility function (5) displays risk aversion towards the consumption *levels.* Risk aversion in total consumption *expenditure* is given for $0 < \sigma < 1$, risk neutrality in consumption expenditure would be represented by $\sigma = 1$.

A household's first period budget constraint equates labor income with savings and expenditure for financial contracts D_t,

$$w_t = s_t + \chi D_t. \tag{7}$$

[2] Some parts of the literature use the terminology forwards if the contract relates to foreign exchange and futures if the contract relates to a commodity (see Kawai and Zilcha (1986) for example). Since in our model there is no trade with this asset between purchase and maturity, we call them forwards even though it relates to a commodity.

[3] If, in contrast, D_t represented options, domestic agents would not be obliged to buy and thus only draw on the contract in favourable situations.

Savings are used to buy capital goods s_t/p^X. There is the implicit assumption of a market in which today's old, being the owners of the capital stock sell it to today's young in exchange for consumption good X, which in turn constitutes the wage of today's young. The sum over all individual savings equal the current capital stock (i.e. after depreciation) plus additional aggregate investment (which might be negative)

$$K_{t+1} = (1-\delta) K_t + I_t = \frac{s_t}{p^X} L. \tag{8}$$

In the second period, households use all of their wealth and other income for financing consumption expenditure e_{t+1}. End of second period wealth amounts to $p^X (1-\delta) \frac{s_t}{p^X} = (1-\delta) K_{t+1}$. Factor rewards for wealth amount to $p^X \frac{\partial X_{t+1}}{\partial K_{t+1}} \frac{s_t}{p^X}$. Income from forward contracts is $\left(p^Y_{t+1} - \bar{p}^Y\right) D_t$, which might be negative. Hence

$$e_{t+1} \equiv p^X C^X + p^Y_{t+1} C_Y = (1 + r_{t+1}) p^X \frac{w_t - \chi D_t}{p^X} + \left(p^Y_{t+1} - \bar{p}^Y\right) D_t, \tag{9}$$

where we defined

$$1 + r_{t+1} \equiv 1 + \frac{\partial X_{t+1}}{\partial K_{t+1}} - \delta \tag{10}$$

and savings s_t were replaced by using the first period budget constraint (7).

The second period budget constraint (9) nicely shows that payoffs $\left(p^Y_{t+1} - \bar{p}^Y\right) D_t$ from forward contracts are positive and therefore a second period source of income when the price p^Y_{t+1} of good Y is sufficiently high relative to its price $\bar{p}^Y$ specified one period before. Forward contracts imply a loss in the case of low price of good Y. Of course, bad terms of trade shocks leading to income and good terms of trade shocks leading to losses from forward contracts are the reason why forwards exist: they insure against terms of trade shocks.

This budget constraint also shows that households *cannot* insure fully against terms of trade risk. Forward contracts refer to a certain amount of goods that can be purchased at this fixed price $\bar{p}^Y$. As the actual amount of goods consumed depends on the realization p^Y_{t+1} of the price, some uncertainty always remains. This is the crucial departure of our model from the classic setups in the hedging literature Ethier (1973, pp. 496) and Benninga et al. (1985, pp. 540). There, firms decide today in t how much they will produce tomorrow in $t+1$. This allows them to fully insure against uncertainty in the price of their output good. The well-known separation theorem of no uncertainty after hedging results. If our agents knew how much they will consume tomorrow, full hedging would be possible as well. They will never know, however, as price uncertainty has an income effect as well.

2.2.2 A no-bankruptcy constraint

In order to avoid insolvency on parts of the agents in our model, we have to introduce a no-bankruptcy constraint. Point of departure is the expenditure equation (9). It goes without saying that a negative expenditure is not possible, hence we argue that the worst that can happen to the budget of our agents is:

$$e_t = (1 + r_{t+1})\, w_t + \left(p^Y_{t+1} - (1 + r_{t+1})\, \chi - \bar{p}^Y\right) D_t = 0$$

Solving for D_t yields

$$D_t = \frac{(1 + r_{t+1})\, w_t}{(1 + r_{t+1})\, \chi + \bar{p}^Y - p^Y_{t+1}}.$$

Regarding our forward, the worst that can happen is $p^Y_{t+1} = 0$. Prudence thus demands that the amount of D_t an agent is allowed to purchase shall never be any greater than:

$$D_t \leq \frac{(1 + r_{t+1})\, w_t}{(1 + r_{t+1})\, \chi + \bar{p}^Y} \tag{11}$$

This condition makes intuitively sense: the greater the contracted $\overline{p}^Y$, the smaller the amount of forwards the agents are allowed to buy. Similar lines of reasoning hold for the other variables.

3 Solving the model

3.1 The maximization problem of households

The maximization problem of households consists in choosing the amount D_t of forward contracts and optimal consumption levels C_X and C_Y such that expected utility $Ev\,(u\,(C_X, C_Y))$ is maximized, given the budget constraint (9).

Conceptually, maximization can be subdivided into two steps. The second step consists in allocating consumption expenditure to goods X and Y, taking consumption expenditure as given. This second sub-problem is solved after realization of terms of trade. It is therefore a choice under certainty. The Cobb-Douglas specification (5) would imply

$$C^X_{t+1} = \frac{\alpha e_{t+1}}{p^X}, \tag{12}$$

$$C_{t+1}^{Y} = \frac{(1-\alpha)\, e_{t+1}}{p_{t+1}^{Y}}. \tag{13}$$

These equations hold at each point in time and determine consumption levels after uncertainty has been resolved.

The first step consists in choosing the optimal amount D_t of forward contracts by solving

$$Ev\left(\frac{e_{t+1}}{P\left(p^{X}, p_{t+1}^{Y}\right)}\right) \rightarrow \max_{D_t} \tag{14}$$

where $v\left(e_{t+1}/P\left(p^{X}, p_{t+1}^{Y}\right)\right)$ is utility where consumption levels in the homothetic utility function $u\left(C_X, C_Y\right)$ have been replaced by optimal consumption levels. Utility $u\left(C_X, C_Y\right)$ can then be written as expenditure divided by the price index. In the Cobb-Douglas case, the price index reads

$$P\left(p^{X}, p_{t+1}^{Y}\right) = \Phi p_X^{\alpha} p_Y^{1-\alpha},$$

where Φ is a constant. Expenditure is given by (9).

This two-step solution to our maximization problem is made possible by assuming that consumption takes place only when agents are old. If consumption were to take place in both periods, the consumption choice in the first period would be linked to the saving decision. The system that would have to be analyzed would be more complicated (as an intertemporal consumption rule would have to be added).

The solution to this problem is then given by

$$E\left(v'\left(\frac{e_{t+1}}{P\left(p^{X}, p_{t+1}^{Y}\right)}\right)\frac{p_{t+1}^{Y} - (1+r_{t+1})\,\chi - \bar{p}^{Y}}{P\left(p^{X}, p_{t+1}^{Y}\right)}\right) \stackrel{!}{=} 0 \tag{15}$$

This first order condition consists, logically, of two parts. The first is marginal utility, here expressed in the form of the indirect utility function. Marginal utility is positive but decreasing in consumption levels, or as stated here, increasing in expenditure and decreasing in prices. The second term in the bracket represents the return from the forwards, which will always be negative under actuarially fair forwards, i.e. $E\left(p_{t+1}^{Y}\right) = \bar{p}^{Y}$, since the term $(1+r_{t+1})\,\chi$ representing the opportunity costs of entering the forward market enters negatively. If forwards could be obtained without costs, clearly these opportunity costs would vanish and actuarially fair forwards would have an expected return of zero. Once the meaning of the two components of (15) is clear, the intuition of this first order condition is more easy to see. The sum that constitutes the expected value has a negative and a positive component. Marginal utility is, as long as $p_{t+1}^Y < (1 + r_{t+1})\,\chi + \bar{p}^Y$ - the "loss" region - multiplied by a negative number and hence in this interval contributes negatively. Concavity of the utility function with respect to the amount of forwards implies that as long as (15) is negative, the agents have too many forwards and hence should decrease holdings. On the other hand, as soon as $p_{t+1}^Y > (1 + r_{t+1})\,\chi - \bar{p}^Y$ - the "win region" - marginal utility contributes positively. Again, concavity tells us, as long as (15) is positive agents should increase holdings of D_t. However, given positive costs to obtain forward cover, i.e. $\chi > 0$, increasing D_t will increase r_t, hence opportunity costs will rise as well, up to a point where marginal utility will fall in D_t.Hence the optimal amount of D_t is such that the positive and the negative components of the sum simply cancel out.

3.2 Reduced form

The reduced form of the model consists of two equations. The capital stock in the next period is given by savings today times the number L of individuals and divided by the price of one unit of capital and is given by (8). With the first-period budget constraint (7) giving individual savings, we obtain

$$K_{t+1} = \frac{p_t^X \partial X_t / \partial L - \chi D_t}{p^X} L, \tag{16}$$

where the wage rate was replaced by its value marginal product (2).

The amount of forward contracts is determined by the first order condition (15). When consuming, the old consume the current capital stock, interest payments on the current capital stock plus income (or losses) from forward contracts. Expenditure in (15) therefore equals

$$e_{t+1} = (1 + r_{t+1})\, p^X \partial X\,(K_t, L)\,/\partial L + \left(p_{t+1}^Y - (1 + r_{t+1})\,\chi - \bar{p}^Y\right) D_t \tag{17}$$

which formally follows from the budget constraint (9) where nominal wages w_t were replaced according to (2).

Equations (15) and (16), given (17), determine the two variables K_t and D_t, given an initial capital stock K_0.

Equation (16) determining the evolution of capital shows that next periods capital is known in t. By contrast, expenditure (17) is uncertain when some forward contracts are signed. This makes consumption levels of both goods and exports and imports uncertain. If no forward contracts are signed ($D = 0$), expenditure is deterministic, consumption of good X would be deterministic but consumption of good Y would be stochastic.

3.3 Steady-state

In the steady state, the capital stock is the same in each period. Variables that are constant are printed without a time subscript. All stochastic variables are denoted by a tilde ($\tilde{}$).The capital stock is then determined by

$$K = \frac{p^X \partial X/\partial L - \chi D}{p^X} L \tag{18}$$

and is therefore a deterministic variable. Domestic production (1) is then deterministic as well, $X = F(K, L)$. Expenditure (17)

$$\tilde{e} = (1+r)\, p^X \partial X/\partial L + \left(\tilde{p}^Y - (1+r)\,\chi - \bar{p}^Y\right) D \tag{19}$$

remains stochastic and D follows implicitely from (15)

$$E\left(v'\left(\frac{\tilde{e}}{P\left(p^X, \tilde{p}^Y\right)}\right)\frac{\tilde{p}^Y - (1+r)\,\chi - \bar{p}^Y}{P\left(p^X, \tilde{p}^Y\right)}\right) \overset{!}{=} 0 \tag{20}$$

with (19).

4 Equilibrium properties

The two reduced form equations yield an unique equilibrium, if they cross once in R_{++}. Equation (16) describing the evolvement of the capital stock is a non-linear first order difference equation. Even though it is not possible to derive an analytical solution, the properties of this type of schedule are well understood[4]. The optimal amount of D to be purchased is given by (15). In principle this equation can be understood as an integral. Nevertheless it is not possible to analytically derive the shape of this schedule, for the sign of the derivative $\frac{dEv}{dK}$ remains ambiguous. It is, however, possible to analytically determine whether or not the agents are willing to hold forwards and hence we turn to this issue first. To determine the equilibrium points we have to resort to numerical methods. This will constitute the second part of our equilibrium discussion. In the last section we introduce and discuss an option contract. By deriving several equilibrium properties of this type of contract the properties of the forwards become more clear also.

[4] See any basic mathematics for economists textbook on that issue, a good example being Chiang (1984).

4.1 The equilibrium demand for forwards

We now present three important results with respect to the existence of interior solutions, i.e. a positive demand for $D_{t.}$

Theorem 1 *Risk averse agents will not buy forward cover at fair prices, i.e.* $E(p_Y) = \overline{p}^Y$.[5]

This result is illustrated in the following figure:

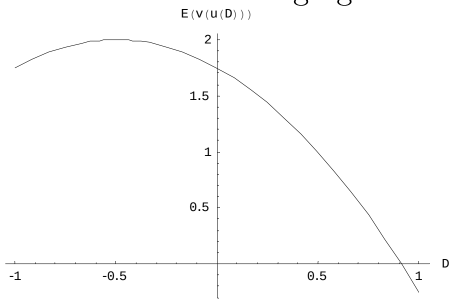

Figure 1: An example for a global concave function in D

We see here an example for a global concave function in a variable D. Even though the picture above does not represent our utility function, expected utility is also a global concave function in D[6] and hence we can use this property here. Since our function is globally concave in D the sign of the first derivative of this function with respect to D at the point $D = 0$ determines whether or not there is an interior solution.

In the light of the existing literature on the topic this result is rather surprising. The standard result is[7], that if an unbiased forward market exists, the agents use this market to avoid all uncertainty, i.e. obtain full cover of their position. The crucial difference of our model to the literature lies in the timing structure. The main body[8] of the literature assumes that all decisions are made *before* uncertainty is resolved. In contrast, we assume that although the agents decide on the optimal amount of forward cover before uncertainty is resolved, their consumption decision is made after the resolution of the uncertainty. Under this setup buying forward contracts amounts to no less than restricting ones possibilities to adjust to price realizations. Risk averse

[5] See appendix 6.1 for the proof.

[6] The formal argument is presented in the Appendix 6.2.

[7] See, inter alia, Ethier (1973) Benninga et al. (1985) Kawai and Zilcha (1986) Eldor and Zilcha (1987) Viaene and de Vries (1992) Zilcha and Broll (1992) Viaene and Zilcha (1998) and Adam-Müller (2000).

[8] There are a few papers that discuss the theoretical possibility of a different timing structure, an example being Perée and Steinherr (1989). We are, however, not aware of any work that explicitely models this.

agents will not give away this opportunity. It is clear, that there are some decisions that will be made in advance and for this part the analysis of the existing literature would be appropriate. We believe, however, that most of consumption decisions are made when actual consumption takes place and prices are clear.

Theorem 2 *Risk averse agents will only buy forward cover for sufficiently low $\overline{p}_Y$, i.e. $E(p_Y) > \overline{p}_Y$.*[9]

Note that this result follows from the first theorem, in which we relied on the negativity of the covariance term. Further this condition is implied by utility maximization of the agents. One possible interpretation would be that if $\overline{p}_Y$ is lower than the expected value of the price uncertainty in period two, the average return of a forward position is positive. Thus the agent will be compensated for giving up their possibility to adjust their consumption bundle according to the price realizations in the next period. Hence the agents are willing to hold a forward position. .

Theorem 3 *If $\overline{p}^Y = \frac{E(p_Y^{\alpha})}{E(p_Y^{\alpha-1})}$, i.e. the price risk neutral households would offer, risk averse agents will buy forward contracts.*[10]

To illustrate the third result we resort again to the figure above. Clearly, since the exponent $c = \sigma(1-\alpha)$ is smaller for risk averse agents than for risk neutral ones and the derivative $\frac{d\xi}{dc}$ is negative, a decrease in c - thus moving from risk neutrality to risk aversion - increases the slope of the function at the intersection with the vertical axis. As we are moving from a point where this slope is zero, we in effect move the whole function to the right.

Note that these results may be somewhat surprising, given the "full-hedge theorem" we normally encounter in the literature (see Ethier (1973) and Kawai and Zilcha (1986) for example). The reason for this is that our model differs from the usual models in the way that agents always will have uncertainty through the price-index channel, whereas in the former models there is the possibility to avoid all uncertainty, for agents completely decide upon their plans in period one.[11] Risk averse agents do not want to lose the ability to adjust to price shocks in the next period, whereas risk neutral

[9] See appendix 6.1 for the proof.

[10] See appendix 6.1 for the proof.

[11] There is one notable exeption. Clark (1973, Section II, pp.308) deals with the case where exporters cannot fully hedge away the exchange rate risk, even though there are perfect forward markets. His reason, however, is a different one, for he considers the effects of limited maturities in that markets.

agents are indifferent towards this opportunity. This is the reason why risk neutral households would be willing to offer forward contracts.

The convexity of the indirect utility function with respect to the prices is illustrated in the figure below.

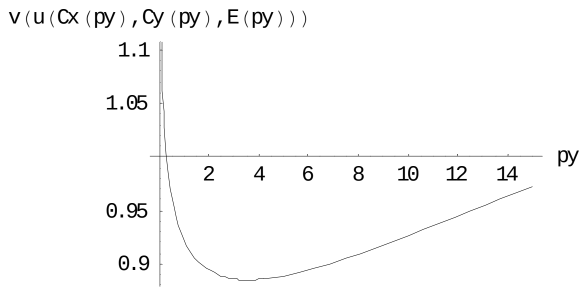

Figure 2: An example for the convexity of the indirect utility function using log-normal distributed price uncertainty

Secondly, we have another factor at work here. By buying forward contracts the agents trade one risk against the other. Holding a forward position means that risk now enters directly nominal income. This can be easily seen from (9). Risk aversion regarding nominal income and the uncertainty through the price-index channel are the reasons for the agents asking for more than actuarially fair forwards.

Another interesting point here is the behavior of the risk neutral agents. By offering forwards at a more than actuarially fair rate, they, on average, incur losses with this asset. Their compensation, on the other hand, is the augmented capital stock and its returns.

4.2 Calibrating the model

In this section we will briefly present some numerical solutions of the model. Further this section gives some insights into the comparative static behavior of our model. We begin with discussing the chosen values. Solving the model numerically involves computing values of both D and K which satisfy (16) and simultaneously (15). To get numerical results we need to specify a couple of parameters and the underlying distribution. As far as possible this has been achieved by drawing on real world data.

Parameter	L	α	β	δ	χ	σ	Φ	S
Value	100	$\frac{4}{5}$	$\frac{3}{10}$	0	$\frac{1}{100}$	$\frac{1}{2}$	$\frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$	1
Parameter	p_X	$\overline{p}_Y$						
Value	1	$\frac{E\left(p_Y^{\alpha}\right)-(1+r)\chi E\left(p_Y^{\alpha-1}\right)}{E\left(p_Y^{\alpha-1}\right)}$						

Table 1: Parameter values used for calibrating the model

As a first step we have to specify the production technology. We will use a Cobb-Douglas form:

$$X(K,L) = SK^{\beta}L^{1-\beta}.$$

Equation (16) relies on three exogenously given parameters. These are L, χ and β. P_X is the numeraire and thus can be set one. Depreciation is assumed to be zero, hence we have $\delta = 0$ and the scale parameter for the technology, S, is set to one. Now χ represents in some way the costs of the forward cover, even though, strictly speaking, χ is the market price of the forward contract. This two concepts are in fact quite different. In reality the market price of the forward cover is quite small, whereas the real costs of obtaining forward cover may very well be substantial.[12] This leaves some room for determining the value of χ and thus we will set this value arbitrarily, but close to zero. In our calibration we used $\frac{1}{100}$. The size of the population is just a scale parameter and therefore no further elaboration is necessary. We set $L = 100$. The beta parameter of our production function reflects relative shares of capital (and, by our specification also labour) and is commonly found e.g.Maddison (1987, p. 658) and thus in this context assumed to be around 0.3.

The other reduced form equation comes with three parameters to specify as well. These are $\overline{p_Y}$, α and the underlying distribution. The $\overline{p}_Y$ is determined by the price at which risk neutral individuals would offer the forwards, i.e.

$$\overline{p}^Y = \frac{E\left(p_Y^{\alpha}\right) - (1+r)\,\chi E\left(p_Y^{\alpha-1}\right)}{E\left(p_Y^{\alpha-1}\right)}. \tag{21}$$

Equation (21) is determined by using the first order condition (20), setting $\sigma = 1$ and solving for $\overline{p_Y}$. The parameter of the utility function, α, determines in what ratio domestic and foreign products are consumed. Using data from

[12] Think of a firm which has to hire expertise to contract such cover and thus may have substantial costs. In terms of transfers, like the χDs are, think of margin requirements.

Statisches Bundesamt, we obtained the empirically observed share foreign products had in aggregate German consumption. This led us come to an estimate for α of approximately 0.80. To determine the most appropriate distribution, we obtained monthly price index data for both import prices and export prices over the period January 1962 until January 2002, leaving us with 482 observations. Calculating $\frac{\text{index import}}{\text{index export}}$ amounts in terms of our model to get the price series p_t^Y. The shape of the histogram suggested choosing a lognormal distribution, which is an assumption commonly made, for example in the finance literature.[13] The parameters of the distribution were obtained by maximum likelihood estimation.[14] The estimates were

Distribution	$E(p_Y)$	$E(p_Y)^2$
lognormal	0.1149	0.0071
underlying normal	1.12611	0.0103

Table 2: The parameters of the lognormal distribution and the related normal.

This completes the discussion of the parameters. For computational purposes we made use of Mathematica 4.1.[15]

4.3 A numerical solution

We now present a simulation result for a small country. Under lognormal distributed price uncertainty, using the parameter specification we presented above, we found that the economy will buy a total amount of 2.03 units of forward contracts, given the price risk neutral agents would offer. The capital stock and thus GDP of the economy can be calculated and using the mean on the distribution as the realization of the price in period two, the economy will import 10.2 units of good Y. This means, that the forward cover to import ratio is in this case approximately 20%. This is in accordance to surveys on the topic. For example Carse et al. (1980) found that firms that import or export and thus face terms of trade risk, only cover between 15-30% of their open positions.

Some caveats are in order here. First, the actual terms of trade variance may well be underestimated with our proxy used. If this is true, the calculated amount of forwards is too high as well. Second, the costs of forwards

[13] The Black-Scholes formula relies on lognormality of prices. Even in international macro this assumption is often used, see for example Obstfeld and Rogoff (1998).

[14] For this purpose we made use of R and the function fitdistr which is icluded in the MASS package.

[15] The programme code is available from the authors upon request.

we used are to some degree arbitrary. They are, however, close to the actual transaction fees charged by banks but would not incorporate such items as information costs and fixed costs for setting up the appropriate institutions, letting alone deliberation costs. To the extent to which the actual costs are higher, our result overestimates the amount of forwards purchased. Lastly there is the issue of the degree if risk aversion with respect to wealth. In the literature there is no consensus on that parameter. We choose to set this parameter, $1 - \sigma$ in our model, to $\frac{1}{2}$, which is a conservative choice in the sense that a broad range of publications support this choice. It also turns out that this particular parameter is the least influential in altering our results. The aformentioned qualifictations notwithstanding, this numerical exercise recaps our analytical results and shows that the model is able to fit the actual data for reasonable parameter values.

4.4 Comparative statics

There are a couple of interesting questions arising when considering changing the parameters. We begin with the terms of trade variance. If there is an exogenously induced increase in the variance of the foreign price we observe a fall in the demand for forwards. At our calculated equilibrium point we observe a decrease of 4.7% in demand for forwards if we increase the variance by 1%. This is accordance with the intuition for our results. Risk averse agents are not willing to give up the possibility to adjust themselves to a terms of trade shock. The greater the likelihood of a terms of trade shock, the more they have to be compensated for holding forward contracts.

Next consider the costs of the forwards. If costs decrease, demand will increase. At the point of our interior solution a 1% decrease in the costs would induce a 16% rise in the demand for forward contracts.

Lastly we look at the degree of risk aversion. A society which is more risk averse than another will demand less forward cover than the less risk averse society. A 1% increase of the degree of risk aversion, i.e. a 1% fall in σ, reduces demand for forwards by 0.4%. The comparative static result for an increase in the variance is covered in the picture below:

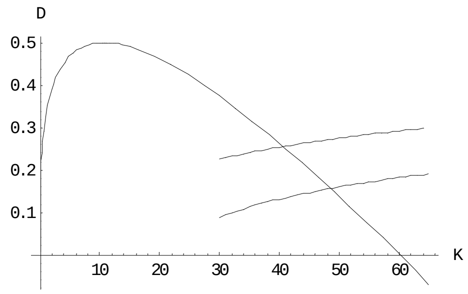

Figure 3: An increase in the terms of trade variance decreases demand for forwards

An increase in the variance of p_Y, an increase in the costs χ and an increase in the degree of risk aversion will ceteris paribus decrease the demand for forward cover by shifting the schedule implied by (15) downwards. Note that in the case of changing costs, the capital schedule will also shift.

4.5 Options

In order to give additional insights into the workings of our model, we will in this section examine what the optimal hedging behavior would be if the agents could buy options instead of forward contracts to insure against the uncertainty regarding the price of the foreign good. An (call) option, as opposed to a forward contract, does not oblige to buy the underlying asset (or commodity), instead the buyer can choose whether or not he will exercise his option. If we are to keep our notation we can extend our model very easily to model an option instead of a forward contract, by observing that in the event

$$p_t^Y \leq \overline{p}^Y$$

the buyer of that option would simply not exercise it. To model options we only have to change the expenditure equation:

$$\begin{aligned} e_{t+1} &= (1 + r_{t+1})(w_t - \chi D_t) \;\; \forall \; p_{t+1}^Y \leq \overline{p}^Y \\ e_{t+1} &= (1 + r_{t+1})(w_t - \chi D_t) + \left(p_{t+1}^Y - \overline{p}^Y\right) D_{t+1} \;\forall\; p_{t+1}^Y > \overline{p}^Y. \end{aligned}$$

Where D_t now denotes now the amount of *options* instead of forward contracts, the strike price being $\overline{p}^Y$. Hence by buying one option for the price χ an agent is entitled to buy one unit of good Y in the next period for the price $\overline{p}^Y$. It follows that the first order condition (15) is changed as well:

$$\int_0^{\overline{p}^Y} v'\left(\frac{(1+r_{t+1})(w_t-\chi D_t)}{P\left(p^X,p^Y_{t+1}\right)}\right)\frac{-\chi(1+r_{t+1})}{P\left(p^X,p^Y_{t+1}\right)}dP_Y + \tag{22}$$
$$\int_{\overline{p}^Y}^{\infty} v'\left(\frac{e_{t+1}}{P\left(p^X,p^Y_{t+1}\right)}\right)\frac{p^Y_t-\overline{p}^Y-\chi(1+r_{t+1})}{P\left(p^X,p^Y_{t+1}\right)}dP_Y \overset{!}{=} 0.$$

Three results emerge[16]:

Theorem 4 *If options are costless, i.e. $\chi = 0$, the optimal amount of $D_t = \infty$.*

This is probably the most straightforward result. Of course rational agents, being offered a free lunch, will happily accept this. Here the free lunch comes as a free lottery ticket, without any risk of loosing. We present this otherwise not very surprising result to make the structure of the decision problem more clear.

Theorem 5 *If agents can choose between options and forwards at the same costs they will always choose options.*

To facilitate the comparison between forwards and options we present the second result. It constitutes, again, a standard property of the utility function of the agents. Forwards will always be dominated by options, as long as the price is the same for both.

Corollary 6 *For options and forward contracts to exist jointly forwards either have to cost less or be more than actuarially fair (or both).*

To have in our world what we observe in reality, the joint existence of options together with forwards, necessitates the latter being cheaper than the former, a result that directly follows from above two theorems.

Theorem 7 *If options are actuarially fair, i.e. $E\left(p^Y_t\right) = \overline{p}^Y$, agents will demand a positive amount of options, for a given positive cost of doing so χ.*

Our last results highlights again the difference between forwards and options. In contrast to forward contracts there exist a positive demand, depending on the price χ, of "actuarially fair options", that is options that have a strike price that equals the expected value of the price in the next period.

[16]The proofs can be found in the appendix 4.1.

5 Conclusion

One largely debated issue in international economics is the question whether or not volatility in exchange rates and terms of trade depresses trade levels. There is an extensive literature on that question, both theoretical and empirical. The main body of the theoretical literature claims that terms of trade and/or exchange rate uncertainty does not matter as long as well developed forward/futures markets exist. This literature further predicts that agents fully hedge the existing risks. The empirical work done in this field fails to unambiguously support these findings.

We model a small open economy that is subject to terms of trade risk completely stemming from abroad. We show that under this setup there is no demand for terms of trade insurance, a direct effect of the convexity of the indirect utility index with respect to prices. Risk aversion with respect to consumption levels and expenditure levels is not sufficient a motive to buy forwards. Further we derive the condition under which, on part of the risk averters, a positive demand for forwards will exist. In any world where different degrees of risk aversion up to risk neutrality jointly exists, there will be a positive demand for the kind of forwards which we modeled, for the agents with differing attitude towards risk would offer a more than actuarially fair insurance so that most risk averse agents would be willing to enter this contracts. The motive for the demand, however, will not be hedging but pure investment. We calibrate our model with data for Germany to obtain numerical solutions. The equilibrium amount of forwards contracted in relation to the equilibrium amount of imports closely resembles the empirical observed values, thus providing a rationale for the apparent underhedging of domestic agents against price level and/or exchange rate uncertainty. We then showed that options, as opposed to forwards, will be demanded as means of insurance. If prices are equal, options strictly dominate forward contracts. This straightforward result may help explain why the market for options has grown exponentially over the last decade or so.

The main contribution of our analysis, however, is that the "price-convexity" effect should be incorporated in the existing models, which could be achieved by giving up the assumption that all plans are irrevocably made in the period which precedes the resolution of the uncertainty. This should alter dramatically the strong theoretical predictions of this literature with respect to forward markets and should thus provide a better understanding of the effects at work here. Since forwards are unattractive and options perhaps too expensive, our analysis may also provide an additional argument *in favour* of international capital flows, and hence capital account liberalization, as a means of insuring the economy.

Our work can be extended in some promising ways. First, to understand the implications of covariance effects so often at work in the hedging process money and thus a nominal exchange rate should be brought into the model. This would also allow a comparison between our modeling approach and the existing literature that has proceeded with considering multiple sources of risk. Another interesting extension would be the to allow for heterogenous agents explicitly. In doing that we could render endogenous the offered forward price $\overline{p}^Y$.

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6 Appendix

6.1 Derivation of the Theorems

This appendix provides the proofs of all results we presented in section 4.1.

Theorem 1: Risk averse agents will not buy forward cover at fair prices, i.e. $E(p_Y) = \overline{p}^Y$.

Proof of Theorem 1. Consider the first-condition[17] (15) and assume away any costs of forward cover, i.e. $\chi = 0$. Further we are looking at the

[17]For this proof we supressed time indices to ease notational burden. Since we are looking at steady-state values, time indices contain no additional information, in the case of the price uncertainty we always take expectations in t of next periods p_Y.

point $D = 0$. Equation (15) is thus negative, iff

$$E\left(\left(\frac{(1+r)\,w}{p_Y^{1-\alpha}}\right)^{\sigma-1}\frac{p_Y-\overline{p}^Y}{p_Y^{1-\alpha}}\right)<0 \Leftrightarrow \tag{23}$$

$$E\left(((1+r)\,w)^{\sigma-1}\left(p_Y^{-\sigma(1-\alpha)}(p_Y-\overline{p}^Y)\right)\right)<0.$$

Since $-\sigma(1-\alpha)<0$ we can define $E\left(p_Y^{-\sigma(1-\alpha)}\right)\equiv E\left(p_Y^{-c}\right)$, $c>0$ and can once more rewrite (23) :

$$((1+r)\,w)^{\sigma-1}\left[E\left(p_Y^{-c}\right)E\left(p_Y\right)+Cov\left(p_Y^{-c},p_Y\right)-\overline{p}^Y E\left(p_Y^{-c}\right)\right]<0.$$

Two results emerge. First, since $Cov[p_Y^{-c},p_Y]$ is negative[18], we have for $E\left(p_Y\right)=\overline{p}^Y$:

$$((1+r)\,w)^{\sigma-1}*Cov\left(p_Y^{-c},p_Y\right)<0$$

Together with the result[19]

$$\frac{d^2EU}{dD^2}<0$$

we know that there cannot be an interior solution with $D>0$. ■

Theorem 2: Risk averse agents will only buy forward cover for sufficiently low $\overline{p}_Y$, i.e. $E\left(p_Y\right)>\overline{p}_Y$.

Proof of Theorem 2. For any interior solution we need the first order condition to be fulfilled. For that, at the point $D=0$ we need

$$E\left(p_Y^{-c}\right)E\left(p_Y\right)+Cov\left(p_Y^{-c},p_Y\right)-\overline{p}^Y E\left(p_Y^{-c}\right)=0$$

to hold. This implies $E\left(p_Y\right)+\frac{Cov\left(p_Y^{-c}p_Y\right)}{E\left(p_Y^{-c}\right)}=\overline{p}^Y<E\left(p_Y\right)$ which is our second result. ■

Theorem 3: If $\overline{p}^Y=\frac{E\left(p_Y^{\alpha}\right)}{E\left(p_Y^{1-\alpha}\right)}$, i.e. the price risk neutral households would offer, risk averse agents will buy forward contracts.

Proof of Theorem 3. The final result may be approached in a slightly different manner. Define a function $\xi(c)$ which gives the sign of (23) at point $D=0$. This function is simply given by

$$\xi(c)=E\left(p_Y^{1-c}\right)-\overline{p}^Y E\left(p_Y^{-c}\right).$$

[18]This follows from the fact that in our case we have $f'(p_Y)*g'(p_Y)\leq 0\ \forall p$ where $f(p_Y)=p_Y$ and $g(p_Y)=p_Y^{-c}$.An application of Chebychevs second inequality brings the result that $Cov\left(p_Y,p_Y^{-c}\right)\leq 0$. See also Hardy, Littlewood and Polya (1952, pp. 43 and p.168) for reference.

[19]See Appendix (6.2)

Surely, a price $\overline{p}^Y$ for which there is an interior solution is then given by

$$\overline{p}^Y = \frac{E\left(p_Y^{1-c}\right)}{E\left(p_Y^{-c}\right)}.$$

Consider now risk neutral households. Their optimization problem will, in principle, be the same as treated above. In particular, since $\sigma = 1$ for risk neutrality, we have

$$\overline{p}^Y = \frac{E\left(p_Y^{\alpha}\right)}{E\left(p_Y^{\alpha-1}\right)}{}^{20} \tag{24}$$

as the price for which there is an interior solution.

Now differentiate $\xi(c)$ with respect to c:

$$\begin{aligned}
\frac{d\xi}{dc} &= -E\left(p_Y^{1-c}\ln p_Y\right) + \overline{p}^Y E\left(p_Y^{-c}\ln_Y\right) \\
&= -E\left(p_Y^{1-c}\right)E\left(\ln p_Y\right) - Cov(p_Y^{1-c}, \ln p_Y) \\
&\quad + \frac{E\left(p^{1-c}\right)}{E\left(p^{-c}\right)} E\left(p_Y^{-c}\right)E\left(\ln p_Y\right) + Cov\left(p_Y^{-c}, \ln p_Y\right) \\
&= -Cov(p_Y^{1-c}, \ln p_Y) + Cov\left(p_Y^{-c}, \ln p_Y\right)
\end{aligned}$$

For values of $0 < c < 1$ clearly the derivative is negative, for both terms are negative then. This, however, implies that at $\overline{p}^Y = \frac{E\left(p_Y^{\alpha}\right)}{E\left(p_Y^{1-\alpha}\right)}$ risk averse agents will buy forward contracts, since, by definition of risk aversion and risk neutrality we have $c_A < c_N = 1 - \alpha$. ■

Theorem 4: If options are costless, i.e. $\chi = 0$, the optimal amount of $D_t = \infty$.

Proof of Theorem 4. If we have $\chi = 0$ we will always have

$$E\left(v'\left(e, P\left(p^X, p_{t+1}^Y\right)\right)\right) > 0,$$

regardless of the choice of D. Since utility is increasing in consumption and consumption is increasing in D_t it is optimal to demand an infinite amount. ■

Theorem 5: If agents can choose between options and forwards at the same cost they will always choose options.

Proof of Theorem 5. We prove this by contradiction. First note that for an interior solution to the optimal choice of D_t we need to have the first order conditions fulfilled. If we substract (15) from (22), we arrive at the following expression

$$
\begin{aligned}
&\int_{o}^{\overline{p}^Y} v'\left(\frac{(1+r_{t+1})(w_t-\chi D_t)}{P\left(p^X,p_{t+1}^Y\right)}\right)\frac{-\chi(1+r_{t+1})}{P\left(p^X,p_{t+1}^Y\right)}dP_Y \\
=\; &\int_{o}^{\overline{p}^Y} v'\left(\frac{e_{t+1}}{P\left(p^X,p_{t+1}^Y\right)}\right)\frac{\left(p_{t+1}^Y-\overline{p}^Y\right)-\chi(1+r_{t+1})}{P\left(p^X,p_{t+1}^Y\right)}dP_Y
\end{aligned}
$$

which cannot be true for the same set of parameters. This establishes that the two first order conditions cannot hold simultaneously. Moreover the above makes clear that

$$
\begin{aligned}
&\int_{o}^{\overline{p}^Y} v'\left(\frac{(1+r_{t+1})(w_t-\chi D_t)}{P\left(p^X,p_{t+1}^Y\right)}\right)\frac{-\chi(1+r_{t+1})}{P\left(p^X,p_{t+1}^Y\right)}dP_Y \\
>\; &\int_{o}^{\overline{p}^Y} v'\left(\frac{(1+r_{t+1})(w_t-\chi D_t)+\left(p_{t+1}^Y-\overline{p}^Y\right)D_t}{P\left(p^X,p_{t+1}^Y\right)}\right)\frac{\left(p_{t+1}^Y-\overline{p}^Y\right)-\chi(1+r_{t+1})}{P\left(p^X,p_{t+1}^Y\right)}dP_Y
\end{aligned}
$$

for the same set of parameters. It follows that

$$
\int_{0}^{\overline{p}^Y} v'\left(\frac{(1+r_{t+1})(w_t-\chi D_t)}{P\left(p^X,p_{t+1}^Y\right)}\right)\frac{-\chi(1+r_{t+1})}{P\left(p^X,p_{t+1}^Y\right)}dP_Y + \tag{25}
$$

$$
\int_{\overline{p}^Y}^{\infty} v'\left(\frac{e_{t+1}}{P\left(p^X,p_{t+1}^Y\right)}\right)\frac{\left(p_{t+1}^Y-\overline{p}^Y\right)-\chi(1+r_{t+1})}{P\left(p^X,p_{t+1}^Y\right)}dP_Y = 0
$$

$$
\Longrightarrow\quad E\left(v'\left(\frac{e_{t+1}}{P\left(p^X,p_{t+1}^Y\right)}\right)\frac{p_{t+1}^Y-(1+r_{t+1})\chi-\overline{p}^Y}{P\left(p^X,p_{t+1}^Y\right)}\right)<0 \tag{26}
$$

This together with concavity of utility in D_t this is enough to establish the result. ■

Theorem 7: If options are actuarially fair, i.e. $E\left(p_t^Y\right)=\overline{p}^Y$, agents will demand a positive amount of options, for a given positive cost of doing so χ.

Proof of Theorem 7. The proof follows directly from (22). The first integral enters negatively, the second positively. In general, there is a χ small enough to render the overall sum zero. ■

6.2 Concavity of expected utility with respect to D

In this section we give a short proof of the concavity of the indirect expected utility function with respect to the forwards.

Our first order condition, i.e. the first derivative of indirect expected utility with respect to D is given by:

$$G\left(K_t, D_t, \psi\right) =$$
$$E\left(v'\left(\frac{e_{t+1}}{P\left(p_x, p_Y\right)}\right)\frac{p^Y_{t+1} - \overline{p}^Y - \left(1 + r_{t+1}\right)\chi}{P\left(p_x, p_Y\right)}\right) = 0.$$

The second derivative is then simply:

$$\frac{\partial G}{\partial D} = E\left(v''\left(\frac{e_{t+1}}{P\left(p_x, p_Y\right)}\right)\frac{\left(p^Y_{t+1} - \overline{p}^Y - \left(1 + r_{t+1}\right)\chi\right)^2}{P\left(p_x, p_Y\right)}\right) < 0$$

since we have, by definition:

$$v''\left(\frac{e_{t+1}}{P\left(p_x, p_Y\right)}\right) < 0.$$

and thus an (possibly) infinite sum over negative values, which cannot be other than negative as well.

6.3 Balance of payment

This appendix checks consistency of the model by validating that the saldo of the balance of payments equals zero. Formally

$$EX_t - IM_t + CF_t = 0$$

must hold. Then

$$\begin{aligned}
&p^X X_t - \alpha e_t L - p^X I_t - (1-\alpha)\, e_t L + \left(p^Y_t - \overline{p}^Y\right) D_{t-1} L - \chi D_t L = 0 \Leftrightarrow \\
&p^X \left[X_t + K_t - I_t\right] - e_t L - \chi D_t L = -\left(p^Y_t - \overline{p}^Y\right) D_{t-1} L
\end{aligned}$$

Using (9) for expenditure brings about:

$$\begin{aligned}
&p^X \left[X_t - I_t\right] - \left(1 + r_t\right)\left(w_{t-1} - \chi D_{t-1}\right) L - \left(p^Y_t - \overline{p}^Y\right) D_{t-1} L - \chi D_t L \\
&= -\left(p^Y_t - \overline{p}^Y\right) D_{t-1} L \Leftrightarrow \\
&p^X \left[X_t - I_t\right] - \left(1 + r_t\right)\left(w_{t-1} - \chi D_{t-1}\right) L - \chi D_t L = 0
\end{aligned}$$

We complete our proof by replacing X_t and I_t.Nominal investment in our model is by (7) and (8) simply first period income reduced by first period spending,

$$p^X I_t = \left(w_t - \chi D_t\right) L - (1-\delta)\, p^X K_t \tag{27}$$

The capital stock in the period after saving is given by (4), specifically

$$p^X K_{t+1} = (1-\delta)\, p^X K_t + p^X I_t$$

Noting further, by the assumption of constant returns to scale, our output in period t can be written as the sum of the factor payments, i.e.

$$p^X X_t = w_t^L L + w_t^K K_t \tag{28}$$

we have everything we need to proceed:

$$\begin{aligned}
& w_t^L L + w_t^K K_t - (w_t - \chi D_t)\, L + (1-\delta)\, p^X K_t \\
& - (1 + r_t)\, (w_{t-1} - \chi D_{t-1})\, L - \chi D_t L = 0 \Leftrightarrow \\
& w_t^K K_t + (1-\delta)\, p^X K_t \\
& - \left(1 + \frac{\partial X_t}{\partial K_t} - \delta\right) (w_{t-1} - \chi D_{t-1})\, L = 0 \Leftrightarrow \\
& p^X \frac{\partial X_t}{\partial K_t} K_t - \left(\frac{\partial X_t}{\partial K_t} - \delta\right) p^X K_t - \delta p^X K_t = 0 \Leftrightarrow 0 = 0
\end{aligned}$$

where we made use of (2), (16) and (10).