Investment Strategies in Incomplete Markets: Sufficient Conditions for a Closed Form Solution*

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Abstract

This paper analyses the portfolio problem of an investor who wants to maximize the expected power utility of his terminal wealth both in a complete and an incomplete financial market. We derive sufficient conditions for having a closed form solution. These conditions must hold on a suitable combination of the drift and diffusion coefficients of the stochastic processes describing the state variables and the asset prices. In particular, we show that our framework leads to two cases: (i) the case solvable thorough a log-linear value function, and (ii) the case solvable thorough a log-quadratic value function.

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1 Introduction

In this paper we contribute to the analysis of the optimal portfolio rules for an investor wanting to maximize the expected power utility function (CRRA) of his terminal wealth. Some closed form solutions for this problem have been found in the literature. In particular, we refer to the works of Kim and Omberg (1996), Wachter (1998), Chacko and Viceira (1999), Deelstra et al. (2000), and Boulier et al. (2001). In all these works the market structure is as follows: (i) there exists only one state variable (the riskless interest rate or the risk premium) following the Vasiček (1977) model or the Cox et al. (1985) model, (ii) there exists only one risky asset, (iii) a bond may exist. Some works consider a complete financial market (Wachter, 1998; Deelstra et al., 2000; and Boulier et al., 2001) while others deal with an incomplete market (Kim and Omberg, 1996; and Chacko and Viceira, 1999).

In spite of these efforts for finding a closed form solution for the optimal asset allocation in more and more complicated framework, we stress that (at least at our knowledge) no author has tried both: (i) to establish some general properties that can be checked on the market structure for verifying if a closed form solution exists, and (ii) to present the algebraic form of this exact solution. Generally, the most technical approaches try to establish the existence (and uniqueness) of a viscosity solution to the Hamilton-Jacobi-Bellman equation deriving from the stochastic optimal control problem (see for instance Crandall et al., 1992; and Buckdahn and Ma, 2001a, 2001b).

Here, we want to analyse this particular issue disregarded by the literature. In particular, we find sufficient conditions for easily checking if there exists a closed form solution to an optimal portfolio problem where there exists both a set of (stochastic) state variables and a set of risky assets. Furthermore, we provide the algebraic form of this exact solution.

These sufficient conditions must hold on some combinations of drift and diffusion coefficients of the state variables and the risky assets. More precisely, we are able to distinguish two different cases in which a closed form solution can be found: (i) the kind of problem which can be solved by a log-linear value function and (ii) the kind of problem which can be solved by a log-quadratic value function. The second set of solutions, which will be defined "quadratic case", is suitable only for a market structure "similar" to that presented in Kim and Omberg (1996), while all the other above-mentioned exact solutions can be lead back to the "linear case".

Our analysis does not need the hypothesis of complete financial market and so, the derived sufficient conditions are useful even for the much more general case of an incomplete market. Under the hypothesis of completeness, in Menoncin (2002) a closed form solution for the asset allocation problem is found when: (i) there exist both a stochastic background risk and a stochastic inflation risk following two general Itô processes, (ii) there exists a set of risky assets whose prices are affected by a set of state variables, both following generic Itô processes, and (iii) the investor maximizes the expected value of his terminal exponential utility function (CARA). This approach is very general indeed but, unfortunately, it is not able to mimic the closed form solutions found in the above-mentioned literature since there does not exist a suitable change in the utility function parameters for transforming a CARA utility function into a CRRA utility function.

After the general presentation, we compute the explicit solution for both the "linear" and the "quadratic" case when the coefficients of the stochastic processes driving the state variables and the asset prices do not depend on time. In fact, this is the case analysed in the above-mentioned literature. We derive the properties of the "linear" solution. In particular, we show that the absolute values of portfolio composition are monotonic functions of time. The direction of this monotonicity can be easily checked looking at the sign of a parameter. Since only the model after Kim and Omberg (1996) lies in the "quadratic" case, then all the properties they expose for their solution are valid in our case. Thus, the reader is referred to their work for a more complete analysis of these properties.

In this paper, we follow the traditional stochastic dynamic programming technique (Merton, 1969, 1971) leading to the Hamilton-Jacobi-Bellman (HJB) equation (Øksendal, 2000; and Björk, 1998 offer a complete derivation of the HJB equation). As regard the "martingale approach" the reader is referred to Cox and Huang (1989, 1991), and Lioui and Poncet (2001).

Through this work we consider agents trading continuously in a frictionless, arbitrage-free market until time H, which is the horizon of the economy. Furthermore, we analyse both a complete and an incomplete financial market.

The paper is structured as follows. Section 2 details the general economic framework and exposes the stochastic differential equations describing the behaviour of asset prices and state variables. In Section 3, both the implicit form of the optimal portfolio and the HJB equation are computed. Section 4 presents our main result, that is to say the sufficient conditions that must hold so that the optimal portfolio composition has a closed form solution. In Section 5, the optimal asset allocation is computed in some particular cases which can be lead back to the closed form solutions already found in the literature. Section 6 concludes. A presentation of the market structures analysed in the literature and the passages for computing the above-mentioned closed form solutions are left to the appendices.

2 The market structure

The financial market is supposed to have the following structure:

$$\begin{cases} \underset{s\times 1}{dX} = \underset{s\times 1}{f}(t,X)dt + \underset{s\times k}{g}(t,X)'\underset{k\times 1}{dW}, \qquad X(t_0) = X_0, \\ \underset{n\times 1}{dS} = \underset{n\times n}{I_S}\left[\underset{n\times 1}{\mu}(t,X,S)dt + \underset{n\times k}{\Sigma}(t,X,S)'\underset{k\times 1}{dW}\right], \qquad S(t_0) = S_0, \\ dG = Gr(t,X)\,dt, \qquad G(t_0) = G_0, \end{cases} \tag{1}$$

where X is a vector containing all the state variables affecting the asset whose values are contained in vector S. For a review of all variables which can affect the asset prices the reader is referred to Campbell (2000) who offers a survey of the most important contributions in this field. We have indicated with G the value of a riskless asset paying the instantaneous riskless interest rate r. Finally, I_S is a diagonal matrix containing the elements of vector S. Hereafter, the prime denotes transposition.

All the functions $f(t,X)$, $g(t,X)$, $\mu(t,X,S)$, $\Sigma(t,X,S)$, and $r(t,X)$ are supposed to be $\mathcal{F}_t-$measurable. The $\sigma-$algebra $\mathcal{F}$ is defined on a set Θ wherethrough the complete probability space $(\Theta,\mathcal{F},\mathbb{P})$ is defined. Here, $\mathbb{P}$ can be considered as the "historical" probability measure.

The stochastic equations in System (1) are driven by a set of risks represented by dW which is the differential of a $k-$dimensional Wiener process whose components are independent.[1]

The set of risk sources is the same for the state variables and for the asset prices. This hypothesis is not restrictive because thanks to the elements of matrices g and Σ we can model a lot of different frameworks. For instance, if we consider $dW = \begin{bmatrix} dW_1 & dW_2 \end{bmatrix}$, $g' = \begin{bmatrix} g_1 & 0 \end{bmatrix}$, and $\Sigma' = \begin{bmatrix} 0 & \sigma_2 \end{bmatrix}$ then the processes of X and S are not correlated even if they formally have the same risk sources.

We recall the main result concerning completeness and arbitrage in this kind of market (for the proof of the following theorem see Øksendal, 2000).

Theorem 1 *A market $\{S(t,X)\}_{t\in[t_0,H]}$ is arbitrage free (complete) if and only if there exists a (unique) $k-$dimensional vector $u(t,X)$ such that*

$$\Sigma(t,X)'\,u(t,X) = \mu(t,X) - r(t,X)\,S(t,X)\,,$$

and such that

$$\mathbb{E}\left[e^{\frac{1}{2}\int_{t_0}^{H}\|u(t,X)\|^2 dt}\right] < \infty.$$

If on the market there are less assets than risk sources ($n < k$), then the market cannot be complete even if it is arbitrage free. In this work we assume that $n \leq k$ and that the rank of matrix Σ is maximum (i.e. it equals n). Thus, the results we obtain in this work are valid for a financial market which is incomplete and stay valid for a complete market ($n = k$).

[1] This condition can be imposed without loss of generality because a set of independent Wiener processes can always be transformed into a set of correlated Wiener processes thanks to the Cholesky decomposition. For an application see Appendices A.3 and A.4.

3 The optimal portfolio

After defining the market structure as in System (1), if we indicate with $w(t) \in \mathbb{R}^{n\times 1}$ the vector containing the wealth amount invested in each risky asset, then the growth in investor's wealth (dR) is given by

$$dR = w' I_S^{-1} dS + (R - w'\mathbf{1}) \frac{dG}{G},$$

where we have applied the self-financing condition and $\mathbf{1}$ is a vector of ones (of dimension $n \times 1$).

Now, after substituting the differentials from System (1) into the wealth differential equation, we have:

$$dR = (Rr + w'(\mu - r\mathbf{1}))\, dt + w'\Sigma' dW. \tag{2}$$

Accordingly, the problem for an investor wanting to maximize the expected power utility of his terminal wealth can be written as follows:

$$\begin{cases} \max\limits_{w} \mathbb{E}_{t_0} \left[\alpha R(H)^{\beta} \right] \\ d \begin{bmatrix} z \\ R \end{bmatrix} = \begin{bmatrix} \mu_z \\ Rr + w'M \end{bmatrix} dt + \begin{bmatrix} \Omega' \\ w'\Sigma' \end{bmatrix} dW, \\ z(t_0) = z_0, R(t_0) = R_0, \quad \forall t_0 \le t \le H, \end{cases} \tag{3}$$

where

$$\begin{aligned} \underset{(s+n)\times 1}{z} &\equiv \begin{bmatrix} X' & S' \end{bmatrix}', \\ \underset{(s+n)\times 1}{\mu_z} &\equiv \begin{bmatrix} f' & \mu' \end{bmatrix}', \\ \underset{k\times(s+n)}{\Omega} &\equiv \begin{bmatrix} g & \Sigma \end{bmatrix}, \\ \underset{n\times 1}{M} &\equiv (\mu - r\mathbf{1}), \end{aligned}$$

and the parameters α and β must be such that the function αR^{β} is increasing and concave (thus, $\alpha\beta > 0$ and $\beta < 1$).

The vector z contains all the state variables but the wealth. From Problem (3) we have the Hamiltonian

$$\begin{aligned} \mathcal{H} = \; & J_z' \mu_z + J_R (Rr + w'M) + \frac{1}{2} \operatorname{tr}(\Omega'\Omega J_{zz}) \\ & + w'\Sigma'\Omega J_{zR} + \frac{1}{2} J_{RR} w'\Sigma'\Sigma w, \end{aligned} \tag{4}$$

where $J(R, z, t)$ is the value function solving the Hamilton-Jacobi-Bellman partial differential equation (see Section 3.1), verifying

$$J(R, z, t) = \sup_{w} \mathbb{E}_t \left[\alpha R(H)^{\beta} \right],$$

and the subscripts on J indicate the partial derivatives. The system of the first order conditions on $\mathcal{H}$ is

$$\frac{\partial \mathcal{H}}{\partial w} = J_R M + \Sigma' \Omega J_{zR} + J_{RR} \Sigma' \Sigma w = 0.$$

The second order conditions hold if the Hessian matrix of $\mathcal{H}$

$$\frac{\partial \mathcal{H}}{\partial w' \partial w} = J_{RR} \Sigma' \Sigma,$$

is negative definite. Since $\Sigma'\Sigma$ is a quadratic form it is always positive definite and so the second order conditions are satisfied if and only if $J_{RR} < 0$, that is if the value function is concave in R. The reader is referred to Stockey and Lucas (1989) for the assumptions that must hold on the objective function for having a strictly concave value function.

From the first order conditions we obtain the optimal portfolio composition

$$w^* = \underbrace{-\frac{J_R}{J_{RR}} \left(\Sigma'\Sigma\right)^{-1} M}_{w^*_{(1)}} \underbrace{- \frac{1}{J_{RR}} \left(\Sigma'\Sigma\right)^{-1} \Sigma' \Omega J_{zR}}_{w^*_{(2)}}. \tag{5}$$

We recall that in this framework the matrix $\Sigma'\Sigma$ is invertible. In fact, $\Sigma'\Sigma$ is an $n \times n$ matrix and we suppose that $\Sigma' \in \mathbb{R}^{n \times k}$ has maximum rank. Because we put $n \leq k$, then Σ has rank n and $\Sigma'\Sigma$ is invertible. This means that, even in an incomplete market, there exists a *unique* optimal portfolio.

We just outline that $w^*_{(1)}$ increases if the risk premium increases, while it decreases if the risk aversion or the asset variance increase. From this point of view, we can argue that this component of the optimal portfolio has just a speculative role. The second part $w^*_{(2)}$ is the only optimal portfolio component explicitly depending on the diffusion terms of the state variables.

3.1 The value function

For finding a closed form solution to the optimal portfolio problem we need to compute the value function $J(R, z, t)$. By substituting the optimal value of w from (5) into the Hamiltonian (4) we have

$$\begin{aligned} \mathcal{H}^* &= J_z' \mu_z + J_R R r + \frac{1}{2} \operatorname{tr}\left(\Omega' \Omega J_{zz}\right) - \frac{J_R}{J_{RR}} M' \left(\Sigma'\Sigma\right)^{-1} \Sigma' \Omega J_{zR} \\ &\quad - \frac{1}{2} \frac{J_R^2}{J_{RR}} M' \left(\Sigma'\Sigma\right)^{-1} M - \frac{1}{2} \frac{1}{J_{RR}} J_{zR}' \Omega' \Sigma \left(\Sigma'\Sigma\right)^{-1} \Sigma' \Omega J_{zR}, \end{aligned}$$

from which we can formulate the PDE whose solution is the value function. This PDE is called the Hamilton-Jacobi-Bellman equation (hereafter HJB) and can be written as follows:

$$\begin{cases} J_t + \mathcal{H}^* = 0, \\ J(H, R, z) = \alpha R (H)^{\beta}. \end{cases} \tag{6}$$

The most common approach for solving this kind of PDE is to try a separability condition. Here, we try for $J(z,R,t) = \alpha R^{\beta} e^{h(z,t)}$. After substituting this functional form into the HJB equation (6) and dividing by J we obtain:

$$\begin{cases} h_t + a(z,t)' h_z + b(z,t) + \frac{1}{2}\operatorname{tr}(C(z,t) h_{zz}) + \frac{1}{2} h_z' D(z,t) h_z = 0, \\ \hfill h(z,H) = 0, \end{cases} \tag{7}$$

where

$$\begin{aligned} \underset{(s+n)\times 1}{a(z,t)} &\equiv \mu_z - \frac{\beta}{\beta-1}\Omega'\Sigma(\Sigma'\Sigma)^{-1} M, \\ b(z,t) &\equiv \beta r - \frac{1}{2}\frac{\beta}{\beta-1} M'(\Sigma'\Sigma)^{-1} M, \\ \underset{(s+n)\times(s+n)}{C(z,t)} &\equiv \Omega'\Omega, \\ \underset{(s+n)\times e(s+n)}{D(z,t)} &\equiv \Omega'\left(I - \frac{\beta}{\beta-1}\Sigma(\Sigma'\Sigma)^{-1}\Sigma'\right)\Omega, \end{aligned}$$

and the subscripts on h indicate partial derivates.

Thus, the choice of a power utility function implies that the optimal portfolio has the following composition:

$$w^* = \frac{R}{1-\beta}(\Sigma'\Sigma)^{-1} M + \frac{R}{1-\beta}(\Sigma'\Sigma)^{-1}\Sigma'\Omega h_z, \tag{8}$$

where the function $h(z,t)$ solves the PDE (7).

In the following section we derive some sufficient conditions that must hold for having a closed form solution to the Equation (7). We underline that in Menoncin (2002) a closed form solution for this kind of equation can be found (thanks to the Feynman-Kač theorem), if a stochastic inflation risk is introduced in a complete financial market and the investor maximizes a CARA utility function. Unfortunately, there does not exist a suitable change in the preference parameters for transforming a CARA utility function into a CRRA utility function.

4 Sufficient conditions for an exact solution

After looking at the literature where a closed form solution to the portfolio problem is found, we can see that the functions $a(z,t)$, $b(z,t)$, $C(z,t)$, and $D(z,t)$ can always be represented as polynomials in z in the following way:

$$\begin{cases} \underset{m\times 1}{a(z,t)} = \underset{m\times 1}{a_0(t)} + \underset{m\times m}{A_1(t)'} \underset{m\times 1}{z}, \\ b(z,t) = b_0(t) + \underset{1\times m}{b_1(t)'} \underset{m\times 1}{z} + \underset{1\times m}{z'} \underset{m\times m}{B_2(t)} \underset{m\times 1}{z}, \\ \underset{m\times m}{C(z,t)} = \underset{m\times m}{C_0(t)} + \underset{m\times m}{C_1(t)'} \underset{m\times m}{I_z} \underset{m\times m}{C_1(t)}, \\ \underset{m\times m}{D(z,t)} = \underset{m\times m}{D_0(t)} + \underset{m\times m}{D_1(t)'} \underset{m\times m}{I_z} \underset{m\times m}{D_1(t)}, \end{cases} \tag{9}$$

	BHT	**DGK**	**CV**	**KO**
$a(z,t)$	$a_0 + a_1 z$	$a_0 + a_1 z$	$a_0 + a_1 z$	$a_0 + a_1 z$
$b(z,t)$	$b_0 + b_1 z$	$b_0 + b_1 z$	$b_0 + b_1 z$	$b_0 + b_2 z^2$
$C(z,t)$	c_0	$c_1 z$	$c_1 z$	c_0
$D(z,t)$	d_0	$d_1 z$	$d_1 z$	d_0

Table 1: The structure of four models with an exact solution

where I_z is a diagonal matrix containing the elements of vector z and $m = n + s$ (we recall that s is the number of state variables while n is the number of risky assets). The particular form for functions $C(z,t)$ and $D(z,t)$ comes from the need to have two symmetric matrices.

We show in Table 1 that the already cited closed form solutions found in the literature always lie on Structure (9). For the particular forms of the parameters in Table 1 the reader is referred to Appendix A. Hereafter, we indicate with KO the model presented in Kim and Omberg (1996), with CV the model in Chacko and Viceira (1999), with DGK the model in Deelstra et al. (2000), and with BHT the model in Boulier et al. (2001).

We immediately see that the function $a(z,t)$ is always a first degree polynomial in z while there exist some differences for what concerns the other functions. From Table 1 it is clear that the model analysed by Deelstra et al. (2000) is identical to the model presented in Chacko and Viceira (1999).

Now, we try to solve the general problem by considering a polynomial for the function $h(z,t)$. In particular, given the form of the functions $a(z,t)$, $b(z,t)$, $C(z,t)$, and $D(z,t)$, we try

$$h(z,t) = y_0(t) + \underset{1\times m}{y_1(t)'}\underset{m\times 1}{z} + \underset{1\times m}{z'}\underset{m\times m}{Y_2(t)}\underset{m\times 1}{z}, \tag{10}$$

where $y_0(t), y_1(t)$, and $Y_2(t)$ are functions whose forms must be determined.

After substituting all these functional forms into the HJB equation (7) we obtain the following differential equation where, for the sake of simplicity, the functional dependences have been omitted:

$$\begin{aligned}
0 &= \frac{\partial y_0}{\partial t} + z'\frac{\partial y_1}{\partial t} + z'\frac{\partial Y_2}{\partial t}z + a_0' y_1 + z' A_1 y_1 + 2a_0' Y_2 z + 2z' A_1 Y_2 z \\
&+ b_0 + b_1' z + z' B_2 z + \operatorname{tr}(C_0 Y_2) + \operatorname{tr}(C_1' I_z C_1 Y_2) \\
&+ \frac{1}{2} y_1' D_0 y_1 + 2y_1' D_0' Y_2 z + 2z' Y_2' D_0 Y_2 z \\
&+ \frac{1}{2} y_1' D_1' I_z D_1 y_1 + 2y_1' D_1' I_z D_1 Y_2 z + 2z' Y_2' D_1' I_z D_1 Y_2 z.
\end{aligned}$$

After recalling the following properties:[2]

$$\begin{aligned}
\operatorname{tr}(C_1' I_z C_1 Y_2) &= \operatorname{tr}(I_z C_1 Y_2 C_1') = z' \operatorname{diag}(C_1 Y_2 C_1'), \\
y_1' D_1' I_z D_1 y_1 &= \operatorname{tr}(y_1' D_1' I_z D_1 y_1) = z' \operatorname{diag}(D_1 y_1 y_1' D_1'),
\end{aligned}$$

we can write the previous polynomial differential equation as a system of differential equations in the following way:

$$\begin{cases}
2z' Y_2' D_1' I_z D_1 Y_2 z = 0, \\
z' \frac{\partial Y_2}{\partial t} z + 2z' A_1 Y_2 z + z' B_2 z + 2z' Y_2' D_0 Y_2 z + 2y_1' D_1' I_z D_1 Y_2 z = 0, \\
z' \frac{\partial y_1}{\partial t} + z' A_1 y_1 + 2z' Y_2' a_0 + z' b_1 + z' \operatorname{diag}(C_1 Y_2 C_1') \\
+2z' Y_2' D_0 y_1 + \frac{1}{2} z' \operatorname{diag}(D_1 y_1 y_1' D_1') = 0, \\
\frac{\partial y_0}{\partial t} + a_0' y_1 + b_0 + \operatorname{tr}(C_0 Y_2) + \frac{1}{2} y_1' D_0 y_1 = 0,
\end{cases} \tag{11}$$

while the boundary condition changes into the system

$$\begin{cases}
y_0(H) = 0, \\
y_1(H) = \mathbf{0}, \\
Y_2(H) = \mathbf{0},
\end{cases}$$

where we have indicated with $\mathbf{0}$ a vector of suitable dimension, containing only zeros.

The first equation in System (11) holds only in two cases: (i) $Y_2(t) = \mathbf{0}$ or (ii) $D_1(t) = \mathbf{0}$. Let us analyse these two cases. If the matrix $Y_2(t)$ contains only zeros, then System (11) becomes:

$$\begin{cases}
z' B_2 z = 0, \\
z' \frac{\partial y_1}{\partial t} + z' A_1 y_1 + z' b_1 + \frac{1}{2} z' \operatorname{diag}(D_1 y_1 y_1' D_1') = 0, \\
\frac{\partial y_0}{\partial t} + a_0' y_1 + b_0 + \frac{1}{2} y_1' D_0 y_1 = 0,
\end{cases}$$

from which we can see that also the condition $B_2(t) = \mathbf{0}$ must hold.

[2] Given a square matrix, the diag operator transforms it into a column vector containing the elements of the main diagonal of the matrix.

Instead, when the matrix $D_1(t)$ contains only zeros, then the HJB equation can be written as the following system:

$$\begin{cases} z'\frac{\partial Y_2}{\partial t}z + 2z'A_1Y_2z + z'B_2z + 2z'Y_2'D_0Y_2z = 0, \\ \\ z'\frac{\partial y_1}{\partial t} + z'A_1y_1 + 2z'Y_2'a_0 + z'b_1 \\ +z'\operatorname{diag}(C_1Y_2C_1') + 2z'Y_2'D_0y_1 = 0, \\ \\ \frac{\partial y_0}{\partial t} + a_0'y_1 + b_0 + \operatorname{tr}(C_0Y_2) + \frac{1}{2}y_1'D_0y_1 = 0, \end{cases}$$

We underline that in both cases, if the function $h(z,t)$ has the form (10), then the optimal portfolio (8) depends only on y_1 and Y_2, while y_0 does not play any role in determining the optimal asset allocation. Furthermore, in both the previous systems, we can see that the value of y_0 is uniquely determined by the last equation. Accordingly, we can forget about this last equation and the only relevant equations for both the above-mentioned cases can be written in the following way:

$$\begin{aligned} B_2(t) &= \mathbf{0} \Rightarrow \begin{cases} Y_2 = \mathbf{0}, \\ \\ \frac{\partial y_1}{\partial t} + b_1 + A_1y_1 + \frac{1}{2}\operatorname{diag}(D_1y_1y_1'D_1') = 0, \end{cases} \\ D_1(t) &= \mathbf{0} \Rightarrow \begin{cases} \frac{\partial Y_2}{\partial t} + B_2 + 2A_1Y_2 + 2Y_2'D_0Y_2 = 0, \\ \\ \frac{\partial y_1}{\partial t} + (2Y_2'D_0 + A_1)y_1 + 2Y_2'a_0 + b_1 \\ +\operatorname{diag}(C_1Y_2C_1') = 0. \end{cases} \end{aligned}$$

In other words, a quadratic form for the function $b(z,t)$ can be considered only if the function $D(z,t)$ does not depend on z. Actually, this is the case analysed in Kim and Omberg (1996) in which $b(z,t)$ is a second degree polynomial but $D(z,t)$ is a constant. We underline that when $B_2(t) = \mathbf{0}$ and $D_1(t) = \mathbf{0}$ the two cases here considered coincide.

We can see that in both cases we have to solve a matrix Riccati differential equation where the main difference is given by the matrix dimensions. In the second case, we also have to solve a linear first order differential equation which has a solution under very general conditions on its parameters. Accordingly, we can state what follows.

Proposition 1 *The optimal portfolio solving Probelm (3) has a closed form solution if one of the two following cases holds:*

1. *(the linear case) the following equalities hold:*

$$\begin{cases} \mu_z - \frac{\beta}{\beta-1}\Omega'\Sigma(\Sigma'\Sigma)^{-1}M = \underset{m\times 1}{a_0(t)} + \underset{m\times m}{A_1(t)'}\underset{m\times 1}{z}, \\ \beta r - \frac{1}{2}\frac{\beta}{\beta-1}M'(\Sigma'\Sigma)^{-1}M = b_0(t) + \underset{1\times m}{b_1(t)'}\underset{m\times 1}{z}, \\ \Omega'\Omega = \underset{m\times m}{C_0(t)} + \underset{m\times m}{C_1(t)'}\underset{m\times m}{I_z}\underset{m\times m}{C_1(t)}, \\ \Omega'\left(I - \frac{\beta}{\beta-1}\Sigma(\Sigma'\Sigma)^{-1}\Sigma'\right)\Omega = \underset{m\times m}{D_0(t)} + \underset{m\times m}{D_1(t)'}\underset{m\times m}{I_z}\underset{m\times m}{D_1(t)}, \end{cases}$$

and the functions $A_1(t)$, $b_1(t)$, and $D_1(t)$ are such that the matrix Riccati differential equation

$$\frac{\partial y_1(t)}{\partial t} + b_1(t) + A_1(t)\, y_1(t) + \frac{1}{2}\operatorname{diag}\left(D_1(t)\, y_1(t)\, y_1(t)'\, D_1(t)'\right) = 0,$$

has a solution;

2. *(the quadratic case) the following equalities hold:*

$$\begin{cases} \mu_z - \frac{\beta}{\beta-1}\Omega'\Sigma\left(\Sigma'\Sigma\right)^{-1} M = \underset{m\times 1}{a_0(t)} + \underset{m\times m}{A_1(t)'}\,\underset{m\times 1}{z}, \\ \beta r - \frac{1}{2}\frac{\beta}{\beta-1} M'\left(\Sigma'\Sigma\right)^{-1} M = b_0(t) + \underset{1\times m}{b_1(t)'}\,\underset{m\times 1}{z} + \underset{1\times m}{z'}\,\underset{m\times m}{B_2(t)}\,\underset{m\times 1}{z}, \\ \Omega'\Omega = \underset{m\times m}{C_0(t)} + \underset{m\times m}{C_1(t)'}\,\underset{m\times m}{I_z}\,\underset{m\times m}{C_1(t)}, \\ \Omega'\left(I - \frac{\beta}{\beta-1}\Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'\right)\Omega = \underset{m\times m}{D_0(t)}, \end{cases}$$

and the functions $A_1(t)$, $B_2(t)$, and $D_0(t)$ are such that the matrix Riccati differential equation

$$\frac{\partial Y_2(t)}{\partial t} + B_2(t) + 2A_1(t)\, Y_2(t) + 2Y_2(t)'\, D_0(t)\, Y_2(t) = 0,$$

has a solution.

For a complete review of all theorems and properties about the matrix Riccati differential equations, the reader is referred to Freiling (2002) who offers an interesting review of the main results in this field.

Contrary to the most common literature, Proposition 1 clarifies a subset of cases in which a closed form solution can be found for the optimal portfolio composition.

We underline that, if the market is complete, then the matrix Σ^{-1} does exist and so the Proposition 1 implies that the functions $C(z,t)$ and $D(z,t)$ must be two polynomials of the same order in z and, furthermore, $(1-\beta)\, C(z,t) = D(z,t)$.

In the following section we compute the explicit solutions for the functions $y_1(t)$ and $Y_2(t)$ in both cases shown in Proposition 1, but when the functions $a(z,t)$, $b(z,t)$, $C(z,t)$, and $D(z,t)$ are all scalar and do not depend on time.

5 Some particular cases

In this section we analyse in detail some particular cases which arise in the literature and for which an exact solution has already been found. In particular, we refer to the works of Kim and Omberg (1996), Chacko and Viceira (1999), Deelstra et al. (2000), and Boulier et al. (2001). We underline that all the parameters of the stochastic processes considered in the cited papers do not depend on time. Thus, the solutions of the differential equations derived from the HJB equation are much easier to compute. A further simplification comes from considering just one state variable (the riskless interest rate or the risk premium) following a Vasiček (1977) process or a Cox et al. (1985) process.

In the following subsections we present the computations for the two cases presented in Proposition 1, but when the functions $a(z,t)$, $b(z,t)$, $C(z,t)$, and $D(z,t)$ do not depend on time. With respect to the classification of solutions mentioned in Proposition 1, the structure presented in Kim and Omberg (1996) lies in the "quadratic" case while all the others lie in the "linear" case. The general solutions we present for both the quadratic and the linear cases are able to mimic the particular solutions already found in the above-mentioned literature.

We just sum up the hypotheses that must hold on the System (9) for replicating the cited works.

Hypothesis 1 *In System (9) all the functions do not depend on time and z is a scalar.*

Furthermore, when the matrix dimension m equals 1, we adopt the following notation: $A_1 = a_1$, $B_2 = b_2$, $C_0 = c_0$, $C_1 = c_1$, $D_0 = d_0$, and $D_1 = d_1$. We recall that m can be equal to 1 only when $n = 0$ and $s = 1$, that is to say when there exists just one state variable and the return of the risky assets do not depend on the risky asset value itself. In this case, in fact, the n state variables representing the risky asset returns disappear from Problem (3) since their role is exhausted in computing the growth in investor's wealth.

5.1 The linear case

When the vector z shrinks to a scalar and all the functions in System (9) are affine transformation of z (i.e. $b_2 = 0$), then the exact solution for the optimal portfolio is given by

$$w^* = \frac{R}{1-\beta}\left(\Sigma'\Sigma\right)^{-1} M + \frac{R}{1-\beta}\left(\Sigma'\Sigma\right)^{-1}\Sigma'\Omega y_1(t),$$

where

$$\begin{cases} \frac{\partial y_1(t)}{\partial t} + b_1 + a_1 y_1(t) + \frac{1}{2} d_1^2 y_1(t)^2 = 0, \\ y_1(H) = 0. \end{cases}$$

Accordingly, we can state the following proposition.

	BHT	DGK	CV
a_1	$-a_r$	$\frac{\beta}{\beta-1}\lambda_r\sigma_r - b_r$	$\frac{\beta}{1-\beta}\left(\mu-r\right)\sigma_{Sz}-\lambda$
b_1	β	$\beta-\frac{1}{2}\frac{\beta}{\beta-1}\lambda_r^2$	$\frac{1}{2}\frac{\beta}{1-\beta}\left(\mu-r\right)^2$
d_1	0	$-\frac{\sigma_r^2}{\beta-1}$	$\sigma^2-\frac{\beta}{\beta-1}\sigma_{Sz}^2$
Δ	a_r^2	$b_r^2+2\frac{\beta}{\beta-1}\sigma_r^2$ $+\frac{\beta}{\beta-1}\sigma_r^2\left(\lambda_r^2-2\frac{b_r}{\sigma_r}\lambda_r\right)$	$\lambda^2+2\frac{\beta}{\beta-1}\left(\mu-r\right)\sigma_{Sz}\lambda$ $+\frac{\beta}{\beta-1}\left(\mu-r\right)^2\sigma^2$

Table 2: The sign of the discriminant

Proposition 2 *Under Hypothesis 1, if* $b_2 = 0$*, then the optimal portfolio solving Problem (3) is*

$$w^* = \frac{R}{1-\beta}\left(\Sigma'\Sigma\right)^{-1}M + \frac{R}{1-\beta}\left(\Sigma'\Sigma\right)^{-1}\Sigma'\Omega y_1\left(t\right),$$

where, after defining $\Delta \equiv a_1^2 - 2d_1^2 b_1$*,*

$$y_1\left(t\right) = \begin{cases} 2b_1\left(\frac{\sqrt{\Delta}}{\tanh\left(\frac{1}{2}\sqrt{\Delta}(H-t)\right)} - a_1\right)^{-1}, & \Delta \geq 0 \\ 2b_1\left(\frac{\sqrt{-\Delta}}{\tan\left(\frac{1}{2}\sqrt{-\Delta}(H-t)\right)} - a_1\right)^{-1}. & \Delta \leq 0 \end{cases}.$$

Proof. See Appendix B. ■

When $\Delta = 0$ both solutions in Proposition 2 are valid. In fact their limits coincide:

$$\lim_{\Delta\to 0} y_1\left(t\right) = 2b_1\left(\frac{2}{H-t} - a_1\right)^{-1}.$$

When $\Delta < 0$, the presence of a tangent function makes the optimal portfolio behave periodically, and this behaviour is quite difficult to explain from an economic point of view. Thus, we want to check if the condition $a_1^2 - 2d_1^2 b_1 < 0$ can be neglected by looking at the above-cited literature.

In Table 2 the value of Δ for the models BHT, DGK, and CV is computed (the detailed computations can be found in Appendix A). We can immediately see that the structure analysed by Boulier et al. (2001) has a Δ which is always positive. In the DGK model, the value of Δ is always positive with respect to

b_r, σ_r, and λ_r if $\beta < 0$. In fact, in DGK Δ is a quadratic function with respect to these parameters and its determinants are all negative. Furthermore, if $\beta < 0$ all the coefficients of the quadratic terms are positive and so the positivity of Δ follows. In the CV model we find a similar behaviour. The value of Δ is always positive with respect to λ, $(\mu - r)$, and σ if $\beta < 0$ and $\sigma_{Sz} > 0$. In fact, as in the previous case, Δ is a quadratic function with respect to the cited parameters and all the coefficients of the quadratic terms are positive if $\sigma_{Sz} > 0$ and $\beta < 0$. Furthermore, under the same conditions, the determinants of these quadratic functions (with respect to the considered variables) are all negative and the positivity of Δ follows.

It is quite interesting to underline that an investor with an infinite time horizon has the following optimal portfolio:

$$\lim_{H\to\infty} w\left(t\right)^* = \frac{R}{1-\beta}\left(\Sigma'\Sigma\right)^{-1} M + \frac{R}{1-\beta}\frac{2b_1}{\sqrt{\Delta}-a_1}\left(\Sigma'\Sigma\right)^{-1}\Sigma'\Omega,$$

whose weights (w^*/R) are constant through time.

Furthermore, we can see from the solution in Proposition 2 that the behaviour of the second optimal portfolio component is monotonic through time. In fact, the derivative of $y_1\left(t\right)$ with respect to t (when $\Delta > 0$) is

$$\frac{\partial y_1\left(t\right)}{\partial t} = -4\Delta b_1 \frac{e^{-\sqrt{\Delta}(H-t)}}{\left(-a_1 + \sqrt{\Delta} + \left(a_1 + \sqrt{\Delta}\right) e^{-\sqrt{\Delta}(H-t)}\right)^2},$$

form which we immediately see that

$$\text{signum}\left(\frac{\partial y_1\left(t\right)}{\partial t}\right) = \text{signum}\left(-b_1\right).$$

Since we know from the boundary conditions that $y_1\left(t\right)$ must be equal to zero when t tends to the time horizon H, then we are able to easily check the sign of $y_1\left(t\right)$. In fact, since this function is monotonic and must reach the value 0 at $t = H$, then we can state the following proposition.

Proposition 3 *Under Hypothesis 1, if* $b_2 = 0$ *and* $\Delta > 0$*, then*

1. *if* $b_1 < 0$ *then* $\frac{\partial y_1(t)}{\partial t} > 0$ *and* $y_1\left(t\right) < 0$, $\forall t < H$*;*
2. *if* $b_1 > 0$ *then* $\frac{\partial y_1(t)}{\partial t} < 0$ *and* $y_1\left(t\right) > 0$, $\forall t < H$*;*
3. *if* $b_1 = 0$ *then* $\frac{\partial y_1(t)}{\partial t} = 0$ *and* $y_1\left(t\right) = 0$, $\forall t < H$*.*

Furthermore, we can conclude what follows.

Corollary 1 *Under Hypothesis 1, if $b_2 = 0$ and $\Delta > 0$, then the second optimal portfolio component never increases the absolute value of its weight through time.*

From Table 2, it can be easily checked that when $\beta < 0$ all the three models show a negative value of b_1. Accordingly, we can conclude that in all the models here analysed, the function $y_1(t)$ takes negative values and so

$$\operatorname{signum}\left(w^*_{(2)}\right) = \operatorname{signum}\left(-\Sigma'\Omega\right).$$

We recall that Σ contains the volatility terms of the asset prices while Ω contains the volatility terms of the state variables. Thus, we can conclude that when an asset is positively correlated with a given state variable (i.e. when the product of their volatility terms is positive) the weight of the second optimal portfolio component for this asset is negative. Actually, for hedging a portfolio against the risk of a state variable, it is necessary to buy an asset which is negatively correlated with this variable and sell an asset which is positively correlated with it.

5.2 An application of the linear case

In this subsection we want to create a market structure from conditions stated in Proposition 1. We consider a market with only one state variable and with only one risky asset whose return does not depend on the asset value. Accordingly, the market structure should be as follows:

$$\begin{aligned} dz &= \mu_z(z,t)\,dt + \underset{1\times k}{\omega_z(z,t)'}\underset{k\times 1}{dW}, \\ \frac{dS}{S} &= \mu_S(z,t)\,dt + \underset{1\times k}{\omega_S(z,t)'}\underset{k\times 1}{dW}. \end{aligned}$$

If there exists only one risk source (i.e. $k = 1$) then this market is complete and the two random variables z and S are perfectly correlated. We do not want to make such a restrictive assumption and so, we suppose to have two risk sources (i.e. $k = 2$). In particular, the market structure can be written as

$$\begin{aligned} dz &= \mu_z(z,t)\,dt + \omega_z(z,t)\,dW_z, \\ \frac{dS}{S} &= \mu_S(z,t)\,dt + \omega_{Sz}(z,t)\,dW_z + \omega_{SS}(z,t)\,dW_S. \end{aligned}$$

Thus, the foundamental matrices $a(z,t)$, $b(z,t)$, $C(z,t)$, and $D(z,t)$ are given by:

$$\begin{aligned}
a(z,t) &\equiv \mu_z(z,t) - \frac{\beta}{\beta-1}\Omega'\Sigma(\Sigma'\Sigma)^{-1}M \\
&= \mu_z(z,t) - \frac{\beta}{\beta-1}\frac{\omega_{Sz}(z,t)\,\omega_z(z,t)}{\omega_{Sz}^2(z,t)+\omega_{SS}^2(z,t)}\left(\mu_S(z,t)-r\right), \\
b(z,t) &\equiv \beta r - \frac{1}{2}\frac{\beta}{\beta-1}M'(\Sigma'\Sigma)^{-1}M \\
&= \beta r - \frac{1}{2}\frac{\beta}{\beta-1}\frac{1}{\omega_{Sz}^2(z,t)+\omega_{SS}^2(z,t)}\left(\mu_S(z,t)-r\right)^2, \\
C(z,t) &\equiv \Omega'\Omega = \omega_z^2(z,t), \\
D(z,t) &\equiv \Omega'\left(I - \frac{\beta}{\beta-1}\Sigma(\Sigma'\Sigma)^{-1}\Sigma'\right)\Omega \\
&= \left(1 - \frac{\beta}{\beta-1}\frac{\omega_{Sz}^2(z,t)}{\omega_{Sz}^2(z,t)+\omega_{SS}^2(z,t)}\right)\omega_z^2(z,t).
\end{aligned}$$

Since we want $C(z,t)$ to be a first order polynomial in z, then we must put

$$\omega_z(z,t) = \sqrt{\sigma_0+\sigma_1 z},$$

and, accordingly, we must chose ω_{Sz} and ω_{SS} such that

$$\frac{\omega_{Sz}^2(z,t)}{\omega_{Sz}^2(z,t)+\omega_{SS}^2(z,t)} = \text{constant},$$

which means that

$$\frac{\omega_{Sz}(z,t)}{\omega_{SS}(z,t)} = \text{constant}. \tag{12}$$

Furthermore, both $a(z,t)$ and $b(z,t)$ are polynomial of first degree in z only if the following equations are verified:

$$\begin{aligned}
\sqrt{\sigma_0+\sigma_1 z}\frac{\mu_S(z,t)-r}{\omega_{SS}(z,t)} &= \phi_0+\phi_1 z, \\
\left(\frac{\mu_S(z,t)-r}{\omega_{SS}(z,t)}\right)^2 &= \theta_0+\theta_1 z,
\end{aligned}$$

for real values of ϕ_0, ϕ_1, θ_0, and θ_1. After computing the square of the first equation and substituting the second equation, we obtain the system

$$\begin{cases}
\phi_0^2 = \sigma_0\theta_0, \\
2\phi_0\phi_1 = \sigma_1\theta_0+\sigma_0\theta_1, \\
\phi_1^2 = \theta_1\sigma_1,
\end{cases}$$

where we have one degree of freedom. For instance, for a free value of ϕ_0 the solution is

$$\theta_0 = \frac{\phi_0^2}{\sigma_0}, \qquad \theta_1 = \sigma_1\frac{\phi_0^2}{\sigma_0^2}, \qquad \phi_1 = \sigma_1\frac{\phi_0}{\sigma_0}.$$

Accordingly, we have

$$\frac{\mu_S(z,t)-r}{\omega_{SS}(z,t)}=\frac{\phi_0}{\sigma_0}\sqrt{\sigma_0+\sigma_1 z}.$$

From this equation we can see that it is possible to freely chose either $\mu_S(z,t)$ or $\omega_{SS}(z,t)$. Now, if we suppose

$$\omega_{SS}(z,t)=\sigma_S\sqrt{\sigma_0+\sigma_1 z},$$

where σ_S is a real constant, then we must chose

$$\mu_S(z,t)=r+\frac{\phi_0}{\sigma_0}\sigma_S(\sigma_0+\sigma_1 z),$$

and, because of Condition (12),

$$\omega_{Sz}(z,t)=\sigma_{Sz}\sqrt{\sigma_0+\sigma_1 z},$$

where σ_{Sz} is a real constant. Finally, $\mu_z(z,t)$ can be an affine transformation of z as

$$\mu_z(z,t)=\mu_0+\mu_1 z,$$

and, thanks to Proposition 2, we are able to compute the closed form solution for the optimal portfolio when the market structure is as follows:

$$\begin{aligned}dz &= (\mu_0+\mu_1 z)\,dt+\sqrt{\sigma_0+\sigma_1 z}dW_z,\\ \frac{dS}{S} &= (r+\lambda(\sigma_0+\sigma_1 z))\,dt+\sigma_{Sz}\sqrt{\sigma_0+\sigma_1 z}dW_z+\sigma_S\sqrt{\sigma_0+\sigma_1 z}dW_S,\end{aligned}$$

where λ is a real constant which can be interpreted as a risk price.

5.3 The quadratic case

When the vector z shrinks to a scalar and in System (9) $d_1=0$, then the exact solution for the optimal portfolio is given by

$$w^*=\frac{R}{1-\beta}\left(\Sigma'\Sigma\right)^{-1}M+\frac{R}{1-\beta}\left(\Sigma'\Sigma\right)^{-1}\Sigma'\Omega\left(y_1(t)+2y_2(t)z\right),$$

where

$$\begin{cases}\frac{\partial y_2(t)}{\partial t}+b_2+2a_1y_2(t)+2d_0y_2(t)^2=0,\\ y_2(H)=0,\\ \\ \frac{\partial y_1(t)}{\partial t}+(2d_0y_2(t)+a_1)\,y_1(t)+b_1+2a_0y_2(t)=0,\\ y_1(H)=0.\end{cases}$$

After solving the first differential equation for computing $y_2(t)$, the second one has the following simple solution:

$$y_1(t)=\int_t^H(b_1+2a_0y_2(s))\,e^{-\int_s^t(2d_0y_2(\tau)+a_1)d\tau}ds.$$

Accordingly, we can conclude with the following proposition.

Proposition 4 *Under Hypothesis 1, if $d_1 = 0$, then the optimal portfolio solving Problem (3) is*

$$w^* = \frac{R}{1-\beta} \left(\Sigma'\Sigma\right)^{-1} M + \frac{R}{1-\beta} \left(\Sigma'\Sigma\right)^{-1} \Sigma'\Omega \left(y_1(t) + 2y_2(t) z\right),$$

where, after defining $\Delta \equiv a_1^2 - 2d_0 b_2$,

$$y_2(t) = \begin{cases} b_1 \left(\frac{\sqrt{\Delta}}{\tanh\left(\sqrt{\Delta}(H-t)\right)} - a_1 \right)^{-1}, & \Delta \geq 0 \\ b_1 \left(\frac{\sqrt{-\Delta}}{\tan\left(\sqrt{-\Delta}(H-t)\right)} - a_1 \right)^{-1}, & \Delta \leq 0 \end{cases},$$

$$y_1(t) = \int_t^H \left(b_1 + 2a_0 y_2(s)\right) e^{-\int_s^t (2d_0 y_2(\tau) + a_1) d\tau} ds.$$

Proof. See Appendix B. ■

As in the previous paragraph, both solutions for function $y_2(t)$ are valid even when $\Delta = 0$. If the investor's time horizon tends to infinity the function $y_2(t)$ has the limit

$$\lim_{H\to\infty} y_2(t) = \frac{b_1}{\sqrt{\Delta} - a_1},$$

which is constant through time, while the function $y_1(t)$, after direct computation of the integral in Proposition 4, takes the following value:

$$\begin{aligned} \lim_{H\to\infty} y_1(t) &= -\frac{b_1}{\sqrt{\Delta} - a_1} \left(1 + \frac{2a_0 - a_1}{\sqrt{\Delta}}\right) \lim_{H\to\infty} \left(e^{-\sqrt{\Delta}(H-t)} - 1\right) \\ &= \frac{b_1}{\sqrt{\Delta} - a_1} \left(1 + \frac{2a_0 - a_1}{\sqrt{\Delta}}\right). \end{aligned}$$

At our knowledge, the only market structure laying in the "quadratic case" is presented in the model after Kim and Omberg (1996). Thus, the reader is referred to these authors for a complete exposition of the properties of the solution presented in Proposition 4.

6 Conclusion

In this paper we have analysed the optimal portfolio problem for an investor maximizing the expected CRRA utility function of his terminal wealth.

Contrary to other results found in the literature, which care either about finding a particular closed form solution to the asset allocation problem or about determining existence and uniqueness of this solution, we present some sufficient conditions that must hold for having a closed form solution and we present the algebraic form of this solution which can be computed by solving a matrix Riccati differential equation.

We do not specify the functional form for the drift and diffusion coefficients of the stochastic processes driving the state variables and the asset prices. Instead, we look for the form that a suitable combination of these drift and diffusion coefficients must have for guaranteeing the existence of a closed form solution.

In particular, we distinguish two kinds of settings: (i) the ones which can be solved through a log-linear value function and (ii) the ones solvable by means of a log-quadratic value function.

We have explicitly computed the closed form solution for the optimal asset allocation in both these cases, when there exists only one state variable and the coefficients of the stochastic processes driving this state variable and the asset prices do not depend on time. In fact, this is the case always considered in the literature where a closed form solution to the asset allocation problem is computed. We derive the properties of the "linear" solution. In particular, we show that the absolute value of the optimal portfolio component hedging against the risk represented by the state variable is a decreasing function of time. The sign of this optimal portfolio component depends on the signs of the product between volatility terms of asset prices and state variable.

A The market structure of some exact solutions

A.1 The structure of Boulier, Huang, and Taillard (2001)

Boulier et al. (2001) consider a market structure in which there is only one state variable (the riskless interest rate r) following the Vasiček (1977) model, and two assets: a stock (S) and a bond (B). In particular, they have:

$$\begin{cases} dr = a_r \left(b_r - r\right) dt - \sigma_r dW_r, \\ \frac{dS}{S} = \left(r + \sigma_1 \lambda_1 + \sigma_2 \lambda_r\right) dt + \sigma_1 dW_S + \sigma_2 dW_r, \\ \frac{dB}{B} = \left(r + \left(1 - e^{a_r (H-t)}\right) \frac{\lambda_r \sigma_r}{a_r}\right) dt + \left(1 - e^{a_r (H-t)}\right) \frac{\sigma_r}{a_r} dW_r, \\ \frac{dG}{G} = r dt, \end{cases}$$

where all the parameters take positive values.

Thus, under their model the matrices introduced in this work assume the following values:

$$
\begin{aligned}
z &= r, \\
\mu_z &= a_r (b_r - r), \\
M &= \begin{bmatrix} \sigma_1\lambda_1 + \sigma_2\lambda_r & \left(1 - e^{a_r(H-t)}\right) \frac{\lambda_r \sigma_r}{a_r} \end{bmatrix}', \\
\Sigma &= \begin{bmatrix} \sigma_2 & \left(1 - e^{a_r(H-t)}\right) \frac{\sigma_r}{a_r} \\ \sigma_1 & 0 \end{bmatrix}, \\
\Omega &= \begin{bmatrix} -\sigma_r & 0 \end{bmatrix}'.
\end{aligned}
$$

Accordingly, we can write:

$$
\begin{aligned}
a(z,t) &\equiv \mu_z - \frac{\beta}{\beta-1}\Omega'\Sigma'^{-1}M = a_r b_r + \frac{\beta}{\beta-1}\lambda_r\sigma_r - a_r r, \\
b(z,t) &\equiv \beta r - \frac{1}{2}\frac{\beta}{\beta-1}M'\left(\Sigma'\Sigma\right)^{-1}M = -\frac{1}{2}\frac{\beta}{\beta-1}\left(\lambda_1^2 + \lambda_r^2\right) + \beta r, \\
C(z,t) &\equiv \Omega'\Omega = \sigma_r^2, \\
D(z,t) &\equiv \Omega'\left(I - \frac{\beta}{\beta-1}\Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'\right)\Omega = -\frac{\sigma_r^2}{\beta-1}.
\end{aligned}
$$

A.2 The structure of Deelstra, Grasselli, and Koehl (2000)

Deelstra et al. (2000) consider a market structure in which there is only one state variable (the riskless interest rate r) following the Cox et al. (1985) model, and two assets: a stock and a bond. In particular, they have:

$$
\begin{cases}
dr = a_r (b_r - r)\,dt - \sigma_r\sqrt{r}dW_r, \\
\frac{dS}{S} = (r + \sigma_1\lambda_1 + \sigma_2\lambda_r r)\,dt + \sigma_1 dW_S + \sigma_2\sqrt{r}dW_r, \\
\frac{dB}{B} = (r + r\lambda_r g(H-t)\,\sigma_r)\,dt + g(H-t)\,\sigma_r\sqrt{r}dW_r, \\
\frac{dG}{G} = r dt,
\end{cases}
$$

where all the parameters take positive values and

$$
\begin{aligned}
g(\tau) &= \frac{2\left(e^{\delta\tau} - 1\right)}{2\delta + \left(e^{\delta\tau} - 1\right)\left(\delta + b_r - \sigma_r\lambda_r\right)}, \\
\delta &\equiv \sqrt{\left(b_r - \sigma_r\lambda_r\right)^2 + 2\sigma_r^2}.
\end{aligned}
$$

Thus, under their model the matrices introduced in this work assume the following values:

$$
\begin{aligned}
z &= r, \\
\mu_z &= (a_r - b_r r), \\
M &= \left[\begin{array}{cc} \sigma_1\lambda_1 + \sigma_2\lambda_r r & g(H-t)\lambda_r\sigma_r r \end{array} \right]', \\
\Sigma &= \left[\begin{array}{cc} \sigma_2\sqrt{r} & g(H-t)\sigma_r\sqrt{r} \\ \sigma_1 & 0 \end{array} \right], \\
\Omega &= \left[\begin{array}{cc} -\sigma_r\sqrt{r} & 0 \end{array} \right]'.
\end{aligned}
$$

Accordingly, we can write:

$$
\begin{aligned}
a(z,t) &\equiv \mu_z - \frac{\beta}{\beta-1}\Omega'\Sigma'^{-1}M = a_r + \left(\frac{\beta}{\beta-1}\lambda_r\sigma_r - b_r\right) r, \\
b(z,t) &\equiv \beta r - \frac{1}{2}\frac{\beta}{\beta-1}M'\left(\Sigma'\Sigma\right)^{-1}M = -\frac{1}{2}\frac{\beta}{\beta-1}\lambda_1^2 + \left(\beta - \frac{1}{2}\frac{\beta}{\beta-1}\lambda_r^2\right) r, \\
C(z,t) &\equiv \Omega'\Omega = \sigma_r^2 r, \\
D(z,t) &\equiv \Omega'\left(I - \frac{\beta}{\beta-1}\Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'\right)\Omega = -\frac{\sigma_r^2}{\beta-1} r.
\end{aligned}
$$

A.3 The structure of Chacko and Viceira (1999)

Chacko and Viceira (1999) consider a market structure with only one risky asset (S), and only one state variable given by the inverse of the volatility of the risky asset and following the Cox et al. (1985) model. In particular, the model is:

$$
\begin{cases}
dz = \lambda(\theta - z)\,dt + \sigma\sqrt{z}d\widetilde{W}_z, \\
\frac{dS}{S} = \mu dt + \frac{1}{\sqrt{z}}d\widetilde{W}_S, \\
dG = Grdt,
\end{cases}
$$

where all the parameters take positive values. We outline that the stochastic differentials $d\widetilde{W}_z$ and $d\widetilde{W}_S$ are correlated:

$$
Cov\left(dz, \frac{dS}{S}\right) = \left[\begin{array}{cc} \sigma^2 z & \sigma_{Sz} \\ \sigma_{Sz} & \frac{1}{z} \end{array} \right].
$$

We can lead this case back to our approach by using the Cholesky decomposition. Because the variance and covariance matrix is always positive semidefinite, we can write:

$$
\left[\begin{array}{cc} \sigma\sqrt{z} & 0 \\ \frac{\sigma_{Sz}}{\sigma\sqrt{z}} & \frac{1}{\sqrt{z}}\sqrt{1 - \frac{\sigma_{Sz}^2}{\sigma^2}} \end{array} \right]
\left[\begin{array}{cc} \sigma\sqrt{z} & \frac{\sigma_{Sz}}{\sigma\sqrt{z}} \\ 0 & \frac{1}{\sqrt{z}}\sqrt{1 - \frac{\sigma_{Sz}^2}{\sigma^2}} \end{array} \right]
= \left[\begin{array}{cc} \sigma^2 z & \sigma_{Sz} \\ \sigma_{Sz} & \frac{1}{z} \end{array} \right].
$$

Thus, the previous problem can be written in the following way:

$$\begin{cases} dz = \lambda(\theta - z)\,dt + \sigma\sqrt{z}dW_z, \\ \frac{dS}{S} = \mu dt + \frac{\sigma_{Sz}}{\sigma\sqrt{z}}dW_z + \frac{1}{\sqrt{z}}\sqrt{1-\frac{\sigma_{Sz}^2}{\sigma^2}}dW_S, \\ \frac{dG}{G} = rdt, \end{cases}$$

where dW_z and dW_S are the differentials of two independent Wiener processes.

Thus, under their model the matrices introduced in this work assume the following values:

$$\begin{aligned} \mu_z &= \lambda(\theta - z), \\ M &= \mu - r, \\ \Sigma &= \begin{bmatrix} \frac{\sigma_{Sz}}{\sigma\sqrt{z}} & \frac{1}{\sqrt{z}}\sqrt{1-\frac{\sigma_{Sz}^2}{\sigma^2}} \end{bmatrix}', \\ \Omega &= \begin{bmatrix} \sigma\sqrt{z} & 0 \end{bmatrix}'. \end{aligned}$$

Accordingly, we can write:

$$\begin{aligned} a(z,t) &\equiv \mu_z - \frac{\beta}{\beta-1}\Omega'\Sigma'^{-1}M = \lambda\theta + \left(\frac{\beta}{1-\beta}(\mu - r)\,\sigma_{Sz} - \lambda\right)z, \\ b(z,t) &\equiv \beta r - \frac{1}{2}\frac{\beta}{\beta-1}M'(\Sigma'\Sigma)^{-1}M = \beta r + \frac{1}{2}\frac{\beta}{1-\beta}(\mu - r)^2 z, \\ C(z,t) &\equiv \Omega'\Omega = \sigma^2 z, \\ D(z,t) &\equiv \Omega'\left(I - \frac{\beta}{\beta-1}\Sigma(\Sigma'\Sigma)^{-1}\Sigma'\right)\Omega = \left(\sigma^2 - \frac{\beta}{\beta-1}\sigma_{Sz}^2\right)z. \end{aligned}$$

A.4 The structure of Kim and Omberg (1996)

Kim and Omberg (1996) consider a market structure with only one state variable (the risk premium z) following the Vasiček (1977) model, and only one risky asset (S). In particular they have:

$$\begin{cases} dz = \lambda(\theta - z)\,dt + \sigma_z d\widetilde{W}_z, \\ \frac{dS}{S} = (r + \sigma_S z)\,dt + \sigma_S d\widetilde{W}_S, \\ \frac{dG}{G} = rdt, \end{cases}$$

where all the parameters take positive values and $\widetilde{W}_z$ and $\widetilde{W}_S$ are two correlated Wiener processes:

$$Cov\left(dz, \frac{dS}{S}\right) = \begin{bmatrix} \sigma_z^2 & \sigma_S\sigma_z\rho_{Sz} \\ \sigma_S\sigma_z\rho_{Sz} & \sigma_S^2 \end{bmatrix}.$$

We can lead this case back to our approach by using the Cholesky decomposition. Because the variance and covariance matrix is always positive semidefinite, we can write:

$$\left[\begin{array}{cc}\sigma_z & 0\\ \sigma_S\rho_{Sz} & \sigma_S\sqrt{1-\rho_{Sz}^2}\end{array}\right]\left[\begin{array}{cc}\sigma_z & \sigma_S\rho_{Sz}\\ 0 & \sigma_S\sqrt{1-\rho_{Sz}^2}\end{array}\right]=\left[\begin{array}{cc}\sigma_z^2 & \sigma_S\sigma_z\rho_{Sz}\\ \sigma_S\sigma_z\rho_{Sz} & \sigma_S^2\end{array}\right].$$

Thus, the previous market structure can be written in the following way:

$$\begin{cases} dz=\lambda\left(\theta-z\right)dt+\sigma_z dW_z,\\ \frac{dS}{S}=\left(r+\sigma_S z\right)dt+\sigma_S\rho_{Sz}dW_z+\sigma_S\sqrt{1-\rho_{Sz}^2}dW_S,\\ \frac{dG}{G}=rdt,\end{cases}$$

where W_S and W_x are two independent Wiener processes.

Thus, under their model the matrices introduced in this work assume the following values:

$$\begin{aligned}\mu_z &= \lambda\left(\theta-z\right),\\ M &= z\sigma_S,\\ \Sigma &= \left[\begin{array}{cc}\sigma_S\rho_{Sz} & \sigma_S\sqrt{1-\rho_{Sz}^2}\end{array}\right]',\\ \Omega &= \left[\begin{array}{cc}\sigma_z & 0\end{array}\right]'.\end{aligned}$$

Accordingly, we can write:

$$\begin{aligned} a\left(z,t\right) &\equiv \mu_z-\frac{\beta}{\beta-1}\Omega'\Sigma'^{-1}M=\lambda\theta-\left(\lambda+\frac{\beta}{1-\beta}\rho_{Sz}\sigma_z\right)z,\\ b\left(z,t\right) &\equiv \beta r-\frac{1}{2}\frac{\beta}{\beta-1}M'\left(\Sigma'\Sigma\right)^{-1}M=\beta r-\frac{1}{2}\frac{\beta}{1-\beta}z^2,\\ C\left(z,t\right) &\equiv \Omega'\Omega=\sigma_z^2,\\ D\left(z,t\right) &\equiv \Omega'\left(I-\frac{\beta}{\beta-1}\Sigma\left(\Sigma'\Sigma\right)^{-1}\Sigma'\right)\Omega=\sigma_z^2\left(1-\frac{\beta}{\beta-1}\rho_{Sz}^2\right).\end{aligned}$$

B Riccati differential equation with constant coefficients

In this appendix we show how to solve a Riccati differential equation having the following form:

$$\frac{\partial f\left(t\right)}{\partial t}+\gamma_0+\gamma_1 f\left(t\right)+\gamma_2 f\left(t\right)^2=0, \tag{13}$$

where $\gamma_i\in\mathbb{R}, i\in\{0,1,2\}$, and with the boundary condition

$$f\left(H\right)=\gamma_H\in\mathbb{R}.$$

Since the coefficients are constant, we know two particular solutions of this equation:

$$f^*\left(t\right)=\frac{-\gamma_1\pm\sqrt{\Delta}}{2\gamma_2},$$

where $\Delta \equiv \gamma_1^2 - 4\gamma_2\gamma_0$. Nevertheless, because we want our general solution to be valid even when $\gamma_2 = 0$, then we chose the solution with the positive sign. In fact, in this case

$$\lim_{\gamma_2 \to 0} f^*(t) = \lim_{\gamma_2 \to 0} -\frac{\gamma_0}{\sqrt{\gamma_1^2 - 4\gamma_2\gamma_0}} = -\frac{\gamma_0}{\gamma_1},$$

which is a particular solution of the differential equation

$$\frac{\partial f(t)}{\partial t} + \gamma_1 f(t) + \gamma_0 = 0.$$

Now we consider the following transformation:

$$\phi(t) = \frac{1}{f(t) - f^*(t)} \Leftrightarrow f(t) = f^*(t) + \frac{1}{\phi(t)},$$

and, after substituting it into (13), we have:

$$\gamma_2 f^*(t)^2 + \gamma_1 f^*(t) + \gamma_0 - \frac{1}{\phi(t)^2}\frac{\partial \phi(t)}{\partial t} + \gamma_2 \frac{1}{\phi(t)^2} + 2\gamma_2 f^*(t)\frac{1}{\phi(t)} + \gamma_1 \frac{1}{\phi(t)} = 0.$$

The first three terms vanish and we have a linear first order differential equation:

$$\frac{\partial \phi(t)}{\partial t} - \sqrt{\Delta}\phi(t) - \gamma_2 = 0,$$

whose boundary condition is

$$\phi(H) = \frac{1}{f(H) - f^*(H)} = \frac{2\gamma_2}{2\gamma_2\gamma_H + \gamma_1 - \sqrt{\Delta}}.$$

The solution of the ODE in $\phi(t)$ is

$$\phi(t) = -\frac{\gamma_2}{\sqrt{\Delta}}\left(1 - \frac{\gamma_1 + 2\gamma_2\gamma_H + \sqrt{\Delta}}{\gamma_1 + 2\gamma_2\gamma_H - \sqrt{\Delta}} e^{-\sqrt{\Delta}(H-t)}\right),$$

and so we can use our initial transformation for obtaining the final result:

$$f(t) = \frac{-\gamma_1 + \sqrt{\Delta}}{2\gamma_2} - \frac{\sqrt{\Delta}}{\gamma_2}\left(1 - \frac{\gamma_1 + 2\gamma_2\gamma_H + \sqrt{\Delta}}{\gamma_1 + 2\gamma_2\gamma_H - \sqrt{\Delta}} e^{-\sqrt{\Delta}(H-t)}\right)^{-1},$$

which can be simplified to[3]

$$f(t) = \frac{\gamma_H + (2\gamma_0 + \gamma_1\gamma_H)\frac{1}{\sqrt{\Delta}}\tanh\left(\frac{1}{2}\sqrt{\Delta}(H-t)\right)}{1 - (\gamma_1 + 2\gamma_2\gamma_H)\frac{1}{\sqrt{\Delta}}\tanh\left(\frac{1}{2}\sqrt{\Delta}(H-t)\right)}, \tag{14}$$

[3] We recall that

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

from which, after substituting for the suitable values (and putting $\gamma_H = 0$) we can find the solution presented in Proposition 2.

We underline that this solution is asymptotically valid when Δ tends to zero. In this case, it is easy to compute the following limit thanks to De L'Ĥopital's rule

$$\lim_{\Delta \to 0} \frac{\tanh \frac{1}{2}\sqrt{\Delta}\,(H-t)}{\sqrt{\Delta}} = \lim_{\Delta \to 0} \frac{\frac{1}{4}\left(1 - \tanh^2 \frac{1}{2}\sqrt{\Delta}\,(H-t)\right)\frac{H-t}{\sqrt{\Delta}}}{\frac{1}{2\sqrt{\Delta}}} = \frac{1}{2}\,(H-t)\,.$$

Furthermore, Solution (14) is valid even for negative values of Δ. In this case, the function $f(t)$ can be written as follows

$$f(t) = \frac{\gamma_H + (2\gamma_0 + \gamma_1\gamma_H)\frac{1}{i\sqrt{-\Delta}}\tanh\left(\frac{1}{2}i\sqrt{-\Delta}\,(H-t)\right)}{1 - (\gamma_1 + 2\gamma_2\gamma_H)\frac{1}{i\sqrt{-\Delta}}\tanh\left(\frac{1}{2}i\sqrt{-\Delta}\,(H-t)\right)},$$

and, since we know, from the Euler's formulae, that

$$\frac{1}{i}\tanh(iy) = \tan y, \qquad \forall y \in \mathbb{R}$$

then we can conclude

$$f(t) = \frac{\gamma_H + (2\gamma_0 + \gamma_1\gamma_H)\frac{1}{\sqrt{-\Delta}}\tan\left(\frac{1}{2}\sqrt{-\Delta}\,(H-t)\right)}{1 - (\gamma_1 + 2\gamma_2\gamma_H)\frac{1}{\sqrt{-\Delta}}\tan\left(\frac{1}{2}\sqrt{-\Delta}\,(H-t)\right)}.$$

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