# How the Financial Managers' Remuneration Can Affect the Optimal Portfolio Composition*

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**Abstract**

In this paper we analyse the problem of an investor who must decide whether to manage his wealth by himself or give it in outsourcing. Financial managers are supposed to charge a commission composed of a fixed ($A$) and a variable ($x$) part, both deducted from portfolio payoffs. We demonstrate that the optimal portfolio composition crucially depends on the magnitude of $A$ and $x$. We make a general analysis of this dependence and, in particular, we show that high level of $A$ (respectively, $x$) lead to an outsourced portfolio which has a lower (respectively, higher) risk-return profile with respect to the self-managed portfolio.

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Key words: optimal portfolio, outsourcing, managers' remuneration

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# 1 Introduction

In this paper we analyse the problem of a fund that must decide whether to manage its wealth by itself or give it in outsourcing. We suppose that the financial managers charge a fee composed of (i) a fixed part ($A$), which is not affected by the portfolio performances, and (ii) a variable one ($x$), which is strictly proportional to the increase in the fund's wealth. Thus, when the fund's wealth decreases, the financial manager may receive a total fee which is lower than the fixed part (or even negative). The case of a proportional fee which is zero if the fund's wealth decreases is not relevant for our work. In fact, we want to analyse a fee structure letting us maintain the portfolio risk to a relative low level. A remuneration having both a fixed and a strictly positive proportional parts would be an incentive to take an excessively risky position.

Managers are supposed to work just in the interest of the investors, thus we do not take into account the agency problem arising when investors delegate their portfolio choices to managers. We underline that the presence of a part of managers' remuneration which is (strictly) proportional to the increase in investor's wealth allows us to suppose that managers want to maximize the portfolio performance so as any investor would do. The most strong assumption we make is that managers acquire exactly the investor's utility function. Nevertheless, if an investor is free to chose his managers, then it is quite reasonable to suppose he will look for a manager having a utility function close to his.

We demonstrate that, even if the investor's utility function and the manager's one are identical, the optimal portfolio composition changes because of the presence of the manager's remuneration which is deducted from portfolio payoffs. Thus, for an investor, choosing the suitable manager is not sufficient, he has also to choose the suitable manager's remuneration.

In practice, we compare two different problems: the optimal portfolio problem as considered in the most common literature (see for instance, Kim and Omberg [1996], Wachter [1998], Boulier, Huang and Taillard [2001], and Deelstra, Grasselli, and Koehl [2001]), and the optimal portfolio occurring when the manager's remuneration must be considered.

After computing the optimal portfolio in both cases, we demonstrate that there exists a pair of fixed and proportional managers' remuneration such that the two portfolios have the same composition, that is to say the same risk-return profile. We call this pair an "iso-mean-variance" (IMV) pair. Furthermore, we show that the locus of the IMV pairs has a positive slope and an increase in the fixed component must correspond to a higher increase in the proportional component. Actually, we underline that the two parts of managers' remuneration have the following effects: (i) the proportional part is positively correlated with both the risk and the return of the optimal portfolio, while (ii) the fixed part is inversely correlated with them.

It is important to stress that each IMV pair does not depend on the market variables given by the mean and the variance of asset returns. Accordingly, in order to investigate the mean-return profile implied by a proposed managers' remuneration, the investor has not to pay any cost for knowing these variables.

Actually, the avoidance of such a cost represents the gain from outsourcing the portfolio management.

The literature about the optimal portfolio allocation for institutional investors and, in particular, for pension funds (Boulier, Huang, and Taillard [2001]), generally neglects the problem of the managers' remuneration. This paper aims at finding properties that can be taken into account when deciding about the portfolio management outsourcing. In particular, we check the risk-return profile which is implied by this decision.

The approach we use is the classical dynamic programming technique. Such a technique leads to a closed form solution for the optimal portfolio under the hypothesis that the considered utility function belongs to the HARA family (with hyperbolic absolute risk aversion index). For the method called "martingale approach" the reader is referred to Cox and Huang [1989, 1991], and Lioui and Poncet [2001].

As we concentrate on the problem of the optimal portfolio for institutional investors, we do not take into account the consumption problem since the whole asset return is supposed to be reinvested in the financial market. Therefore, we limit ourselves to solve the problem of maximizing the expected utility of the terminal wealth, given a fixed time horizon ($H$). For an application to the case of an insurance company, we refer to Young and Zariphopoulou [2000] while Blake [1998], Blake, Cairns, and Dowd [1998], and Boulier, Huand and Taillard [2001] consider the case of a pension fund.

In studying this problem we consider a simple market structure in which the asset prices follow geometric Brownian motions and the riskless interest rate is deterministic and constant. In particular, we use the well known framework developed in Merton [1969, 1971]. A much more general analysis is carried out in Menoncin [2002] who considers a framework in which the investment opportunities (including the interest rate), the inflation rate, and a background risk set are all stochastic.

Through this work we consider agents trading continuously in a frictionless, arbitrage-free market until time $H$, which is the horizon of the economy. Furthermore, our model is able to deal with both a complete and an incomplete financial market.

The paper is structured as follows. Section 2 details the general economic framework and exposes the stochastic differential equations describing the behaviour of asset prices and fund's wealth. In Section 3 the optimal portfolio is computed in both cases of outsourcing and self-managing. Section 4 shows the main results and, in particular, the way the managers' remuneration affects the portfolio risk-return profile. Some particular cases follow the general analysis of the IMV locus. In particular, we consider: (i) a remuneration without fixed or proportional component, (ii) the case of a CARA utility function, (iii) the case of a zero riskless interest rate, and (iv) the case of a long run investor. Section 5 concludes. Finally, the closed form solution of the optimal portfolio for an investor having a HARA utility function is computed in the Appendix.

## 2 The model

In this paper we consider an economy where the behaviour of asset prices is described by geometric Brownian motions. Accordingly, we define as $\{S(t)\}_{t\in[t_0,H]}$ a market where, given the time horizon $H$, there are $n$ risky assets and one riskless asset ($G$) whose prices follow the stochastic differential equations:

$$\begin{cases} \underset{n\times 1}{dS}(t) = \underset{n\times n}{I_S}\left(\underset{n\times 1}{\mu}\,dt + \underset{n\times k}{\Sigma'}\,\underset{k\times 1}{dW}\right), \\ dG(t) = G(t)\,r dt, \end{cases} \tag{1}$$

where $I_S$ is a diagonal matrix containing the elements of vector $S$, $dW$ is the differential of a $k-$dimensional Wiener process, and $r$ is the instantaneous (and constant) riskless interest rate. Hereafter, the prime denotes transposition. We say that the market $\{S(t)\}_{t\in[t_0,H]}$ is normalized if $G(t) \equiv 1$. This hypothesis means that the riskless asset is the numeraire of the economy. Any market can always be normalized by putting $\overline{S}(t) = G(t)^{-1} S(t)$.

We present the main results concerning completeness and arbitrage in this kind of market (for the proofs of the two following theorems see Øksendal [2000]).

**Theorem 1** *A market $\{S(t)\}_{t\in[t_0,H]}$ is arbitrage free if and only if there exists a $k-$dimensional vector $\xi(t)$ such that:*

$$\Sigma(t)'\,\xi(t) = \mu(t) - r(t)\,S(t),$$

*and such that:*

$$\mathbb{E}\left[e^{\frac{1}{2}\int_{t_0}^{H}\|\xi(t)\|^2 dt}\right] < \infty.$$

**Theorem 2** *A market $\{S(t)\}_{t\in[t_0,H]}$ is complete if and only if there exists a unique $k-$dimensional vector $\xi(t)$ such that:*

$$\Sigma(t)'\,\xi(t) = \mu(t) - r(t)\,S(t),$$

*and such that:*

$$\mathbb{E}\left[e^{\frac{1}{2}\int_{t_0}^{H}\|\xi(t)\|^2 dt}\right] < \infty.$$

If on the market there are less assets than risk sources ($n < k$), then the market cannot be complete even if it is arbitrage free. In this work we assume that $n \leq k$ and that the rank of matrix $\Sigma$ is maximum (i.e. it equals $n$), and we are able to consider, in this way, both the case of a complete and an incomplete market.

## 2.1 The fund's wealth

The fund's wealth $R(t)$ at each time $t$ is given by the values of the assets held in the portfolio. Thus, if we indicate with $w(t) \in \mathbb{R}^{n\times 1}$ and $w_G(t) \in \mathbb{R}$ the number of risky assets and the number of riskless asset respectively, then we can write:

$$R(t) = w(t)' S(t) + w_G(t) G(t), \tag{2}$$

where we recall that $w(t)$ and $w_G(t)$ are stochastic processes (see Øksendal, 2000). Accordingly, for differentiating the budget constraint (2) we must use the Itô's lemma and we have:

$$\begin{aligned} dR(t) &= w(t)' dS(t) + w_G(t) dG(t) + \\ &\quad + dw(t)' dS(t) + dw(t)' S(t) + dw_G(t) G(t). \end{aligned}$$

The self-financing condition requires that the changes in the wealth level due to the changes in portfolio composition must be zero (see Björk, 1998). Thus, algebraically, the following equation must hold:

$$dw(t)' dS(t) + dw(t)' S(t) + dw_G(t) G(t) = 0.$$

Nevertheless, when the portfolio is outsourced, then the self-financing condition must be rewritten in order to take into account the amounts that must be paid to the fund's managers. In particular, these managers are suppose to charge a fee formed by a fixed part ($A \geq 0$) and a variable part ($0 \leq x \leq 1$) proportional to the increase in fund's wealth ($dR$). So, the new self-financing condition can be written as follows:

$$dw(t)' dS(t) + dw(t)' S(t) + dw_G(t) G(t) = -Adt - xdR(t),$$

since we want the portfolio return to finance the managers' remuneration. Accordingly, after substituting the value of $w_G(t)$ from Equation (2), the dynamic budget constraint can be written as:

$$dR(t) = w(t)' dS(t) + \left(R(t) - w(t)' S(t)\right) \frac{dG(t)}{G(t)} - Adt - xdR(t).$$

Now, for simplifying the computations we take into account the amounts of money invested in each asset instead of the number of assets. Thus, let:

$$\theta(t) \equiv I_S w(t),$$

and the dynamic budget constraint, after substituting for the differentials given in System (1), can be written as:

$$(1+x) dR(t) = \left(R(t) r - A + \theta(t)' (\mu - r\mathbf{1})\right) dt + \theta(t)' \Sigma' dW. \tag{3}$$

We underline that, in this framework, the managers' remuneration could also vanish (or be negative) if the value of $A$ is not big enough for compensating the negative term $xdR$ when the fund's wealth decreases ($dR < 0$). We do not take into account the case of a proportional fee which is zero if the fund's wealth decreases. In fact, in this case there would be an excessive incentive to go risky for managers since their remuneration would be affected only by the "positive" realizations of risk.

In the following section we present the derivation of the optimal portfolio.

# 3 The optimal portfolio

We suppose that the aim of the fund is to maximize its terminal wealth given the fee $(Adt + xdR(t))$ that must be paid to the fund's managers. Thus, the problem of a fund having a HARA utility function[1] and a fixed time horizon $H$, can be written as follows:

$$\begin{cases} \max_{\widetilde{\theta}} \mathbb{E}_{t_0} \left[ (\alpha + \gamma R(H))^{1-\frac{\beta}{\gamma}} \right] \\ dR(t) = \left( R(t)\widetilde{r} - \widetilde{A} + \widetilde{\theta}(t)' (\mu - r\mathbf{1}) \right) dt + \widetilde{\theta}(t)' \Sigma' dW, \\ R(t_0) = R_0, \quad R(t) \geq 0, \quad \forall t_0 \leq t \leq H, \end{cases} \tag{4}$$

where:

$$\begin{aligned} \widetilde{r} &\equiv \frac{r}{1+x}, \\ \widetilde{A} &\equiv \frac{A}{1+x}, \\ \widetilde{\theta}(t) &\equiv \frac{1}{1+x}\theta(t). \end{aligned}$$

Since we want the utility function to be increasing and concave in wealth, then the preference parameters must be such that $\gamma, \beta > 0$, $\alpha < 0$, and $\gamma - \beta > 0$. Furthermore, when there exists a non negative value of $R$ (let us say $\hat{R}$) such that the marginal utility tends to infinity when the wealth tends to $\hat{R}$, then in Problem (4) we can neglect the positivity constraint $R(t) \geq 0$, $\forall t_0 \leq t \leq H$. In the case of the HARA utility function, when the parameters verify the previous conditions such a value $\hat{R}$ does exist and it equals $-\alpha/\gamma$. Although optimal

[1] A Hyperbolic Absolute Risk Aversion index utility function has the following form:

$$(\alpha + \gamma R(t))^{1-\frac{\beta}{\gamma}},$$

whose Arrow-Pratt risk aversion index is:

$$\frac{\beta}{\alpha + \gamma R(t)}.$$

Accordingly, this kind of utility function includes the following particular cases:

1. CARA utility function: when $\gamma \to 0$ and $\alpha = 1$;
2. CRRA utility function: when $\alpha \to 0$.

rules violating the non-negativity constraints are not globally feasible, Cox and Huang [1989] show that such rules are asymptotically valid as wealth becomes large. Luckily, this is the case for the institutional investors whose portfolio we are analysing here.

We can state what follows.

**Proposition 1** *The unique portfolio composition solving Problem (4) is given by:*

$$\theta^* = (1+x)\left(\frac{\alpha - A\frac{\gamma}{r}\left(e^{\frac{r}{1+x}(H-t)} - 1\right)}{\beta e^{\frac{r}{1+x}(H-t)}} + \frac{\gamma}{\beta}R(t)\right)(\Sigma'\Sigma)^{-1}(\mu - r\mathbf{1}). \qquad (5)$$

**Proof.** See Appendix A. ■

We can immediately see that the optimal portfolio without any managers' remuneration ($x = A = 0$) is given by:

$$\theta^*|_{x=A=0} = \left(\frac{\alpha}{\beta}e^{-r(H-t)} + \frac{\gamma}{\beta}R(t)\right)(\Sigma'\Sigma)^{-1}(\mu - r\mathbf{1}). \qquad (6)$$

In the following analysis we compare Equations (5) and (6) for studying how $x$ and $A$ affect the risk-return profile of the optimal portfolio. Actually, $\theta^*$ gives the optimal composition for the risky portfolio, while the amount of money that must be invested in the riskless asset is given by $R(t) - \theta^{*\prime}\mathbf{1}$. Thus, by examining Equations (5) and (6) we can understand how the terms $x$ and $A$ modify the amount of money that must be invested in the risky assets.

In particular, we underline that the parts of Equations (5) and (6) depending on risky asset parameters (i.e. $\Sigma$ and $\mu$) are identical and so the choice between a self-managed portfolio and an outsourced portfolio can be made only by investigating: (i) the level of the riskless interest rate, (ii) the magnitude of the coefficients representing the investor's preferences (i.e. $\alpha$, $\beta$, and $\gamma$), (iii) the managers' remuneration ($x$ and $A$), and (iv) the level of the managed wealth $R$. For the investor it should be quite easy to investigate the level of all these variables, while the estimation of the market variables ($\mu$ and $\Sigma$) is a much more costly process. the avoidance of such a cost represents the gain from outsourcing the portfolio management.

With respect to the risk-return profile we can distinguish three different cases:[2]

1. $\|\theta^*\| > \|\theta^*|_{x=A=0}\|$, in this case the self-managed portfolio is less risky but also has a lower expected return;

[2] With $\|v\|$ we indicate the Euclidean norm of the vector $v$, thus:

$$\|v\| = \sqrt{v'v}.$$

2. $\|\theta^*\| = \|\theta^*|_{x=A=0}\|$, in this case the self-managed portfolio and the outsourced one have exactly the same risk and the same expected return;

3. $\|\theta^*\| < \|\theta^*|_{x=A=0}\|$, in this case the outsourced portfolio is less risky and gives a lower expected return.

In what follows we present a detailed analysis of the second case. In fact, we want to determine if there exists (and, in this case what is the value of) a couple $(x, A)$ which allows an outsourced portfolio to have exactly the same mean-variance profile as a self-managed one.

# 4 The effect of managers' remuneration

In this section we compare Equation (5) with the composition of the self-managed optimal portfolio given in Equation (6). In particular, we want to check if there exists a couple $(A, x)$ such that the optimal portfolio compositions given in Equations (5) and (6) are equal. Accordingly, we should solve the following equality:

$$(1+x)\left(\frac{\alpha - A\frac{\gamma}{r}\left(e^{\frac{r}{1+x}(H-t)}-1\right)}{e^{\frac{r}{1+x}(H-t)}} + \gamma R(t)\right) = \alpha e^{-r(H-t)} + \gamma R(t), \qquad (7)$$

where we can see that the parameter $\beta$ does not play any role. The most difficult component to compute form this Equation is the proportional managers' remuneration $x$.

During our analysis we will use the following definition.

**Definition 1** *Each couple of values $(A > 0, 0 < x < 1)$ satisfying Equation (7) is called an "iso-mean-variance" (IMV) pair.*

We can use the implicit function theorem for investigating how the variables $x$ and $A$ must behave in order to satisfy Equation (7). In particular, after defining the following function:

$$F(x, A) = (1+x)\left(\frac{\alpha - A\frac{\gamma}{r}\left(e^{\frac{r}{1+x}(H-t)}-1\right)}{e^{\frac{r}{1+x}(H-t)}} + \gamma R\right) - \alpha e^{-r(H-t)} - \gamma R,$$

we obtain the condition:

$$\frac{\partial x}{\partial A} = -\frac{\frac{\partial F}{\partial A}}{\frac{\partial F}{\partial x}} > 0 \Leftrightarrow$$
$$\left(\frac{\alpha}{\gamma}r + A\right)\left(1 + \frac{r}{1+x}(H-t)\right)e^{-\frac{r}{1+x}(H-t)} + (Rr - A) > 0,$$

where we recall that $\alpha < 0$ and that $-\alpha/\gamma$ is the lowest acceptable level of wealth ($\hat{R}$). Accordingly, two sufficient conditions (even if not necessary) for satisfying this inequality are:

$$\begin{cases} Rr - A > 0, \\ -\hat{R}r + A > 0, \end{cases}$$

which can be written as:

$$\hat{R}r < A < Rr.$$

Thus, during our work, the following hypothesis is supposed to hold.

**Hypothesis 1** *The fixed amount (A) of managers' remuneration falls within the riskless return on the wealth level giving an infinity marginal utility and the riskless return on fund's wealth:*

$$\hat{R}r < A < Rr.$$

Furthermore, after using a second time the implicit function theorem for finding the sign of the second derivative, we obtain the following condition:

$$\frac{\partial^2 x}{\partial A^2} = -\frac{\frac{\partial^2 F}{\partial A^2}\frac{\partial F}{\partial x} - \frac{\partial^2 F}{\partial x \partial A}\frac{\partial F}{\partial A}}{\left(\frac{\partial F}{\partial x}\right)^2} > 0 \Leftrightarrow \frac{\partial^2 F}{\partial x \partial A}\frac{\partial F}{\partial A} > 0 \Leftrightarrow$$
$$1 + \frac{r}{1+x}(H-t) < e^{\frac{r}{1+x}(H-t)},$$

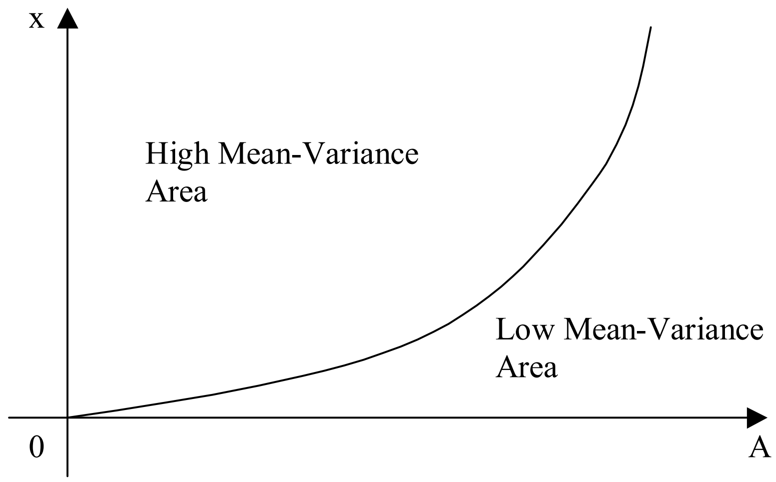
which is verified for all $H > t$, $r > 0$, and $x > 0$, since the continuous-time accumulation factor is always stronger than the simple accumulation factor. Accordingly, we can state the following proposition.

**Proposition 2** *Under Hypothesis 1 and if $r > 0$ and $H > t$, then the locus of the IMV pairs is strictly convex and has a positive slope.*

We can also underline that, given the value of $x$, if $A$ is higher than the level belonging to the IMV locus, then the outsourced portfolio has both a lower risk and a lower return with respect to the self-managed portfolio. Accordingly, the IMV locus divides the first quadrant into two areas: (i) one containing the pairs $(A, x)$ giving a high mean-variance profile, (ii) and one containing the pairs $(A, x)$ giving a low mean-variance profile. This characteristic and the result of Proposition 2 are shown in Figure 1. It is easy to check that the origin belongs to the IMV locus.

We underline that, at each instant, when the wealth level $R(t)$ and the time to horizon $(H-t)$ change, also the locus of the IMV pairs changes. In particular, the magnitude of both its slope and its convexity change even if its

Figure 1: The generic "iso-mean-variance" locus




slope always remains positive and the locus is always non-concave. Accordingly, even if the initial values of $x$ and $A$ belong to the locus, this "equilibrium" is easily unbalanced.

If we solve Equation (7) with respect to $A$ we obtain:

$$A = r\frac{\alpha\left(1+x\right)+\left(x\gamma R-\alpha e^{-r(H-t)}\right)e^{\frac{r}{1+x}(H-t)}}{\gamma\left(1+x\right)\left(e^{\frac{r}{1+x}(H-t)}-1\right)},$$

from which it is easy to see that:

$$\frac{\partial A}{\partial R}=\frac{rx}{\left(1+x\right)\left(1-e^{-\frac{r}{1+x}(H-t)}\right)}>0.$$

This result means that when the fund's wealth increases the fixed component of managers' remuneration should increase in order to remain on the IMV locus. Thus, if the original pair $(A,x)$ is not adjusted, then an increase in the wealth level makes the new pair belong to the new high risk-return area.

We can state what follows.

**Proposition 3** *When the managers' remuneration belongs to the IMV locus, an increase in the wealth level (ceteris paribus) makes the managers' remuneration belong to the high risk-return area (see Figure 1).*

It is a bit more difficult to investigate the sign of the derivative of $A$ with respect to $H-t$. If it were negative, then it could compensate the effect of $\frac{\partial A}{\partial R}$ and so the investor could remain close to the IMV locus.

The sign of this derivative is as follows:

$$\text{signum}\left\{\frac{\partial A}{\partial\left(H-t\right)}\right\}=$$
$$-\,\text{signum}\left\{R-\hat{R}\left(\frac{1+x}{x}\left(1-e^{-r(H-t)\frac{x}{1+x}}\right)+e^{-r(H-t)}\right)\right\},$$

where we recall that $\hat{R}\equiv-\alpha/\gamma$ is the lowest acceptable level of $R$. The function multiplying $\hat{R}$ is increasing in $H$ for $H>t$ and has a minimum in $H=t$. This minimum value is 1 while its maximum value $\left(1+x\right)/x$ is reached when $H$ tends to infinity. This means that the sign of the derivative we are interested in is negative if the following sufficient (even if not necessary) condition holds:

$$R\left(t\right)>\hat{R}\frac{1+x}{x},\qquad\forall t<H.$$

The strength of this hypothesis is difficult to check without knowing the values of $\alpha$, $\gamma$, and $x$. Thus, since it is quite difficult to find more general properties for the IMV pairs, in the following subsections we present some results which are obtained under some simplifying assumptions. It will be easy to check that all the particular results do respect the general properties exposed in Proposition 2.

## 4.1 A remuneration only fixed or proportional

Firstly, we want to examine the cases in which one of the two components of the managers' remuneration lacks. Let us start with the hypothesis that financial managers only charge a proportional fee ($A=0$). In this case, it is easy to show that, for positive values of $x$, the first term of Equation (7) is always greater than the second term. In fact, since:

$$\left.\left(1+x\right)e^{-\frac{r}{1+x}(H-t)}\right|_{x=0}=e^{-r(H-t)},$$

and the derivative of the left hand expression is always positive (for $H>t$):

$$\frac{\partial}{\partial x}\left(\left(1+x\right)e^{-\frac{r}{1+x}(H-t)}\right)=e^{-\frac{r}{1+x}(H-t)}\left(1+\frac{r}{1+x}\left(H-t\right)\right)>0,$$

then for positive values of $x$ the following inequality always holds:

$$\left(1+x\right)e^{-\frac{r}{1+x}(H-t)}>e^{-r(H-t)},$$

and so we can write:

$$\left(1+x\right)\left(\alpha e^{-\frac{r}{1+x}(H-t)}+\gamma R\left(t\right)\right)>\alpha e^{-r(H-t)}+\gamma R\left(t\right).$$

So, we can conclude the following proposition.

**Proposition 4** *Under market structure (1) if the fixed managers' remuneration vanishes ($A = 0$), then the outsourced portfolio is always riskier than the self-managed one and has a higher expected return.*

Now, we consider the opposite case in which the proportional part of managers' remuneration disappears. If $x = 0$, then the risk linked to the outsourced portfolio is always less than the risk linked to the self-managed portfolio. In fact, Equation (7) can be written as the following inequality:

$$\alpha e^{-r(H-t)} + \gamma R(t) - A\frac{\gamma}{r}\left(1 - e^{-r(H-t)}\right) < \alpha e^{-r(H-t)} + \gamma R(t),$$

which holds for each positive value of $A$ (we recall $\gamma > 0$).

Accordingly, we can state the following proposition.

**Proposition 5** *Under market structure (1) if the proportional managers' remuneration vanishes ($x = 0$), then the outsourced portfolio is always less risky than the self-managed one and gives a lower expected return.*

These results mean that, after the investor has found a manager having a utility function equal to his, if the manager charges only a fixed (proportional) fee, then the optimal outsourced portfolio will have a lower (higher) risk and a lower (higher) return than the self-managed one.

In the following subsection we simplify the analysis through the choice of a particular utility function.

## 4.2 The case of a CARA utility function

Now, we want to consider a particular restriction on the preference parameters. When $\gamma$ tends to zero and $\alpha = 1$, that is when we have a CARA utility function, the optimal portfolio is given by:

$$\theta^*|_{\alpha=1,\gamma\to 0} = (1+x)\frac{1}{\beta}e^{-\frac{r}{1+x}(H-t)}\left(\Sigma'\Sigma\right)^{-1}(\mu - r\mathbf{1}),$$

where the fixed remuneration $A$ does not play any role. In fact, since the CARA utility function has a constant absolute risk aversion index, when the managed wealth increases, the risk aversion is not affected. This means that if a constant amount $A$ is withdrawn from the total wealth, then the behaviour of the fund with respect to the risk does not change. In other words, the optimal portfolio composition is independent of the wealth level.

Furthermore, we can see that Equation (7) changes into the following inequality:

$$(1+x)\,e^{-\frac{r}{1+x}(H-t)} > e^{-r(H-t)},$$

which is always true for $x > 0$ and $H > t$ as we have already shown in Note **??**.

Accordingly, we can conclude the following proposition.

**Proposition 6** *Under market structure (1), if the investor maximizes a CARA utility function (in Problem (4), $\alpha = 1$ and $\gamma \to 0$), then the optimal portfolio does not depend on the fixed part of managers' remuneration and it systematically contains higher percentages of risky assets than the self-managed portfolio.*

Accordingly, if an investor has a CARA utility function and he wants the outsourced portfolio to have the same composition as the self-managed one, then he must commit the portfolio management to a manager charging only a fixed fee.

### 4.3 The riskless asset is money

Another interesting case in which the Equation (7) can be simplified arises when the riskless asset has a zero return. This is the case when the riskless asset coincides either with a money account paying no interest, or directly with money. In this case, if we take the limit of Equation (5) for $r \to 0$, we see that the optimal portfolio composition is given by:

$$\theta^*|_{r\to 0} = (1+x)\frac{1}{\beta}\left(\alpha - A\frac{\gamma}{1+x}(H-t) + \gamma R(t)\right)(\Sigma'\Sigma)^{-1}(\mu - r\mathbf{1}).$$

Accordingly, Equation (7) has the following solution:

$$x = A\frac{\gamma}{\alpha + \gamma R(t)}(H-t).$$

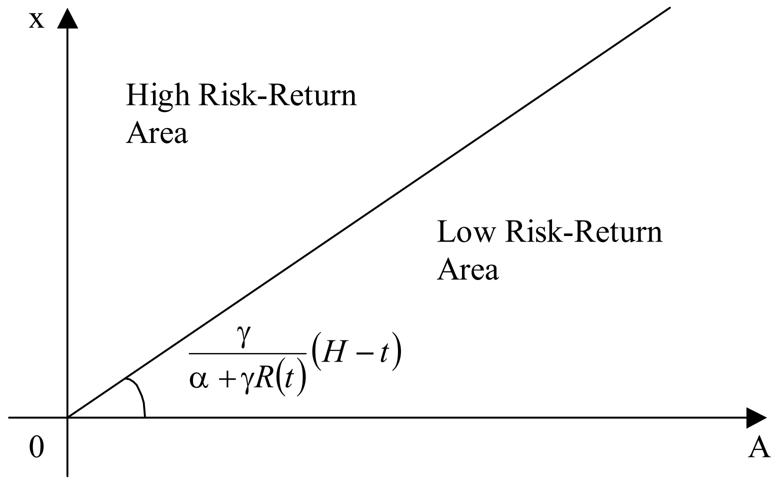
In this case the locus of the IMV pairs is not strictly convex since, according to Proposition 2, one of the conditions for the strictly convexity ($r > 0$) does not hold. Thus, we can state the following proposition.

**Proposition 7** *Under market structure (1), if the riskless interest rate is zero, then each IMV pair must satisfy:*

$$x = A\frac{\gamma}{\alpha + \gamma R(t)}(H-t). \tag{8}$$

It can be interesting to underline that the proportional component $x$ should increase when the time horizon $H$ increases. This means that, when we consider two different investors having the same preference parameters but a different time horizon, then their sets of IMV pairs are represented by different loci. In Figure 2 the straight line (8) is represented. If the proportional managers' remuneration $x$ is higher (lower) with respect to the value given by Equation (8), then the outsourced portfolio is riskier (less risky) than the self-managed one and gives a higher (lower) return.

Figure 2: "Iso-mean-variance" locus with zero riskless interest rate




### 4.4 The long run investor

One of the most interesting simplification is made when the financial horizon $H$ is quite long. In particular, an easy result can be shown when the financial horizon tends to infinity. In this case the optimal portfolio has the following composition:

$$\lim_{H\to\infty}\theta^* = (1+x)\,\frac{\gamma}{r\beta}\,(R\,(t)\,r - A)\,(\Sigma'\Sigma)^{-1}\,(\mu - r\mathbf{1})\,,$$

from which we can see that the signs of the portfolio composition are preserved with respect to the self-managed case if the fixed amount of the managers' remuneration is not greater than the return which could be obtained if the whole wealth were invested in the riskless asset.

Actually, it seems quite unlikely that a fund is willing to accept to remunerate a manager with a fixed part greater than the riskless return on the fund's wealth. In fact, this is not the case under Hypothesis 1.

If we consider again the Equality (7) but when the horizon $H$ tends to infinity, then we have:

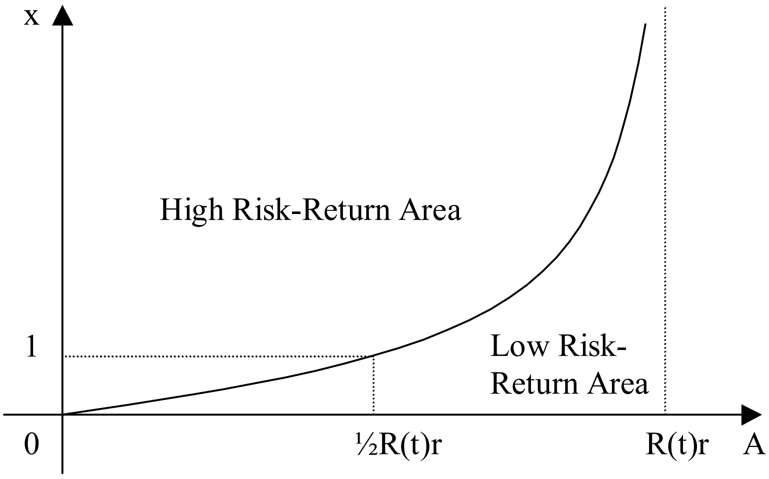
$$(1+x)\,(R\,(t)\,r - A) = R\,(t)\,r,$$

form which we can immediately state what follows.

**Proposition 8** *Under market structure (1), if the time horizon tends to infinity, then each IMV pair must satisfy:*

$$x = \frac{A}{R\,(t)\,r - A}. \tag{9}$$

Figure 3: "Iso-mean-variance" locus for a long run investor




We underline that Equation (9) does not depend at all on the preference parameters. Its behaviour is represented in Figure 3 where the locus of all the IMV pairs satisfying Equation (9) is drawn.

We underline that this locus is convex and has a positive slope as stated in Proposition 2. Since we want $x$ to be a value belonging to $[0,1]$ then, given Hypothesis 1 guaranteeing that in (9) $x>0$, we must also have:

$$A < \frac{1}{2} R(t)\, r.$$

If the proportional remuneration $x$ is higher than the IMV locus, then the outsourced portfolio is riskier than the self-managed one but also have a higher return. Instead, if $x$ lays under the IMV locus, then the risk linked with the self-managed portfolio is lower than the risk implied by the outsourced portfolio but also the return is lower. These areas of high $(HMV)$ and low $(LMV)$ mean-variance profile can be easily computed:

$$\begin{aligned} LMV &= \int_0^{\frac{1}{2}Rr} \frac{A}{Rr-A} dA = Rr\left(\ln 2 - \frac{1}{2}\right), \\ HMV &= \frac{1}{2} Rr - HR = Rr\,(1-\ln 2)\,, \end{aligned}$$

from which we can see that the ratio between the high mean-variance area and the total area depends neither on the level of wealth nor on the riskless interest rate. In particular, we can immediately obtain the following results:

$$\begin{aligned}\frac{LMV}{LMV+HMV} &= \ln 4 - 1 \simeq 40\%, \\ \frac{HMV}{LMV+HMV} &= 2 - \ln 4 \simeq 60\%.\end{aligned}$$

This means that in a market where the managers' remunerations $(A, x)$ are uniformly distributed between both the high and low mean-variance areas, then the probability of finding a remuneration leading to a high risk for investor's portfolio is quite high. Thus, in this case the diversification cannot help. In other words, the investor cannot decide to give his wealth to invest to a lot of managers hoping that, without analysing the remunerations they ask for, on the mean, he will be able to obtain a risk-return mix equal to the profile implied by the self-managed portfolio.

# 5 Conclusion

In this paper we have considered the case of an investor facing the alternative of managing his financial wealth by himself or giving it in outsourcing. In the latter case, the manager's fee is supposed to consist of two parts, one fixed $(A)$, and the other one $(x)$ proportionally computed on the increase in the fund's wealth. The managers' remuneration is deducted from portfolio payoffs.

The paper concentrates on the problem of finding a suitable pair $(A, x)$ such that the outsourced portfolio has the same composition (i.e. the same risk-return profile) than the self-managed one. We show that such a pair always exists and we have called it an "iso-mean-variance" (IMV) pair. We also demonstrate that the locus of all the IMV pairs has a positive slope and is non-concave. This locus divides the first quadrant of the plane $(A, x)$ into two areas. The area above (below) the locus contains all the $(A, x)$ pairs leading to an outsourced portfolio having both a higher (lower) risk and a higher (lower) return with respect to the self-managed one. In fact, the proportional part $(x)$ is positively correlated with both the risk and the return of the optimal portfolio, while the fixed part $(A)$ is inversely correlated with them.

We show that each IMV pair does not depend on the market variables given by the mean and the variance of asset returns. Accordingly, in order to investigate the mean-return profile implied by a proposed managers' remuneration, the investor has not to pay any cost for knowing these variables. The avoidance of this cost justify the outsourcing of portfolio management. Actually, for the investor, it is sufficient to know: (i) the level of his wealth, (ii) his utility function parameters, and (iii) the riskless interest rate, in order to understand if a given remuneration composition $(A, x)$ will lead to a portfolio management implying a higher or a lower risk-return profile with respect to the self-managed one.

# A The optimal portfolio

Given Problem (4) we can derive the following Hamiltonian:[3]

$$\mathcal{H} = J_R \left( R(t)\,\widetilde{r} - \widetilde{A} + \widetilde{\theta}(t)'(\mu - r\mathbf{1}) \right) + \frac{1}{2} J_{RR} \widetilde{\theta}(t)' \Sigma' \Sigma \widetilde{\theta}(t),$$

where $J(R,t)$ is the value function solving the Hamilton-Jacobi-Bellman partial differential equation, verifying:

$$J(R,t) = \sup_{\widetilde{\theta}} \mathbb{E}_t \left[ (\alpha + \gamma R(H))^{1-\frac{\beta}{\gamma}} \right],$$

and the subscripts on $J$ indicate the partial derivatives.

On the Hamiltonian we have the first order conditions:[4]

$$\frac{\partial \mathcal{H}}{\partial \widetilde{\theta}} = J_R(\mu - r\mathbf{1}) + J_{RR}\Sigma'\Sigma\widetilde{\theta}(t) = 0,$$

from which:

$$\widetilde{\theta}^* = -\frac{J_R}{J_{RR}} (\Sigma'\Sigma)^{-1} (\mu - r\mathbf{1}).$$

The Hamilton-Jacobi-Bellman equation has the following form:

$$\begin{cases} J_t + \mathcal{H}^* = 0, \\ J(R,H) = (\alpha + \gamma R(H))^{1-\frac{\beta}{\gamma}}, \end{cases}$$

which can be written as:

$$\begin{cases} J_t + J_R \left( R(t)\,\widetilde{r} - \widetilde{A} \right) - \frac{1}{2}\frac{J_R^2}{J_{RR}}\lambda = 0, \\ J(R,H) = (\alpha + \gamma R(H))^{1-\frac{\beta}{\gamma}}, \end{cases}$$

where $\lambda \equiv (\mu - r\mathbf{1})'(\Sigma'\Sigma)^{-1}(\mu - r\mathbf{1})$.

Since we can suppose that the value function inherits its functional form from the utility function, then we can try the following general solution:

$$J(R,t) = f(t)\,(a(t) + b(t)\,R)^{1-\frac{\beta}{\gamma}},$$

and, accordingly, the boundary conditions become:

$$\begin{cases} f(H) = 1, \\ a(H) = \alpha, \\ b(H) = \gamma. \end{cases}$$

[3] A complete derivation of the Hamilton-Jacobi-Bellman equation can be found in Björk [1998] and Øksendal [2000].

[4] The second order conditions hold if the Hessian matrix of $\mathcal{H}$:

$$\frac{\partial \mathcal{H}}{\partial \widetilde{\theta}' \partial \widetilde{\theta}} = J_{RR}\Sigma'\Sigma,$$

is negative definite. Because $\Sigma'\Sigma$ is a quadratic form it is always positive definite and so the second order conditions are satisfied if and only if $J_{RR} < 0$, that is if the value function is concave in $R$.

From the form of the value function we have:

$$\begin{aligned}
\frac{\partial}{\partial t}J(R,t) &= \frac{\partial f(t)}{\partial t}(a(t)+b(t)R)^{1-\frac{\beta}{\gamma}}+ \\
&\quad + f(t)(a(t)+b(t)R)^{-\frac{\beta}{\gamma}}\left(1-\frac{\beta}{\gamma}\right)\left(\frac{\partial a(t)}{\partial t}+\frac{\partial b(t)}{\partial t}R\right), \\
\frac{\partial}{\partial R}J(R,t) &= f(t)(a(t)+b(t)R)^{-\frac{\beta}{\gamma}}\frac{\gamma-\beta}{\gamma}b(t), \\
\frac{\partial^2}{\partial R^2}J(R,t) &= f(t)(a(t)+b(t)R)^{-\frac{\beta}{\gamma}-1}\frac{\gamma-\beta}{\gamma}\left(-\frac{\beta}{\gamma}\right)b(t)^2, \\
\frac{\left(\frac{\partial}{\partial R}J(R,t)\right)^2}{\frac{\partial^2}{\partial R^2}J(R,t)} &= -f(t)(a(t)+b(t)R)^{\frac{\gamma-\beta}{\gamma}}\frac{\gamma-\beta}{\beta},
\end{aligned}$$

and, after substituting in the HJB equation:

$$\begin{aligned}
0 &= \frac{\partial f(t)}{\partial t}(a(t)+b(t)R)^{1-\frac{\beta}{\gamma}}+ \\
&\quad + f(t)(a(t)+b(t)R)^{-\frac{\beta}{\gamma}}\left(1-\frac{\beta}{\gamma}\right)\left(\frac{\partial a(t)}{\partial t}+\frac{\partial b(t)}{\partial t}R\right)+ \\
&\quad + f(t)(a(t)+b(t)R)^{-\frac{\beta}{\gamma}}\frac{\gamma-\beta}{\gamma}b(t)\left(R(t)\widetilde{r}-\widetilde{A}\right)+ \\
&\quad + \frac{1}{2}f(t)(a(t)+b(t)R)^{\frac{\gamma-\beta}{\gamma}}\frac{\gamma-\beta}{\beta}\lambda.
\end{aligned}$$

If we consider the similar terms, we have the system:

$$\begin{cases}
(a(t)+b(t)R)^{1-\frac{\beta}{\gamma}}\left(\frac{\partial f(t)}{\partial t}+\frac{1}{2}f(t)\frac{\gamma-\beta}{\beta}\lambda\right)=0, \\
f(t)(a(t)+b(t)R)^{-\frac{\beta}{\gamma}}\left(1-\frac{\beta}{\gamma}\right)\left(\frac{\partial a(t)}{\partial t}-b(t)\widetilde{A}\right)=0, \\
f(t)(a(t)+b(t)R)^{-\frac{\beta}{\gamma}}\left(1-\frac{\beta}{\gamma}\right)R\left(\frac{\partial b(t)}{\partial t}+b(t)\widetilde{r}\right)=0,
\end{cases}$$

and, accordingly, we have to solve the three following differential equations:

$$\begin{cases}
\frac{\partial f(t)}{\partial t}+\frac{1}{2}f(t)\frac{\gamma-\beta}{\beta}\lambda=0, \\
f(H)=1,
\end{cases}$$

$$\begin{cases}
\frac{\partial b(t)}{\partial t}+b(t)\widetilde{r}=0, \\
b(H)=\gamma,
\end{cases}$$

$$\begin{cases}
\frac{\partial a(t)}{\partial t}-b(t)\widetilde{A}=0, \\
a(H)=\alpha,
\end{cases}$$

whose solutions are:

$$\begin{aligned}
f(t) &= e^{\frac{1}{2}\lambda\frac{\gamma-\beta}{\beta}(H-t)}, \\
b(t) &= \gamma e^{\widetilde{r}(H-t)}, \\
a(t) &= \alpha-\frac{\gamma}{\widetilde{r}}\widetilde{A}\left(e^{\widetilde{r}(H-t)}-1\right),
\end{aligned}$$

and, accordingly, the optimal portfolio:

$$\widetilde{\theta}^* = -\frac{J_R}{J_{RR}} \left(\Sigma'\Sigma\right)^{-1} (\mu - r\mathbf{1}),$$

can be written as:

$$\widetilde{\theta}^* = \left(\frac{a(t)}{b(t)} + R\right) \frac{\gamma}{\beta} \left(\Sigma'\Sigma\right)^{-1} (\mu - r\mathbf{1}),$$

or:

$$\widetilde{\theta}^* = \left(\frac{\alpha - \frac{\gamma}{\widetilde{r}}\widetilde{A}\left(e^{\widetilde{r}(H-t)} - 1\right)}{\gamma e^{\widetilde{r}(H-t)}} + R\right) \frac{\gamma}{\beta} \left(\Sigma'\Sigma\right)^{-1} (\mu - r\mathbf{1}),$$

and, since:

$$\frac{\widetilde{A}}{\widetilde{r}} = \frac{A/(1+x)}{r/(1+x)} = \frac{A}{r},$$

we can finally write:

$$\theta^* = (1+x) \left(\frac{\alpha - \frac{\gamma}{r}A\left(e^{\widetilde{r}(H-t)} - 1\right)}{\beta e^{\widetilde{r}(H-t)}} + \frac{\gamma}{\beta}R\right) \left(\Sigma'\Sigma\right)^{-1} (\mu - r\mathbf{1}).$$

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