Inequality and Growth: Why Differential Fertility Matters*

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April 26, 2001

Abstract

We argue that inequality and growth are linked through differential fertility and the accumulation of human capital. We build an overlapping-generations model in which dynasties differ in their initial endowment with human capital. Growth, the income distribution, and fertility are endogenous. Due to a quantity-quality tradeoff, families with less human capital decide to have more children and invest less in education. When initial inequality is high, large fertility differentials lower the growth rate of average human capital, since poor families who invest little in education make up a large fraction of the population in the next generation. A calibrated model shows that this fertility-differential effect is quantitatively important. We also provide empirical evidence to confirm the links between inequality, differential fertility and growth suggested by the model.

*We thank Costas Azariadis, Harold Cole, Roger Farmer, Gary Hansen, Lee Ohanian and Martin Schneider for helpful comments.

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1 Introduction

How does the income distribution of a country affect its rate of economic growth? We argue that to answer this question it is essential to account for the fertility differential between the rich and the poor. The fertility differential matters because it affects the accumulation of human capital. Assuming that we identify human capital with education, future human capital is a weighted average of the education of today's children from families in different income groups, with the weights given by income-specific fertility rates. Children from poorer families tend to receive less education. If the fertility differential between the rich and the poor is large, more weight is put on groups with less education. Therefore future human capital will be lower than it would have been otherwise. If the fertility differential in turn increases with inequality, we would expect that countries with higher inequality accumulate less human capital, and therefore grow slower.

We develop a growth model which captures this channel from inequality to growth. Our framework is related to Glomm and Ravikumar (1992), who analyze the effects of public versus private education on growth in a model with fixed fertility. We choose a similar overlapping-generations framework, but model endogenous fertility decisions along the lines of Becker and Barro (1988). Both fertility and education are thus chosen endogenously. Parents face a quality-quantity tradeoff in their decision on children, and we show that education increases with the income of a family, while fertility decreases with income. The aggregate behavior of the model depends on the initial distribution of income. Other things being equal, we find that economies with a less equitable income distribution have higher fertility differentials, accumulate less human capital, and have a lower rate of economic growth. In contrast, if we impose fertility to be constant across income groups, the effects of inequality on human capital and growth are small.

We also find empirical support for the relationships between inequality, differential fertility, and growth postulated by the model. Kremer and Chen (2000) examine the relationship between inequality and differential fertility. Using cross-country data, they find that more inequality tends to be associated with larger fertility differentials within a country. This supports the first part of our hypothesis, linking inequality to differential fertility. To examine the second part of our hypothesis, the link from differential fertility to growth, we add a differential-fertility variable to a standard growth regression, and find large significant effects of differential fertility on growth. In the same regressions, the direct effects of inequality as measured by Gini coefficients is insignificant.

In essence, we are proposing a new mechanism that links inequality and growth through fertility and the accumulation of human capital. There is a large existing literature on inequality and growth. The majority of this literature concentrates on channels in which inequality affects growth through the accumulation of physical capital (see Benabou 1996). Althaus (1980) is the only model that we are aware of that works out the effects of differential fertility on growth. However, in Althaus' model fertility is exogenously given, and the role of human capital is not considered. In Galor and Zang (1997), inequality affects growth through its effect on overall fertility and human capital. Borrowing constraints play a crucial role in their analysis, while differential fertility is not considered. Morand (1999) has a model of inequality and fertility in which the sole motive for fertility is old-age support. He concentrates on the possibility of poverty traps when the initial level of human capital is too low. Another related model is Dahan and Tsiddon (1998). They examine a model with two skill levels, endogenous fertility, and capital market imperfections. Since the model does not allow for long-run growth, the analysis concentrates on the transition to the steady state. On the empirical side, Barro (2000) studies of the link between growth and inequality, based on a newly available data set on inequality across countries and over time. His study does not consider the role of differential fertility, however. Perotti (1996) finds that demographic variables are important for understanding the growth effects of the income distribution, but once again differential fertility is not considered directly.

In the following section, we will introduce the model. Section 3 presents theoretical results, and in Section 4 we calibrate and simulate the model. Empirical evidence is discussed in Section 5, and Section 6 concludes.

2 The Model Economy

Time is discrete and goes from 0 to ∞. The economy is populated by overlapping generations of people who live for three periods, childhood, adulthood, and old age. All decisions are made in the adult period of life. People care about adult consumption c_t, old-age consumption d_{t+1}, their number of children n_t, and the human capital of children h_{t+1}. The utility function is given by:

$$\ln(c_t) + \beta \ln(d_{t+1}) + \gamma \ln(n_t h_{t+1}).$$

The parameter $\beta > 0$ is the psychological discount factor and $\gamma > 0$ is the altruism factor. Raising one child takes fraction $\phi \in (0, 1)$ of an adult's time. An adult has to choose a consumption profile c_t and d_{t+1}, savings for old age s_t, number of children n_t, and schooling time per child e_t. The budget constraint for an adult with human capital h_t is:

$$c_t + s_t + e_t n_t w_t \bar{h}_t = w_t h_t (1 - \phi n_t), \tag{1}$$

where w_t is the wage per unit of human capital. We assume that the average human capital of teachers equals the average human capital in the population $\bar{h}_t$, so that education cost per child is given by $e_t w_t \bar{h}_t$. The assumption that teachers instead of parents provide education is crucial for generating fertility differentials. It implies that the cost of education does not depend on the parent's wage. In contrast, since raising each child takes a fixed amount of the parent's time, having many children is more costly for parents who have high wages. Parents with high human capital and high wages therefore substitute child quality for child quantity, and decide to have less children with more education.

The only friction in the model is that children cannot borrow to finance their own education. Instead, education has to be paid for by the parents. This assumption is made in most studies of the joint determination of fertility and education. In the real world, children generally do not finance their own education (at least up to the secondary level), and we do observe a fair amount of intergenerational persistence in education levels.

The budget constraint for the old-age period is:

$$d_{t+1} = R_{t+1} s_t. \tag{2}$$

R_{t+1} is the interest factor. The human capital of the children h_{t+1} depends on human capital of the parents h_t, average human capital $\bar{h}_t$, and education e_t:

$$h_{t+1} = B(\theta + e_t)^{\eta} (h_t)^{\tau} (\bar{h}_t)^{1-\tau}. \tag{3}$$

The parameters satisfy $B, \theta > 0$ and $\eta, \tau \in (0, 1)$. The presence of θ guarantees that human capital remains positive even if parents do not invest in education.

Production of the consumption good is carried out by a single representative firm which operates the technology:

$$Y_t = A K_t^{\alpha} L_t^{1-\alpha},$$

where K_t is aggregate capital, L_t is aggregate labor supply, $A > 0$ and $\alpha \in (0, 1)$. Physical capital totally depreciates in one period. The firm chooses inputs by maximizing profits $Y_t - w_t L_t - R_t K_t$.

Human capital is distributed over the adult population according to the distribution function $F_t(h_t)$. Total population P_t evolves over time according to:

$$P_{t+1} = P_t \int_0^{\infty} n_t \; \mathrm{d}F_t(h_t), \tag{4}$$

and the law of motion for the distribution of human capital is:

$$F_{t+1}(\hat{h}) = \frac{P_t}{P_{t+1}} \int_0^{\infty} n_t \, I(h_{t+1} \leq \hat{h}) \; \mathrm{d}F_t(h_t). \tag{5}$$

Here $I(\cdot)$ is an indicator function, and it is understood that the choice variables n_t and h_{t+1} are functions of the individual state h_t. Average human capital $\bar{h}_t$ is given by:

$$\bar{h}_t = \int_0^\infty h_t \ \mathrm{d}F_t(h_t). \tag{6}$$

The market-clearing conditions for capital and labor are:

$$K_{t+1} = P_t \int_0^\infty s_t \ \mathrm{d}F_t(h_t), \tag{7}$$

and:

$$L_t = P_t \left[\int_0^\infty h_t(1 - \phi n_t) \ \mathrm{d}F_t(h_t) - \int_0^\infty e_t n_t \bar{h}_t \ \mathrm{d}F_t(h_t) \right]. \tag{8}$$

This last condition reflects the fact that the time devoted to teaching is not available for goods production.

Definition 1 *Given an initial distribution of human capital $F_0(h_0)$, an initial stock of physical capital K_0, and an initial population size P_0, an equilibrium consists of sequences of prices $\{w_t,\ R_t\}$, aggregate quantities $\{L_t,\ K_{t+1},\ \bar{h}_t,\ P_{t+1}\}$, distributions $F_{t+1}(h_{t+1})$, and decision rules $\{c_t,\ d_{t+1},\ s_t,\ n_t,\ e_t,\ h_{t+1}\}$ such that:*

1. *the households' decision rules c_t d_{t+1}, s_t, n_t, e_t, h_{t+1} maximize utility subject to the constraints* (1), (2), *and* (3);
2. *the firm's choices L_t and K_t maximize profits;*
3. *the prices w_t and R_t are such that markets clear, i.e., (7) and (8) hold;*
4. *the distribution of human capital evolves according to (5);*
5. *aggregate variables $\bar{h}_t$ and P_t are given by (4) and (6).*

3 Theoretical Results

We first characterize the quality-quantity tradeoff faced by individuals. We next find sufficient conditions under which a balanced growth path exists. Finally we study the dynamics of individual human capital as a function of the parameters.

3.1 The Tradeoff between the Quality and Quantity of Children

We denote the relative human capital of a household as:

$$x_t \equiv \frac{h_t}{\bar{h}_t}.$$

For a household that has enough human capital such that the condition $x_t > \frac{\theta}{\phi\eta}$ holds, there is an interior solution for the optimal education level, and the first-order conditions imply:

$$s_t = \frac{\beta}{1+\beta+\gamma} w_t h_t, \tag{9}$$

$$e_t = \frac{\eta\phi x_t - \theta}{1-\eta}, \tag{10}$$

$$n_t = \frac{(1-\eta)\gamma x_t}{(\phi x_t - \theta)(1+\beta+\gamma)}. \tag{11}$$

The second-order conditions for a maximum are satisfied. Note that

$$\frac{\partial e_t}{\partial x_t} > 0 \text{ and } \frac{\partial n_t}{\partial x_t} < 0,$$

which reflects the well-documented fact that skilled people invest relatively more in the quality of their children than in their quantity. The lowest possible fertility rate is given by:

$$\lim_{x_t \to \infty} n_t = \frac{\gamma(1-\eta)}{\phi(1+\beta+\gamma)}.$$

For a household endowed with a human capital such that $x_t < \frac{\theta}{\phi\eta}$ holds, the optimal choice for education e_t is zero. The first-order conditions imply equation (9) and:

$$e_t = 0, \tag{12}$$

$$n_t = \frac{\gamma}{\phi(1+\beta+\gamma)}. \tag{13}$$

Once the choice for education is zero, fertility no longer increases as the human capital endowment falls.

Fertility as a function of human capital is plotted in Figure 1. The horizontal part of the relationship corresponds to the range of human capital which leads to a choice of zero for education e_t. Fertility depends negatively on human capital, and moves

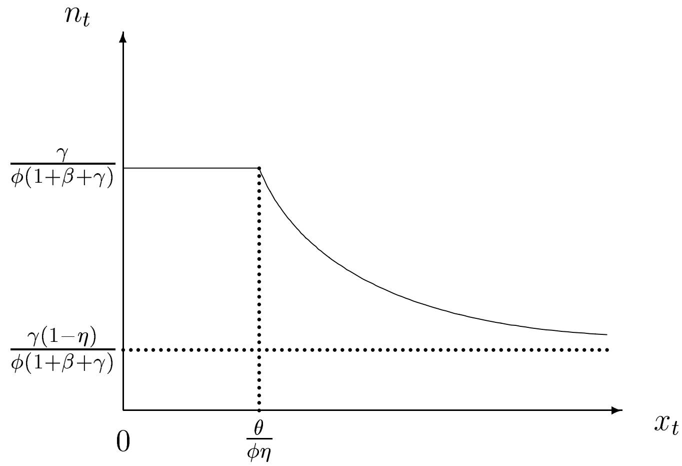

Figure 1: Fertility as a function of human capital

within a finite interval. The upper bound on the fertility differential is given by:

$$\frac{\lim_{x_t \to 0} n_t}{\lim_{x_t \to \infty} n_t} = \frac{1}{1-\eta}.$$

This relationship will turn out to be helpful to interpret the role of the parameter η and to calibrate its value.

The results derived so far reflect the main effects of inequality on growth that we are interested in. Assuming that all dynasties choose positive levels of education, equation (10) shows that education is a linear function of relative human capital. If the dispersion of human capital increases for a given average level of human capital, this linearity implies that the average education choice will still be the same. However, since the production function for human capital is concave in education, future average human capital will be lower if the distribution of human capital is less equal. This would be true even if fertility were constant across families with different human capital levels. The fact that fertility is actually higher for people with low human capital greatly amplifies the negative effect of inequality on human capital accumulation.

3.2 The Balanced Growth Path

To study the long-run behavior of the economy, we rewrite the equilibrium conditions in terms of stationary variables. The capital/labor ratio k_t, the growth rate of average human capital g_t, and the population growth rate N_t are defined by:

$$k_t \equiv \frac{K_t}{L_t}, \qquad g_t \equiv \frac{\bar{h}_{t+1}}{\bar{h}_t}, \qquad N_t \equiv \frac{P_{t+1}}{P_t}.$$

We also need to define the distribution of the relative human capital levels:

$$G_t(x_t) \equiv F_t(x_t \bar{h}_t).$$

Rewriting equations (4), (5) and (6) in terms of the stationary variables leads to:

$$N_t = \int_0^\infty n_t \, \mathrm{d}G_t(x_t), \tag{14}$$

$$G_{t+1}(\hat{x}) = \frac{1}{N_t} \int_0^\infty n_t \, I(x_{t+1} \leq \hat{x}) \;\; \mathrm{d}G_t(x_t), \tag{15}$$

$$1 = \int_0^\infty x_t \;\; \mathrm{d}G_t(x_t). \tag{16}$$

Prices follow from the competitive behavior of firms, which leads to the equalization between marginal costs and productivities:

$$w_t = A(1-\alpha)k_t^\alpha, \tag{17}$$

$$R_t = A\alpha k_t^{\alpha-1}.$$

Schooling and fertility decisions are given by (12) and (13) for $x_t < \theta/(\eta\phi)$ and by (10) and (11) otherwise. The number of children for an adult with relative human capital x_t is thus given by:

$$n_t = \min\left[\frac{(1-\eta)\gamma x_t}{(\phi x_t - \theta)(1+\beta+\gamma)}, \frac{\gamma}{\phi(1+\beta+\gamma)}\right]. \tag{18}$$

From equation (3), the children's human capital is given by:

$$x_{t+1} = \frac{B x_t^\tau}{g_t} \left(\theta + \max\left[0, \frac{\eta\phi x_t - \theta}{1-\eta}\right]\right)^\eta. \tag{19}$$

From equation (8), labor input satisfies:

$$\frac{L_t}{P_t \bar{h}_t} = \int_0^{\frac{\theta}{\eta\phi}} \frac{(1+\beta)x_t}{1+\beta+\gamma} \, \mathrm{d}G_t(x_t) + \int_{\frac{\theta}{\eta\phi}}^\infty \left(1 - \gamma \frac{\phi(1-\eta) + (\eta\phi x_t - \theta)}{(\phi x_t - \theta)(1+\beta+\gamma)}\right) x_t \, \mathrm{d}G_t(x_t).$$

which leads to:

$$\frac{L_t}{\bar{h}_t P_t} = \frac{1+\beta}{1+\beta+\gamma}. \tag{20}$$

Using (7), (9), (17) and (20), the capital stock evolves according to the following law of motion:

$$k_{t+1} = \frac{\beta}{1+\beta}\,\frac{1}{g_t N_t}\,A(1-\alpha)k_t^{\alpha}. \tag{21}$$

Given initial conditions k_0 and $G_0(x_0)$, an equilibrium can be characterized by sequences $\{g_t,\ n_t,\ G_{t+1}(x),\ N_t,\ x_t,\ k_{t+1}\}$ such that (14), (15), (16), (18), (19), and (21) hold at all dates.

This dynamic system is block recursive. Given the initial conditions, we can first use (18) to solve for n_t. Then equations (16) and (19) determine x_{t+1} and g_t. The new distribution of relative human capital is given by equation (15). The aggregate population growth rate N_t is obtained from (14), and the future capital-labor ratio k_{t+1} from (21). Given any initial conditions, an equilibrium exists and is unique. Moreover, from these equations we can deduce that there is a balanced growth path in which everyone has the same human capital:

Proposition 1 *If $\eta\phi > \theta$ there is a balanced growth path characterized by $dG(1) = 1$, i.e. the limiting distribution is degenerate, and the growth rate of output and human capital is:*

$$g^{\star} = B\left(\frac{\eta(\phi-\theta)}{1-\eta}\right)^{\eta} > 0.$$

Proof: Setting $g_t = g^{\star}$ and $x_{t+1} = x_t = 1$, the constant values:

$$N_t = n_t = \frac{(1-\eta)\gamma}{(\phi-\theta)(1+\beta+\gamma)},$$

and:

$$k_{t+1} = k_t = \left(\frac{A\beta(1+\beta+\gamma)(1-\alpha)(\phi-\eta)^{1-\eta}}{B\gamma(1+\beta)\eta^{\eta}(1-\eta)^{1-\eta}}\right)^{\frac{1}{1-\alpha}},$$

solve equations (14), (15), (16), (18), (19) and (21). Q.E.D.

Along this balance growth path, there is no longer any inequality among households. This holds because we have assumed that the only difference across households lies in their initial level of human capital. If we had introduced ability shocks on top of an unequal initial distribution of human capital, inequality would persist along the balanced growth path. However, this inequality would simply reflect the randomness of abilities, which is not an important factor for our purposes.

We will assume $\eta\phi > \theta$ from here on. To analyze the stability of the balanced growth path and the convergence of the distribution of human capital, numerical simulations are required. Before doing that, we consider the dynamics of the human capital of an

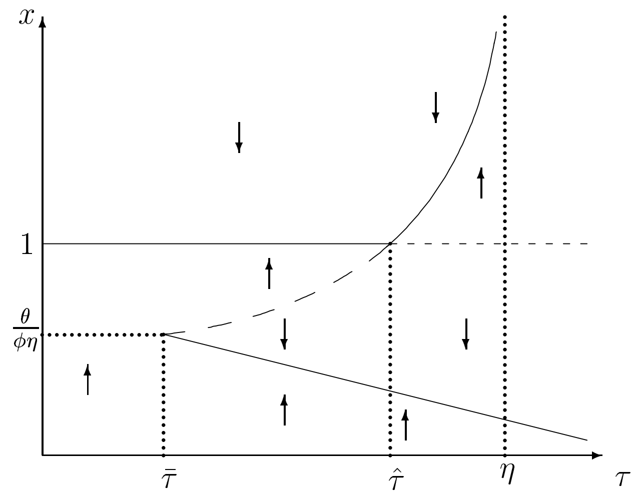

Figure 2: Steady states as a function of τ

individual dynasty (of mass zero) around an aggregate balanced growth path. This will be useful to understand the role of the parameter τ and interpret the numerical results of Section 4.3.

3.3 The Dynamics of Individual Human Capital

We assume that the economy is on a balanced growth path, so that the growth rate of average human capital is constant over time: $g_t = g^\star$. We focus on the effect of the parameter τ on the dynamics of individual human capital. We consider the function $x_{t+1} - x_t = \Psi(x_t; \tau)$ given by:

$$\Psi(x;\tau) = \frac{Bx^\tau}{g^\star}\left(\theta + \max\left[0, \frac{\eta\phi x - \theta}{1-\eta}\right]\right)^\eta - x.$$

A detailed study of this function is performed in the appendix.

A complete characterization of the dynamics of x as a function of the parameter τ is presented in the bifurcation diagram in Figure 2. The steady states x are represented on the vertical axis as a function of τ. For small τ there is only one steady state, $x = 1$, which is globally stable. Once τ reaches a threshold $\bar{\tau}$ (given in the appendix) two additional steady states appear. The lower one is stable and the second is unstable. This threshold arises at the point where the cutoff value for an interior solution is a steady state of the individual dynamics (see Figure 6).

Proposition 2 *At the point:*

$$\hat{\tau} = 1 - \frac{\eta\phi}{\phi - \theta}$$

the dynamics of individual capital described by $x_{t+1} - x_t = \Psi(x_t; \tau)$ undergo a transcritical bifurcation. There are two steady-state equilibria, 1 and $\bar{x}$, near $(1, \hat{\tau})$ for each value of τ smaller or larger than $\hat{\tau}$. The equilibrium 1 (resp. $\bar{x}$) is stable (resp. unstable) for $\tau < \hat{\tau}$ and unstable (resp. stable) for $\tau > \hat{\tau}$.

Proof: We check the five conditions that define such a bifurcation in Wiggins (1990), p. 365:

$$\Psi(1, \hat{\tau}) = 0, \quad \Psi'_x(1, \hat{\tau}) = 0, \quad \Psi'_\tau(1, \hat{\tau}) = 0,$$

$$\Psi''_{xx}(1, \hat{\tau}) = -\frac{\eta\theta\phi}{(\phi - \theta)^2} \neq 0, \quad \Psi''_{x\tau}(1, \hat{\tau}) = 1 \neq 0.$$

Q.E.D.

This bifurcation occurs when an unstable and a stable fixed point collide and exchange stability. That is, the unstable fixed point becomes stable and vice versa.[1] When τ increases beyond $\hat{\tau}$, the high steady state increases and then vanishes once $\tau > \eta$.

This analytical study at given aggregate conditions is helpful to understand the numerical simulations carried in the next section. When the growth rate is not at a steady state, the phaseline of Figure 6 shifts depending on g_t, and the dynamics can be investigated by means of computational experiments.

4 Computational Experiments

The theoretical results in the previous section highlight two channels through which inequality affects growth in this model. First, inequality in human capital leads to inequality in education, and since the production function for human capital is concave, inequality in education lowers future average human capital. Second, people with lower human capital not only choose less education for their children, but also a higher number of children. This differential fertility effect increases the weight in the population on families with little education, which also lowers future human capital. The question arises which effect is more important, and how large the effects are quantitatively. To answer this question, we calibrate our model and provide numerical simulations of the evolution of fertility, inequality, human capital, and income. The main findings are that the effects of inequality on human capital accumulation

[1] Note that beyond the bifurcation point the number of fixed points does not change, whereas in a saddle-node bifurcation two fixed points either appear or disappear.

and growth are sizable, and that the differential fertility effect is crucial for generating this result.

4.1 Calibration

We choose the parameters of the model such that the balanced growth path resembles empirical features of the U.S. economy and population. The model is calibrated under the assumption that one period (or generation) has a length of thirty years. The parameter α is the capital share in the final goods sector, and is set to $1/3$ to match the empirical counterpart. The discount factor β mainly affects the ratio of human capital to physical capital in the balanced growth path. Since this ratio depends on the choice of units, it does not provide a convenient basis for calibrating β. Given that β does not influence qualitative features of the model that we are interested in, we choose a value that is standard in the real-business-cycle literature, $\beta = 0.99^{120}$ (i.e., 0.99 per quarter). The overall productivity B in the production function for human capital governs the growth rate of output per capita, and is set to 7.367. This value implies a growth rate of output per capita of 1.02^{30} or 2% per year, which approximates the average growth rate in the U.S. The weight γ of children in the utility function governs the growth rate of population in the balanced growth path. In the U.S. as in other industrialized countries, fertility rates are close to the reproduction level. Accordingly, we choose γ such that the growth rate of population in the balanced growth path is zero. This is achieved by choosing $\gamma = 0.271$.[2]

The remaining parameters concern the cost of children and the parameters of the production function for human capital. These parameters are harder to calibrate, since they do not correspond to easily observable features of the balanced growth path. The elasticity η of future human capital with respect to education governs the maximum fertility differential in the economy. Specifically, the maximum differential written as a ratio is given by $1/(1-\eta)$. In the data, the highest fertility differential between women at the highest and the lowest education levels in any country is 2.74 (Brazil). This is achieved by choosing $\eta = 0.635$.

The time-cost parameter ϕ for having a child determines the overall opportunity cost of children. Evidence in Haveman and Wolfe (1995) and Knowles (1999) suggests that the opportunity cost of a child is equivalent to about 15% of the parents' time endowment. This cost only accrues as long as the child is living with the parents. If we assume that children live with parents for 15 years and that the adult period lasts 30 years, the overall time cost should be 50% of the time cost per year with the child present. Accordingly, we choose $\phi = 0.075$. The parameter ϕ also sets an upper limit on the number of children a person can have. With our choice, a person spending all

[2]Since convergence to this steady state is slow, the model still allows for substantial population growth for long time periods.

time on raising children would have a little above thirteen children. A family of two could have a little under 27 children.[3]

The parameter θ in the production function for human capital affects the education choice of parents, and therefore determines the aggregate expenditures on education. We choose θ such that in the balanced growth path total education expenditure as a fraction of GDP matches the corresponding value in U.S. data, which is 7.3%.[4] The implied choice is $\theta = 0.0119$. Our choice of η and θ implies an elasticity of human capital with respect to education of 0.6 at steady state, which is consistent with the estimate by Heckman (1976).

The remaining parameter τ determines the weight of parental human capital in the production of children's human capital. For individual dynamics to be stable, τ has to satisfy:

$$\tau < \hat{\tau} = 1 - \frac{\eta\phi}{\phi - \theta}.$$

If τ exceeds this limit, human capital of dynasties with different initial human capital diverges over time. Given our choices for the other parameters, τ should be smaller than 0.246. In principle, a precise value for τ could be calibrated by measuring the influence of parent's earnings on children's earnings in the model and in the data. While such measures are available for a number of countries, they cannot be compared directly to the model. The reason is that we do not model varying ability of children, which also would affect the correlation of parent's and children's earnings. In light of this situation, our strategy is to provide simulation results for a variety of values for τ, in order to highlight the sensitivity of our results to this parameter.

Given our parameter choices, we still need to set the initial conditions for the simulations. The overall size of the population and the average level of human capital are scale parameters which do not affect the results, and are therefore set to one. Likewise, the distribution of physical capital does not matter, since capital is owned by old people who have nothing left to decide. We therefore only specify the aggregate value. The initial distribution of human capital follows a log-normal distribution $F(\mu, \sigma^2)$, where μ and σ^2 are the mean and variance of the underlying normal distribution. The parameter μ is set such that $\bar{h}_t = 1$. We provide simulations for different variances of the distribution, in order to examine the effects of inequality. The initial level of physical capital is chosen such that the ratio of physical to human capital is equal to its value in the balanced growth path. The implied interest rate per year is 3.7%.

[3] If we also modeled a goods cost of having children, the upper bound would be somewhat lower, and close to the maximum human fertility levels observed so far.

[4] This figure (Digest of Education Statistics, 1998, US Department of Education) does not include on-the-job training, since it is not part of the parental investment in children.

4.2 Initial Inequality, Fertility, and Growth

Table 1 presents the initial annualized growth rates of human capital g_0 and population N_0, the initial inequality I_0, and the initial differential fertility D_0 for different variances of the distribution of human capital. Inequality is measured by the Gini coefficient I_0 computed on the earnings of the working population. Differential fertility is the difference between the average fertility of the top quintile and the bottom quintile; this quantity is then multiplied by two to yield a number per woman. For the computations the parameter τ was set to 0.2. To evaluate the role of differential fertility in our model, we also computed results under the assumption of constant, exogenous fertility.

	Endogenous Fertility				Exogenous Fertility			
σ^2	g_0	N_0	I_0	D_0	g_0	N_0	I_0	D_0
0.10	2.00%	0.00%	0.056	0.09	2.00%	0%	0.056	0
0.75	1.26%	0.66%	0.404	1.95	1.87%	0%	0.400	0
1.00	0.80%	1.08%	0.520	2.76	1.78%	0%	0.513	0
1.50	0.01%	1.71%	0.707	2.77	1.53%	0%	0.700	0

Table 1: Initial growth with and without endogenous fertility

The results in Table 1 show that inequality lowers growth both with and without endogenous fertility, but the effects are much larger when fertility is endogenous. When the variance of the distribution of human capital is low ($\sigma^2 = 0.10$), the difference between endogenous and exogenous fertility is small, and the growth rates are close to their values on the balanced growth path. When we increase the initial variance to $\sigma^2 = 0.75$, substantial fertility differentials within the population begin to arise, and the annual growth rate of human capital drops 0.74% below the steady state. With constant exogenous fertility, the drop in the growth rate is six times smaller. Further increases in the initial variance eventually lead to a negative growth rate (for $\sigma^2 > 1.5$) with endogenous fertility, while growth stays positive with exogenous fertility.

The results are robust with respect to the choice of τ. For example, with $\sigma^2 = 0.75$, initial growth with endogenous fertility is 1.22% for $\tau = 0.05$, and 1.32% for $\tau = 0.3$. With exogenous fertility, it is 1.80% for $\tau = 0.05$, and 1.94% for $\tau = 0.3$. We also carried out the same computations with a uniform instead of a log-normal distribution of initial human capital. We still found that growth declines much faster with inequality when fertility is endogenous. For example, when the Gini index goes from 0 to 0.33, growth drops by 0.7% with endogenous fertility and by 0.1% with exogenous fertility.

The initial dispersion of human capital also influences the overall growth rate of population. When the variance of human capital rises, fertility of low-skilled households

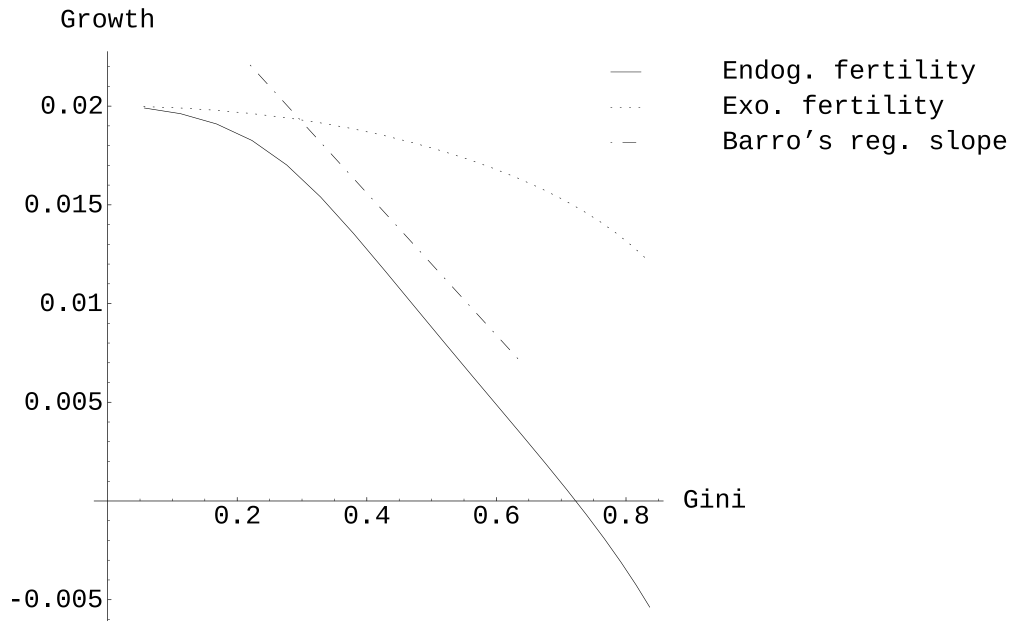

Figure 3: The relationship Growth – Gini

increases, while high-skilled households decide to have fewer children. Because of the shape of the fertility function (see Figure 1), the first effect dominates and aggregate fertility rises. This is in line with empirical studies that report a high positive correlation between aggregate fertility rates and Gini coefficients (see for instance Barro 2000).

Figure 3 depicts the growth rate of human capital as a function of the Gini coefficent. The computed relationships with endogenous and exogenous fertility are drawn, illustrating that the slope is much steeper with endogenous fertility. In the data, income Ginis for a country vary roughly in the range 0.2 to 0.65. In the model, raising the initial Gini from 0.2 to 0.65 lowers the growth rate only by about 0.3% with exogenous fertility, but by 1.4% with endogenous fertility. In a quantitative sense, fertility differentials within the population are essential for the relationship between inequality and growth.

We have also represented the slope of the regression of economic growth on income Ginis run by Barro (2000) when the fertility rate variable is omitted. In this regression, the Gini coefficient captures the intrinsic effect of inequality as well as the one going through fertility. Since actual Gini coefficients lies in the interval $[.21, .64]$, the regression line has been restricted to this interval. Our computational experiment is consistent with the Barro (2000) finding: "A reduction of the Gini coefficient by 0.1 would be estimated to raise the growth rate on impact by 0.4 percent per year." Perotti (1996) reports effects of similar magnitude. Given that the standard errors of the empirical estimates are sizable, not too much should be made out of the ability of the model to match these point estimates. However, it is reassuring that our results are in a range that seems empirically plausible.

4.3 The Dynamics of Inequality, Differential Fertility, and Growth

So far, we have only analyzed the effects of inequality on growth in the initial period. Given a period length of 30 years, the first period matters more than usual in this model, but still the long-run dynamics are of interest. It turns out that the possibility of corner solutions for education can lead to non-monotone behavior.

Figure 4 shows the evolution of the growth rate of human capital, the population growth rate, inequality, and differential fertility for different values of τ. Growth rates are annualized. The initial distribution of human capital is assumed log-normal with $\sigma^2 = 1$. Figure 5 compares the density function of human capital after 18 periods with the initial one.

With a low τ of $.05$, all variables converge monotonically to their steady-state values. Growth starts out low and then rises as inequality decreases. Population growth and fertility differentials decrease as well. If τ is raised to $.2$, inequality still decreases monotonically, but the growth rate of human capital first decreases before converging to the steady state, and population growth initially increases. This behavior is related to the corner solution for education. A fraction of the people in the first period decides not to invest in education. Since these people have the highest fertility rates, they make up an even larger fraction of the population in the next period. This leads to an increase in population growth in the first periods, and a decrease in growth of human capital, since investment in education is low. Through the externalities in human capital accumulation, however, after some time even the dynasties that did not invest in education initially accumulate enough human capital to choose positive education. From this point on, the growth rate of human capital increases, and population growth falls.

This non-monotone behavior only occurs if the initial distribution of human capital is such that some people choose not to invest in education. If everyone is above the threshold initially, convergence to the steady state is monotone. It is tempting to interpret the initial rise and subsequent fall in population growth as a demographic transition. However, this demographic transition only occurs given specific initial conditions. To produce a believable model of a demographic transition (which is not

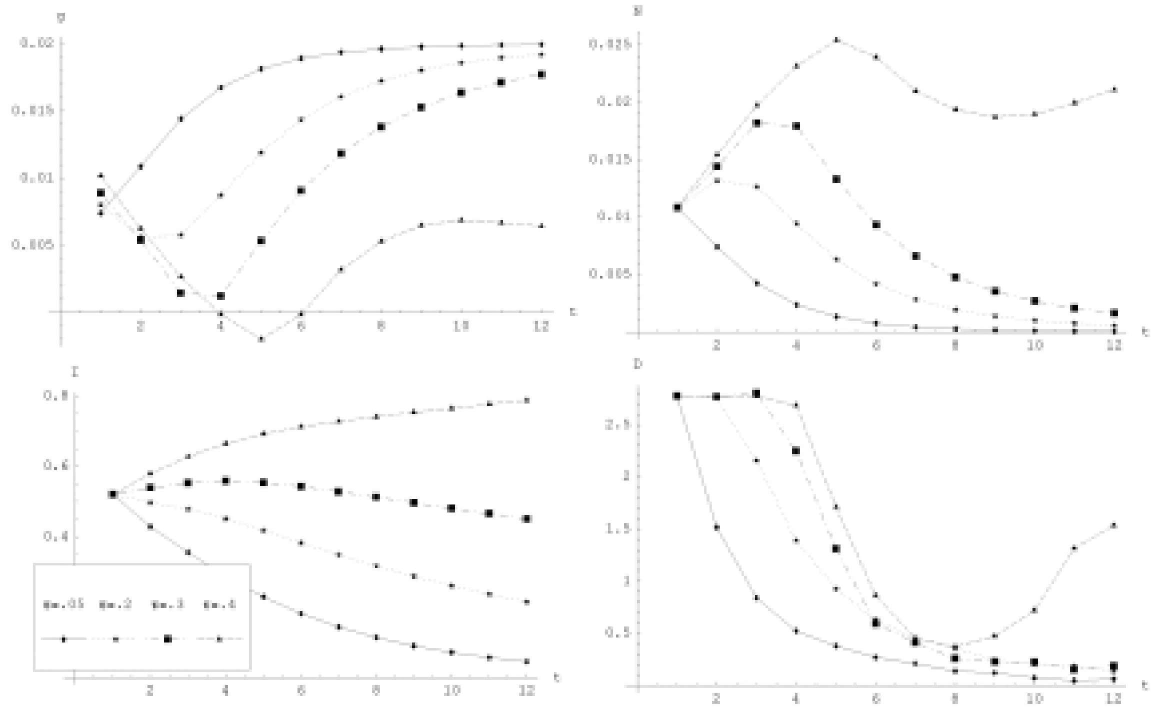

Figure 4: Growth, fertility, Gini, and differential fertility for different τ

our aim here) one would also have to explain why one would expect such an initial distribution in the first place.

If we increase τ above the bifurcation value $\hat{\tau} = .246$, we enter the region of the parameter space in which the dynamics of individual human capital are no longer stable. The returns to parental human capital in the education function are not sufficiently decreasing to compensate the centrifugal force of to the quality-quantity trade-off. Two cases can arise. If τ is smaller than η, there exists a high steady state $\bar{x} > 1$ of the individual dynamics. This steady state is locally stable. The initially high skilled dynasties are attracted there. Those who start out with low human capital get caught in a poverty trap. Since low skilled dynasties have more children, the demographic weight of the high skilled dynasties tends to zero. While the overall dynamics look similar to the $\tau = 0.2$ case, notice that the Gini first increases and then decreases only slowly. Differential fertility stays high for three periods and then declines. With $\tau = 0.4$, there is no high steady state, and the relative human capital of the initially skilled dynasties grows without bound. Inequality increases over time and the Gini ultimately converges to one. As the income distribution diverges, differential fertility increases again. The growth rate of human capital increases over time, but does not reach the value of the balanced growth path with a collapsed income distribution.

Figure 5 shows the distribution over human capital after 18 periods for the four differ-

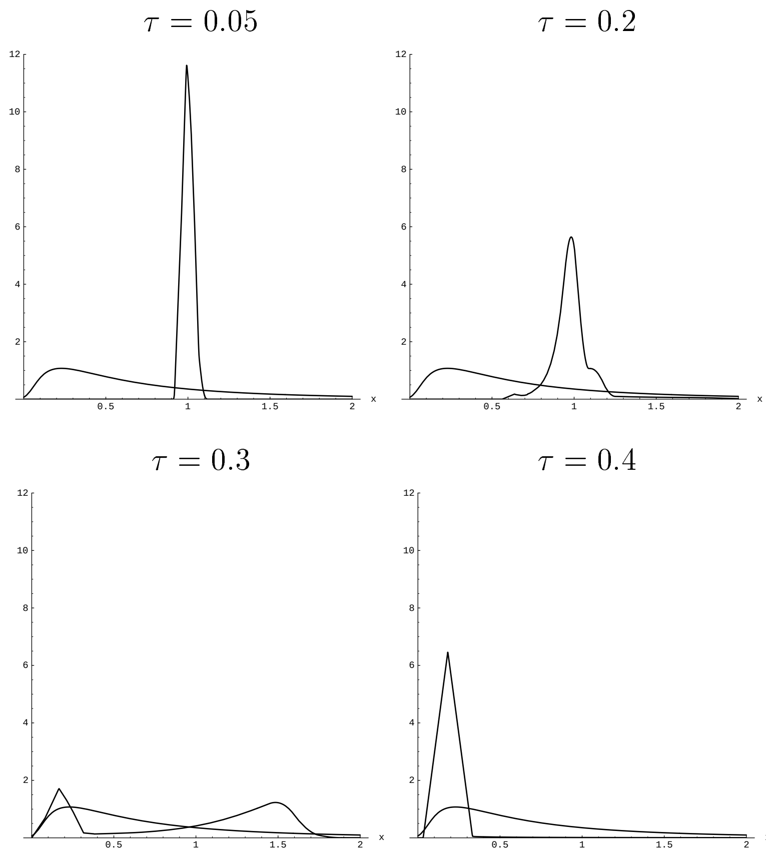

Figure 5: Density functions after 18 periods

ent values for τ. The graphs for $\tau = 0.05$ and $\tau = 0.2$ confirm that the distribution of human capital converges to a degenerate steady state where everyone has the same human capital. In both cases, most of the mass of the distribution is concentrated around $x = 1$. For the case $\tau = 0.2$, the density has a number of kinks. These kinks correspond to the cutoff points for dynasties that were at the corner solution for education in the first periods. The figure for $\tau = 0.3$ shows a bimodal distribution of human capital. This bimodal distribution reflects the case of two stable steady states for the individual dynamics that we discussed in the bifurcation analysis. However, the fact that the individual dynamics are stable around two steady states does not imply that the limiting distribution is bimodal as well. As long as the dynasties at one of the steady states have higher fertility rates, their fraction of the population will converge to one, leading to a unimodal limiting distribution. Indeed, our computations indicate that the limiting distribution is degenerate at $x = 1$ even for the case $\tau = 0.3$. In the case $\tau = 0.4$, the distribution of human capital is unimodal around a point x smaller than one. In this case, there are only few dynasties with high human capital, but since they invest in a lot of education they account for a large fraction of total human capital. The mass of people with above-average human capital tends to zero, but the fraction of human capital accounted for by them tends to one.

To conclude, our dynamic simulations show that convergence to the balanced growth path is slow even for low values of τ (recall that one period lasts 30 years). Even if convergence in the Gini in monotone, fertility differentials and growth may be non-monotone. In our simulations growth follow differential fertility more closely than the Gini.

5 Empirical Evidence

Can the effects postulated by our model be supported by empirical evidence? The first part of our hypothesis, that income inequality leads to high fertility differentials, has been analyzed by Kremer and Chen (2000). In line with our conjecture, they find that Gini coefficients have a significant and sizable positive correlation with fertility differentials. In this section we examine the second part of our hypothesis, the link from fertility differentials to growth. Our approach is to introduce a differential fertility variable into a standard growth regression, and we find that differential fertility has a negative effect on growth. Moreover, when the differential fertility variable is present, the Gini variable is no longer significant in the regression.

5.1 Data

Our sample contains 68 countries for which data on fertility differentials is available. The dependent variable (GR) in all regressions is the average annual growth rate of GDP per capita over the periods 1960 to 1976 or 1976 to 1992 (the period depends on the availability of fertility data). The GDP data is from the Penn World Tables, and growth rates are continuously compounded and expressed as percentages.[5] Since we are interested in long-run growth, we chose the longest sub-sample periods available in the Penn World Tables.

As Kremer and Chen (2000), for fertility differentials we rely on information from the World Fertility Survey and the Demographic and Health Surveys on total fertility rates by women's educational attainment (see Jones 1982, United Nations 1987, Mboup and Saha 1998, and United Nations 1995). For countries that participated in the World Fertility Survey (carried out around 1977), the independent variable is growth in GDP per capita in the first period, and for countries that participated in

[5]For countries where data from 1960 and/or 1992 was not available, we computed growth rates over the closest available interval.

Sample	nobs	data	Mean	S.D.	Min	Max
1960-1976	40	GR	1.95	3.65	-5.75	8.44
		GINI	44.32	11.14	23.38	68.00
		TFR	5.56	1.89	2.02	7.93
		DTFR	2.23	1.56	0.22	5.30
1976-1992	43	GR	0.39	1.89	-3.46	4.97
		GINI	45.91	9.56	28.90	69.00
		TFR	6.06	1.08	3.37	8.00
		DTFR	2.41	0.99	0.10	4.50
Total	83	GR	1.14	2.97	-5.75	8.44
		GINI	45.14	10.32	23.38	69.00
		TFR	5.82	1.54	2.02	8.00
		DTFR	2.32	1.29	0.10	5.30

Table 2: Descriptive statistics

the Demographic and Health Surveys the left-hand side variable is growth over the second period. Our differential fertility variable DTFR is the difference in the total fertility rate between women with the highest and the lowest education level. For some countries we have two observations from the Demographic and Health Surveys, in which case we averaged the two resulting values. For 25 countries we have only observations for the first period, for 28 countries we have only observations for the second period, and for 15 countries there are observations from both periods.

The remaining independent variables are the initial GDP per capita (GDP), the average ratio of investment to GDP (I/GDP), the average ratio of government expenditure to GDP (G/GDP), the initial Gini coefficient for the income distribution (GINI), a dummy variable for African countries (AFR), and the initial total fertility rate (TFR). I/GDP and G/GDP are from the Penn World Tables, and the income Ginis are from Deininger and Squire (1996).[6]

Table 2 provides descriptive statistics for the main variables in our analysis. The

[6]Where possible, the Gini coefficients are from the initial year; otherwise we used the closest available year. For a few countries, no data is available in Deininger and Squire (1996). For Benin, Burundi, Central Africa, and Namibia, we relied on the "Economic Report on Africa 1999: The Challenges of Poverty Reduction and Sustainablity", United Nations. For Haiti and Syria, only land Ginis from (Jazairy, Alamgir, and Panuccio 1992) are available. We regressed the available income Ginis on the land Ginis (correlation: 0.61) and used the predicted values.

two sub-samples are similar, except that the growth rate is much lower in the second sample. Since we will allow for different constant terms in the two sub-samples, this difference will not play a role in the results. These will thus reflect cross-sectional differences among countries, as well as variation over time within countries.

5.2 Estimation Results

One problem with the data is that our observations on fertility differentials are close to the end of the period over which we compute growth rates. Since the fertility observations are five-year averages and result from decisions and actions taken even earlier, the endogeneity problem is not too severe. Still, it would be preferable to observe growth rates for a prolonged period after observing fertility differentials. Given our data, this would be possible only for a few countries. We correct for potential endogeneity of the differentials by using instrumental variables.

Table 3 contains our estimation results. In all cases, the left-hand side variable is the average growth rate of GDP per capita. The regression equation is estimated with the Generalized Method of Moments. For countries that are present in both sample periods, we allow the error term to be correlated across the periods, and we use instrumental variables to correct for possible endogeneity of I/GDP, G/GDP, GINI, and DTFR. We allow the constant to differ across the periods (Constant A for the early period and Constant B for the late period). Our regressions are designed to be comparable to Barro (2000), but we include fewer variable because of the small sample size. Regression (1) reproduces standard results in the growth regression literature: the investment rate has a positive effect, whereas the government share, initial GDP, and the African dummy have negative effects on growth. Regression (2) adds the Gini coefficient. The estimated coefficient is significantly negative and the size is close to Barro's estimate. The value of the parameter implies that an increase in the Gini of 0.4 (roughly the range of variation in the data) lowers growth by about 1.2% per year. However, when the total fertility rate is included (regression 3), the coefficient on the Gini coefficient changes sign and becomes insignificant, which is also in line with Barro (2000).

Regression (4) includes the differential-fertility variable. The coefficient on differential fertility is significantly negative. The point estimate implies that an increase in the fertility differential from one to two would lower growth by 0.7% per year. With differential fertility included, the coefficients on both Gini and TFR are insignificant, and the point estimate on the Gini is positive.

Based on the results in Section 4, our model predicts that Gini, total fertility, and differential fertility are all equally negatively related with growth. It is therefore not clear why the coefficients on the Gini and the total fertility rate become insignificant once differential fertility is introduced. One possibility that the Gini is insignificant

Independent variable	Regression (1)		(2)		(3)		(4)	
Constant A	12.04**	(1.40)	12.01**	(1.48)	15.11**	(1.68)	13.49**	(1.84)
Constant B	10.20**	(1.46)	10.28**	(1.54)	13.28**	(1.70)	11.70**	(1.80)
ln(GDP)	-1.27**	(0.20)	-1.10**	(0.20)	-1.37**	(0.18)	-1.44**	(0.18)
I/GDP	0.14**	(0.02)	0.12**	(0.03)	0.08**	(0.04)	0.07*	(0.04)
G/GDP	- 0.09**	(0.03)	-0.07**	(0.03)	-0.06*	(0.03)	-0.06*	(0.03)
AFR	-1.63**	(0.39)	-1.77**	(0.41)	-1.89**	(0.36)	-2.15**	(0.38)
GINI			-0.03**	(0.01)	0.02	(0.03)	0.05	(0.04)
ln(TFR)					-1.74*	(0.92)	-0.61	(1.05)
ln(DTFR)							-1.08**	(0.48)
J_{test}	15.24	[0.36]	15.11	[0.30]	13.11	[0.36]	9.50	[0.58]
LR_1							5.21	[0.02]
LR_2							1.77	[0.41]

The dependent variable is the growth rate of real per capita GDP. Estimation by GMM. The instruments are: constant, log of per capital GDP at the beginning of the period, investment/GDP and gov. spending/GDP at the beginning of the period, Africa dummy, tropics, and access to the sea variables of Sachs and Warner (1997), fertility and life expectancy at the beginning of the period, and the openness variable from the Penn World Tables, at the beginning of the period.

Standard errors are reported in parentheses. These are based on the heteroscedastic-consistent covariance matrix of Newey-West. One star indicates significance at the 10%, two stars indicate significance at the 5 % level.

J_{test} is the test for over-identifying restrictions of Hansen (1982), asymptotically χ^2 distributed with n degrees of freedom, where n is the number of over-identifying restrictions. Corresponding $P-$values are reported in brackets.

LR_1 is a quasi likelihood ratio test for the absence of the differential fertility in the equation. LR_2 is the test for the absence of both Gini and total fertility. As suggested by Gallant (1987), they are computed as the normalized difference between the constrained objective function and the unconstrained one. The constrained estimation is computed with the weighting matrix provided by the unconstrained estimation. The corresponding $p-$value is reported in brackets.

Table 3: Generalized Method of Moments estimation

because its relation to the growth rate is nonlinear. Our dynamic simulations showed that along the transition path, Gini and growth as not as closely related as differential fertility and growth. Another interpretation is that inequality and total fertility are influenced by other factors which do not affect growth, while differential fertility is observed with less noise. A third possibility is that total fertility and inequality have other effects on growth, which are not present in our model and do not work through differential fertility. If some of these effects on growth are positive and therefore offset the negative effects, it would be possible that the overall effect of total fertility and the Gini is insignificant once the differential-fertility channel is controlled for.

Hansen's J test measures how close the residuals are to being orthogonal to the instrument set. It can be seen as a global specification test. The degrees of freedom equal the number of restrictions imposed by the orthogonality conditions. These restrictions are never rejected at the 5% level. Moreover, there is a significant improvement in the value of the test when differential fertility is introduced. The significance of the differential fertility variable is verified both by its t-statistic and by the quasi-likelihood ratio test LR_1. The test LR_2 of joint insignificance of GINI and TFR is not rejected.

In summary, we find that standard growth regressions detect the effect of differential fertility on growth postulated by our model. The effects implied by the regressions are sizable. At the same time, including differential fertility leaves the direct effect of the Gini coefficient insignificant, with a positive point estimate. While the empirical results are encouraging, they should be taken with a grain of salt. Growth regressions have been shown to be sensitive to the choice of variables included in the regression. For example, if we add initial life expectancy as an independent variable, the effect of differential fertility is no longer significant. This is not surprising given our small sample size, but it shows that results from growth regressions are not always robust.

6 Conclusion

Most of the theoretical literature on inequality and growth has concentrated on channels in which inequality affects growth through the accumulation of physical capital. In this paper we propose a different mechanism which links inequality and growth through differential fertility and the accumulation of human capital. In our model, families with less human capital decide to have more children and invest less in education. When income inequality is high, large fertility differentials lower the growth rate of average human capital, since poor families who invest little in education make up a large fraction of the population in the next generation. A calibration exercise shows that these effects can be fairly large. In the benchmark case, raising the Gini from 0.2 to 0.65 lowers the initial annual growth rate by 1.4%. We also examine the role of differential fertility in the growth-regression framework used, among others, by Perotti (1996) and Barro (2000). In line with the predictions of the theory, we find sizable negative effects of differential fertility on growth.

A natural direction for further research concerns the policy implications of our model. On the one hand, since differential fertility rather than inequality per se is the main source of growth effects in the model, it is not clear that redistribution policies would increase economic growth. Indeed, a typical outcome in models with endogenous fertility is that direct income redistribution tends to increase fertility differentials, which would lower the growth rate (see Knowles 1999). On the other hand, policies aimed directly at the distribution of human capital would be expected to be more effective.

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Appendix. Study of the Individual Transition Function

In order to understand the dynamic behavior of the model, we study the behavior of the function $x_{t+1} - x_t = \Psi(x_t; \tau)$:

$$\Psi(x;\tau) = \frac{Bx^\tau}{g^\star}\left(\theta + \max\left[0, \frac{\eta\phi x - \theta}{1-\eta}\right]\right)^\eta - x.$$

After replacing $g^\star$ by its value from Proposition 1 we have:

$$\Psi(x;\tau) = x^\tau\left(\frac{\theta(1-\eta)}{\eta(\phi-\theta)} + \max\left[0, \frac{\eta\phi x - \theta}{\eta(\phi-\theta)}\right]\right)^\eta - x.$$

Let us first consider the limits of this function. We have: $\Psi(0;\tau) = 0$ and $\Psi'_x(0;\tau) = +\infty$,, and:

$$\lim_{x\to\infty} \Psi(x;\tau) = x^{\tau+\eta}\left(\frac{\phi}{\phi-\theta}\right)^{\eta} - x,$$

which implies:

$$\lim_{x\to\infty} \Psi(x;\tau) = -\infty \text{ if } \tau+\eta<1 \text{ and } \lim_{x\to\infty} \Psi(x;\tau) = +\infty \text{ otherwise.}$$

Hence the function Ψ starts from $(0,0)$ with an infinite slope and goes either to $-\infty$ or $+\infty$ depending on parameter values. The function is plotted in Figure 6.

By definition of the aggregate balanced growth path, $x = 1$ is a steady state and thus $\Psi(1;\tau) = 0$. This steady state is locally stable if and only if $\Psi'_x(1;\tau) < 0$,, i.e.:

$$\tau + \frac{\eta\phi}{\phi-\theta} - 1 < 0.$$

At the point:

$$\hat{\tau} = 1 - \frac{\eta\phi}{\phi-\theta}$$

the dynamics of individual capital described by: $x_{t+1} - x_t = \Psi(x_t;\tau)$ undergo a transcritical bifurcation, as proved in Proposition 2. There are thus two steady-state equilibria, 1 and $\bar{x}$, near $(1,\hat{\tau})$ for each value of τ smaller or larger than $\hat{\tau}$. The equilibrium 1 (resp. $\bar{x}$) is stable (resp. unstable) for $\tau < \hat{\tau}$ and unstable (resp. stable) for $\tau > \hat{\tau}$.

Another point of interest is $x_t = \frac{\theta}{\eta\phi}$. If, at this point, the function Ψ is negative, it implies that it crosses the horizontal axes between 0 and $\frac{\theta}{\eta\phi}$. The existence of this steady state results from the infinite slope of Ψ at 0 and from its continuity; uniqueness results from the concavity of the function in the interval $(0, \frac{\theta}{\eta\phi})$. We evaluate:

$$\Psi\left(\frac{\theta}{\eta\phi},\tau\right) = \left(\frac{\theta(1-\eta)}{\eta(\phi-\theta)}\right)\eta\left(\frac{\theta}{\eta\phi}\right)^{\tau} - \frac{\theta}{\eta\phi}.$$

This is negative if τ is above a threshold $\bar{\tau}$:

$$\bar{\tau} = \frac{(1-\eta)\ln(\theta/\eta) - \ln\phi - \eta\ln((1-\eta)/(\phi-\theta))}{\ln\theta - \ln(\eta\phi)}.$$

We are now able to fully characterize the dynamics of x as a function of the parameter τ. The bifurcation diagram is presented in Figure 2 and the different functions Ψ are plotted in Figure 6.

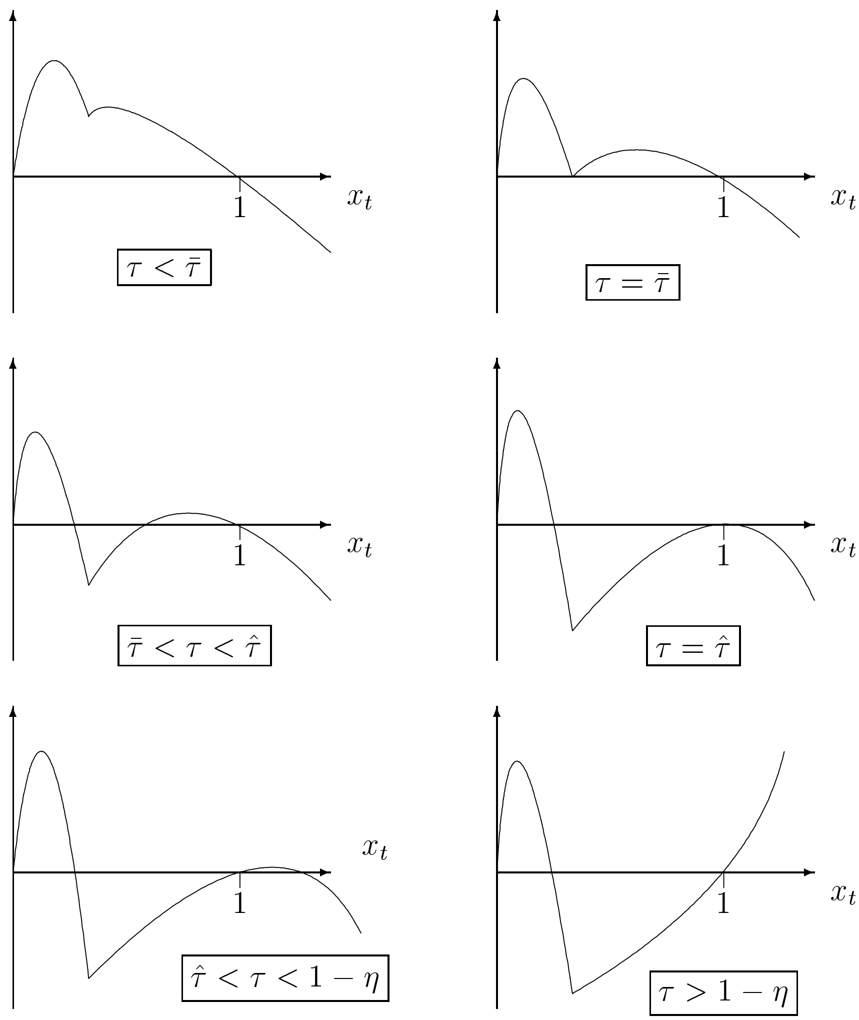

Figure 6: The function $x_{t+1} - x_t = \Psi(x_t; \tau)$