Optimal Portfolio Rules for an Integrated Stock Bond Portfolio*

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June 21, 2001

Abstract

In this paper we critique the famous separation theorem (or mutual fund theorem). We show that, if a portfolio contains stocks and bonds, then bonds generate a dependence of optimal portfolio composition on the investors' temporal horizon. This dependence makes the theorem inapplicable if all investors have different time horizons. Thus, we state a new theorem explaining the behaviour of financial advisors recommending higher percentage of bonds for more risk averse investors. This new theorem considers the separation theorem as a special case. Finally, we propose a solution to the so called "equity premium puzzle".

JEL classification: G11.
Key-words: separation theorem, asset allocation, equity premium puzzle.

*We have received helpful comments from R. Anderson and O. Scaillet. We gratefully acknowledge financial support from the Belgian French Community's program "Action de recherche Concentrée" 99/04-235.

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1 Introduction

The Capital Asset Pricing Model (CAPM) proposed by Sharpe (1964), Lintner (1965), and Mossin (1966) and based on the mean-variance approach used first by Markowitz (1952), has been for a long time the predominant normative and positive theory for determining asset returns and prices in financial markets. In particular, it has been used for solving problems such as the allocation of risky assets, price predictions, creation of performance indices and market efficiency tests.

One of the most important results of the CAPM is the so called *separation theorem* concerning the portfolio allocation problem. The theorem states that, if on financial market there exists a riskless interest rate for lending and borrowing, then all investors should hold the same composition inside the risky subset of their portfolio. Thus, investors' decisions can be different only about the holding percentage between the riskless asset and the risky portfolio.

A lot of empirical efforts have been made for testing the separation theorem. Unfortunately, there does not exist a general consensus about its validity. For instance, we refer to the works of Kroll, Levy, and Rapoport (1988) and of Kroll and Levy (1992) which are, respectively, less and more supportive of separation theorem.

However, a lot of papers (see for example Canner, Mankiw, and Weil (1997) and Campbell and Viceira (1998)) find that financial advisors disregard the separation theorem. In fact, it is well known that financial advisors suggest an asset allocation having a ratio between stocks and bonds which is higher for more risk averse investors. The model presented by Campbell and Viceira (1998) is consistent with this conventional portfolio advice but it fails to complain with the separation theorem.

According to us, the most important source of trouble is the presence of both stocks and bonds into investors' portfolios. In particular, in the most popular asset allocation strategy, investor's wealth is first allocated to indices corresponding to diverse asset classes and then allocated to individual assets using appropriate models for each asset class. Accordingly, two main portfolio strategies have been separately developed for stocks and bonds. In particular, for stocks the most common used technique is the mean-variance approach, while the hedging theory for bonds is based on the principle of equating a weighted mean of coupon maturities (*duration*) to a particular time that we will call *financial horizon* (for a review of this approach see Redington (1952), Fisher and Weil (1971), Cox, Ingersoll and Ross (1979), Fong and Vasicek (1984), Barber and Copper (1997)).

This expensive method of two-stage decision has been eluded in the optimal stochastic control approach (see Merton (1969, 1971) and Poncet and Portait (1993)). In fact, because of a problem of market completeness,[1] in these models no more than one bond is considered. In such a framework, it becomes im-

[1] Øksendal (2000) and Björk (1998) offer an interesting survey about the problem of market completeness.

possible to apply the technique of duration matching because of the lack of a sufficient number of bonds.

A more interesting approach to the problem of unifying the optimal portfolio rules for a set of stocks and bonds is presented by Konno and Kobayashi (1997). The authors determine the allocation of the investor's wealth to each asset in one stage by solving a large-scale mean-variance model. Unfortunately, according to us, they do not pay enough attention to the main economic difference between stocks and bonds: bonds have a maturity date while stocks have not. So, according to us, it is not appropriate to use the same risk measure for both stocks and bonds.

In this work, we use the Konno and Kobayashi approach and we maintain the framework of quadratic programming but we prefer to keep separate the risk measures for different assets. In particular, we use a result presented in Barber and Copper (1998). These authors show that the usual linear programming problem for immunizing a bond portfolio against particular interest rate shifts, can be generalized to a quadratic programming problem immunizing the portfolio against any interest rate shift.

Because in our model we maintain the *duration* approach, we obtain an optimal portfolio depending on investors' financial horizon. In our framework the separation theorem stays valid only if all investors have the same financial horizon. Thus, our result contains the separation theorem as a particular case.

Furthermore, the model developed in this work is consistent with the idea that in the very long run there are no reasons for holding bonds. In fact, when the investor's financial horizon tends to infinity, we find that the optimal amount of wealth invested in bonds must be equal to zero. In this way, we are able to reconcile the inconsistencies stressed by Canner, Mankiw, and Weil (1997) and Campbell and Viceira (1998). In particular, we argue that the asset allocation strategy suggested by financial advisors is consistent with our model if the financial horizon can be considered as an indicator of investors' risk aversion. This idea is supported by Epstein and Zin (1989).

As we have already underlined, our optimal ratio of bonds to stocks is a function of financial horizon. Furthermore, thanks to some algebraic considerations and to a numerical example, we find that this function is not monotonic. In fact, there exists a financial horizon maximizing the bonds to stocks ratio.

This characteristic allows us to explain at least a component of the well known *equity premium puzzle* underlined for the first time by Mehra and Prescott (1985). In particular, if investors' financial horizon stays around its value implying the maximum ratio of bonds to stocks, then any variation in the risk aversion degree always implies the same behaviour on financial market: investors sell bonds. In this way, the great difference between bond and stock returns (known as *equity premium puzzle*) does not vanish as a simple arbitrage argument could imply.

While our work concentrates on the different risk measures for stocks and bonds, we underline that previous attempts at solving the *equity premium puzzle* have been based on: (i) modeling market crashes (Rietz (1988) and Brown, Goetzmann, and Ross (1995)), (ii) modeling utility functions with risk aversion index and elasticity of substitution which are unrelated (Weil (1989), and Epstein and Zin (1989 - 1990)), (iii) restating the calibration methodology for describing first and second moments of asset returns (Cecchetti, Lam, and Mark (1993)), and (iv) considering the risk related to the inflation rate (Bagliano e Beltratti (1997)).

Throughout this work we consider a competitive financial market without frictions, liquidity constraints, taxes or commissions on asset transactions.

The paper is structured as follows. In Section 2 we recall how to derive, in a mean-variance framework, the result of the separation theorem. In Section 3 we expose briefly the classical immunization theory for a bond portfolio, based on the weighted mean of its coupon flow (the so called *duration*). Section 4 exposes two different approaches that can be used in order to solve a simultaneous problem instead of the two ones presented in the previous sections. Accordingly, by solving a unique optimal portfolio problem, Section 5 presents our main results: the confutation of the separation theorem and a proposal for solving the *equity premium puzzle*. Section 6 concludes. A numerical simulation can be found in the Appendix.

2 The classical risk minimization model: the mean-variance approach

The fundamental idea of the mean-variance approach is to describe the behaviour of asset returns through the two first moments of their distribution (Markowitz (1952)). The mean represents the expected return while the standard deviation shows how values are spread out around the mean. From this point of view the standard error can be a measure of risk only if we call "risk" the possibility that asset return goes far from its mean value (in any direction).

Accordingly, we can state that a risk averse investor wants to minimize the standard deviation (or the variance) of his portfolio given a desired expected return. This result implying only the two first moments of return distribution can be obtained if we consider an investor having quadratic utility function in his wealth (the so called "mean-variance utility function"). On the other hand, there exist precise hypotheses on the density functions of asset returns (see Chamberlain (1983)) implying mean-variance utility functions for all risk-averse expected utility maximizers.

Furthermore, de Finetti (1952) found that the risk premium for a small risk can be approximated by its variance. Let $\pi(h)$ denotes the risk premium associated to the risk $h\widetilde{\varepsilon}$ where $\widetilde{\varepsilon}$ is a random variable such that $\mathbb{E}[\widetilde{\varepsilon}] = 0$. If the investor has wealth W_0, and utility function $u(\cdot)$, then it must be true that:

$$\mathbb{E}\left[u\left(W_0 + h\widetilde{\varepsilon}\right)\right] = u\left(W_0 - \pi(h)\right).$$

By differentiating two times this identity with respect to h we have that, because $\pi(0) = 0$, and $\pi'(0) = 0$, thus:

$$\pi''(0) = -\frac{u''(W_0)}{u'(W_0)}\mathbb{E}\left[\widetilde{\varepsilon}^2\right].$$

If we expand $\pi(h)$ in Taylor series around $h = 0$, we obtain:

$$\pi(h) = \frac{1}{2}h^2\pi''(0) + O\left(h^3\right).$$

This is the so called Arrow-Pratt approximation allowing us to disentangle the characteristics of risk and preferences to evaluate the impact of the former on welfare.

Konno and Kobayashi (1997) argue that a mean-variance model can, in principle, be applied to any financial asset as long as one can adequately estimate the expected rate of return and correlation coefficients of the rate of return of each asset. Nevertheless, we outline that, in this framework, the well known Borch-Feldstein paradox (Borch (1969) and Feldstein (1969)) arises when an asset return has a positive asymmetry. In fact, in this case, if we minimize its variance, then we are minimizing the gain possibility and not the risk. The mean-variance approach is paradox-free if and only if the indifference curves on the mean-variance axes have a slope lower than 1 or, alternatively, the Sharpe index for investor's portfolio is greater than 1.

So, under these hypotheses, we recall the main steps for reaching the separation theorem according to which the composition of optimal portfolio can be divided into two funds. In particular, if there exists a riskless asset, one of the two funds contains only the riskless asset, while the other one contains only risky assets.

If k is the number of assets on the financial market (or, alternatively, the number of assets the investor wants to invest in), $x \in \mathbb{R}^{k\times 1}$ contains the percentage of wealth invested in each asset, $\mu \in \mathbb{R}^{k\times 1}$ contains the expected return of each asset, $\Sigma \in \mathbb{R}^{k\times k}$ is the variance and covariance matrix of asset returns, m is the return the investor wants to obtain from his portfolio, $e \in \mathbb{R}^{k\times 1}$ contains only 1s, and W is the investor's wealth (in this framework we can put $W = 1$ without loss of generality), then the one-period variance minimization problem can be written as follows:

$$\begin{array}{l} \min\limits_{x} x'\Sigma x \quad s.t. \\ x'\mu = m, \\ x'e = 1. \end{array} \tag{1}$$

The first constraint indicates that investor wants to obtain a given return (m) from his portfolio while the second constraint only means that investor cannot invest more than his wealth.

Since the matrix Σ is positive semi-definite, then the first order conditions for this problem are necessary and sufficient for a minimum.

It is simple to show that the solution to problem (1) is:[2]

$$
\begin{aligned}
x^* &= \Sigma^{-1}\begin{bmatrix} e & \mu \end{bmatrix}\left(\begin{bmatrix} e' \\ \mu' \end{bmatrix}\Sigma^{-1}\begin{bmatrix} e & \mu \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ m \end{bmatrix} = \\
&= \Sigma^{-1}\left(\frac{e'\Sigma^{-1}e}{\Delta}\mu - \frac{e'\Sigma^{-1}\mu}{\Delta}e\right)m + \Sigma^{-1}\left(-\frac{e'\Sigma^{-1}\mu}{\Delta}\mu + \frac{\mu'\Sigma^{-1}\mu}{\Delta}e\right),
\end{aligned} \tag{2}
$$

where $\Delta = \left(e'\Sigma^{-1}e\right)\left(\mu'\Sigma^{-1}\mu\right) - \left(\mu'\Sigma^{-1}e\right)^2$. We can see that each component x_i^* of vector x^* is an affine function of return m. This means that the portfolio variance (σ^2) is a quadratic polynomial in m. Analytically:

$$
\sigma^2 = x^{*\prime}\Sigma x^* = \frac{e'\Sigma^{-1}e}{\Delta}m^2 - 2\frac{e'\Sigma^{-1}\mu}{\Delta}m + \frac{\mu'\Sigma^{-1}\mu}{\Delta}, \tag{3}
$$

and graphically, on the σ, m plane, we obtain a branch of an equilateral hyperbola.

Now, if we introduce a risk-free rate (r) at which the investor can lend and borrow any amount of money, then the constraints in problem (1) change while the objective function is unaffected. In particular, the mere budget constraint disappears because it is possible to invest more than wealth by borrowing some money. Thus, the new problem is:

$$
\begin{aligned}
&\min_x x'\Sigma x \quad s.t. \\
&x'\mu + (1 - x'e)\, r = m,
\end{aligned} \tag{4}
$$

and its solution is:

$$
x_r^* = \left[(\mu - re)'\,\Sigma^{-1}(\mu - re)\right]^{-1}\Sigma^{-1}(\mu - re)(m - r)\,. \tag{5}
$$

In this case, the variance of the optimal portfolio is given by:

$$
\sigma_r^2 = x_r^{*\prime}\Sigma x_r^* = (m - r)^2\left[(\mu - re)'\,\Sigma^{-1}(\mu - re)\right]^{-1}, \tag{6}
$$

which is no more a branch of an equilateral hyperbola but a straight line on the σ, m plane. It is fundamental to stress that this straight line is tangent to the locus represented by equation (3). Thus, we have two "critical" points in the risk-return plane: the point corresponding to the riskless asset having coordinates $(0; r)$ and the tangency point between the loci (3) and (6) having coordinates $(\sigma_P; m_P)$.[3] The portfolio referring to the point $(\sigma_P; m_P)$ will be called "market portfolio".

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[2] We recall that the solution to the quadratic problem: $\min_x \left(\underset{1\times k}{x'}\ \underset{k\times k}{S}\ \underset{k\times 1}{x}\right)$ s.t. $\underset{1\times k}{x'}\ \underset{k\times g}{v} = \underset{1\times g}{d}$ where g is the number of linear constraints, is as follows: $x = S^{-1}v\left(v'S^{-1}v\right)^{-1}d'$.

[3] Easy computations show that:

$$
\begin{aligned}
\sigma_P &= \frac{\left(\mu'\Sigma^{-1}\mu\right) - 2\left(e'\Sigma^{-1}\mu\right)r + \left(e'\Sigma^{-1}e\right)r^2}{\left[\left(e'\Sigma^{-1}\mu\right) - \left(e'\Sigma^{-1}e\right)r\right](\mu - re)'\,\Sigma^{-1}(\mu - re)}, \\
m_P &= \frac{\left(\mu'\Sigma^{-1}\mu\right) - \left(e'\Sigma^{-1}\mu\right)r}{\left(e'\Sigma^{-1}\mu\right) - \left(e'\Sigma^{-1}e\right)r}.
\end{aligned}
$$

We know that any point belonging to a straight line (like (6)) can be reached through a linear combination of two given points on the same line. In this sense, we can say that any optimal portfolio can be represented as a linear combination of the riskless asset and the market portfolio.

Thus, we have reached the result (see, for instance, Merton (1990)) of the well known:

Theorem 1 (Separation theorem) *: If it is possible to lend and borrow at the same risk-free rate, then all investors will hold the same composition for their portfolio of risky assets and their choices will be different only for the holding percentage between the riskless asset and the risky portfolio.*

This kind of result stays valid also without the riskless asset. Nevertheless, in this case, the theorem must be expressed as follows (see Merton (1990)):

Theorem 2 *Given k assets, there exists a unique (up to a non-singular transformation) pair of "mutual funds" constructed from linear combinations of these assets such that individuals will be indifferent between choosing from a linear combination of these two funds or a linear combination of the original k assets.*

We underline that, if asset prices are log-normally distributed, then these theorems stay valid also in the optimal stochastic control approach where, in particular, the optimal portfolio composition is identical to (5) if the representative investor has a log-utility function (see Merton (1969, 1971)).

Furthermore, under the assumption of log-normal distributed asset prices, the separation result is independent of preferences, wealth distribution or time horizon (Merton (1971)).

3 The classical immunization theory for a bond portfolio

During our work we will refer to the following:

Definition 3 *A bond portfolio is immunized against a liability, occurring at a certain time, if its value, at this time, is at least equal to the value of liability.*

This is the most common definition of "immunization" (see Fisher and Weil (1971) or Barber and Copper (1997)). According to this definition, the easiest way to immunize a portfolio is to buy only a zero coupon bond with the same maturity date as the time at which the liability is foreseen. Nevertheless, we cannot be sure that this kind of zero coupon bond exists. If there are only coupon bonds or zero coupon bonds with maturity dates different from that of liability, then we must find a right composition immunizing investor's portfolio.

The value of a coupon bond (at time 0) is given by the present value of all coupons according to the following formula:[4]

$$V = \sum_{t=0}^{T} C_t \left(1 + r\left(0; 0, t\right)\right)^{-t}, \tag{7}$$

where C_t is the coupon paid at time t, T is the time to maturity (generally C_T equals the bond nominal value), and $r(t; s, T)$ is the forward interest rate, fixed in t, for the period from s to T ($t \leq s < T$). If $t = s$, then $r\left(t; t, T\right)$ is a spot rate.

If all coupons can be reinvested at the same interest rates fixed in 0, then, at time H, the value (V) of a portfolio containing only one bond is given by the formula:

$$V\left(H, r\right) = \sum_{t=0}^{T} C_t \left(1 + r\left(0; t, H\right)\right)^{H-t}.$$

The immunization problem consists in choosing the right time H^* at which it is optimal to sell the bond. Because investor's portfolio must be immunized against changes in r, then H^* is computed as the time at which the value $V\left(H, r\right)$ does not change with respect to the interest rate. Thus, if we consider constant interest rates and coupons, the immunization condition can be written as follows:

$$\frac{\partial}{\partial r} V\left(H, r\right) = \sum_{t=0}^{T} \left(H - t\right) C_t \left(1 + r\right)^{H-t-1} = 0,$$

from which we have:

$$H^* = \frac{\sum_{t=0}^{T} t C_t \left(1 + r\right)^{-t}}{\sum_{t=0}^{T} C_t \left(1 + r\right)^{-t}}. \tag{8}$$

Accordingly, the optimal time horizon is equal to the weighted mean of times at which coupons are paid, where the weights are given by the discounted coupons. This weighted mean is called *duration*.

It is easy to show that this result can be extended to the case in which interest rates are not constant. In particular, we recall the Fisher and Weil's theorem (see Fisher and Weil (1971)):

[4] If we consider a continuous-time interest rate δ, then our discount factor is: $\exp\left(-\int_0^t \delta\left(0; s\right) ds\right)$.

Theorem 4 *Given an interest rate term structure evolving according to parallel shifts, then a portfolio is hedged with respect to a liability occurring at time H, if:*

- the portfolio present value equals the present value of the liability;

- the portfolio duration equals H.

We underline that Redington (1952) had already state this same theorem for a flow of liabilities. Unfortunately, his theorem is valid only for little changes in the interest rate structure. Nevertheless, the limitation to have only one liability can be relaxed because multiple liabilities can be handled as an extension of the single liability case by immunizing separately for each liability cash flow (see Bierwag, Kaufman, and Toevs (1983)).

Furthermore, a more general result can be obtained if we consider not only parallel shifts on interest rates but also convex shifts.[5] We recall the following (see for instance Moriconi (1994)):

Theorem 5 (Hedging General Theorem) *: Given an interest rate term structure evolving according to convex shifts, then a portfolio is hedged with respect to a sequence of liabilities if:*

- the portfolio present value equals the present value of all liabilities;

- the portfolio duration equals the duration of liabilities;

- the MAD (mean absolute deviation) index for liabilities is not greater than the MAD index for portfolio, at each time of payment.[6]

As a corollary we have the:

Theorem 6 (Preservation Theorem) *: A bond portfolio, hedged at time t according to the hedging general theorem, stays hedged until an interest rate shift occurs.*[7]

We underline that this preservation theorem is no more valid if we consider a stochastic interest rate term structure. In fact, in this case, we can immunize the investor's portfolio only instantaneously (Barber and Copper (1997)).

[5] A shift $Y(s)$ on the interest rate structure is convex if the shift factor $f(s) = \exp\left(-\int_t^s Y(u)\,du\right)$ has a negative second order derivative, i.e. $Y^2(s) \geq \frac{\partial Y(s)}{\partial s}$.

[6] The third condition can be algebraically written in the following way:

$$\sum_{i=1}^{k} x_i \sum_{t=0}^{T} |t-h|\, C_{i,t}\,(1+r)^{-t} \geq \sum_{t=0}^{T} |t-h|\, L_t\,(1+r)^{-t}, \quad \forall h = 0,1,2,...,T,$$

where k is the number of bonds in the investor's portfolio, x_i is the percentage of wealth invested in asset i, $C_{i,t}$ is the coupon paid at time t on bond i, and L_t is the liability due at time t.

[7] The preservation theorem is valid also for Fisher-Weil's theorem.

In this context, the bond risk is represented by the changes in its value (V) due to interest rate shocks. If we consider a small shock in r, then the percentage change in V can be written in the following way:

$$\frac{\partial V/\partial r}{V} = -\left(1+r\right)^{-1} \frac{\sum_{t=0}^{T} tC_t \left(1+r\right)^{-t}}{\sum_{t=0}^{T} C_t \left(1+r\right)^{-t}}.$$

Thus, we have obtained the well known result according to which higher the duration higher the volatility of bond values. That is to say that long duration bonds react more widely to interest shocks than short duration bonds.

4 How to unify the stock and bond problems

In the previous sections we have shown that the risk reduction policy implies the solution of a quadratic problem for stocks and of a linear system for bonds. Thus, the most common portfolio strategy is to solve a two stage problem: in the first stage the wealth percentages to be invested in stocks and bonds are computed and, in the second stage, the optimal composition of each portfolio subset is determined.

Our work is aimed at eliminating this double choice. The same problem is considered by Konno and Kobayashi (1997) who determine the allocation of the investor's wealth to each asset in one stage by solving a large-scale mean-variance model. They underline that the return structure of bond changes as a function of time (see also Barber and Copper (1997)) and, nevertheless, they do not render explicit this dependence.

Thus, in the following sections, we use the Konno and Kobayashi approach maintaining the framework of quadratic programming but we prefer to keep separate the risk measures for different assets. For stocks we use the classical mean-variance approach and for bonds we use the technique of duration matching. By using the *duration* approach, the model we present is able to point out how optimal portfolio composition depends on investors' financial horizon.

In the following sections, we expose two possible ways to transform the linear programming problem for bonds into a quadratic programming problem. For this purpose, a result presented in Barber and Copper (1998) is useful. These authors show that the usual linear programming problem for immunizing a bond portfolio against particular interest rate shifts, can be generalized to a quadratic programming problem immunizing the portfolio against any interest rate shift.

4.1 The minimax risk strategy

The most common immunization techniques are based on the assumption that interest rate shifts have a precise behaviour (parallel for Fisher and Weil's theorem or convex for the General Hedging Theorem). Barber and Copper (1998) consider a complete set of feasible interest rate shifts and they propose to choose the portfolio minimizing its maximum sensitivity over this set of shocks (*minimax* strategy).

They consider a set of feasible shocks consisting of all directions (that is to say they present the situation for an investor having no prior information) and they find a result that is very interesting for our purpose. In their framework, if the portfolio cash flow stream is discrete, then the minimax strategy requires minimizing a quadratic form in a symmetric positive definite matrix. This result is exactly what we need in order to integrate stock and bond problems. Thus, for our purpose, the model proposed by Barber and Copper gives strong bases to the idea of using quadratic programming even for the immunization of a bond portfolio.

The most important steps for reaching this minimax strategy are as follows. Let α be a Lebesgue-Stieltjes measure on the real number line standing for the cash flows for a portfolio of assets and liabilities. If we consider any shift on the forward interest rate structure whose direction is $f(t)$ and whose magnitude is x, then the evolution of the current instantaneous forward interest rate can be represented as:

$$\delta(t,x) = \delta(t,0) + f(t)\,x.$$

Accordingly, after a shift the present value of a monetary unit due in t is given by:

$$B(t,x) = B(t)\exp\left(-x\int_0^t f(s)\,ds\right),$$

where $B(t) = \exp\left(-\int_0^t \delta(s,0)\,ds\right)$. Thus, the present value of cash flows is $\beta(t) = \int_t^\infty B d\alpha$ and the portfolio value after a shift is:

$$V(x) = \int_0^\infty \exp\left(-x\int_0^t f(s)\,ds\right) d\beta(t).$$

The authors compute the local change in the portfolio value for a shift in a given direction:

$$\left.\frac{\partial V(x)}{\partial x}\right|_{x=0} = \int_0^\infty \beta(t)\,f(t)\,dt,$$

then, they use the Cauchy-Schwarz inequality:

$$\left|\left.\frac{\partial V(x)}{\partial x}\right|_{x=0}\right| \leq \|\beta(t)\|,$$

and, finally, they state the following proposition:

$$\|\beta(t)\|^2 = \int_0^\infty \int_0^\infty \min(s,t)\,d\beta(s)\,d\beta(t).$$

Thanks to this result they claim that the minimax strategy can be expressed as the following minimization problem (where Θ is a set of feasible cash flows):

$$\min_{\alpha \in \Theta} \left\{ \int_0^{\infty} \int_0^{\infty} \min(s,t)\, d\beta(s)\, d\beta(t) \right\}, \tag{9}$$

and we can immediately see that if the cash flow stream is discrete, then this result consists in minimizing a quadratic objective function based on a symmetric positive definite matrix.

4.2 The maximum diversification strategy

The minimum variance problem for stocks implies a diversification in portfolio composition. This diversification, in fact, is able to reduce the non systematic risk of the investor's portfolio. Thus, even for a bond portfolio, we can use an objective function allowing to obtain a diversification result. This kind of function must have the form of a weighted mean of the square of portfolio composition. The weights can be represented, for instance, by bond prices and, in this case, the diversification problem can be written as follows:[8]

$$\begin{array}{l} \min\limits_{y} y' \Delta_q y \quad s.t. \\ y' \Delta_V D = HL \left[1 + r(t,H)\right]^{-(H-t)}, \\ y' V = L \left[1 + r(t,H)\right]^{-(H-t)}, \end{array} \tag{10}$$

where $y \in \mathbb{R}^{k_B \times 1}$ contains bond portfolio composition, $q \in \mathbb{R}^{k_B \times 1}$ contains bond prices, $\Delta_q \in \mathbb{R}^{k_B \times k_B}$ is a diagonal matrix containing the elements of vector q, $V \in \mathbb{R}^{k_B \times 1}$ contains bond values V_i (as in formula (7)),[9] $\Delta_V \in \mathbb{R}^{k_B \times k_B}$ is a diagonal matrix containing the elements of vector V, $D \in \mathbb{R}^{k_B \times 1}$ contains bond durations D_i (as in formula (8)), L is the liability occurring at time H and, accordingly, $L[1 + r(t,H)]^{-(H-t)}$ is the liability present value.

The two constraints in problem (10) are the duration constraint and the budget constraint respectively. The solution to this problem[10] is given by formula (2) where we have to consider the following changes:

$$\begin{array}{rcl} \mu & \rightarrow & \Delta_V D, \\ e & \rightarrow & \dfrac{V}{L} \left[1 + r(t,H)\right]^{(H-t)}, \\ m & \rightarrow & HL \left[1 + r(t,H)\right]^{(H-t)}, \\ \Sigma & \rightarrow & \Delta_q. \end{array}$$

We underline that if there were transaction costs, then this diversification should not be optimal but, since in our analysis there are no market frictions,

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[8] Written in scalar notation the objective function is: $\sum_{i=1}^{k_B} q_i y_i^2$.

[9] If the financial market is efficient, then $V = q$.

[10] Also in this case the first order conditions are necessary and sufficient for a minimum.

then this result can be considered consistent with the approach used in the previous sections.

Thanks to above-mentioned results, we can argue that the linear programming problem for a bond portfolio can be restated as a quadratic programming problem. This result allows us to put together the stock and bond problems in a very easy way. The following section analyzes the solution to the unified problem.

5 An asset allocation problem

Canner, Mankiw and Weil (1997) consider the allocation problem for an investor who can hold cash (a riskless asset), bonds and stocks. They find that the investment strategy commonly used by financial advisors is inconsistent with the separation theorem. In particular, financial advisors recommend aggressive investors to hold a lower ratio of bonds to stocks than conservative investors. Instead, according to the separation theorem, the optimal portfolio allocation between stocks and bonds should be the same for all investors.

Here, we analyze the same problem as these authors. We consider a subject who can invest his wealth in stocks, bonds, and a riskless asset. The riskless asset return (r) is the rate at which investor can lend and borrow any amount of money. Furthermore, we suppose investor wants to achieve the two objectives of minimizing the variance of the stock subset of his portfolio and of immunizing the bond subset of his portfolio against a liability occurring at a given time H.

Thus, we define our framework as follows:

Definition 7 *The M-F-W (Markowitz-Fischer-Weil) framework is the set of hypotheses:*

1) investors can lend and borrow any amount of money at the same risk free interest rate (r);

2) investors minimize the variance of the stock subset of their portfolio;

3) investors immunize the bond subset of their portfolio with respect to a liability occurring at a given time H, under parallel shifts of the interest rate structure;

4) there are no short selling constraints.

Accordingly, in our framework the investor has two different objective functions for stocks and bonds. We use the well known property according to which we can obtain a Pareto optimal result by optimizing the weighted sum of these functions (see for example Duffie (1996)). If bond and stock subsets have the same importance for investors (i.e. they have the same weights in the objective function), then the optimization problem can be written in the following way:

$$\begin{array}{l} \min_{x,y} \left[\begin{array}{cc} x' & y' \end{array}\right] \left[\begin{array}{cc} \Sigma & \underline{0} \\ \underline{0} & \Gamma \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] \quad s.t. \\ \left[\begin{array}{cc} x' & y' \end{array}\right] \left[\begin{array}{cc} \Delta_p\mu - pr & \underline{0} \\ \Delta_q\eta - qr & \Delta_V D - Hq \end{array}\right] = \left[\begin{array}{cc} m-r & 0 \end{array}\right], \end{array} \tag{11}$$

where the variables not already defined in the previous sections are as follows: $x \in \mathbb{R}^{k_A \times 1}$ contains stock weights, $p \in \mathbb{R}^{k_A \times 1}$ contains stock prices, $\Gamma \in \mathbb{R}^{k_B \times k_B}$ is a symmetric and positive semidefinite matrix (measuring the risk of bonds),[11] and $\eta \in \mathbb{R}^{k_B \times 1}$ contains bond returns.

In problem (11) we have formulated the return constraint for stocks and bonds together and we have maintained the duration constraint for bonds.

We underline that our hypothesis of equal importance for stock and bond objective functions is just made for simplifying the computations. In fact, the case of different weights for stocks (λ_A) and bonds (λ_B) can be implemented in an easy way by considering the following matrices: $\widetilde{\Sigma} = \lambda_A\Sigma$ and $\widetilde{\Gamma} = \lambda_B\Gamma$. Thus, in the following analysis we put $\lambda_A = \lambda_B = 1$ without loss of generality.

By solving the quadratic problem (11) we have:[12]

Proposition 8 *The optimal portfolio compositions for stocks (x^*) and bonds (y^*) in the M-F-W framework are given by:*

$$\left[\begin{array}{c} x^* \\ y^* \end{array}\right] = \frac{1}{\delta_r} \left[\begin{array}{c} f(H)'\,\Gamma^{-1} f(H) \left(A - \Sigma^{-1} pr\right) \\ \left[f(H)'\,\Gamma^{-1} f(H)\, I - \Gamma^{-1} f(H)\, f(H)'\right] \left(B - \Gamma^{-1} qr\right) \end{array}\right] (m-r)\,, \tag{12}$$

where I is the identity matrix, $A = \Sigma^{-1}\Delta_p\mu$, $B = \Gamma^{-1}\Delta_q\eta$, $f(H) = \Delta_V D - Hq$, and, finally, δ_r is the determinant of the following 2×2 matrix:

$$\left[\begin{array}{cc} A'\Sigma A + B'\Gamma B - (p'A - q'B)\,r + \left(p'\Sigma^{-1}q + q'\Gamma^{-1}q\right) r^2 & \left(B' - q'\Gamma^{-1} r\right) f(H) \\ f(H)' \left(B - \Gamma^{-1} qr\right) & f(H)'\,\Gamma^{-1} f(H) \end{array}\right].$$

We can see that the optimal portfolio compositions x^* and y^* are linear functions of m and are proportional to the ratio between two quadratic polynomials in H.

We have to underline that the vector function $f(H)$ has a precise meaning: it measures the distance between the duration of each bond and the investor's financial horizon H. This distance can be negative and it is measured in "time-money" intensity. From the second constraint in problem (11)[13] we can see that the optimal composition y^* must equate this distance to zero.

---

[11] As we have shown in the previous section, matrix Γ can stands for the diagonal matrix of bond prices or for the discrete time version of Barber and Copper matrix in formula (9).

[12] The following proposition can be checked by applying the formula in the footnote 2 to problem (11).

[13] This constraint can be written as: $y'f(H) = 0$.

5.1 The portfolio with one stock and one bond

In this section we point out the case in which the objective function no more matters. In problem (11), we have two equality constraints and so, if there exist only one bond and one stock, then the objective function is irrelevant. Indeed, the solution can be found just by inverting the matrix of the constraints and the problem becomes as follows:

$$\begin{bmatrix} p_1(\mu_1 - r) & q_1(\eta_1 - r) \\ 0 & V_1 D_1 - H q_1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} m - r \\ 0 \end{bmatrix},$$

whose solution is:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \frac{m-r}{p_1(\mu_1 - r)} \\ 0 \end{bmatrix}.$$

Because we cannot match the duration of the unique bond with the duration of other bonds, it is senseless to buy this bond (unless its duration equals H, and in this case the investor must choose the asset with the highest return). The optimal stock weight has just to satisfy the constraint: $x_1 p_1(\mu_1 - r) = m - r$.

5.2 The separation theorem: only a particular case

In order to check if the separation theorem holds also in our framework, we can write the optimal portfolio composition as a linear function of the return m, in the following way:

$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \alpha(H) + \beta(H)\, m,$$

where the Greek coefficients are computed from matrices in (12).

By representing in the same way the case in which there does not exist any risk-free interest rate, we obtain the same qualitative result that can be written as:

$$\begin{bmatrix} x^*_{\setminus r} \\ y^*_{\setminus r} \end{bmatrix} = \alpha_{\setminus r}(H) + \beta_{\setminus r}(H)\, m.$$

Given a fixed value for H (i.e. all investors have the same horizon), then the coefficients of this two linear functions are constant. Thus, as in Section 2, we can find a unique value of m for which the two vectors $\begin{bmatrix} x^{*\prime} & y^{*\prime} \end{bmatrix}'$ and $\begin{bmatrix} x^{*\prime}_{\setminus r} & y^{*\prime}_{\setminus r} \end{bmatrix}'$ coincide. This value is the abscissa of tangent point between loci (3) and (6) and it is given by:

$$m_P = \frac{\alpha(H) - \alpha_{\setminus r}(H)}{\beta_{\setminus r}(H) - \beta(H)},$$

and, accordingly, the market portfolio has the following composition:

$$\left[\begin{array}{c} x_P^* \\ y_P^* \end{array}\right] = \alpha(H) + \beta(H)\frac{\alpha(H) - \alpha_{\backslash r}(H)}{\beta_{\backslash r}(H) - \beta(H)} = \alpha_{\backslash r}(H) + \beta_{\backslash r}(H)\frac{\alpha(H) - \alpha_{\backslash r}(H)}{\beta_{\backslash r}(H) - \beta(H)}. \tag{13}$$

In this case, we have reached the result of the separation theorem for which, given the return m_P of the market portfolio, all investors choose the same composition for their risky portfolio. Nevertheless, this result is true only if H is fixed. In fact, we can see that m_P is no more a function of given parameters but it becomes a function of the preference parameter H. So, we can state the following:

Proposition 9 (General Separation Theorem) *: In the M-F-W framework all investors hold the same composition (equation 13) for their stock and bond portfolio only if they have the same investment horizon (H) for their bond portfolio.*

In this way, we claim that the original separation theorem is only a particular case of a more general one. It holds if we divide investors into different categories according to their horizon H, and take different optimal portfolio for each category.

5.3 Asymptotic behaviour of optimal portfolio composition

In this section we compute the algebraic solutions for the two limit cases concerning the horizon H:

$$\begin{array}{lccc} \text{a)} & H & = & 0, \\ \text{b)} & H & \to & \infty. \end{array}$$

After long but easy computations we obtain the following results:

$$\begin{aligned} \lim_{H\to 0}\left[\begin{array}{c} x^* \\ y^* \end{array}\right] &= \frac{m-r}{g(r)\,D'\Delta_V\Gamma^{-1}\Delta_V D - \left[(\eta'\Delta_q - q'r)\,\Gamma^{-1}\Delta_V D\right]^2} \qquad (14) \\ &\cdot\left[\begin{array}{c} D'\Delta_V\Gamma^{-1}\Delta_V D\Sigma^{-1}(\Delta_p\mu - pr) \\ \left[D'\Delta_V\Gamma^{-1}\Delta_V DI - \Gamma^{-1}\Delta_V DD'\Delta_V\right]\Gamma^{-1}(\Delta_q\eta - qr) \end{array}\right], \\ \lim_{H\to\infty}\left[\begin{array}{c} x^* \\ y^* \end{array}\right] &= \frac{m-r}{g(r)\,q'\Gamma^{-1}q - \left[(D'\Gamma - q'r)\,\Gamma^{-1}q\right]^2} \qquad (15) \\ &\cdot\left[\begin{array}{c} q'\Gamma^{-1}q\Sigma^{-1}(\Delta_p\mu - pr) \\ \left[q'\Gamma^{-1}qI - \Gamma^{-1}qq'\right]\Gamma^{-1}(\Delta_q\eta - qr) \end{array}\right], \end{aligned}$$

where:

$$g(r) = \mu'\Delta_p\Sigma^{-1}\Delta_p\mu + \eta'\Delta_q\Gamma^{-1}\Delta_q\eta - 2(p'\Sigma^{-1}\Delta_p\mu - q'\Gamma^{-1}\Delta_q\eta)r + \\ +(p'\Sigma^{-1}p + q'\Gamma^{-1}q)r^2.$$

By comparing the two limit cases (14) and (15) we see that they can be equal only in a very particular case. In fact, formulae (14) and (15) coincide if:

$$\begin{cases} q = \Delta_V D, \\ D = \Gamma^{-1}\Delta_q\eta, \end{cases}$$

but, since in an efficient market the condition $q = V$ must hold, then the first equation of the system becomes $D = e$, that is to say that all bonds should have a duration of one period. This condition is very unlikely. Accordingly, we can state:

Proposition 10 *In the M-F-W framework the investment strategy for an investor with infinite time horizon equals the strategy for an investor with zero time horizon only if each bond has one period duration.*

In the following section we will investigate the sign of portfolio compositions (14) and (15) and we will try to answer the question: does it exist a time H for which there is no interest in holding bonds?

5.4 The investment horizon as a measure of risk aversion (a solution to the asset allocation puzzle)

We have not yet introduced any "risk aversion" measure, nevertheless, we can find in our framework something that can be viewed as an indicator of risk aversion: the variable H. In fact, it is quite intuitive to imagine that:

1. investors who are more risk averse choose little H, that is to say they prefer to immunize their portfolio for a short period of time in order to have their money available as soon as possible;

2. investors who are less risk averse choose big values for H because they are less afraid of the economic future and they are more willing to bet on higher and riskier returns.

We underline that, under the hypotheses of the M-F-W framework, the horizon H is the time at which the portfolio uncertainty is resolved. In fact, the bond subset of the optimal portfolio is immunized until the time H. Thus, we can find a support for the hypotheses in points 1. and 2. in the work of Epstein and Zin (1989). They develop a class of recursive preferences over intertemporal consumption permitting risk attitudes to be disentangled from the degree of intertemporal substitution. The authors use a utility function of the form:

$$U(c,z) = \frac{1}{\alpha}\left[c^{\rho} + \beta(\alpha z)^{\frac{\rho}{\alpha}}\right]^{\frac{\alpha}{\rho}},$$

where c is the present consumption while z is the present expected value of future consumption. The positive parameter α can be interpreted as a risk aversion parameter and ρ as an index of intertemporal substitution. Epstein and Zin find that the curvature of $U(c,z)$ is the determinant of attitudes towards timing with indifference towards timing prevailing only if $U(c,z)$ is linear, while early (late) resolution is preferred if α is less (greater) than ρ. This means that higher the risk aversion lower the investor's time horizon.

From equation (12) we can compute the amount of money invested in stocks and bonds: respectively $x^{*\prime}p$ and $y^{*\prime}q$. Then, in order to explain the investor's behaviour analyzed by Canner, Mankiw and Weil (1997), we should find a ratio of bonds to stocks decreasing with respect to H. We call this function "preference for bonds" (P_B) whose value can be computed from (12) as follows:

$$P_B(H) = \frac{q'y^*}{p'x^*} = \frac{q'\left[f(H)'\Gamma^{-1}f(H)I - \Gamma^{-1}f(H)f(H)'\right]\left(B - \Gamma^{-1}qr\right)}{f(H)'\Gamma^{-1}f(H)p'\left(A - \Sigma^{-1}pr\right)}.$$

Because we want to outline the dependence of P_B on the investor's horizon H, then this ratio can be simplified in the following way:

$$P_B(H) = \frac{n_3}{d_3} - \frac{n_1 + n_2H + n_3H^2}{d_1 + d_2H + d_3H^2}, \tag{16}$$

where:

$$\begin{aligned}
n_1 &= (q'\Gamma^{-1}\Delta_V D)D'\Delta_V\Gamma^{-1}(\Delta_q\eta - qr),\\
n_2 &= -[(q'\Gamma^{-1}\Delta_V D)q' + (q'\Gamma^{-1}q)D'\Delta_V]\Gamma^{-1}(\Delta_q\eta - qr),\\
n_3 &= (q'\Gamma^{-1}q)q'\Gamma^{-1}(\Delta_q\eta - qr),\\
d_1 &= D'\Delta_V\Gamma^{-1}\Delta_V Dp'\Sigma^{-1}(\Delta_p\mu - pr),\\
d_2 &= -[D'\Delta_V\Gamma^{-1}q + q'\Gamma^{-1}\Delta_V D]p'\Sigma^{-1}(\Delta_p\mu - pr),\\
d_3 &= (q'\Gamma^{-1}q)p'\Sigma^{-1}(\Delta_p\mu - pr).
\end{aligned}$$

Obviously, if $P_B > 1$ then the investor prefers bonds in the sense that he invests more money in bonds than in stocks. On the contrary, if $P_B < 1$ then the investor puts more money in stocks than in bonds.

It is interesting to underline that the ratio P_B does not depend on m, and this means that the relative amount of money that must be invested in bonds with respect to stocks is not affected by the return the investor wants to obtain from his portfolio. Accordingly, we can state:

Proposition 11 *In the M-F-W framework the ratio of wealth invested in bonds to wealth invested in stocks* ($q'y^*/p'x^*$) *is independent of the rate of return investor wants to obtain from his portfolio* (m).

Unfortunately, the derivative of P_B with respect to H has not a clear sign (and no more the second derivative). In fact, this indetermination can be seen in the following formula:

$$\frac{\partial P_B}{\partial H} = \frac{(n_1 d_2 - n_2 d_1) + 2\,(n_1 d_3 - n_3 d_1)\,H + (n_2 d_3 - n_3 d_2)\,H^2}{\left(d_1 + d_2 H + d_3 H^2\right)^2},$$

from which we are not able to determine if the function $P_B\,(H)$ is monotonic. Thus, it is interesting to check the behaviour of index P_B in the two limits $H \to 0$ and $H \to \infty$. From (16) we obtain:

$$\begin{aligned}
\lim_{H\to 0} P_B &= \frac{n_3}{d_3} - \frac{n_1}{d_1} = \qquad (17)\\
&= \frac{q'\Gamma^{-1}\left(\Delta_q \eta - qr\right)}{p'\Sigma^{-1}\left(\Delta_p \mu - pr\right)} - \frac{\left(q'\Gamma^{-1}\Delta_V D\right) D'\Delta_V \Gamma^{-1}\left(\Delta_q \eta - qr\right)}{\left(D'\Delta_V \Gamma^{-1}\Delta_V D\right) p'\Sigma^{-1}\left(\Delta_p \mu - pr\right)},\\
\lim_{H\to \infty} P_B &= \frac{n_3}{d_3} - \frac{n_3}{d_3} = 0. \qquad (18)
\end{aligned}$$

By recalling our result in Proposition 10, according to which the two limits $\lim_{H\to 0}\left[\begin{array}{cc} x^{*\prime} & y^{*\prime}\end{array}\right]'$ and $\lim_{H\to\infty}\left[\begin{array}{cc} x^{*\prime} & y^{*\prime}\end{array}\right]'$ cannot be equal unless each bond has the same one period duration, then result (17) implies that: $\lim_{H\to 0} P_B \neq 0$. Accordingly, we can state:

Proposition 12 *In the M-F-W framework when the investor's financial horizon equals zero* ($H = 0$) *he never holds an arbitrage bond subset portfolio* ($q'y^* \neq 0$).

Obviously, Proposition 12 must not be misinterpret: when we state that for $H = 0$ it is true that $q'y^* \neq 0$, we are not stating that $q'y^*$ is high enough to be non-negligible.

Result (18) means that if the investor does not care about his financial horizon (that is to say he does not want to obtain a certain amount of money at a fixed date), then he puts all his money in stocks and uses an arbitrage portfolio for bonds. Thus, we can state:

Proposition 13 *In the M-F-W framework if the time horizon of an investor tends to infinity, then he holds an arbitrage bond subset portfolio* ($q'y^* = 0$) *and he invests all his wealth in stocks.*

There is an obvious consideration to underline: Proposition 13 holds if it is possible to buy short, if not we can restate the proposition in the following way:

Proposition 14 *In the M-F-W framework, if it is not possible to buy short and if investor's time horizon tends to infinity, then he does not hold any bond* ($y^* = 0$) *and he invests all his wealth in stocks.*

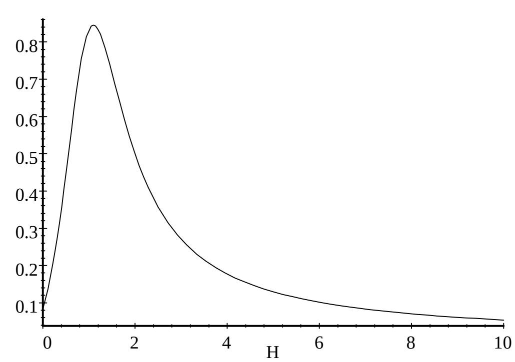
Accordingly, for an investor whose financial horizon tends to infinity, we can restate the maximization problem (11) in the following way:

$$\begin{array}{l} \min_{x,y} \left[\begin{array}{cc} x' & y' \end{array} \right] \left[\begin{array}{cc} \Sigma & \underline{0} \\ \underline{0} & \Gamma \end{array} \right] \left[\begin{array}{c} x \\ y \end{array} \right] \quad s.t. \\ \left[\begin{array}{cc} x' & y' \end{array} \right] \left[\begin{array}{cc} \Delta_p \mu - pr & \underline{0} \\ \Delta_q \eta - qr & q \end{array} \right] = \left[\begin{array}{cc} m - r & 0 \end{array} \right], \end{array} \tag{19}$$

where we have substituted q for $\Delta_V D - Hq$.

Now, thanks to Propositions 13 and 14, we are able to explain the asset allocation puzzle found by Canner, Mankiw and Weil (1997). They claim that financial advisors suggest aggressive investors hold a lower ratio of bonds to stocks than conservative investors and this advice is inconsistent with Theorem 1 (separation theorem). If H can be considered as a good proxy for the investor's aggressiveness, then our result stated in Proposition 9 is consistent with financial advisors' suggestion. In fact, when H is high enough (that is to say when we consider an aggressive investor) the ratio P_B measuring the preference for bonds is low.

Figure 1: Index of bond preference $P_B(H)$



A numerical example can confirm this result. By putting, for instance, $n_1 = 1$, $n_2 = -3$, $n_3 = 1$, $d_1 = 4$, $d_2 = -5$, and $d_3 = 3$, we find the behaviour of

P_B represented in Figure 1. We can see that the preference for bonds has a maximum level corresponding to a time horizon we will call H^*.

A more articulated numerical example is considered in Appendix A where the behaviour of function P_B shown in Figure 1 is confirmed.

5.5 The "equity premium puzzle": a proposal for a solution

According to Proposition 11 in the previous section, we can propose a solution for the well known "equity premium puzzle". This puzzle has been outlined by Mehra and Prescott (1985) who find that from 1889 to 1978 on the New York Stock Exchange, the average annual return for the S&P500 index is about 7%, while the average annual real return for bonds is about 1%. The authors claim that this difference (about 6%) cannot be explained through a reasonable level of risk aversion and a lot of attempts have been made in order to solve this puzzle. One of the most fruitful attempt was aimed at defining more accurate forms for investors' utility function (see Weil (1989), and Epstein and Zin (1990 - 1991)). Other attempts concerned the restatement of the calibration methodology for describing first and second moments of asset returns (Cecchetti, Lam, and Mark (1993)) or the inclusion of the inflation risk into the asset risk measure (Bagliano e Beltratti (1997)). Also the problem of market crashes has been considered (Rietz (1988) and Brown, Goetzmann, and Ross (1995)). In particular, Brown, Goetzmann, and Ross (1995) find that all markets survived to wars and other political or economic troubles, have an index characterized by mean reversion. For a short review of this literature see Campbell (2000) who outlines the importance of each contribution.

In our framework, we are able to present at least a reason for which the great difference between returns on stocks and bonds does not shrink as a simple arbitrage argument could imply.

In fact, in the M-F-W framework, after choosing their optimal allocation between stocks and bonds, investors never change it even if they want to obtain a greater return (m) from their portfolio (see Proposition 11). So, for every long time period, if the time horizon of investors (H) and the market parameters ($\mu, \eta, \Sigma, \Gamma, p, q, V, D$) remain stable, also the ratio P_B does not change (that is to say for investors there are no reasons for altering their portfolio composition).

Thus, by using Proposition 11 we claim that, if the objective of investors is described by the M-F-W hypotheses and the problem parameters do not change (at least do not significantly change from a statistical point of view), then the equity premium behaviour is consistent with our framework developed so far. In fact, even if on the market there is a high difference between stock and bond returns, it doesn't vanish.

Unfortunately, the data after Mehra and Prescott show a great variation in parameters during the considered period (as it can be seen in Table 2), and because the 6% difference is not constant during the period 1889-1978, then we cannot directly apply Proposition 11. Nevertheless, thanks to the functional

Figure 2: Returns on stocks and bonds, and consumption growth rate

Period	*Growth (%) for real per capita*		*Return (%) on bonds (B)*		*Return on S&P 500 index (A)*		*Difference (A) - (B)*	
	Mean	*Std.Dev.*	*Mean*	*Std.Dev.*	*Mean*	*Std.Dev.*	*Mean*	*Std.Dev.*
1889-1978	1.83	3.57	0.8	5.67	6.98	16.54	6.18	16.67
Std. Error	(0.38)		(0.6)		(1.74)		(1.76)	
1889-1989	2.30	4.90	5.80	3.23	7.58	10.02	1.78	11.57
1899-1908	2.55	5.31	2.62	2.59	7.71	17.21	5.08	16.86
1909-1918	0.44	3.07	-1.63	9.02	-0.14	12.81	1.49	9.18
1919-1928	3.00	3.97	4.30	6.61	18.94	16.18	14.64	15.94
1929-1938	-0.25	5.28	2.39	6.50	2.56	27.90	0.18	31.63
1939-1948	2.19	2.52	-5.82	4.05	3.07	14.67	8.89	14.23
1949-1958	1.48	1.00	-0.81	1.89	17.49	13.08	18.30	13.20
1959-1968	2.37	1.00	1.07	0.64	5.58	10.59	4.50	10.17
1969-1978	2.41	1.40	-0.72	2.06	0.03	13.11	0.75	11.64

Source: Mehra and Prescott [1985], p 147.

form of index P_B, we are still able to find a solution to the equity premium puzzle.

In particular, we suppose to start in a situation where the investor's financial horizon H equals H^* (the horizon giving the maximum P_B), and the changes in parameters are such that the maximum of the curve P_B oscillates around the investor's financial horizon. Under these hypotheses, we can see from Figure 1 that in both cases when $H > H^*$ and $H < H^*$ the optimal behaviour for investor is to sell bonds for buying stocks. In this way, the high difference between returns on stocks and bonds is maintained in spite of any arbitrage argument.

The above-mentioned hypotheses are not too restrictive. Thanks to the numerical example presented in the appendix, we can see that the point H^* is near the portfolio duration. Because this duration reflects the preference of investors with respect to the immunization length for the bond subset of their portfolio, then, we can suppose that the hypothesis of having a value of H^* oscillating around the investors' actual financial horizon H is quite likely. In fact, the optimal portfolio composition we have computed represents the investors' asset demand, and if supply conforms itself to demand, then, in equilibrium, it must be true that $H = H^*$. Each shift from this equilibrium implies that investors sell bonds maintaining the high difference between stock and bond returns.

6 Conclusion

In our work we have presented a model aimed at considering in a single problem the two portfolio decisions concerning investment in stocks and bonds.

We consider a framework in which investors hold bonds for immunizing, at a certain time, a part of their portfolio against interest rate fluctuations. In particular, we use the approach of duration matching.

We introduce an index for measuring the investors' preference for bonds. We find that this index is not a monotonic function of investors' financial horizon (H) but it is an increasing function for low values of H and it becomes a decreasing function for high values of H. Thus, if time horizon is considered as a proxy of investors' risk aversion, then, for a sufficiently high value of H, our model is consistent with the financial advisors' strategies which, instead, are inconsistent with the separation theorem. In fact, we have shown that in our approach the separation theorem is valid only if all investors have the same financial horizon but this condition is even more demanding than that prescribing homogeneous expectations on asset prices.

Thus, our work confutes the separation theorem and solves the inconsistency between practice and theory often underlined by the literature.

Finally, thanks to the index of preference for bonds developed in this paper, we can also propose a possible solution for the equity premium puzzle. A simple arbitrage argument shows that the financial horizon giving the maximum value for the bond preference index should oscillate around investors' actual financial horizon. This allows us to argue that after each shift in market parameters, investors are willing to sell their bonds. This behaviour contributes to maintain a high difference between returns on stocks and bonds.

A A numerical example

In this appendix we consider a numerical example for investment problem (11). In order to avoid inconsistencies, we consider the year as the time measure unit. For the stock subset we take four stocks whose values can be represented, for eight periods, in the pay-off matrix Y (Y_{ti} = pay-off of stock i in period t) to which corresponds the return matrix R containing the percentage changes in the elements of Y (and more precisely: $R_{ti} = Y_{t+1,i}/Y_{ti} - 1$):

$$
\begin{aligned}
Y &= \begin{bmatrix} 100 & 100 & 100 & 100 \\ 108 & 105 & 107 & 106 \\ 113 & 111 & 118 & 111 \\ 110 & 110 & 115 & 108 \\ 121 & 119 & 120 & 119 \\ 118 & 119 & 118 & 124 \\ 130 & 124 & 125 & 130 \\ 137 & 136 & 130 & 138 \end{bmatrix}, \\
R &= \frac{1}{100}\begin{bmatrix} 8.0 & 5.0 & 7.0 & 6.0 \\ 4.6296 & 5.7143 & 10.28 & 4.717 \\ -2.6549 & -.9009 & -2.5424 & -2.7027 \\ 10.0 & 8.1818 & 4.3478 & 10.185 \\ -2.4793 & 0 & -1.6667 & 4.2017 \\ 10.169 & 4.2017 & 5.9322 & 4.8387 \\ 5.3846 & 9.6774 & 4.0 & 6.1538 \end{bmatrix}.
\end{aligned}
$$

If we consider the first period as period zero, then we can see that in period 3 there is a financial crisis during which the price growth is broken. Instead, in period 5 the price of stock 4 increases while the other prices do not increase (in this case stock 4 is very appealing). The vector μ in problem (11) contains the mean value of each column of matrix R and the matrix Σ is the variance and covariance matrix associated with R. Accordingly:

$$
\begin{aligned}
\mu &= \frac{1}{100}\begin{bmatrix} 4.7213 & 4.5535 & 3.9073 & 4.7705 \end{bmatrix}', \\
\Sigma &= \frac{1}{1000}\begin{bmatrix} 2.9166 & 1.5836 & 1.8447 & 1.5521 \\ 1.5836 & 1.5262 & 1.1783 & 1.1758 \\ 1.8447 & 1.1783 & 2.1191 & .95806 \\ 1.5521 & 1.1758 & .95806 & 1.482 \end{bmatrix}.
\end{aligned}
$$

For the first subset of investor's portfolio we still lack stock prices. Since the minimization problem for stocks is based on past values, then we can consider the last row in matrix Y as the (transposed) vector of last prices:

$$p = \begin{bmatrix} 137 & 136 & 130 & 138 \end{bmatrix}'.$$

Now, for bond portfolio we can consider the following (spot) interest rate structure: $r(0,t) = 0.03 + 0.001t$, and we can define a cash flow matrix (C) in which each element C_{it} represents the coupon paid on bond i at time t. For instance, we can have:

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 \\ 0 & 12 & 0 & 12 & 0 & 12 & 0 & 112 \\ 5 & 5 & 5 & 5 & 5 & 105 & 0 & 0 \end{bmatrix},$$

where we have four bonds and eight periods. The first two bonds are zero coupon bonds. By applying the following formulae:

$$\begin{aligned} V_i &= \sum_{t=1}^{T} C_{it} \left[1 + r(0,t)\right]^{-t}, \\ D_i &= \frac{\sum_{t=1}^{T} t C_{it} \left[1 + r(0,t)\right]^{-t}}{\sum_{t=1}^{T} C_{it} \left[1 + r(0,t)\right]^{-t}}, \end{aligned}$$

we can compute bond values and durations:

$$\begin{aligned} V &= \begin{bmatrix} 84.1973 & 74.2030 & 114.5781 & 107.5884 \end{bmatrix}', \\ D &= \begin{bmatrix} 5 & 8 & 6.8741 & 5.3532 \end{bmatrix}'. \end{aligned}$$

In order to find the vector η of bond returns, we have to solve the following equations:

$$V_i = \sum_{t=1}^{T} C_{it} \left[1 + \eta_i\right]^{-t}.$$

For the two zero coupon bonds returns coincide with the interest rates $r(0, D_i)$ and so:

$$\eta = \frac{1}{100} \begin{bmatrix} 3.5 & 3.8 & 3.747 & 3.5651 \end{bmatrix}'.$$

We suppose that the market is efficient and so: $q = V$ and in order to complete the bond parameters let $\Gamma = \Delta_q \, (= \Delta_V)$. We suppose that the riskless interest rate is $r = 0.02$. Thus, in this framework: $\mu_i > \eta_j > r \quad \forall i, j \in [1, 4]$. Now, we have all the elements in order to compute the solution for problem

(11). This solution is given by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \delta \begin{bmatrix} -15085 + 4674H - 373.98H^2 \\ 31881 - 9878.1H + 790.37H^2 \\ -265.89 + 82.386H - 6.5919H^2 \\ 69882 - 21653H + 1732.5H^2 \\ 628 - 3.0288H - 11.511H^2 \\ -837.28 + 88.628H + 11.324H^2 \\ -104.54 - 2.4058H + 7.2844H^2 \\ 546.9 - 20.542H - 6.5596H^2 \end{bmatrix},$$

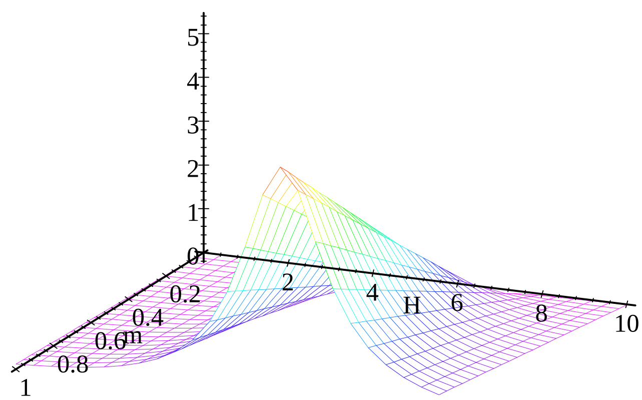
$$\delta = \frac{50m - 1}{160.69 - 49.692H + 3.9811H^2} 10^{-5}.$$

Graphically, for the bond subset of optimal portfolio we can see, from figure 3, that:

- there exists a critical value of H (say H^*); when $H \neq H^*$ bond portfolio rapidly falls to zero and even when $H = H^*$ the bond weight is very low;

- the weight of bond portfolio is an increasing function of the expected return on the whole portfolio (m).

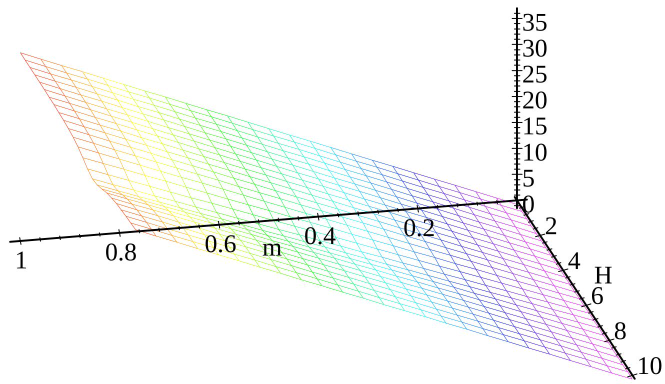
Figure 3: Weight of bond subset portfolio ($q'y^*$)



For the stock subset of the optimal portfolio the result is summarized by Figure 4 where we can see that its weight does not heavily depend on the time horizon H.

We can see from Figure 5 (where $m = 0.04$) that $p'x^*$ depends on H according to a relation that is the inverse of those describing the behaviour of $q'y^*$.

Figure 4: Weight of stock subset portfolio ($p'x^*$)



Nevertheless, H affects $p'x^*$ in a negligible way (as we can observe from the ordinate values).

Finally, in Figure 6, we can see the behaviour of index P_B (preference for bonds). Here we confirm the behaviour described in Figure 1. In particular, there exists (due to the behaviour of bond portfolio) a critical value of time horizon (H^*) for which the total amount of wealth invested in bonds is maximum (and in this example wealth invested in bonds must not exceed 13.79%[14] of wealth invested in stocks and bonds).

[14] If we call W_A and W_B the wealth invested respectively in stocks and bonds, then $P_B = W_B/W_A$ and so, for a given value of P_B ($\overline{P_B}$) the ratio between the wealth invested in stocks and the wealth invested in stocks and bonds is: $\frac{W_A}{W_A+W_B} = \frac{1}{1+\overline{P_B}}$.

Figure 5: Weight of stock subset portfolio (as a function of H)

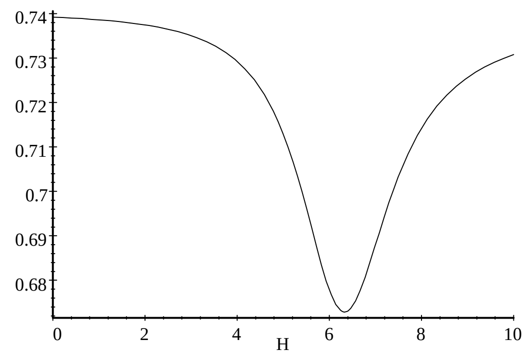
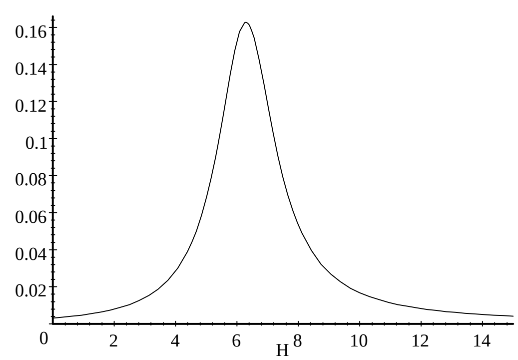


Figure 6: Index of bond preference $P_B\left(H\right)$



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