Altruism and the poverty risk in old-age

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Abstract

This paper modifies Blanchard's (1985) model to build an OLG model with an increasing probability of death, endogenous growth and a bequest motive. The motivation is to obtain a more rich, realistic and flexible framework to reproduce -using numerical methods- some stylised facts of the age-profiles of education, consumption, and wealth variables. Moreover, this model can be used like Auerbach and Kotlikoff's (1987) to simulate the impact of economic shocks in the transition between steady states.

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This paper presents an OLG model with an increasing death probability, endogenous growth and a bequest motive to generate age-profiles for education, consumption, and wealth variables. The model is based on Blanchard (1985), Auerbach and Kotlikoff (1987), and Azariadis and Drazen (1990).

Blanchard (1985) assumes a constant probability of death, whereas in this paper the probability of death is age-dependent. The former assumption was made to obtain reduced forms for aggregate variables, whereas the latter is necessary to derive realistic age-profiles for consumption and wealth variables, although requiring numeric solutions. As it is well-known since Yaari (1965), the assumption of lifetime uncertainty raises a difficulty, because an individual can die with negative financial wealth. Blanchard (1985) solves this problem by introducing competitive insurance companies that raise the interest rate by an amount equal to the (constant) death probability in exchange for the financial wealth of the deceased. In the present model, with a bequest motive operating from the "parent's" to the respective children's generation, the competitive insurance companies are assumed, instead, to redistribute the financial wealth positions of the deceased among the children of that generation.

The present model builds a general equilibrium framework, which can be used to simulate the transition between steady states, like in Auerbach and Kotlikoff's model (AK, 1987). It makes two significant improvements on the AK's setting: (1) human capital is endogenously determined, whereas in the AK's model it is exogenous; and, (2) there is lifetime uncertainty and altruism, whereas in the AK's model, an individual lives for a fixed number of periods and there is no bequest motive. However, in order to obtain a closed form equation for the accumulation of human capital, it is necessary to assume that the period utility function is isoelastic on consumption levels only, whereas the AK's model uses a nested CES utility function defined on consumption and leisure. The latter model generates an elastic labour supply, whereas the present model substitutes the margin of decision between work and leisure by the education/work margin.

As in Lucas (1988) and in Azariadis and Drazen (1990), the engine of growth is education, and balanced growth is possible, because human capital is produced using a CRS production function and is transmitted across generations. The latter generates an externality in the education sector.

Summing-up, at birth an individual inherits a financial endowment resulting from a (voluntary) bequest, and a human capital endowment, which is assumed to be a fraction of the per capita average stock of human capital in the active population.

This OLG model generates a declining age-profile for the length of time spent in human capital activities. This fits the stylised facts which were highlighted in the seminal work of Becker (1965) and Ben-Porath (1967) ([1]).

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[1] Becker writes: "...The amount of time spent investing in human capital would tend to decline with age for two reasons. One is that the number of remaining periods (*in the labour market*), and thus the present value of future returns, would decline with age. The other is that the (*opportunity*) cost of investment would tend to rise with age as human capital rose because foregone earnings would rise (1975, pp. 64-65).

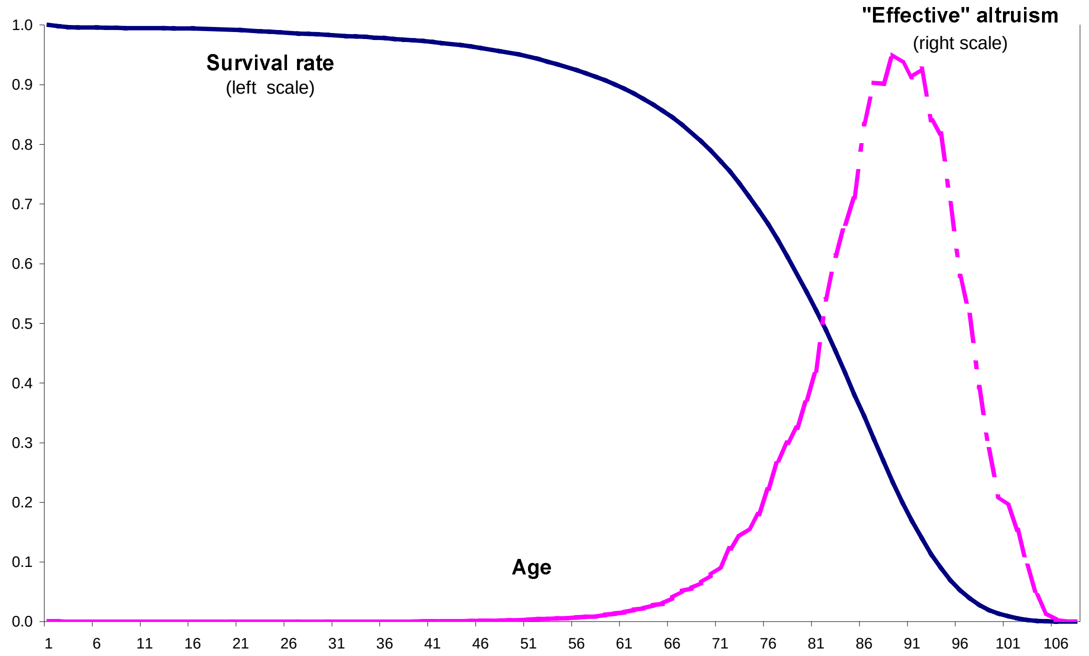
As in Mincer (1974), the present model assumes that an individual (while in the labour force) allocates time between education and the production of goods. Therefore, human capital can be accumulated either through formal education or productivity-augmenting investments in human capital carried out after completion of schooling (e.g. vocational training). This broad definition of investment in human capital is necessary to explain the variability of age-earning profiles in actual economies.

In the current model, human capital/the amount of time allocated to education is determined by two factors: (1) the ratio between the present value of future labour earnings and the opportunity cost of investing in education (the wage rate); and, (2) the externality in education that is captured by the ratio between the per capita average stock of human capital in the active population and the individual stock. Ageing reduces both ratios, and with it, the optimal amount of time spent in education, because on the one hand, the number of periods to recoup the investment in education declines, and on the other, the ratio between the per capita average stock of human capital in the active population and the individual stock falls as an individual accumulates human capital, thereby reducing the strength of the externality affecting education.

One main insight provided by this model is that the decline in old-age consumption -due to an increasing mortality rate- can be offset provided that an individual cares sufficiently about the welfare of a member of her children's generation. If there was no or only a weak bequest motive, consumption and financial wealth holdings would become unrealistically low in old-age. A reasonably strong bequest motive suffices to counteract the effect due to the decline in the probability of survival, protecting people against the risk of running out of financial wealth in old-age (i.e. the longevity risk).

For a given level of the bequest motive, the impact of altruism on the profile of consumption raises with age (Graph 1) -particularly between 60 and 90 years-, reflecting the fact that, with age, the (marginal) value of a bequest raises in relation to that of financing future own consumption due to an increasing mortality rate. Nevertheless, notice that as the probability of survival goes to zero when age approaches the maximum possible age, an individual reduces again the (effective) weight of her bequest motive, because the probability of still being alive (in very old-age) to pass on her wealth becomes negligible.

Graph 1 : The survival rate and "effective" altruism ([2])



This model can be used to examine the consequences of a liquidity constraint, and the policy implications that follow.

This paper is organised as follows. Section 2 derives the individual's optimal programme. Section 3 states the usual conditions characterising a competitive equilibrium, and presents the general equilibrium conditions. In section 4, calibrations are calculated for the bequest and no bequest cases that reproduce some stylised facts. Section 5 generates age-profiles. Section 6 concludes. Annex I shows that the individual's optimisation programme is separable. Annex II derives a sufficient condition for the existence of the value function. Annex III develops the optimal conditions for the allocation of human capital. The Appendix derives the individual's optimal programme, corresponding to a liquidity constraint and no altruism.

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The present model develops a lifecycle framework to generate age-profiles for education, consumption, and wealth variables. The major assumptions are: (i) lifetime uncertainty with an age-dependent death probability (an individual can live for a maximum of T years); (ii) human capital is determined endogenously; and, (iii) altruism.

An individual maximises an objective function subject to a wealth constraint and a production function for human capital. The objective function incorporates a bequest motive: an individual maximises a weighted sum of her objective function, and that of her "descendant".

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[2] The two curves are based on death tables. As it will be shown below, the impact of altruism on the age profile of consumption follows closely the curb: $d(n)^2 \prod_{l=1}^{n-1} [1 - d(l)]$, where *d(n)* is the probability of death at age *n*.

An individual has children at the fixed age ι, and leaves her financial wealth to the generation of her children when she dies. Given that each age cohort is made-up of a large number of individuals, the initial child's endowment is determined by the choice of the financial wealth profile of a representative parent.

Because leisure is not an argument of the period utility function, the individual's maximisation problem can be divided in two stages (see Annex I): first, the profile of consumption is chosen for a given level of total resources; and, second, time is allocated between education and the production of goods to maximise total resources ([3]). Moreover, separability is necessary to obtain a closed form equation for the accumulation of human capital.

2.1. The death risk and the financial endowment at birth

As it is well-known since Yaari (1965), the assumptions of lifetime uncertainty and perfect capital markets raise a difficulty, because an individual can die with negative financial wealth. Blanchard (1985) solves this problem, in a selfish model with a constant death probability, by introducing competitive insurance companies that earn a premium (on negative financial wealth positions) or pay an interest rate supplement (on positive positions) in exchange for receiving the financial balance of the insured when they die. This insurance scheme raises the interest rate by an amount equal to the (constant) death probability.

In the present model, with a bequest motive operating from the "parent's" to the "descendant's" generation, the competitive insurance companies are assumed, instead, to redistribute the financial wealth positions of the deceased among the children of that generation. Given that each generation is made-up of a large number of individuals, the financial wealth profile of a representative "parent" determines the bequest received by an "anonymous" child.

Let $f(t,v)$ denote the financial wealth in period t of an individual of generation v. The financial endowment at birth is the (per child) expected discounted value of the financial wealth of a representative "parent" when she dies:

$$f(v-1,v)=\frac{1}{b\,(v-\iota)}\sum_{n=1}^{\mathrm{T}}\frac{d\,(n) f\,(n+v-\iota,v-\iota)\left\{\prod_{l=1}^{n-1}\left[1-d\,(l)\,\right]\right\}}{\left\{\prod_{l=v}^{n+v-\iota}R\,(l)\right\}} \tag{1}$$

where $R(l)$ is the interest rate factor between periods $l\text{-}1$ and l; $d(n)$ is the probability of death at age n, with $d(0)=1$, $d(T)=1$, and $0\le d(n)\le 1$; and, $b(v\text{-}\iota)$ (lowercase) is the average number of children per (single) "parent" of generation $v\text{-}\iota$.

2.2. The resource constraints

The flow of income of an adult depends on the amount of time spent at work and the amount of human capital previously accumulated. Let us denote $\varepsilon(t,v)$ as the

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[3] Or equivalently, to maximise human wealth: the discounted value of future labour income net of lump-sum taxes.

fraction of time allocated to education (or retirement), and $h(t,v)$ as human capital. Furthermore, let us introduce the exogenous human capital efficiency factor $e(t-v)$ to facilitate the reproduction of the stylised facts regarding the age-profile of labour earnings ([4]).

The individual's dynamic budget constraint can then be written as:

$$f(t,v) = R(t)\ f(t-1,v) + w(t)\left[1-\varepsilon(t,v)\ \right]e(t-v)\ h\,(t,v) - c(t,v) - \tau(t) \quad for \ \ t \le v + \mathrm{T}$$

where **(2)**

$f(v-1,v)$ is determined by (1)

where $w(t)$ is the wage rate; $c(t,v)$ is consumption; and, $\tau(t)$ is lump-sum taxes.

Financial wealth increases with interest and labour income, and declines with consumption and lump-sum taxes. Labour income is the product of human capital used in production (in efficiency units) and the wage rate. Labour force participation begins/ends at the fixed ages $\underline{a}$ and $\overline{a}$, respectively ([5]); thereby, the previous expression applies in the interval $t \in \left]v+\underline{a}, v+\overline{a}\right]$. In young age and in retirement, the dynamic equation of financial wealth is obtained from (2) by setting $\varepsilon(t,v)$=1.

A no-Ponzi-game (NPG) condition is used to prevent an individual from leaving assets after the maximum possible duration of life (T):

$$f(v+\mathrm{T}, v) = 0 \tag{3}$$

Using the NPG condition, the individual's budget constraint (2) can be integrated forwards to obtain the intertemporal budget constraint:

$$\omega\ (t,v) = \sum_{q=t+1}^{v+\mathrm{T}} \frac{c\,(q,v)}{\left\{\prod_{n=t+1}^{q} R\,(n)\right\}} \tag{4}$$

where $\omega(t,v)$ is total wealth net of lump-sum taxes –the sum of financial and human wealth:

$$\omega\,(t,v) \equiv f\,(t,v) + l\,(t,v)$$

and $l(t,v)$ is human wealth -the discounted value of future labour earnings minus lump-sum taxes.

$$l\,(t,v) \equiv \sum_{q=t+1}^{v+\mathrm{T}} \frac{w\,(q)\left[1-\varepsilon\,(q,v)\ \right]e\,(q-v)\ h\,(q,v) - \tau\,(q)}{\left\{\prod_{n=t+1}^{q} R\,(n)\right\}}$$

Equation (4) states that total resources (ω) equal the present value of lifetime consumption.

The dynamic equation of total wealth net of lump-sum taxes (4) is:

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[4] It is a well-known fact that the typical age profile of labour earnings rises until about 50–55 years of age, declining thereafter. The distinction between human capital (in physical units) and human capital in efficiency units facilitates the reproduction of the above-mentioned stylised fact.

[5] Age intervals are assumed to be left opened and right closed.

$$\omega(t,v) = R(t)\,\omega(t-1,v) - c(t,v)$$
with **(5)**
$$\omega\,(v+\mathrm{T},v) = 0$$

2.3. The value function

An individual maximises the sum of current utility and a discounted weighted sum of two value functions: i) her value function weighted by the probability of survival; and, ii) the value function of a child (fathered by her generation) weighted by the product of the death probability and the strength of the bequest motive.

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with

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$$\max \quad \sum_{t=v}^{v+\mathrm{T}} \beta^{t} \left\{ \prod_{l=v}^{t-1} \left[1 - d\,(l-v)\right] \right\} \frac{c(t,v)^{1-\sigma}}{1-\sigma} \qquad s.t.\ (5) \tag{6a}$$

Problems (6a) and (8) are very similar due to the assumption of an isoelastic utility function and the closeness between the feasibility sets. Therefore, it is natural to expect that the value function that solves (6) is close to that used in Levhari and Srinivasan (1969). An educated guess is ([6]):

$$V\{\omega(t,v)\} = \Xi\,(t,v)\,\frac{\omega\,(t,v)^{1-\sigma}}{1-\sigma} \tag{9}$$

In the present model, the value function is affected by the variable $\Xi(t,v)$ -instead of a constant as in Levhari and Srinivasan (1969)- to generate an increasing age profile for the propensity to consume out of total wealth ([7]). In fact, in models

Using (1), the relation between an anonymous child's total wealth and her "parent's" is ([8]):

$$\frac{d\omega(t,v+\iota)}{d\omega(t,v)}=\frac{d(t-v)\prod_{l=1}^{t-v-1}\left[1-d(l)\right]}{b(v)} \qquad \textbf{(12)}$$

The link between the total wealth of a member of generation $v+\iota$ and that of a member of generation v increases with the ex-ante death probability of the latter -the numerator- and decreases with the number of children per "parent" -the denominator.

Substituting (12) in (11) yields:

$$\begin{aligned}\mathrm{M}\left(t,v,\iota\right)=\ &\beta\left[1-d\left(t-v\right)\right]\Xi(t,v)\\ &+\beta\eta\, d(t-v)^2\frac{\prod_{l=1}^{t-v-1}\left[1-d(l)\right]}{b(v)}\left[\frac{\omega(t,v)}{\omega(t,v+\iota)}\right]^{\sigma}\Xi(t,v+\iota)\end{aligned} \qquad \textbf{(13)}$$

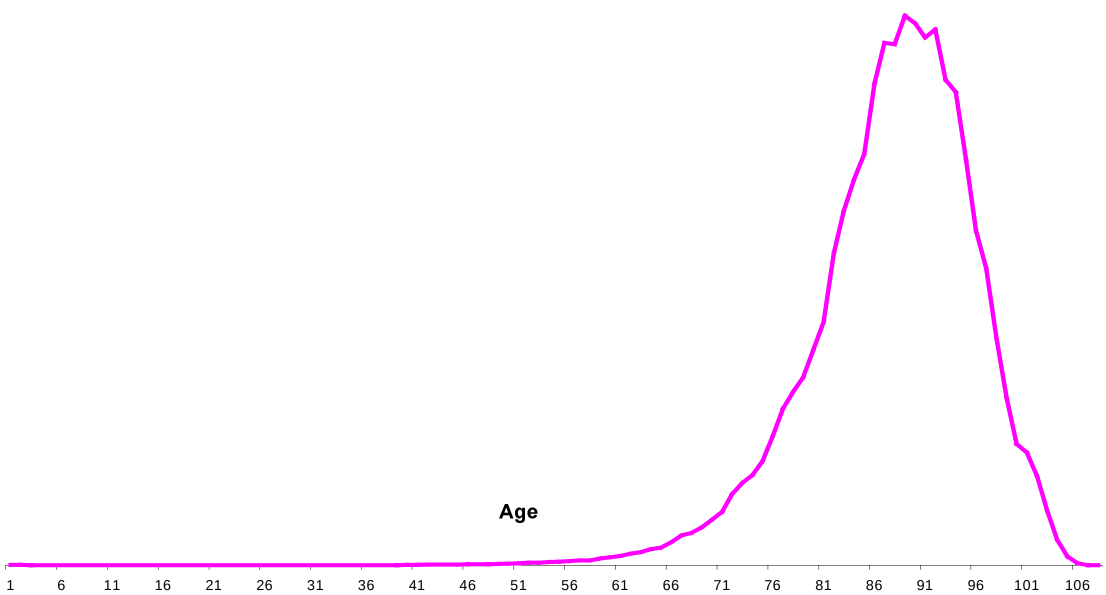
The first and second terms in the right-hand-side of the previous expression reflect the "selfish" and "altruistic" elements in the objective function, respectively. The first term in (13) differs from the fixed value in Levhari and Srinivasan (1969) due to the assumption of an increasing probability of death.

The second term in $\mathrm{M}(t,v,\iota)$, reflecting altruism, is basically determined by two factors: (1) the (overall) strength of the bequest motive (η); and (2) the "demographic" variable: $d(n)^2\prod_{l=1}^{n-1}\left[1-d(l)\right]$. The latter is the product of the (conditional) death probability *d(n)*, and the ex-ante death probability $d(n)\prod_{l=1}^{n-1}\left[1-d(l)\right]$. The first reflects the weighting in (6) of the child's value function by her "parent's" (conditional) probability of death, and the second, according to (12), measures the link between the total wealth of a child and that of a member of her "parent's" generation.

The "demographic" variable gives a pronounced age-profile to the impact of altruism on consumption (Graph 2).

[8] Given that the maximisation problem is separable, the profile of human capital/wealth is given when determining the optimal consumption profile. Therefore, variations in financial wealth (*f*) are equivalent to variations in total wealth (ω).

Graph 2: The "demographic" variable: $d(n)^2 \prod_{l=1}^{n-1} [1 - d(l)]$ ([9])



Therefore, the impact of altruism in the age-profile of consumption increases with age, becoming particularly strong in (very) old-age. However, as the probability of survival goes to zero when age approaches the maximum possible age (T), an individual reduces again the weight of her bequest motive, because the probability of still being alive to pass on her wealth becomes negligible.

Altruism reduces the propensity to consume out of total wealth, countering the opposite effect due to the increasing -with age- probability of death. The bequest motive, if reasonably strong, can eliminate the risk of poverty in old-age (i.e. the longevity risk).

Using (13), when $\sigma \to 1$ and η=0, $\Xi(t,v)$ is closely related to (the inverse of) the propensity to consume out of total wealth: $\mathrm{M}(t,v,\iota) = \beta[1-d(t-v)]\Xi(t,v)$.

It is straightforward to show that when an individual ages (i.e. $t \to v + \mathrm{T}$), the propensity to consume out of total wealth goes to infinity (i.e. $\mathrm{M}(v+\mathrm{T},v,\iota) = 0$), because both $d(\mathrm{T}) = 1$ and $\omega(v+\mathrm{T},v) = 0$ ([10]).

In the general case, replacing the optimal value of consumption (10) in (6), $\Xi(t,v)$ satisfies the following first order difference equation:

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[9] Graph 2 uses the death tables published by the Belgian *Institut National de Statistique* and the *Bureau Féderal du Plan* in "Perspectives de Population 1995-2050", which are adjusted to obtain an average life expectancy at birth (equal for men and women) of 78 years.

[10] The latter is the NPG condition (3).

$$\Xi(t-1,v)\left[\frac{1+\mathrm{M}(t,v,\iota)^{-\frac{1}{\sigma}}}{R(t)}\right]^{1-\sigma} = \mathrm{M}(t,v,\iota)^{-\frac{1-\sigma}{\sigma}}$$

$$+\ \beta\left[1-d(t-v)\right]\Xi(t,v)+\beta\eta\, d(t-v)\,\Xi(t,v+\iota)\left[\frac{\omega(t,v+\iota)}{\omega(t,v)}\right]^{1-\sigma} \tag{14}$$

In the logarithmic case (i.e. $\sigma \to 1$) and solving forwards (14), $\Xi(t,v)$ simplifies to:

$$\Xi(t,v) = \sum_{j=1}^{v+\mathrm{T}-t}\beta^{j-1}\prod_{n=1}^{j-1}\left[1-d(t+n-v)\right]$$

$$+\,\eta\sum_{j=1}^{v+\mathrm{T}-t}\Xi(t+j,v+\iota)\,\beta^{j}d(t+j-v)\prod_{n=1}^{j-1}\left[1-d(t+n-v)\right] \tag{15}$$

where

$$\Xi(t+\mathrm{T},v)=0$$

In the no bequest case (η=0), the first term in (15) can be interpreted as the present value (discounted by the time discount factor β) of an annuity that pays one unit. The second term measures the impact of altruism on (t,v)

$$h\,(t+1,v)=\left(1-\delta_h\right)h\,(t,v)+\xi\left[\,\varepsilon(t,v)e(t-v)h(t,v)\,\right]^{\alpha}\;H(t)^{1-\alpha}$$

where **(16)**

$$0\le\varepsilon(t,v)\le 1\;,\;0\le\delta_h<1\;,\;0<\alpha<1\;,\;\xi>0$$

where δ_h is the depreciation rate of human capital; ξ is a multiplicative factor; and, α is the elasticity of the production function with respect to the private input.

2.6. The optimal profile of human capital

The optimal (private) level of education maximises the discounted value of lifetime earnings net of lump-sum taxes subject to the production function of human capital (16):

$$\max_{\{\varepsilon\}}\; l(v,v)\equiv\sum_{u=v+\underline{a}}^{v+\bar{a}}\frac{w(u)\left[1-\varepsilon(u,v)\right]e(u-v)\,h(u,v)-\tau(u)}{\left\{\prod_{n=v}^{u-1}R(n)\right\}}\qquad s.t.\quad(16)\qquad\textbf{(17)}$$

where $l(v,v)$ is human wealth at birth (i.e. lifetime earnings net of lump-sum taxes).

In an interior solution, the optimal (private) level of education/or the fraction of human capital resources allocated to education/formation activities (ε) is:

$$\varepsilon(t,v)=\left[\frac{\alpha\xi\,\mu(t,v)}{w(t)}\right]^{\frac{1}{1-\alpha}}\frac{H(t)}{e(t-v)\,h(t,v)}$$

with **(18)**

$$\mu\,(v+\bar{a},v)=0$$

where $\mu(t,v)$ is the shadow price of human capital evaluated in period t ([15]).

The amount of time spent in education (ε) varies directly with the ratio of the shadow price of human capital -$\mu(t,v)$- to the opportunity cost of investing in education –$w(t)$ ([16]). The ratio of the per capita average stock of human capital in the active population over the individual stock of human capital (in efficiency units) also increases ε, which reflects the positive externality present in the production function of human capital (16).

Using (18), the optimal (private) age-profile of the amount of time spent in education declines with age for two reasons: (i) the ratio between the shadow price of human capital and the opportunity cost of education decreases with the remaining number of periods in the labour market to recoup the investment in education; and, (ii) the strength of the externality falls with age, because an individual progressively

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[15] See Annex III for the expression of $\mu(t,v)$.

[16] The fraction of time spent in education increases also with ξ (a measure of the productivity of resources used in education), and with α (the elasticity of the production function of human capital with respect to the private input).

accumulates human capital, reducing the ratio between the per capita average stock of human capital in the active population and the individual stock ([17]).

In an interior solution and using equations (16) and (18), the optimal level of human capital evolves according to:

$$h(t,v) = h(v,v)\,(1-\delta_h)^{t-v} + \sum_{n=v+\underline{a}+1}^{t-1} (1-\delta_h)^{t-1-n}\,\Omega(n,v)\,H(n) \qquad \textbf{(19)}$$

where $h(v,v)$ is the level of human capital at birth, which is assumed to be a strictly positive fraction (γ) of the per capita average stock of human capital in the active population ([18]):

$$\begin{aligned} &h\,(v,v) = \gamma\,H\,(v) \\ &0 < \gamma \le 1 \end{aligned} \qquad \textbf{(20)}$$

Other things being equal, a low value for γ reduces the initial value of human capital, and by (18) increases the strength of the externality; and, thereby, the amount of time spend in education (early in life).

$\Omega(n,v)$ is the period optimal (private) increment in the stock of human capital per unit of *H(n)*. It is given by:

$$\Omega\,(t,v) = \xi\left[\frac{\xi\,\alpha\,\mu\,(t,v)}{w\,(t)}\right]^{\frac{\alpha}{1-\alpha}} \qquad \textbf{(21)}$$

Again it can be noticed that the accumulation of human capital varies directly with the ratio of the shadow price of human capital -$\mu(t,v)$- to the opportunity cost of investing in education -*w(t)*.

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3.1. The competitive equilibrium

Firms produce a homogeneous good which can be used either for consumption or investment. Firms employ a CRS technology *F(K,L)* using capital (*K*) and labour in efficiency units (*L*). The production function in intensity form is $f(k(t)) \equiv F(k(t),1)$, where $k \equiv \frac{K}{L}$ is the capital intensity ratio. There are a large number of firms operating in a competitive setting. Therefore, gross rates of return equal marginal productivities, and the capital intensity ratio is identical across firms:

[17] Provided that (in old-age) a decline in the exogenous human capital efficiency factor or the depreciation of human capital does not reverse the rise in human capital due to education/formation.

[18] Sustained growth is possible because the production function (16) is a CRS function and human capital is transmitted across generations.

$$w(t) = \frac{\partial F(K(t), L(t))}{\partial L(t)} = f\big(k(t)\big) - k(t) f^1\big(k(t)\big) \quad (i)$$

$$r(t) + \delta_k = \frac{\partial F(K(t), L(t))}{\partial K(t)} = f^1\big(k(t)\big) \quad (ii) \tag{22}$$

where δ_k is the depreciation rate of physical capital; and, *r(t)* is the interest rate.

The aggregate supply of labour in efficiency units used in production is:

$$L(t) = \sum_{v=t-\bar{a}}^{t-a-1} [1-\varepsilon(t,v)]\, e(t-v)\, h(t,v)\, B(v) \left\{ \prod_{n=v}^{t-1} [1-d(n-v)] \right\} \tag{23}$$

where *B(v)* is the number of individuals born in period *v*.

It is assumed that physical capital becomes available for production in the period after its purchase. Thereby, physical capital evolves according to:

$$K(t+1) = I(t) + (1-\delta_k) K(t)$$

where **(24)**

$$0 \le \delta_k \le 1$$

where *I* is gross investment.

3.2. The general equilibrium conditions

The equilibrium condition in the goods market equals gross output to gross investment plus aggregate private and public consumption. Government consumption equals aggregate lump-sum taxes in every period. The only role of public consumption is to facilitate the calibration of the model.

$$F(K(t), L(t)) = C(t) + I(t) + G(t) \quad (i)$$

$$G(t) = \Gamma(t) \quad (ii)$$

where **(25)**

$$C(t) = \sum_{v=t-T}^{t} c(t,v)\, B(v) \left\{ \prod_{n=v}^{t-1} [1-d(n-v)] \right\}$$

$$\Gamma(t) = \tau(t) \sum_{v=t-T}^{t} B(v) \left\{ \prod_{n=v}^{t-1} [1-d(n-v)] \right\}$$

where *C* is aggregate private consumption; *G* is public consumption; and, Γ is aggregate lump-sum taxes.

The equilibrium condition in the financial assets market can be derived using the set of equations in the model.

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The process of model calibration consists of selecting a number of parameter values to describe: individual's preferences, human capital formation, production, and the age-profiles of education and consumption variables. The parameters are chosen in such a way as to guarantee the existence of balanced growth, and to make it consistent with important stylised facts.

The forward-looking simulator in *Troll* (version 1.07) -which employs a stacked time algorithm- is used in the calculations ([19]).

The model is calibrated using annual data.

Three calibrations are calculated: one corresponds to the absence of a bequest motive; and, two represent a weak and a strong bequest motive, respectively. The values used in the calibrations coincide unless stated otherwise.

In the steady state, all aggregate variables growth at the rate κ_h:

$$1+\kappa_h = (1+\kappa_v)(1+\kappa_a) \tag{26}$$

where κ_v is the growth rate of per capita variables; and, κ_a is the growth rate of demographic variables.

In the steady state, the age-profile of any variable "indexed" on human capital -*x(t,v)*- differs across generations only by a factor of proportionality, which is the per capita growth rate (κ_v):

$$x(t+v_2-v_1, v_2) = (1+\kappa_v)^{v_2-v_1}\, x(t\,, v_1) \tag{27}$$

where v_1 and v_2 are any two generations.

The corresponding aggregate variable X is defined as:

$$X(t) = \sum_{v=t-\mathrm{T}}^{t} x(t,v)\; B(v) \left\{ \prod_{n=v}^{t-1} \left[1-d(n-v) \right] \right\} \tag{28}$$

In the steady state and using (27) in (28), the aggregate value X is calculated from the age-profile x as:

$$X(t) = B(t) \sum_{y=0}^{\mathrm{T}} \frac{x(y)}{(1+\kappa_h)^y} \left\{ \prod_{z=0}^{y-1} \left[1-d(z) \right] \right\}$$

where **(29)**

$$y \equiv t-v$$

[19] The "Stacked-Time" algorithm stacks all time periods into one large system of equations and then solves them simultaneously using Newton-Raphson iterations. Newton-Raphson is usually an efficient and robust method. Each iteration in Newton-Raphson requires solving a matrix equation involving the Jacobian matrix. In a stacked system, the Jacobian matrix can be very large. However, the large stacked Jacobian matrix has a repetitive structure of nonzero blocks along its diagonal, and each of these blocks is sparse. The "Stacked-Time" simulator in Troll uses methods developed by Laffargue (1990), Boucekkine (1995) and Juillard (1996) to take advantage of the repetitive block structure, and introduces computational techniques to explore the sparsity within the blocks.

The resulting algorithm is generally more accurate and usually faster than Fair-Taylor's. However, it does require more memory than the latter, and that is possibly the limiting factor for large models with many lags and leads.

4.1. Mortality rates

Mortality rates are based on the death tables published by the Belgian *Institut National de Statistique* and the *Bureau Féderal du Plan* in "Perspectives de Population 1995-2050", which are adjusted to obtain an average life expectancy at birth (equal for men and women) of 78 years. The use of actual death tables is paramount in the generation of age-profiles for consumption.

4.2. The age-profile of human capital efficiency

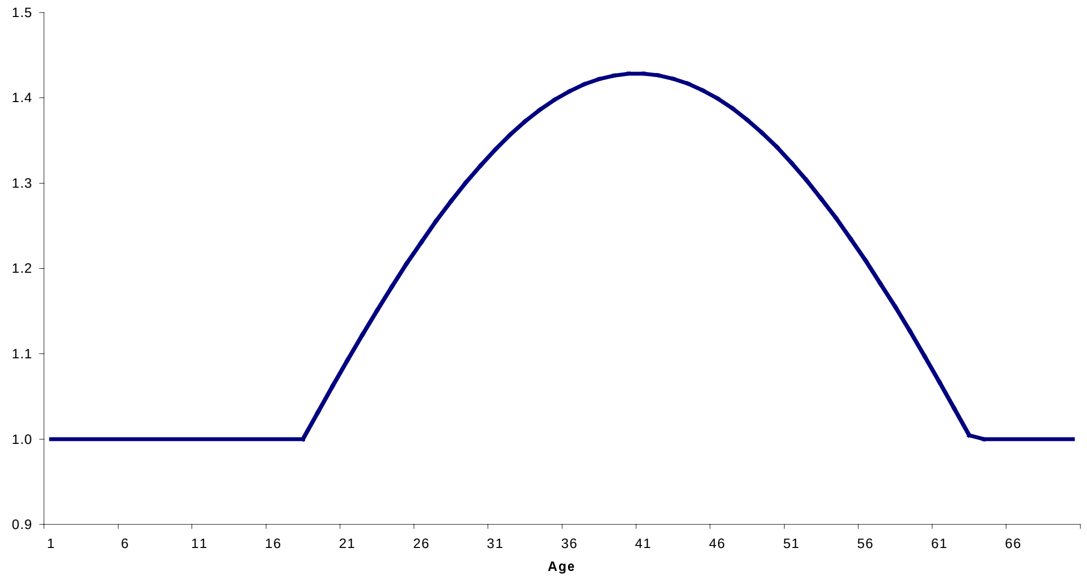
The exogenous factor of human capital efficiency evolves according to the parabola used in Auerbach and Kotlikoff (1987) (Graph 3), which is based on estimates obtained by Welch (1979) from a cross-section regression of weekly labour earnings of full-time workers on personal variables, including experience and experience squared ([20]) ([21]):

$$e(t) = \frac{\exp\left\{4.47 + 0.033\,[t - \hat{a} + 1] - 0.0007\,[t - \hat{a} + 1]^2\right\}}{\exp\left\{4.47 + 0.033 - 0.0007\right\}} \qquad t \geq \hat{a}$$

$$e(t) = 1 \qquad t < \hat{a}$$

where $\hat{a}$ is the starting age of the parabola describing the human capital efficiency factor.

Graph 3: The age-profile of the human capital efficiency factor (*e*)



4.3. The relevant ages in the calibrations

Recall that all relevant ages are parametrically fixed.

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[20] In Auerbach and Kotlikoff (1987), *e* is the exogenous profile of household's human capital, because human capital *h* (in physical units) is fixed to 1.

[21] Japelli and Pagano (1989) argue that age earning profiles vary considerably across countries, which could partly reflect differences in the age profile of the human capital efficiency factor (*e*).

4.5. Major conditions used in the calibrations

Table 4-3

Description	*Symbol*	*Value*
The per capita/productivity growth rate	κ_v	0.02
The population growth rate	κ_a	0.005
The implied economic growth rate	κ_h	0.0251
The equilibrium interest rate	r	0.08
The intertemporal elasticity of substitution in consumption	$\frac{1}{\sigma}$	1
The aggregate consumption over output ratio	$\frac{C}{F(K,L)}$	0.60
The strength of the bequest motive	η	{¼,½}
The child bearing age	ι	28
The elasticity of substitution in production		1
The depreciation rate of physical capital	δ_k	0.1
The labour share in output		2/3

The calibrations use the stylised facts that characterise the European Union economy (EU) ([23]).

In the 1970-1997 period, output per capita grew at the (geometric) annual average rate of 2.1%. The calibrations assume a per capita growth rate of 2%. In the same period, the average annual growth rate of the population was 0.4%. The annual growth rate of the population is set to 0.5%.

The time discount factor (β) is adjusted so that the equilibrium interest rate (r) equals 8%. The interest rate has a major impact, affecting the profile of consumption (10), (13), and (14); and, the allocation of human capital between education and production activities (18), through the shadow price of human capital (see Annex III). Other things being equal, a rise in the interest rate increases the steepness of the consumption profile (i.e. consumers become less impatient); and, reduces the fraction of human capital allocated to education, because future labour earnings are discounted more. In conjunction with other parameters, namely the intergenerational transmission of human capital (γ), an interest rate of 8% limits the fraction of aggregate human capital resources used in education to about 1/3 (see Table 4-4).

An equilibrium interest rate of 8% for riskless investments could be considered as being too high. However, this rate is also used to discount future labour earnings (i.e. the return on human capital investments). Estimates of the return on human capital investments tend to find rates above 10% for all levels of education. Becker (1975) states that the return on human capital investments should be higher than on

[23] Using the European Commission's AMECO database, average data for the period 1970-1997 are calculated for the EU-14 (the EU-15, excluding Germany due to the break in series caused by the unification).

nonhuman capital, because of financing difficulties and inadequate knowledge of opportunities.

The intertemporal elasticity of substitution in consumption ($\frac{1}{\sigma}$) is 1 ([24]). The choice of the logarithmic utility function linearises the equations that determine the profile of consumption: (10), (13), and (14).

In industrialised economies, the ratio of aggregate consumption to GDP is between 0.6 and 2/3. The lower limit is closer to both the European and Japanese economies, whereas the upper limit describes better the U.S. situation. In the calibrations, the ratio of aggregate consumption over output is set to 60% –the average of EU-14 in the 1970-1997 period is 60.3% ([25]).

The parameter defining the strength of the bequest motive (η) is set to ¼ in the “weak” bequest motive case, and to ½ in the “strong” bequest motive case. Both values guarantee the existence of the value function (see Annex II).

The child-bearing age (ι) is 28 years, which corresponds to the mean age of women at childbirth in the EU-15 in the 1970-1993 period (Eurostat).

The elasticity of substitution in production is 1 (i.e. a Cobb-Douglas production function) ([26]).

The annual depreciation rate of physical capital is 10%. In the 1970-1997 period, the average annual depreciation rate in the EU-14 was approximately 12%.

The labour share in output is 2/3, which corresponds to the average in the EU-14 (65.9% in the 1970-1997 period).

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[24] Estimates of the intertemporal elasticity of substitution in consumption suggest that it is likely to be lower than 0.5. Econometric work based on U.S. and U.K. time series data by Patterson and Pesaran (1992) suggest that it is between 0.1 and 0.3. Faruqee and al. (1996) also argue that a more plausible range for the intertemporal elasticity of substitution is centred around ¼. Unfortunately, as pointed out in Evans (1991) and Faruqee and al. (1996), there is a lower limit for the intertemporal elasticity of substitution in the CRRA type of models, beyond which, a negative value for the time preference rate ($1-\beta$) is necessary -given a reasonable baseline for interest rates- in order to reproduce the actual ratio of aggregate consumption over output.

[25] The ratio of aggregate consumption over output increases when: the intertemporal elasticity of substitution in consumption rises; the interest rate increases; or, the time preference rate ($1-\beta$) decreases.

[26] Auerbach and Kotlikoff (1987) also use a *Cobb-Douglas* production function. Estimates of the elasticity of substitution in the manufacturing industry, and in aggregate production functions usually find values of 1 or slightly less.

4.6. Major results of the calibrations

Table 4-4

Description	*Symbol*	*Value*
The physical capital over output ratio	$\frac{K}{F(K,L)}$	1.85
The time preference rate in the no bequest case (η=0)	$1\text{-}\beta$	0.0044
The time preference rate in the "weak" bequest case (η=¼)	$1\text{-}\beta$	0.012
The time preference rate in the "strong" bequest case (η=½)	$1\text{-}\beta$	0.018
The ratio of the (involuntary) inheritance over human wealth in the no bequest case at birth (η=0)		0.208
The ratio of the inheritance over human wealth in the "weak" bequest case at birth (η=¼)		0.303
The ratio of the inheritance over human wealth in the "strong" bequest case at birth (η=½)		0.433
The gross investment over output ratio	$\frac{I}{F(K,L)}$	0.232
The public consumption over output ratio	$\frac{G}{F(K,L)}$	0.168
The average fraction of human capital used in education in efficient units		0.303

The steady state ratio of physical capital over output is approximately 2. Actual ratios observed in Europe are around 3.

A time preference rate around 1½% in the bequest motive case, and about ½% in the no bequest motive case secure a consumption over GDP ratio of 60%.

The ratios of the inheritance over human wealth at birth increase with the strength of the bequest motive.

The gross investment over output ratio is about 23%, which is close to the average in the EU-14 in the 1970-1997 period (21%).

The public consumption value satisfies the equilibrium conditions in (25).

In the balanced growth, about 30% of aggregate human capital (in efficiency units) is used (on average) in education/formation. Although on the high side, this seems to be an acceptable value for industrialised countries ([27]). For an individual entering the labour market/education at 8 years of age and retiring at 65, this is equivalent to full time education until about 25 years of age ([28]).

---

[27] A lower value requires a higher interest rate.

[28] A high degree of uncertainty surrounds the "true" values of the parameters related to human capital formation, namely α (the elasticity of the production function of human capital with respect to the private input in education), and γ (the fraction of human capital inherited at birth). A value for γ is selected such as that early in life all time is spent in education. Other things being equal, for a given

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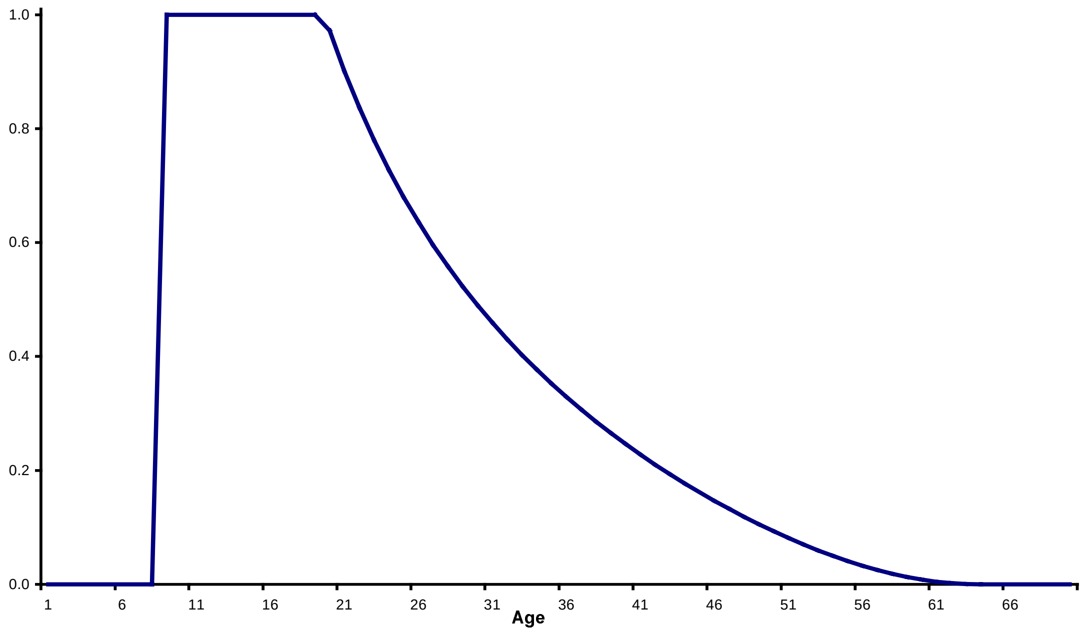
The age-profile for the (private) optimal length of time spent in human capital activities is:

Graph 4: The fraction of time spent in human capital activities (ε)



As in Mincer (1974), this paper assumes a broad definition of human capital activities, comprising both formal education and all forms of human capital investments carried out after completion of schooling (e.g. vocational training). A broader definition is necessary to explain the variability of age-earning profiles in actual economies.

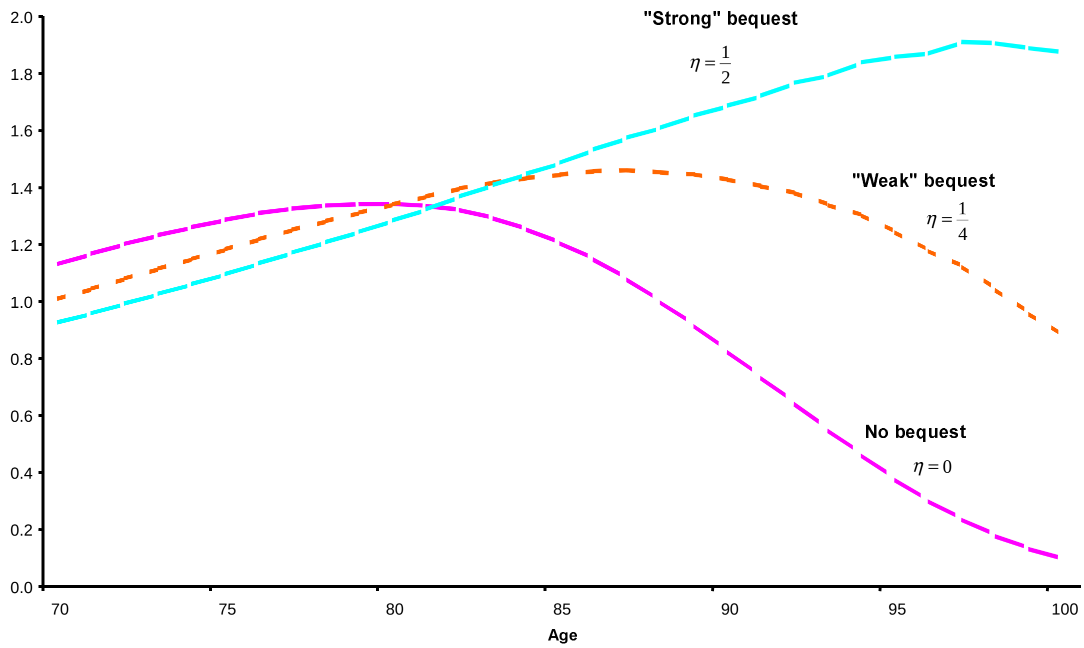
Between ages 9 and 25, an individual spends nearly all their time in human capital activities; between 30 and 50 years, the production of goods becomes progressively their main activity, although investment in education/formation remains significant; and, after 50 years, investment in human capital falls rapidly to zero as the retirement age approaches at 65. The divisibility of time explains the high levels of investment in human capital after the "normal" age for leaving school.

A crucial determinant of ε is the shadow price of human capital:

fraction of the average stock of human capital used in education, there is a direct relation between α and the equilibrium interest rate *r*.

utility is discounted by the probability of survival. Altruism slows the running down of financial assets and the decline in consumption in old-age, and if reasonably strong, consumption and financial wealth can actually increase, because the utility derived from leaving a bequest might exceed that from own consumption. Therefore, in case of a reasonably strong bequest motive, an individual is protected against poverty in old-age (i.e. the longevity risk). Otherwise, this can be achieved by setting annuity markets or a means-tested public pension.

Graph 7: Consumption in old-age

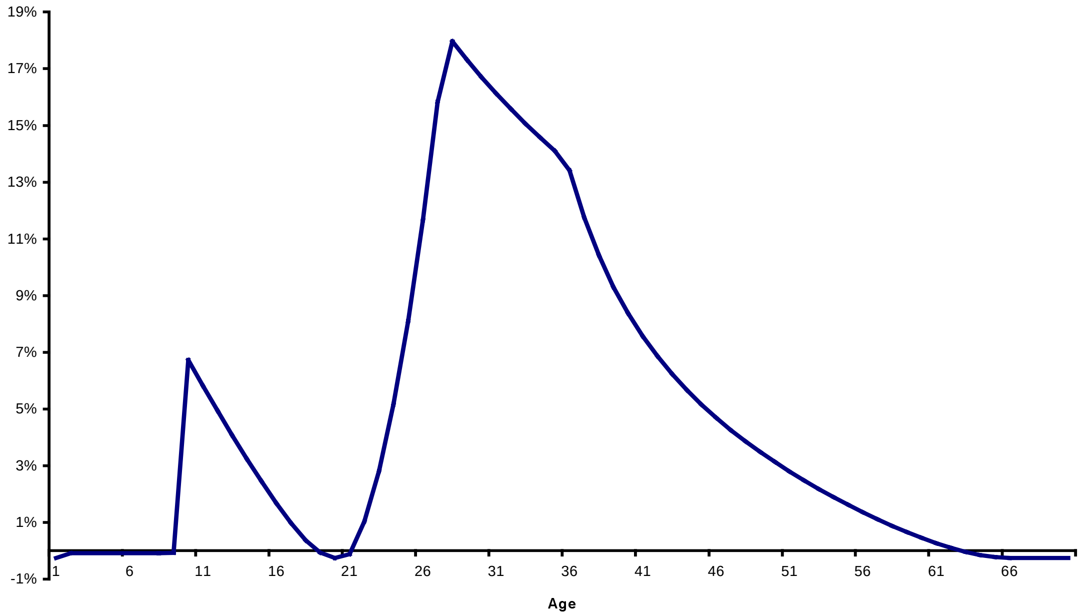


The longevity risk results from lack of annuity markets, which could reflect problems of adverse selection (Abel, 1986). Setting a **mandatory** defined-contribution pension fund, which had to be exchanged for annuities at retirement, would solve this problem. In fact, the existence of annuity markets would make the growth rate of consumption independent of death probabilities, and the profile of consumption would not exhibit a declining profile in old-age (Yaari 1965, and Eckstein et al. 1985).

In a perfect foresight model, public pensions can increase consumption in old-age only if an individual is liquidity constrained, otherwise the profile of consumption remains basically unchanged. This calls for **means-testing** income support.

profiles, significantly reducing lifetime utility in relation to the perfect capital markets case.

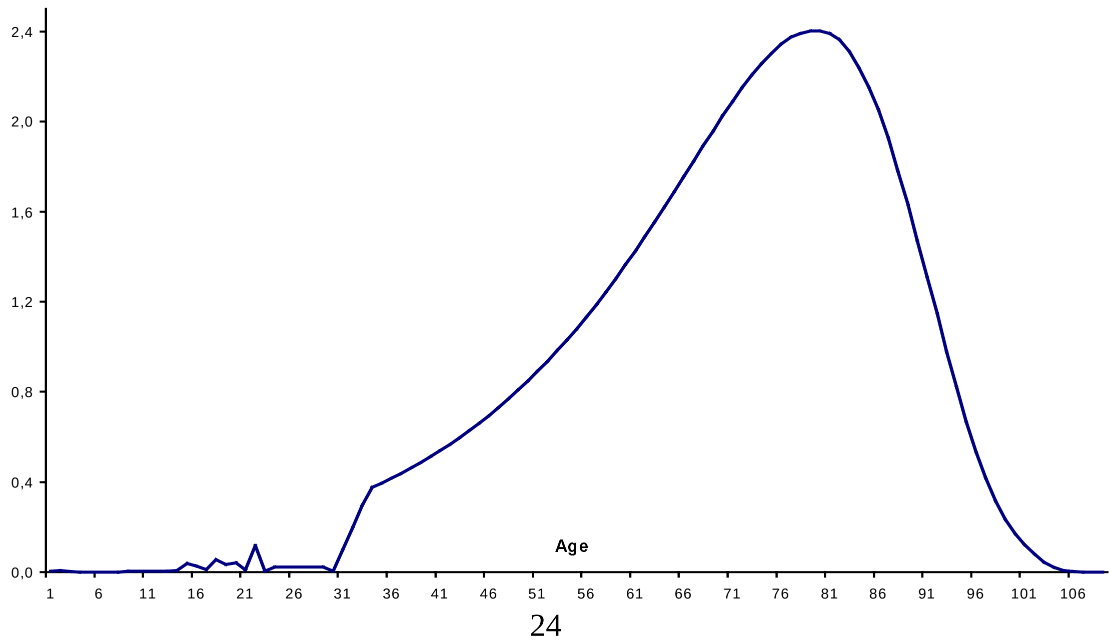
Graph 9: The fraction of time spent in human capital activities (ε)



In the liquidity constraint case, the length of time spent in human capital activities is low in young-age (when the liquidity constraint exerts maximum pressure), raising in middle-age. This profile is in conflict with stylised facts, probably because in actual economies human capital is heavily subsidised and time is far from being perfectly divisible. The significant difference in the profiles of education in the perfect capital markets and the liquidity constraint cases (Graphs 4 and 9, respectively) highlights the potential impact of a liquidity constraint in the absence of human capital subsidies.

A liquidity constraint also affects the profiles of consumption and financial wealth.

Graph 10: Consumption (*c*)



constraint forces an individual to: (1) reduce the total amount of time spend in human capital activities, below the unconstrained optimal (private) level; and, (2) postpone part of the investment in human capital to middle-age (i.e. substituting vocational training for formal education), which have a significant negative impact on growth and lifetime utility. In these circumstances, human capital subsidies can yield potentially large welfare gains.

In middle- old-age, an individual has positive financial wealth, depending on the strength of altruism. However, in the presence of no or only a weak bequest motive, the age-dependent mortality rate may excessively reduce financial wealth and consumption in (very) old-age (i.e. the longevity risk), because utility is discounted using survival probabilities. Altruism reduces the propensity to consume out of total wealth, in particular in (very) old-age, countering the opposite effect due to an increasing death probability. Therefore, in the absence of liquidity constraints, a reasonably strong bequest motive suffices to avoid the longevity risk. Otherwise, it would be necessary to set-up annuity markets or a means-tested public pension.

This model can be used to simulate the impact of economic shocks in the transition between steady states. In future research, it will be used to assess the impact of a fertility shock. The results will be compared with those obtained using the exogenous growth model of Auerbach and Kotlikoff (1987).

Annex I The individual's maximisation programme is separable

The individual's optimal programme consists in the maximisation of the right-hand-side of (6) subject to the financial wealth constraint (2), the total wealth constraint (5), and the dynamic equation for human capital accumulation (16).

$$\begin{aligned}
&\underset{\{c,f,\omega,h,\varepsilon\}}{max}\ U\left(c(t,v)\right)+\beta[1-d(t-v)]V\{\omega(t,v)\}+\beta\eta\, d(t-v)\,V\{\omega(t,v+\iota)\}\\
&s.t\\
&f(t,v)=R(t)\,f(t-1,v)+w(t)\,[1-\varepsilon(t,v)\,]e(t-v)\,h\,(t,v)-c(t,v)-\tau(t)\\
&\omega(t,v)=R(t)\omega(t-1,v)-c(t,v)\\
&h(t+1,v)=(1-\delta_h)h(t,v)+\xi[\varepsilon(t,v)e(t-v)h(t,v)]^{\alpha}H(t)^{1-\alpha}
\end{aligned}\qquad \mathbf{(30)}$$

After using the total wealth constraint (5) to replace consumption in the maximand function, the first order conditions of the *Lagrangean* derived from (30) are:

$$\begin{aligned}
&-U^1(c(t,v))+\beta[1-d(t-v)]V^1\{\omega(t,v)\}+\beta\eta\, d(t-v)V^1\{\omega(t,v+\iota)\}\frac{d\omega(t,v+\iota)}{d\omega(t,v)}=0 && (i)\\
&-\lambda(t,v)+R(t+1)\lambda(t+1,v)=0 && (ii)\\
&w(t)[1-\varepsilon(t,v)]e(t-v)\lambda(t,v)\\
&+\left\{(1-\delta_h)+\xi\alpha[\varepsilon(t,v)e(t-v)h(t,v)]^{\alpha-1}\varepsilon(t,v)e(t-v)H(t)^{1-\alpha}\right\}\mu(t,v)-\mu(t-1,v)=0 && (iii)\\
&-w(t)e(t-v)h(t,v)\lambda(t,v)+\xi\alpha[\varepsilon(t,v)e(t-v)h(t,v)]^{\alpha-1}e(t-v)h(t,v)H(t)^{1-\alpha}\mu(t,v)=0 && (iv)
\end{aligned}\qquad \mathbf{(31)}$$

where $\lambda(t,v)$ is the shadow price of financial wealth; and, $\mu(t,v)$ is the shadow price of human capital.

Equation (31) allows to solve the maximisation problem in two stages, because (31*i*) is separable from (31*ii*) to (31*iv*). Condition (31*i*) and constraint (5) define the optimal consumption profile, corresponding to problem (6). Conditions (31*ii*) to (31*iv*) derive the optimal profile of human capital investments and are equivalent to problem (17).

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The value function $V\{\omega(t,v)\}$ can be seen as an age-profile:

Graph 12 : The value function $V(\omega(t,v))$ in the bequest motive case

Problem (6) is well-defined iff the value function $V\{\omega(t,v)\}$ is (for an arbitrary large v) bounded. Obviously, convergence of the value function to an age-profile is a sufficient condition to secure boundness:

$$\lim_{t \to \infty} V\{\omega(t, t-p+1)\} = V\{\omega(p)\} \quad \text{where } p \in [1, \mathrm{T}] \tag{32}$$

The solution to (6) can be expressed as:

$$V\{\omega(t,v+1)\} = b(t,v,\iota)\, V\{\omega(t-1,v)\}$$

where **(33)**

$$b(t,v,\iota) = \frac{1}{\left[\dfrac{\mathrm{M}(t,v,\iota)^{\frac{1-\sigma}{\sigma}}}{\Xi(t,v)} + \beta[1-d(t-v)]\right]\dfrac{V\{\omega(t,v)\}}{V\{\omega(t,v+1)\}} + \beta\eta d(t-v)\dfrac{V\{\omega(t,v+\iota)\}}{V\{\omega(t,v+1)\}}}$$

Equation (33) can be interpreted as a set of T difference equations indexed on the age parameter $p \in [1, \mathrm{T}]$:

$$V\{\,\omega(t\,;p-1)\,\} = b(t,\iota\,;p)\, V\{\,\omega(t-1;p-1)\,\}$$

where **(34)**

$$b(t,\iota\,;p) = \frac{1}{\left[\dfrac{\mathrm{M}(t,\iota\,;p)^{\frac{1-\sigma}{\sigma}}}{\Xi(t\,;p)} + \beta[1-d(p)]\right]\dfrac{V\{\,\omega(t\,;p)\,\}}{V\{\,\omega(t\,;p-1)\,\}} + \beta\,\eta\, d(p)\dfrac{V\{\,\omega(t\,;p-\iota)\,\}}{V\{\,\omega(t\,;p-1)\,\}}}$$

The value function in (6) converges to an age-profile if the set of T difference equations in (34) converges, which requires:

$$\lim_{t\to\infty} \; b(t,\iota\,;p) \le 1 \qquad \forall \;\; p \in [1,\mathrm{T}] \tag{35}$$

In a steady state and using (27), (33) implies:

$$\lim_{t\to\infty} \; b(t,\iota\,;p) = (1+\kappa_v)^{1-\sigma} \qquad \forall \;\; p \in [1,\mathrm{T}] \tag{36}$$

Combining (35) and (36), a sufficient condition for the value function to be bounded is for the elasticity of substitution in consumption ($\frac{1}{\sigma}$) be lower or equal to 1. As argued in footnote 24, the intertemporal elasticity of substitution in consumption is likely to be lower than ½.

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This Annex derives the optimal conditions for the (private) allocation of human capital between education/formation and production activities.

An individual maximises the value of human wealth at birth (i.e. the discounted value of lifetime labour earnings minus lump-sum taxes) subject to the production function of human capital (16).

$$\max_{\{\varepsilon\}} \; l(v,v) \equiv \sum_{u=v+\underline{a}}^{v+\bar{a}} \frac{w(u)\,[1-\varepsilon(u,v)]\,e(u-v)\,h(u,v) - \tau(u)}{\left\{\prod_{n=v}^{u-1} R(n)\right\}} \qquad s.t. \quad (16) \tag{37}$$

The first order conditions of the *Lagrangean* associated with (37) are:

$$\frac{w(t)\,[1-\varepsilon(t,v)]\,e(t-v)}{\prod_{n=v}^{t-1} R(n)} - \lambda(t-1,v)$$

$$+\left[(1-\delta_h)+\alpha\xi\;\varepsilon(t,v)\,e(t-v)\,[\,\varepsilon(t,v)\,e(t-v)\,h(t,v)\,]^{\alpha-1}\,H(t)^{1-\alpha}\right]\lambda(t,v)=0 \qquad (i)$$

(38)

$$\frac{w(t)\,e(t-v)\,h(t,v)}{\prod_{n=v}^{t-1} R(n)} - \alpha\xi\,e(t-v)\,h(t,v)\,[\,\varepsilon(t,v)\,e(t-v)\,h(t,v)\,]^{\alpha-1}\,H(t)^{1-\alpha}\lambda(t,v)=0 \qquad (ii)$$

where $\lambda(t,v)$ is the shadow price of human capital evaluated at birth.

After solving the dynamic equation for the shadow price of human capital (38*i*) and rearranging (38*ii*), the optimal conditions describing the private choice of human capital investment are (in an interior solution):

$$\mu(t,v)=\sum_{n=t+1}^{v+\bar{a}}\left\{ w(n)\,[\,1-\varepsilon(n,v)\,]\,e(n-v)\frac{\prod_{m=t+1}^{n-1}\left\{\alpha\frac{h(m+1,v)}{h(m,v)}+(1-\alpha)(1-\delta_h)\right\}}{\prod_{p=t}^{n-1}\{R(p)\}}\right\} \qquad (i)$$

$$\varepsilon(t,v)\;=\left[\frac{\alpha\xi\,\mu(t,v)}{w(t)}\right]^{\frac{1}{1-\alpha}}\frac{H(t)}{e(t-v)\,h(t,v)} \qquad (ii)$$

with the terminal condition

$$\mu(v+\bar{a},v)=0$$

where

$$\mu(t,v)\equiv\lambda(t,v)\prod_{n=v}^{t-1}R(n)\;,\;\;0\le\varepsilon(t,v)\le 1$$

$$H(t)\equiv\frac{\sum_{v=t-\bar{a}}^{t-\underline{a}-1} e(t-v)\,h(t,v)B(v)\left\{\prod_{n=v}^{t-1}[1-d(n-v)]\right\}}{\sum_{v=t-\bar{a}}^{t-\underline{a}-1} B(v)\left\{\prod_{n=v}^{t-1}[1-d(n-v)]\right\}} \qquad \textbf{(39)}$$

where $\mu(t,v)$ is the shadow price of human capital evaluated in period *t*; and, *H(t)* is the per capita average stock of human capital in the active population in efficiency units.

The price of human capital (39*i*) is the discounted value of future labour earnings corresponding to an investment decision in human capital. Its value can be calculated in two steps: first, future labour earnings (in period *n*) per unit of human capital invested today (in period *t*) are given by

$w(n)\left[1-\varepsilon(n,v)\right]e(n-v)\prod_{m=t+1}^{n-1}\left\{\alpha\frac{h(m+1,v)}{h(m,v)}+(1-\alpha)(1-\delta_h)\right\}$ ([31]), and second, they are discounted by the accumulated interest factor $\prod_{p=t}^{n-1}R(p)$.

Appendix The OLG model with a liquidity constraint and no altruism

The OLG model without altruism can be solved imposing a liquidity constraint. In the liquidity constraint case, only financial wealth can be used as collateral for borrowing. Whereas, in the perfect capital market case, total wealth (the sum of financial and human wealth) can. In the former case, financial wealth is restricted to be non-negative (i.e. $f\geq 0$). In the latter case, total wealth is restricted to be non-negative (i.e. $\omega\geq 0$).

In an OLG model, Buiter and Kletzer (1995) assume that households are unable to borrow against future income (human wealth) to finance education expenditure when young. This capital market imperfection constraints investment in human capital (and consumption) to be financed out of financial wealth and current labour earnings ([32]).

Households may be unable to borrow against future labour earnings because of laws restricting the ability to attach wages to loan payments, incomplete financial markets, or inadequate knowledge of opportunities. Therefore, institutional factors and/or market imperfections can reduce or eliminate the use of human wealth for loan collateral. Also asymmetric information about labour quality can turn human capital into an inferior collateral.

In the current model without a bequest motive, an individual maximises the intertemporal utility function:

$$U(t,v) = \sum_{q=t}^{v+T}\left[\frac{c(q,v)^{1-\sigma}-1}{1-\sigma}\right]\beta^{q-t}\left\{\prod_{l=t}^{q-1}\left[1-d(l-v)\right]\right\} \qquad \textbf{(40)}$$

The optimal solution is derived maximising (40) subject to the financial wealth constraint (2), the liquidity constraint ($f\geq 0$), the total wealth constraint (5), and the production function of human capital (16). The total wealth constraint is considered to calculate the relative (shadow) price of financial wealth over total wealth, which is

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[31] The product $\prod_{m=t+1}^{n-1}\left\{\alpha\frac{h(m+1,v)}{h(m,v)}+(1-\alpha)(1-\delta_h)\right\}$ is close to the accumulated increase in human capital multiplied by α (the elasticity of the production function of human capital with respect to the private input). When α tends to 1 (and the externality vanishes), labour earnings are capitalised by the accumulated increase in human capital: $\prod_{m=t+1}^{n-1}\frac{h(m+1,v)}{h(m,v)}$.

[32] The belief that capital market imperfections constrain human capital accumulation is widespread and supported by empirical studies of education attainment. Becker (1975) discusses borrowing constraints in models of human capital accumulation (pp 78-80).

greater than one when the liquidity constraint is binding. In the liquidity constraint case, the profiles of education and consumption are determined simultaneously.

The individual's optimisation programme is given by the *Lagrangean*:

$$
\begin{aligned}
L(t,v) = & \sum_{q=t}^{v+T} \left[\frac{c(q,v)^{1-\sigma}-1}{1-\sigma} \right] \beta^{q-t} \left\{ \prod_{l=t}^{q-1} [1-d(l-v)] \right\} \\
& + \sum_{q=t}^{v+T} \lambda(q,v) \{ -f(q,v) + R(q) f(q-1,v) \\
& + w(q) [1-\varepsilon(q,v)] e(q-v) h(q,v) - c(q,v) - \tau(q) \} \\
& + \sum_{q=t}^{v+T} \varsigma(q,v) f(q,v) \\
& + \sum_{q=t}^{v+T} \nu(q,v) \{ -\omega(q,v) + R(q)\,\omega(q-1,v) - c(q,v) \} \qquad \textbf{(41)} \\
& + \sum_{q=t}^{v+T} \mu(q,v) \{ -h(q+1,v) + (1-\delta_h) h(q,v) \\
& + \xi [\, \varepsilon(q,v)\, e(q-v)\, h(q,v) \,]^{\alpha} H(q)^{1-\alpha} \}
\end{aligned}
$$

$$c(q,v) \ge 0 \;,\; h(q,v) \ge 0 \;,\; 0 \le \varepsilon(q,v) \le 1 \;,\; f(q,v) \ge 0 \;,\; \omega(q,v) \ge 0$$

where $\lambda(q,v)$ is the shadow price of financial wealth (f); $\zeta(q,v)$ is the shadow price of the liquidity constraint; $\nu(q,v)$ is the shadow price of total wealth (ω); and, $\mu(q,v)$ is the shadow price of human capital (h).

10.1. Determination of the optimal age-profile of human capital

Assuming that the impact of individual choices on aggregate variables is negligible, the first order optimal conditions derived from (41) for the allocation of human capital resources are (for an interior solution):

$$
\begin{aligned}
& \frac{\partial L(t,v)}{\partial h(q,v)} = 0 \Rightarrow \mu_\nu(q-1,v) = \frac{\mu_\nu(q,v)\left[\alpha \dfrac{h(q+1,v)}{h(q,v)} + (1-\alpha)(1-\delta_h)\right]}{R(q)} \\
& \qquad + \frac{\lambda_\nu(q,v)[w(q)\{1-\varepsilon(q,v)\} e(q-v)]}{R(q)} \qquad (i) \\
& \frac{\partial L(t,v)}{\partial \varepsilon(q,v)} = 0 \Rightarrow \varepsilon(q,v) = \left[\frac{\xi\alpha\;\mu_\nu(q,v)}{w(q)\,\lambda_\nu(q,v)}\right]^{\frac{1}{1-\alpha}} \frac{H(q)}{e(q-v)\,h(q,v)} \qquad (ii) \qquad \textbf{(42)} \\
& \lambda(q,v) \ge 0 \;,\; \nu(q,v) \ge 0 \;,\; \mu(q,v) \ge 0 \quad 0 < \varepsilon(q,v) < 1 \qquad (iii)
\end{aligned}
$$

where

$$\lambda_\nu(q,v) \equiv \frac{\lambda(q,v)}{\nu(q,v)} \;;\; \mu_\nu(q,v) \equiv \frac{\mu(q,v)}{\nu(q,v)}$$

The optimal education levels in the perfect capital markets case can be obtained from (42) by setting λ_v (the relative price of financial wealth over that of total wealth) equal to 1.

The relevant corner solution to be considered corresponds to the situation where all time is used in human capital investment (i.e. ε=1). The following condition applies instead of (42*ii*):

$$\varepsilon(q,v)=1$$

$$\frac{\partial L(t,v)}{\partial \varepsilon(q,v)}>0 \Rightarrow \xi \alpha \mu_v(q,v)\left[\frac{\varepsilon(q,v)e(q-v)h(q,v)}{H(q)}\right]^{\alpha-1} > w(q)\ \lambda_v(q,v) \qquad \textbf{(43)}$$

10.2. Determination of the optimal age-profile of consumption

The optimal conditions derived from (41) for the age-profile of consumption are:

$$\frac{\partial L(t,v)}{\partial c(q,v)}=0 \Rightarrow c(q,v)^{-\sigma}\beta^{q-t}\left\{\prod_{l=u}^{q-1}[1-d(l-v)]\right\}=\lambda(q,v)+\nu(q,v) \quad (i)$$

$$\frac{\partial L(t,v)}{\partial \omega(q,v)}=0 \Rightarrow \frac{\nu(q,v)}{\nu(q-1,v)}=\frac{1}{R(q)} \quad (ii) \qquad \textbf{(44)}$$

Using 44 *i* and *ii*, the growth rate of consumption -$\kappa(q,v)$- is:

$$\kappa(q,v)\equiv\frac{c(q,v)}{c(q-1,v)}=\left[\frac{1+\lambda_V(q-1,v)}{1+\lambda_V(q,v)}R(q)\,\beta\left\{1-d(q-1-v)\right\}\right]^{\frac{1}{\sigma}} \qquad \textbf{(45)}$$

Using (45), the level of consumption can be written as:

$$c(t,v)=c(q,v)\left\{\prod_{n=q+1}^{t}\kappa(n,v)\right\} \qquad t\geq q \qquad \textbf{(46)}$$

Substituting (46) into (4), the optimal level of consumption is:

$$c(t,v)=\frac{\omega(t,v)}{\Phi(t,v)} \quad (i)$$

$$\textbf{(47)}$$

$$\Phi(t,v)=\sum_{l=t+1}^{v+T}\left[\prod_{n=t+1}^{l}\left\{R(n)^{\frac{1-\sigma}{\sigma}}\left\{\frac{1+\lambda_v(n-1,v)}{1+\lambda_v(n,v)}\beta\left[1-d(n-v-1)\right]\right\}^{\frac{1}{\sigma}}\right\}\right] \quad (ii)$$

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