A Tribute to Paul Van Dooren

D.C. Sorensen

DYSCO Workshop Honoring M. Gevers and P. van Dooren

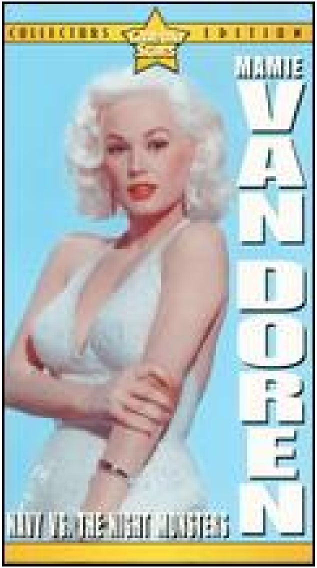
UCL, Louvain-la-Neuve, 2010

Meeting Paul

- Householder Symposium 1981
  - Paul wins Householder Prize
- Organized Series of NATO Advanced Study Meetings (1980's)
  - Great Food and ABUNDANT Wine!
  - Paul is unflappable! - Nothing bothers him.

    Persistently calm, whether when facing difficulties or experiencing success; not easily upset or excited

Google: Photos of Paul Van Dooren



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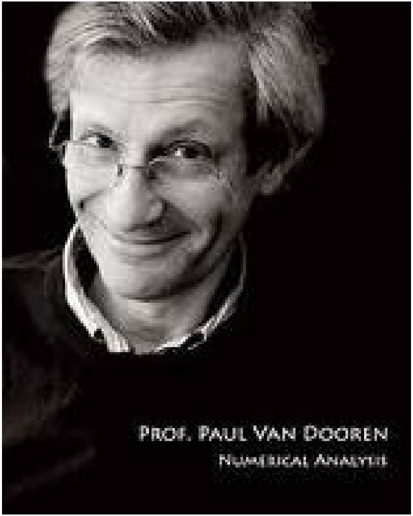


Paul Introduces Me to Model Reduction 1994

- Visits CRPC at Rice U 1994
- Introduces me to A.C. Antoulas
  - This collaboration is still active today
  - Thank you Paul !
- With Eric Grimme, we developed paper on IRA for MOR via NS Lanczos during this visit.

Two Papers in 1996

- E. Grimme, D. Sorensen, and P. Van Dooren, Model reduction of state space systems via an implicitly restarted Lanczos method, *Numerical Algorithms*, **12**,1–31, (1996).
- K. Gallivan, E. Grimme, and P. Van Dooren, A rational Lanczos algorithm for model reduction, *Numerical Algorithms*, **12**, 33–63, (1996).

MOR via IRA Lanczos

Main Ideas: Given Large System

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \mathbf{y} = \mathbf{c}\mathbf{x},$$

$$\begin{aligned}
\mathbf{A}\mathbf{V} &= \mathbf{V}\mathbf{T} + \mathbf{f}\mathbf{e}_k^T \\
\mathbf{A}^T\mathbf{W} &= \mathbf{W}\mathbf{T}^T + \mathbf{g}\mathbf{e}_k^T \\
\mathbf{W}^T\mathbf{V} &= \mathbf{I}, \quad \mathbf{V}\mathbf{e}_1 = \mathbf{b}, \quad \mathbf{W}\mathbf{e}_1 = \mathbf{c}
\end{aligned}$$

Take

$$\hat{\mathbf{A}} = \mathbf{W}^T\mathbf{A}\mathbf{V} = \mathbf{T}, \quad \hat{\mathbf{b}} = \mathbf{W}^T\mathbf{b}, \quad \hat{\mathbf{c}} = \mathbf{c}\mathbf{V}$$

Produce Reduced System

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{b}}u, \quad \hat{\mathbf{y}} = \hat{\mathbf{c}}\hat{\mathbf{x}}$$

Nearly the same response: Moments matched at ∞

Stablize Unstable Reduced System

Problem: $\mathbf{T}$ can be unstable!

Used hyperbolic rotations

Implicitly Shifted HR- Algorithm (Bunse-Gerstner)

Obtain Implicitly Restarted NS Lanczos

$$
\begin{aligned}
\mathbf{A}(\mathbf{VH}) &= (\mathbf{VH})(\mathbf{H}^{-1}\mathbf{TH}) + \mathbf{f}\mathbf{e}_k^T\mathbf{H} \\
\mathbf{A}^T(\mathbf{WH}^{-T}) &= (\mathbf{WH}^{-T})(\mathbf{H}^{-1}\mathbf{TH})^T + \mathbf{g}\mathbf{e}_k^T\mathbf{H}^{-T}
\end{aligned}
$$

$\mathbf{H}$ is S-orthogonal ($\mathbf{H}^T\mathbf{SH} = \mathbf{S}$)
$\mathbf{T}$ is S-symmetric ($\mathbf{T}^T\mathbf{S} = \mathbf{ST}$)

Stablize Unstable Reduced System

Truncate

$$\mathbf{A}\hat{\mathbf{V}} = \hat{\mathbf{V}}\hat{\mathbf{T}} + \hat{\mathbf{f}}\mathbf{e}_{k-p}^T$$

$$\mathbf{A}^T\hat{\mathbf{W}} = \hat{\mathbf{W}}\hat{\mathbf{T}}^T + \hat{\mathbf{g}}\mathbf{e}_{k-p}^T$$

$$\sigma(\mathbf{T}) = \{\theta_1, \theta_2, \ldots, \theta_{k-p}\} \cup \{\mu_1, \mu_2, \ldots, \mu_p\}$$

$\mu's$ - unstable eigenvalues

IDEA: Purge Unstable Poles from Reduced System

$$\sigma(\hat{\mathbf{T}}) = \{\theta_1, \theta_2, \ldots, \theta_{k-p}\}$$

Expand NS Lanczos back to k-steps and repeat
Get k-th order stable system

Result:

Quite Successful

Cited Often

Didn't Catch On in Practice

Got me hooked on Model Reduction

Rational Krylov Methods for MOR

IDEA: Match moments of Transfer Function at selected points

$$
\begin{aligned}
(\mathbf{A} - \sigma\mathbf{I})^{-1}\mathbf{V} &= \mathbf{V}\mathbf{T} + \mathbf{f}\mathbf{e}_k^T \\
(\mathbf{A} - \sigma\mathbf{I})^{-T}\mathbf{W} &= \mathbf{W}\mathbf{T}^T + \mathbf{g}\mathbf{e}_k^T
\end{aligned}
$$

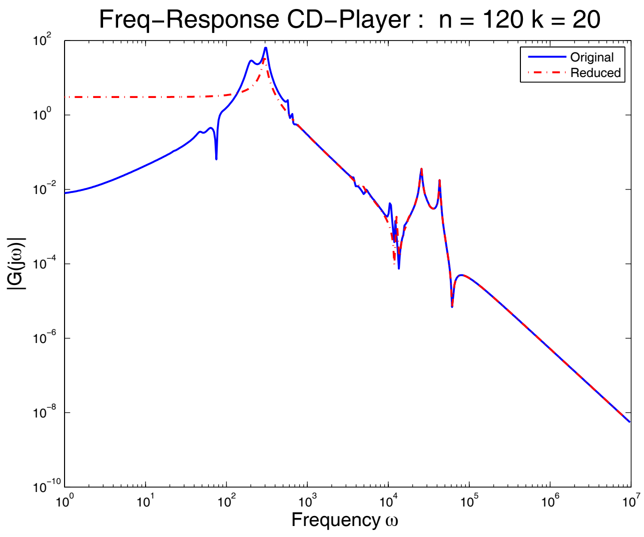
Matches leading k moments at the point σ
Transfer Function

$$\mathbf{G}(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$$

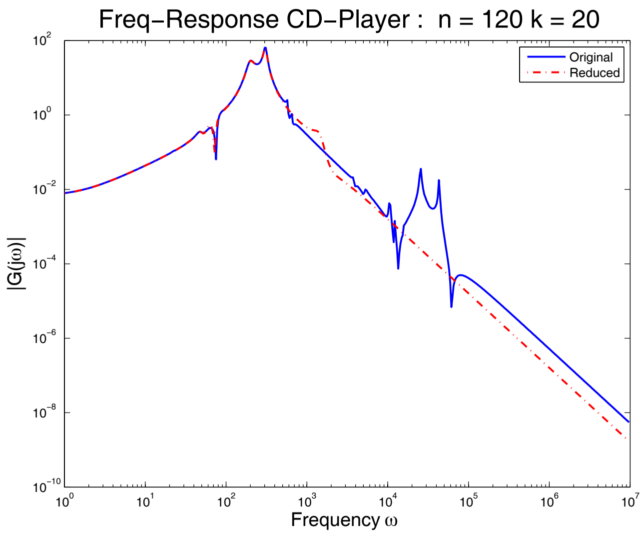
Interpolated at point σ to high order (k) by
Reduced Transfer Function

$$\hat{\mathbf{G}}(s) = \hat{\mathbf{c}}(s\mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{b}}$$

CD Player Freq Resp: Arnoldi, Match at infinity



CD Player Freq Resp: Arnoldi, Match at 0



Interpolation via Rational Krylov

Villemagne and Skelton(1986), Grimme's Thesis (1996)

Theorem

Consider the generalized rational Krylov spaces

$$\mathcal{S}_1 = \mathrm{span}\{\cup_{j=1}^{k_1} \mathcal{K}_{p_j}(\mathbf{A}_{\sigma_j}^{-1}, \mathbf{b})\} \text{ and } \mathcal{S}_2 = \mathrm{span}\{\cup_{i=1}^{k_2} \mathcal{K}_{q_i}(\mathbf{A}_{\mu_i}^{-*}, \mathbf{c}^*)\}.$$

Suppose the columns of $\mathbf{V}, \mathbf{W} \in \mathbb{C}^{n \times k}$ *provide biorthogonal bases for* $\mathcal{S}_1$ *and* $\mathcal{S}_2$*; i.e.,* $\mathcal{S}_1 = \mathrm{Ran}\mathbf{V}$ *and* $\mathcal{S}_2 = \mathrm{Ran}\mathbf{W}$ *with* $\mathbf{W}^*\mathbf{V} = \mathbf{I}_k$*. Put* $\widehat{\mathbf{A}} = \mathbf{W}^*\mathbf{A}\mathbf{V}$*,* $\widehat{\mathbf{b}} = \mathbf{W}^*\mathbf{b}$ *and* $\widehat{\mathbf{c}} = \mathbf{c}\mathbf{V}$*. Then*

$$\mathbf{c}\mathbf{A}_{\mu}^{-i}\mathbf{A}_{\sigma}^{-j}\mathbf{b} = \widehat{\mathbf{c}}\widehat{\mathbf{A}}_{\mu}^{-i}\widehat{\mathbf{A}}_{\sigma}^{-j}\widehat{\mathbf{b}},$$

for all possible values μ, σ, i, j*.*

Optimal $\mathcal{H}_2$ reduction via rational interpolation

Lemma

Suppose $\sigma \in \mathbb{C}$, $\sigma \notin \sigma(\mathbf{A}) \cup \sigma(\mathbf{A}_r)$. *Then*

$$(\sigma \mathbf{I} - \mathbf{A})^{-1}\mathbf{b} \in \mathcal{V} \Rightarrow \hat{\mathbf{G}}(\sigma) = \mathbf{G}(\sigma), \tag{1}$$

$$(\mathbf{c}(\sigma \mathbf{I} - \mathbf{A})^{-1})^* \in \mathcal{W} \Rightarrow \hat{\mathbf{G}}(\sigma) = \mathbf{G}(\sigma), \tag{2}$$

and if both (1) *and* (2) *hold, then*

$$\mathbf{G}'(\sigma) = \hat{\mathbf{G}}'(\sigma). \tag{3}$$

VanDooren, Gallivan and Absil (2008)
Gugercin, Beattie and Antoulas (2008)

Optimal $\mathcal{H}_2$ reduction: Iterated Rational Interpolation

1. Select $\sigma_i,\ i = 1, ..., k$ closed under conjugation.
2. **while** (relative change in $\{\sigma_i\} > tol$),
   - 2.1 Construct $n \times k$ matrices $\mathbf{V}, \mathbf{W}$ with
     $\text{Range}(\mathbf{V}) = \text{Span}\{(\sigma_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \ldots, (\sigma_k\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\}$
     $\text{Range}(\mathbf{W}) = \text{Span}\{(\sigma_1\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{c}, \ldots, (\sigma_k\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{c}\}$
     and $\mathbf{W}^*\mathbf{V} = \mathbf{I}$;
   - 2.2 $\mathbf{A}_k = \mathbf{W}^*\mathbf{A}\mathbf{V}$; $\mathbf{b}_k = \mathbf{W}^*\mathbf{b}$; $\mathbf{c}_k = \mathbf{c}\mathbf{V}$;
   - 2.3 Set $\sigma_i = -\lambda_i(\mathbf{A}_k)$ for $i = 1, \ldots, k$;
3. **end**

On Convergence:

$[\mathbf{A}_k, \mathbf{b}_k, \mathbf{c}_k]$ satisfy the first order necessary conditions for optimal approximation of $[\mathbf{A}, \mathbf{b}, \mathbf{c}]$ in the $\mathcal{H}_2$ - norm.

Summary

- Paul always has deep insights and elegant results
- He continues to this day
- Many times I have said:

I wish I had done it!

CONGRATULATIONS

PAUL