Sparsity and decomposition in semidefinite optimization

Lieven Vandenberghe
ECE Department, UCLA

Joint work with Joachim Dahl, Martin S. Andersen, Yifan Sun, Xin Jiang

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Semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) \\ \text{subject to} & \mathbf{tr}(A_i X) = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0 \end{array}$$

variable X is $n \times n$ symmetric matrix; $X \succeq 0$ means X is positive semidefinite

- matrix inequalities arise naturally in many areas (for example, control, statistics)
- used in convex modeling systems (CVX, YALMIP, CVXPY, ...)
- relaxations of nonconvex quadratic and polynomial optimization

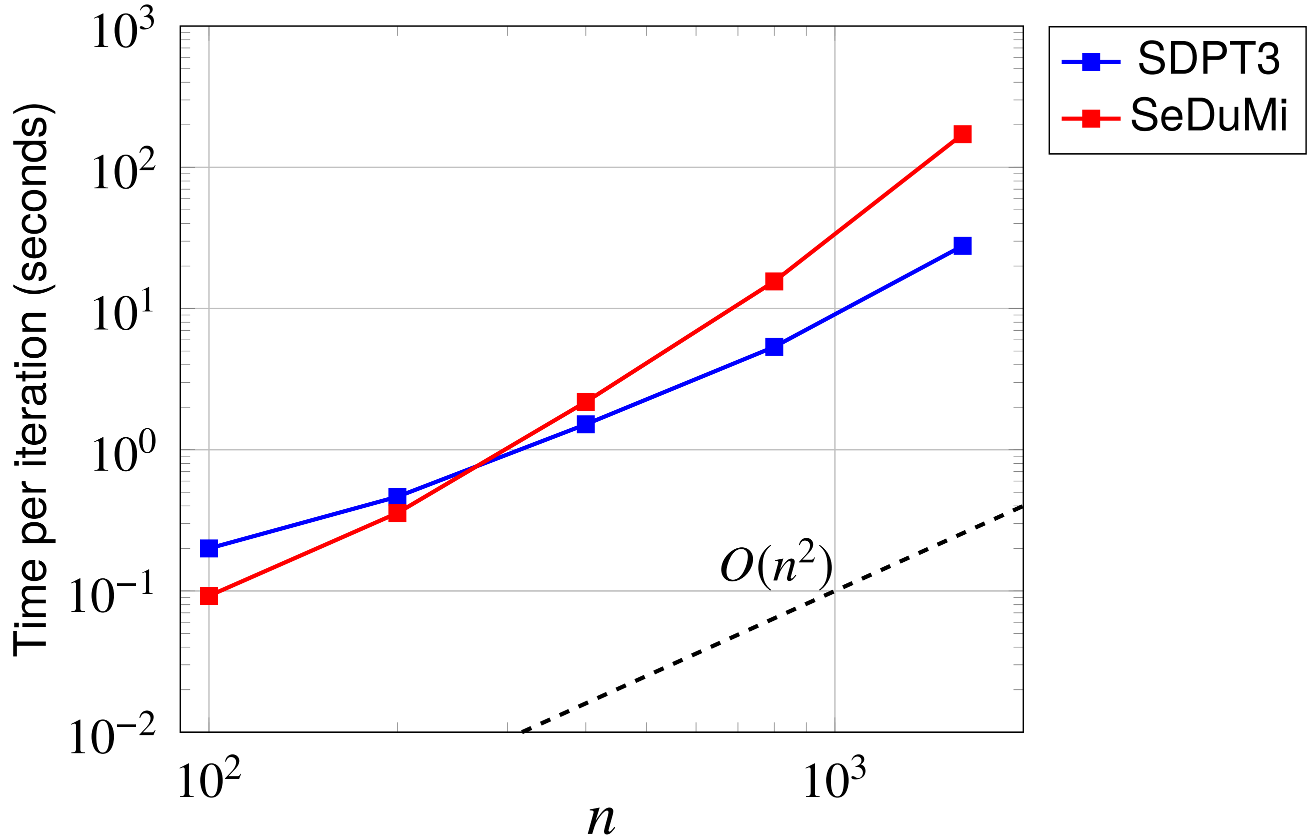
Algorithms

- primal-dual interior-point algorithms (used in SeDuMi, SDPT3, MOSEK)
- nonlinear programming methods based on parameterization $X = YY^T$
- first order methods

This talk: structure in solution X that results from sparsity in coefficients A_i, C

Band structure

cost of solving SDP with banded matrices (bandwidth 11, 100 constraints)



- for bandwidth 1 (linear program), cost/iteration is linear in n
- for bandwidth > 1, cost grows as n^2 or faster

[Andersen, Dahl, Vandenberghe 2010]

Power flow optimization

an optimization problem with non-convex quadratic constraints

Variables

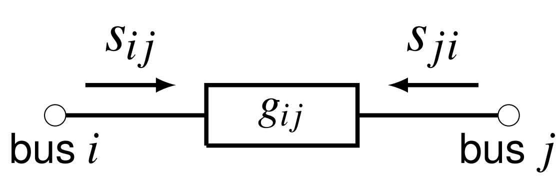
- complex voltage v_i at each node (bus) of the network
- complex power flow s_{ij} entering the link (line) from node i to node j

Non-convex constraints

- (lower) bounds on voltage magnitudes

$$v_{\min} \leq |v_i| \leq v_{\max}$$

- flow balance equations:



$$s_{ij} + s_{ji} = \bar{g}_{ij}|v_i - v_j|^2$$

g_{ij} is admittance of line from node i to j

Semidefinite relaxation of optimal power flow problem

- introduce matrix variable $X = \mathrm{Re}(vv^H)$, *i.e.*, with elements $X_{ij} = \mathrm{Re}\,(v_i \bar{v}_j)$
- voltage bounds and flow balance equations are convex in X:

$$
\begin{array}{lcl}
v_{\min} \leq |v_i| \leq v_{\max} & \longrightarrow & v_{\min}^2 \leq X_{ii} \leq v_{\max}^2 \\
s_{ij} + s_{ji} = \bar{g}_{ij}|v_i - v_j|^2 & \longrightarrow & s_{ij} + s_{ji} = \bar{g}_{ij}(X_{ii} + X_{jj} - 2X_{ij})
\end{array}
$$

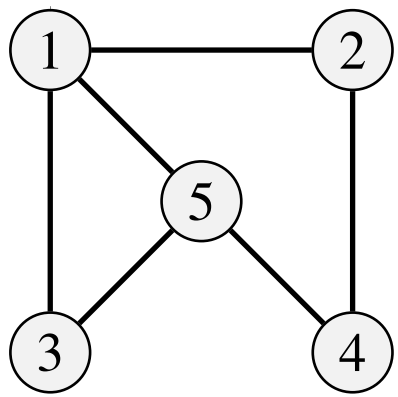
- replace constraint $X = \mathrm{Re}(vv^H)$ with weaker constraint $X \succeq 0$
- relaxation is exact if optimal X has rank two

Sparsity in SDP relaxation:

off-diagonal X_{ij} appears in constraints only if there is a line between buses i and j

[Jabr 2006] [Bai et al. 2008] [Lavaei and Low 2012], [Molzahn et al. 2013], . . .

Sparsity graph



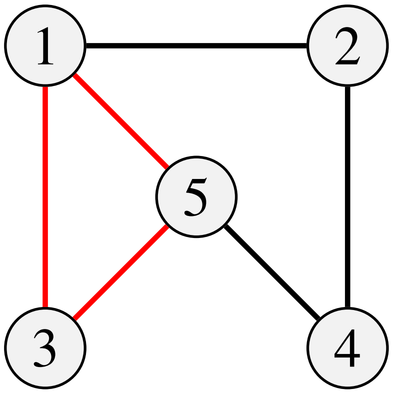
$$A = \begin{bmatrix} A_{11} & A_{21} & A_{31} & 0 & A_{51} \\ A_{21} & A_{22} & 0 & A_{42} & 0 \\ A_{31} & 0 & A_{33} & 0 & A_{53} \\ 0 & A_{42} & 0 & A_{44} & A_{54} \\ A_{51} & 0 & A_{53} & A_{54} & A_{55} \end{bmatrix}$$

- sparsity pattern of symmetric $n \times n$ matrix is set of 'nonzero' positions

$$E \subseteq \{\{i, j\} \mid i, j \in \{1, 2, \ldots, n\}\}$$

- A has sparsity pattern E if $A_{ij} = 0$ if $i \neq j$ and $\{i, j\} \notin E$
- notation: $A \in \mathbf{S}^n_E$
- represented by undirected graph (V, E) with edges E, vertices $V = \{1, \ldots, n\}$
- clique (maximal complete subgraph) forms maximal 'dense' principal submatrix

Sparsity graph



$$A = \begin{bmatrix} \mathbf{A_{11}} & A_{21} & \mathbf{A_{31}} & 0 & \mathbf{A_{51}} \\ A_{21} & A_{22} & 0 & A_{42} & 0 \\ \mathbf{A_{31}} & 0 & \mathbf{A_{33}} & 0 & \mathbf{A_{53}} \\ 0 & A_{42} & 0 & A_{44} & A_{54} \\ \mathbf{A_{51}} & 0 & \mathbf{A_{53}} & A_{54} & \mathbf{A_{55}} \end{bmatrix}$$

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Sparse matrix cones

we define two convex cones in $\mathbf{S}^n_E$ (symmetric $n \times n$ matrices with pattern E)

- positive semidefinite matrices

$$\mathbf{S}^n_+ \cap \mathbf{S}^n_E = \{X \in \mathbf{S}^n_E \mid X \succeq 0\}$$

- matrices with a positive semidefinite completion

$$\Pi_E(\mathbf{S}^n_+) = \{\Pi_E(X) \mid X \succeq 0\}$$

  Π_E is projection on $\mathbf{S}^n_E$

Properties

- two cones are convex
- closed, pointed, with nonempty interior (relative to $\mathbf{S}^n_E$)
- form a pair of dual cones (for the trace inner product)

Sparse semidefinite program

Standard form SDP and dual (variables $X, S \in \mathbf{S}^n$, $y \in \mathbf{R}^m$)

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) \\ \text{subject to} & \mathbf{tr}(A_i X) = b_i, \;\; i = 1, \ldots, m \\ & X \succeq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & \sum_{i=1}^m y_i A_i + S = C \\ & S \succeq 0 \end{array}$$

Equivalent pair of conic linear programs (variables $X, S \in \mathbf{S}^n_E$, $y \in \mathbf{R}^m$)

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) \\ \text{subject to} & \mathbf{tr}(A_i X) = b_i, \;\; i = 1, \ldots, m \\ & X \in K \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & \sum_{i=1}^m y_i A_i + S = C \\ & S \in K^* \end{array}$$

- E is union of sparsity patterns of C, A_1, …, A_m
- $K = \Pi_E(\mathbf{S}^n_+)$ is cone of p.s.d. completable matrices with sparsity pattern E
- $K^* = \mathbf{S}^n_+ \cap \mathbf{S}^n_E$ is cone of positive semidefinite matrices with sparsity pattern E

Outline

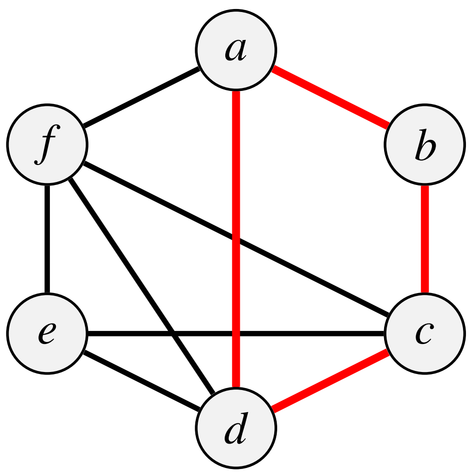
1. Sparse semidefinite programs
2. **Chordal graphs**
3. Decomposition of sparse matrix cones
4. Multifrontal algorithms for logarithmic barrier functions
5. Minimum rank positive semidefinite completion

Chordal graph

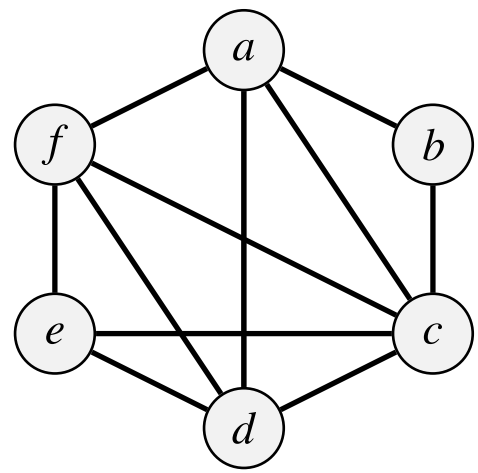
- undirected graph with vertex set V, edge set $E \subseteq \{\{v, w\} \mid v, w \in V\}$

$$G = (V, E)$$

- a **chord** of a cycle is an edge between non-consecutive vertices
- G is **chordal** if every cycle of length greater than three has a chord



not chordal



chordal

also known as triangulated, decomposable, rigid circuit graph, . . .

History

chordal graphs have been studied in many disciplines since the 1960s

- combinatorial optimization (a class of *perfect* graphs)
- linear algebra (sparse factorization, completion problems)
- database theory
- machine learning (graphical models, probabilistic networks)
- nonlinear optimization (partial separability)

first used in semidefinite optimization by Fujisawa, Kojima, Nakata (1997)

Chordal sparsity and Cholesky factorization

Cholesky factorization of positive definite $A \in \mathbf{S}^n_E$:

$$PAP^T = LDL^T$$

P a permutation, L unit lower triangular, D positive diagonal

- if E is chordal, then there exists a permutation for which

$$P^T(L + L^T)P \in \mathbf{S}^n_E$$

  A has a 'zero fill' Cholesky factorization

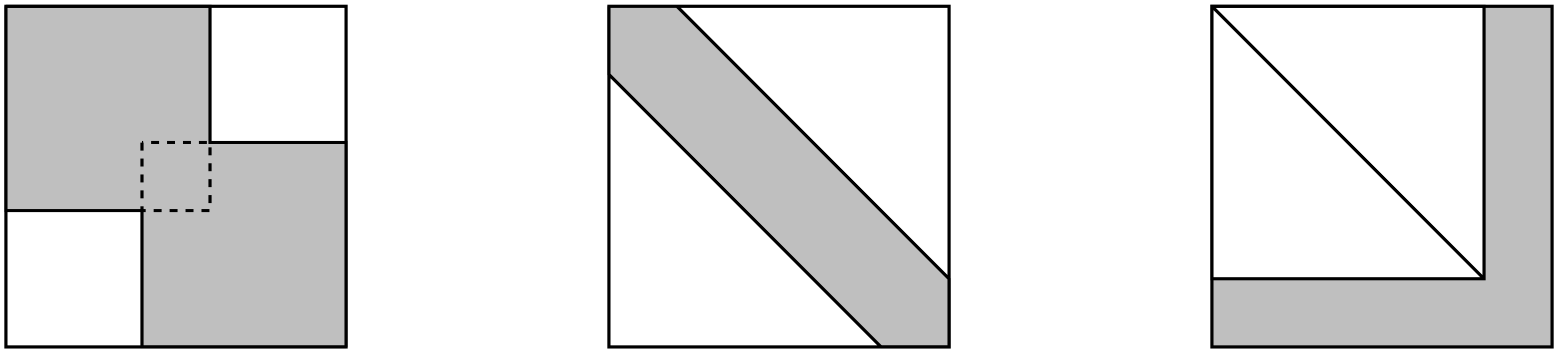
- if E is not chordal, then for every P there exist positive definite $A \in \mathbf{S}^n_E$ for which

$$P^T(L + L^T)P \notin \mathbf{S}^n_E$$

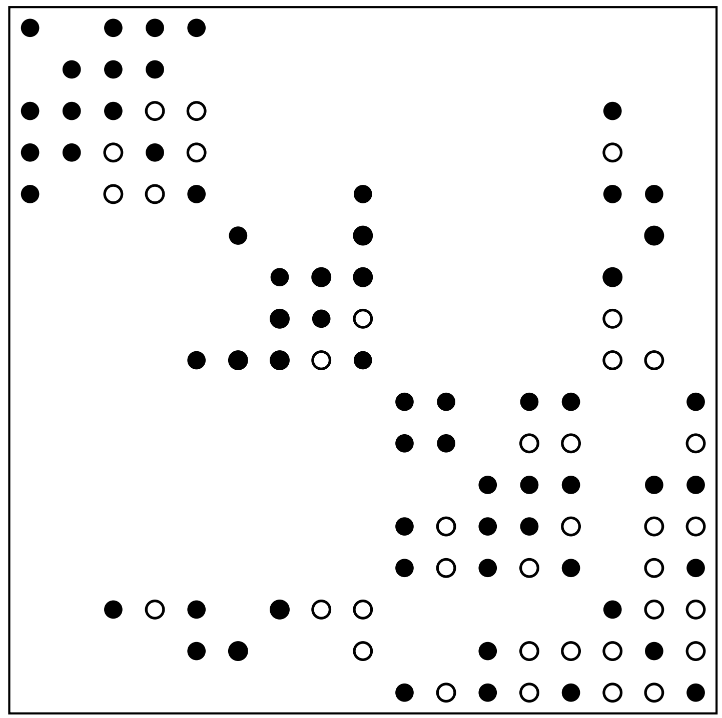
[Rose 1970]

Examples

Simple patterns



Sparsity pattern of a Cholesky factor

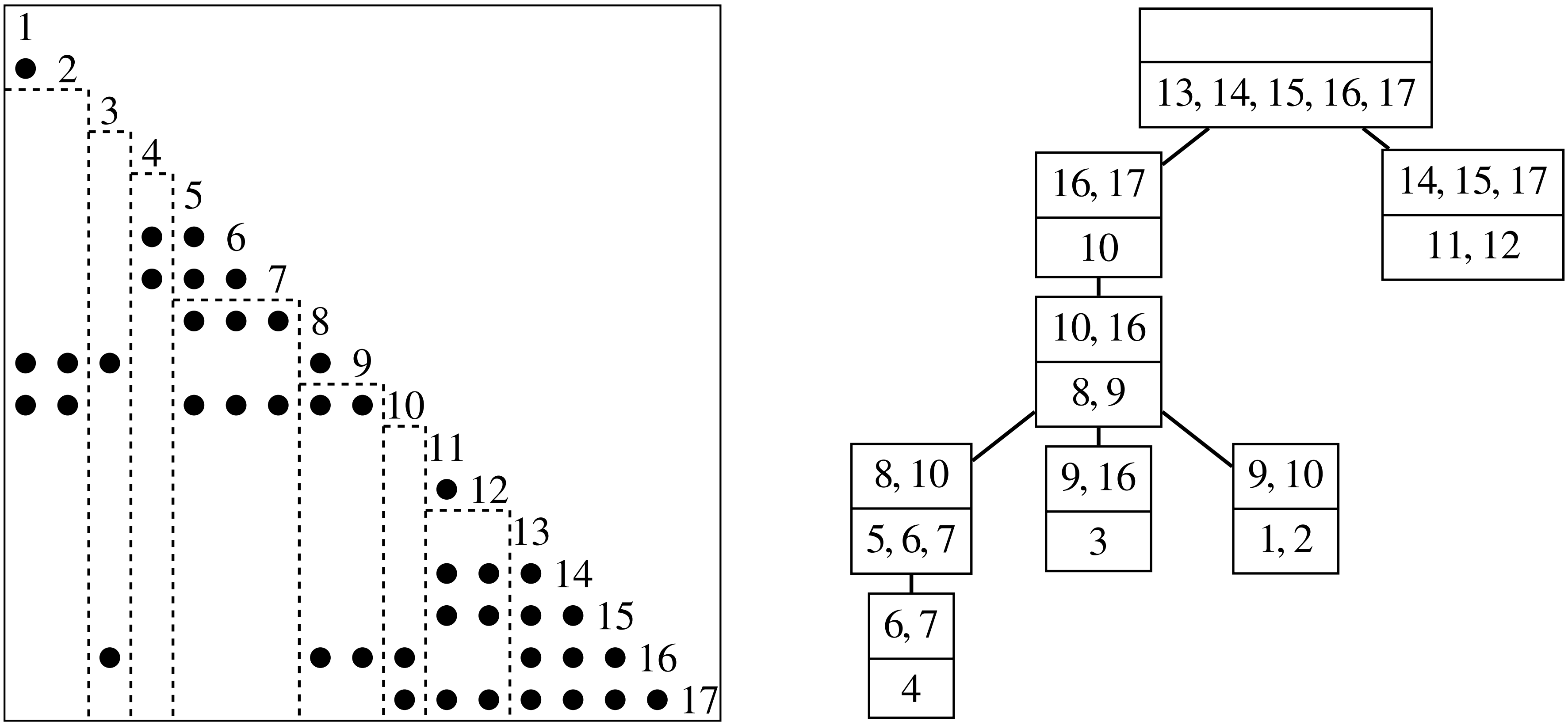


●: edges of non-chordal sparsity pattern

○: fill entries in Cholesky factorization

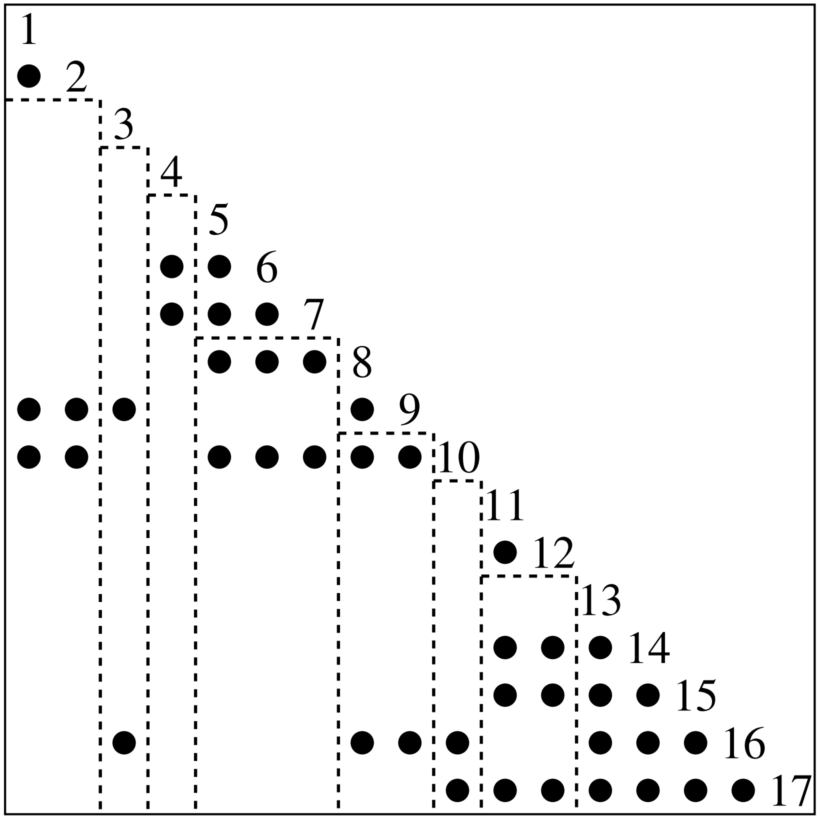
a chordal extension of non-chordal pattern

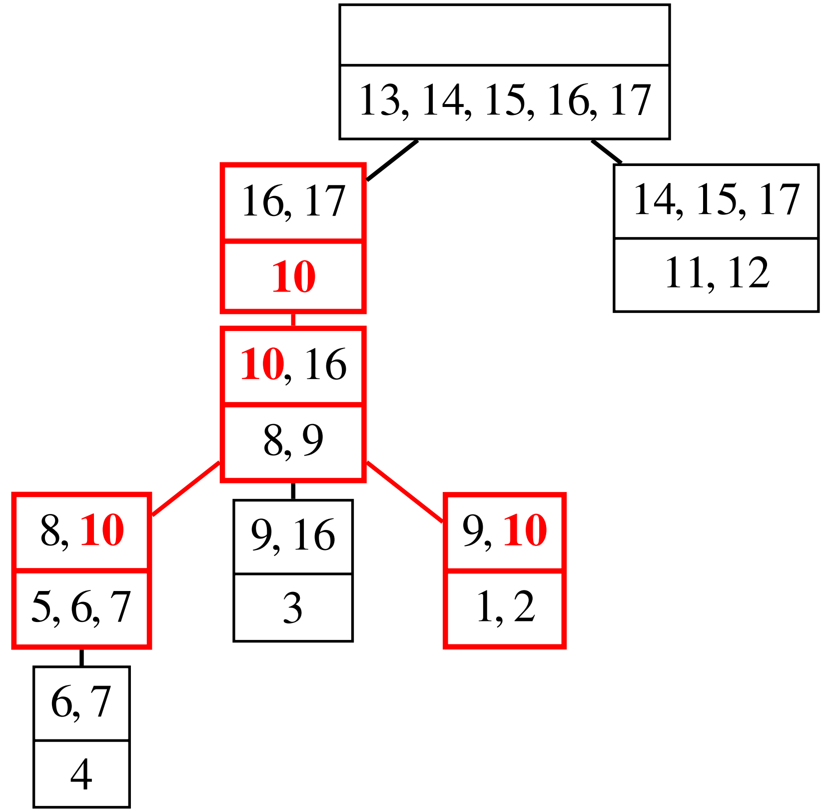
Supernodal elimination tree (clique tree)



- vertices of tree are cliques of chordal sparsity graph
- top row of each block is intersection of clique with parent clique
- bottom rows are (maximal) *supernodes*; form a partition of $\{1, 2, \ldots, n\}$
- for each v, cliques that contain v form a subtree of elimination tree

Supernodal elimination tree (clique tree)





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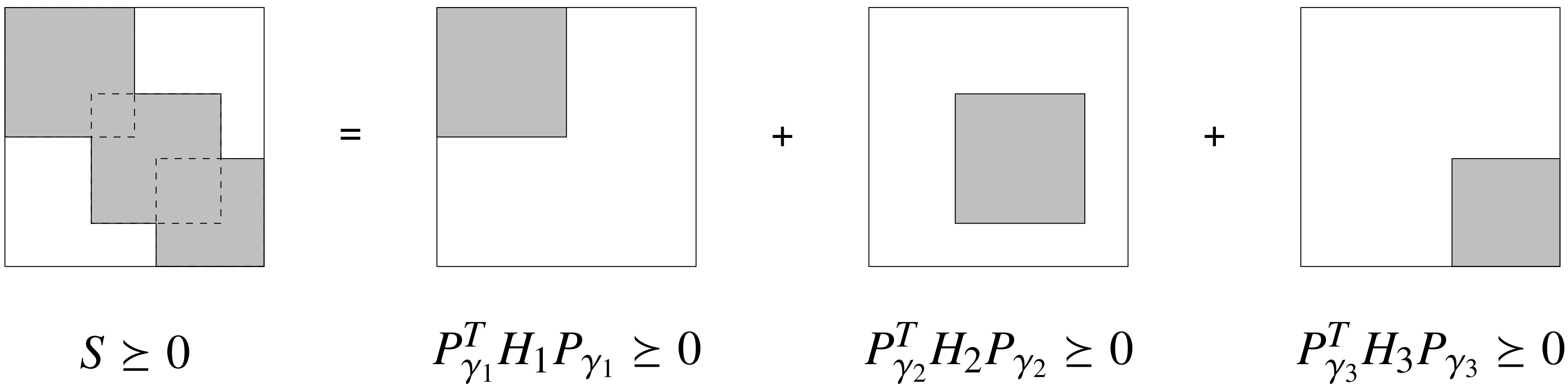
1. Sparse semidefinite programs
2. Chordal graphs
3. **Decomposition of sparse matrix cones**
4. Multifrontal algorithms for logarithmic barrier functions
5. Minimum rank positive semidefinite completion

Positive semidefinite matrices with chordal sparsity pattern

$S \in \mathbf{S}_E^n$ is positive semidefinite if and only if it can be expressed as

$$S = \sum_{\text{cliques } \gamma_i} P_{\gamma_i}^T H_i P_{\gamma_i} \qquad \text{with } H_i \succeq 0$$

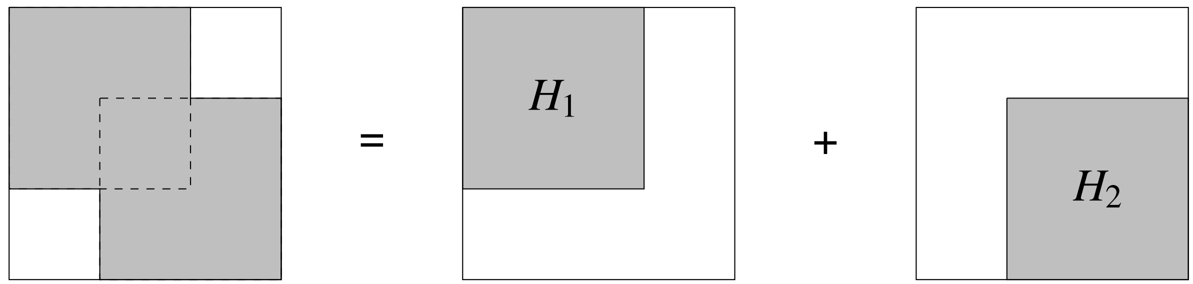
(for an index set β, P_β is 0-1 matrix of size $|\beta| \times n$ with $P_\beta x = x_\beta$ for all x)



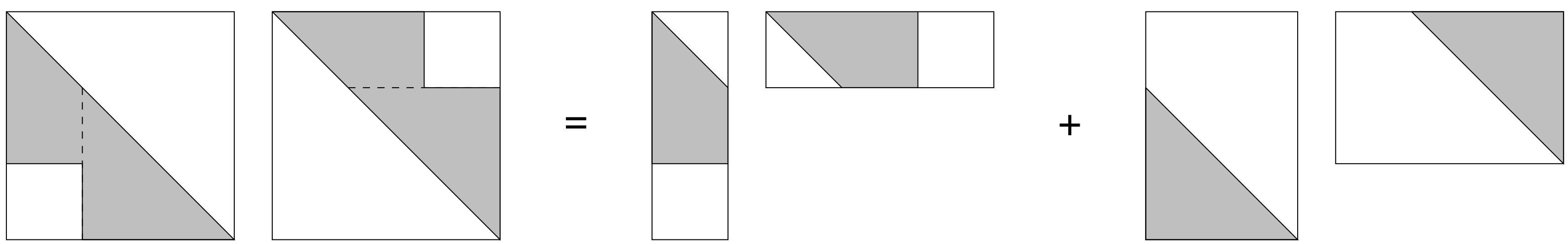
[Griewank and Toint 1984] [Agler, Helton, McCullough, Rodman 1988]

Decomposition from Cholesky factorization

- example with two cliques:



H_1 and H_2 follow by combining columns in Cholesky factorization



- readily computed from update matrices in multifrontal Cholesky factorization

PSD completable matrices with chordal sparsity

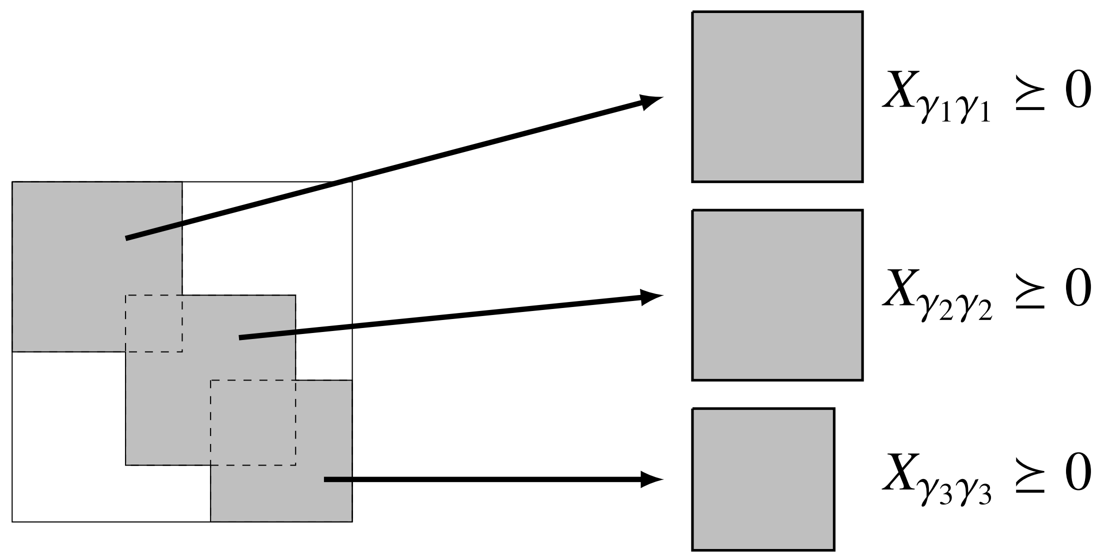
$X \in \mathbf{S}^n_E$ has a positive semidefinite completion if and only if

$$X_{\gamma_i \gamma_i} \succeq 0 \quad \text{for all cliques } \gamma_i$$

follows from duality and clique decomposition of positive semidefinite cone

Example (three cliques $\gamma_1, \gamma_2, \gamma_3$)

PSD completable X



[Grone, Johnson, Sá, Wolkowicz, 1984]

Sparse semidefinite optimization

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) \\ \text{subject to} & \mathbf{tr}(A_i X) = b_i, \quad i = 1, \ldots, m \\ & X \in K \end{array}$$

- E is union of sparsity patterns of C, A_1, ..., A_m
- $K = \Pi_E(\mathbf{S}^n_+)$ is cone of p.s.d. completable matrices
- without loss of generality, can assume E is chordal

Decomposition algorithms

- cone K is intersection of simple cones ($X_{\gamma_i \gamma_i} \succeq 0$ for all cliques γ_i)
- first used in interior-point methods
  [Fukuda et al. 2000] [Nakata et al. 2003]
- first order, splitting, and dual decomposition methods
  [Lu, Nemirovski, Monteiro 2007] [Lam, Zhang, Tse 2011] [Sun et al. 2014, 2015]
  [Pakazad et al. 2017] [Zheng, Fantuzzi, Papachristodoulou, Goulart, Wynn 2017], ...

Example: sparse nearest matrix problems

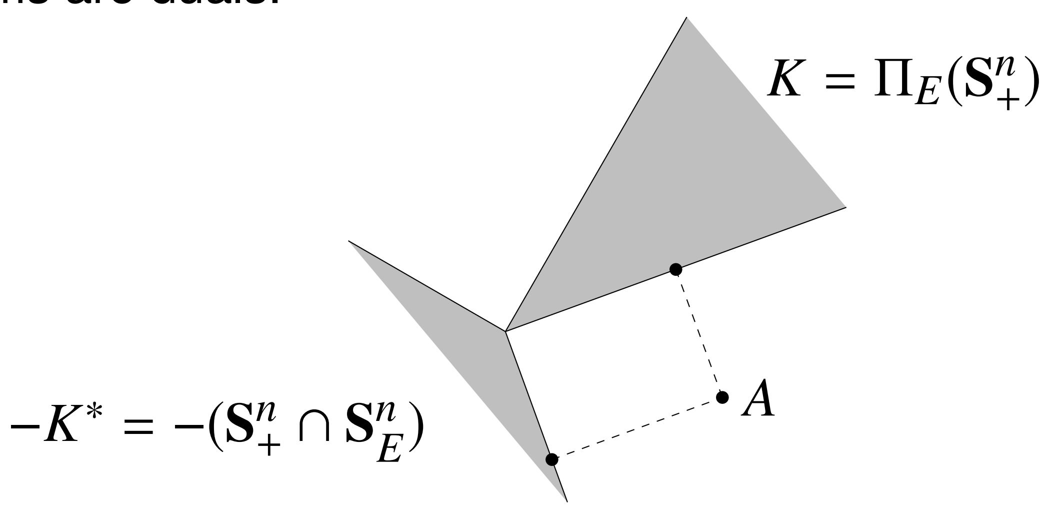
- find nearest sparse PSD-completable matrix with given sparsity pattern

$$\begin{array}{ll} \text{minimize} & \|X - A\|_F^2 \\ \text{subject to} & X \in \Pi_E(\mathbf{S}_+^n) \end{array}$$

- find nearest sparse PSD matrix with given sparsity pattern

$$\begin{array}{ll} \text{minimize} & \|S + A\|_F^2 \\ \text{subject to} & S \in \mathbf{S}_+^n \cap \mathbf{S}_E^n \end{array}$$

these two problems are duals:



Decomposition methods

from the decomposition theorems, the two problems can be written

$$\begin{array}{lll} \text{Primal:} & \text{minimize} & \|X - A\|_F^2 \\ & \text{subject to} & X_{\gamma_i \gamma_i} \succeq 0 \quad \text{for all cliques } \gamma_i \end{array}$$

$$\begin{array}{lll} \text{Dual:} & \text{minimize} & \|A + \sum_i P_{\gamma_i}^T H_i P_{\gamma_i}\|_F^2 \\ & \text{subject to} & H_i \succeq 0 \quad \text{for all cliques } \gamma_i \end{array}$$

Algorithms

- Dykstra's algorithm (dual block coordinate ascent)
- (fast) dual projected gradient algorithm (FISTA)
- Douglas-Rachford splitting, ADMM

sequence of projections on PSD cones of order $|\gamma_i|$ (eigenvalue decomposition)

Results

sparsity patterns from University of Florida Sparse Matrix Collection

n	density	#cliques	avg. clique size	max. clique
20141	2.80e-3	1098	35.7	168
38434	1.25e-3	2365	28.1	188
57975	9.04e-4	8875	14.9	132
79841	9.71e-4	4247	44.4	337
114599	2.02e-4	7035	18.9	58

	total runtime (sec)			time/iteration (sec)		
n	FISTA	Dykstra	DR	FISTA	Dykstra	DR
20141	2.5e2	3.9e1	3.8e1	1.0	1.6	1.5
38434	4.7e2	4.7e1	6.2e1	2.1	1.9	2.5
57975	> 4hr	1.4e2	1.1e3	3.5	5.7	6.4
79841	2.4e3	3.0e2	2.4e2	6.3	7.6	9.7
114599	5.3e2	5.5e1	1.0e2	2.6	2.2	4.0

[Sun and Vandenberghe 2015]

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5. Minimum rank positive semidefinite completion

Sparse SDP as nonsymmetric conic linear program

Standard form SDP

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) \\ \text{subject to} & \mathbf{tr}(A_iX) = b_i, \;\; i = 1, \ldots, m \\ & X \succeq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & \sum_{i=1}^m y_i A_i + S = C \\ & S \succeq 0 \end{array}$$

Equivalent conic linear program

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) \\ \text{subject to} & \mathbf{tr}(A_iX) = b_i, \;\; i = 1, \ldots, m \\ & X \in K \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & \sum_{i=1}^m y_i A_i + S = C \\ & S \in K^* \end{array}$$

- $K \in \Pi_E(\mathbf{S}^n_+)$ is cone of p.s.d. completable matrices with pattern E
- $K^* \in \mathbf{S}^n_+ \cap \mathbf{S}^n_E$ is cone of p.s.d. matrices with pattern E
- optimization problem in a lower-dimensional space $\mathbf{S}^n_E$
- K is not self-dual; no symmetric primal-dual interior-point methods

Barrier function for positive semidefinite cone

$$\phi(S) = -\log\det S, \qquad \operatorname{dom}\phi = \{S \in \mathbf{S}_E^n \mid S \succ 0\}$$

- gradient (negative projected inverse)

$$\nabla\phi(S) = -\Pi_E(S^{-1})$$

requires entries of dense inverse S^{-1} on diagonal and for $\{i,j\} \in E$

- Hessian applied to sparse $Y \in \mathbf{S}_E^n$:

$$\nabla^2\phi(S)[Y] = \frac{d}{dt}\nabla\phi(S+tY)\Big|_{t=0} = \Pi_E\left(S^{-1}YS^{-1}\right)$$

requires projection of dense product $S^{-1}YS^{-1}$

Multifrontal algorithms

assume E is a chordal sparsity pattern (or chordal extension)

Cholesky factorization [Duff and Reid 1983]

- factorization $S = LDL^T$ gives function value of barrier: $\phi(S) = -\sum_i \log D_{ii}$
- computed by a recursion on elimination tree in topological order

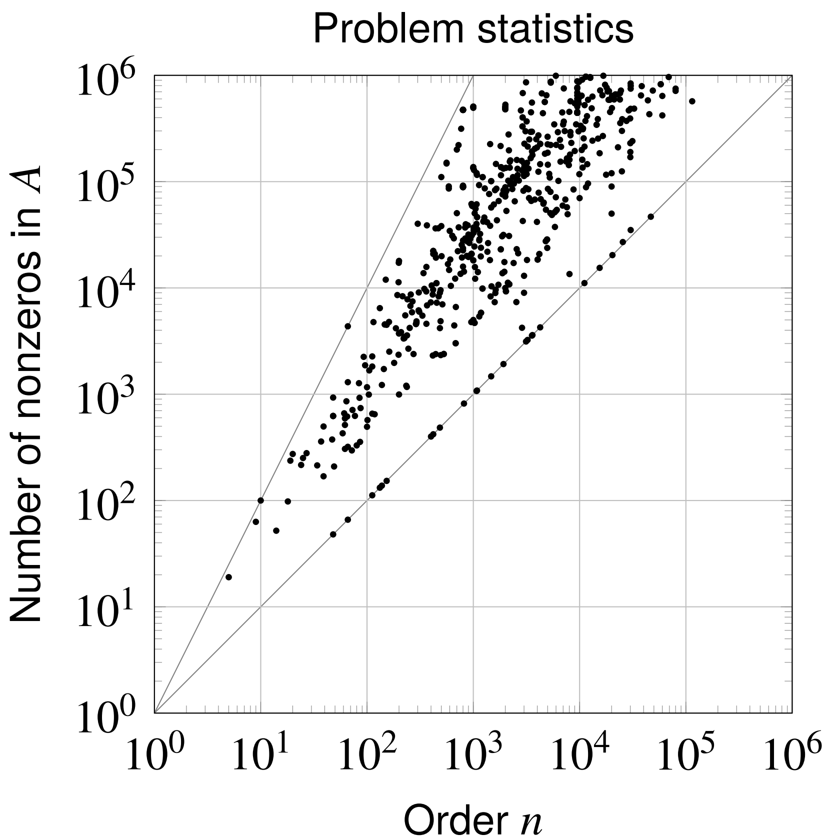
Gradient [Campbell and Davis 1995] [Andersen et al. 2013]

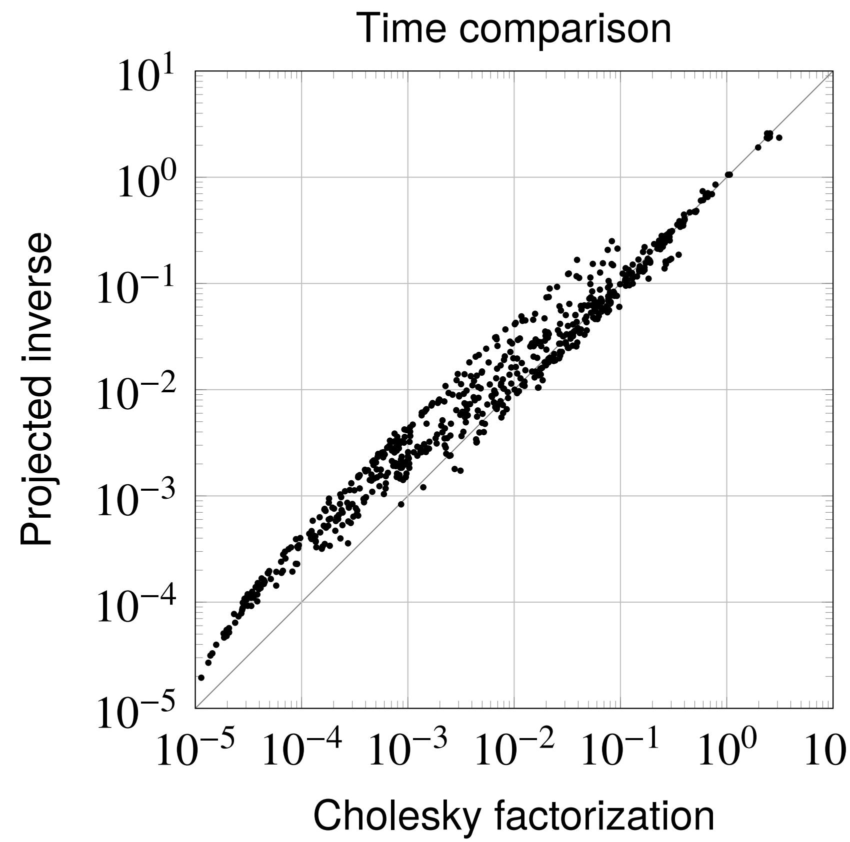
- compute $\nabla\phi(S) = -\Pi_E(S^{-1})$ from equation $S^{-1}L = L^{-T}D^{-1}$
- recursion on elimination tree in inverse topological order

Hessian

- compute $\nabla^2\phi(S)[Y] = \Pi_E(S^{-1}YS^{-1})$ by linearizing recursion for gradient
- two recursions on elimination tree (topological and inverse topological order)

Projected inverse versus Cholesky factorization





- 667 patterns from University of Florida Sparse Matrix Collection
- time in seconds for projected inverse and Cholesky factorization
- code at `github.com/cvxopt/chompack`

Barrier for positive semidefinite completable cone

$$\phi_*(X) = \sup_S \left(-\mathbf{tr}(XS) - \phi(S)\right), \qquad \mathrm{dom}\,\phi_* = \{X = \Pi_E(Y) \mid Y \succ 0\}$$

- this is the conjugate of the barrier $\phi(S) = -\log\det S$ for the sparse p.s.d. cone
- inverse $Z = \widehat{S}^{-1}$ of optimal $\widehat{S}$ is maximum determinant PD completion of X:

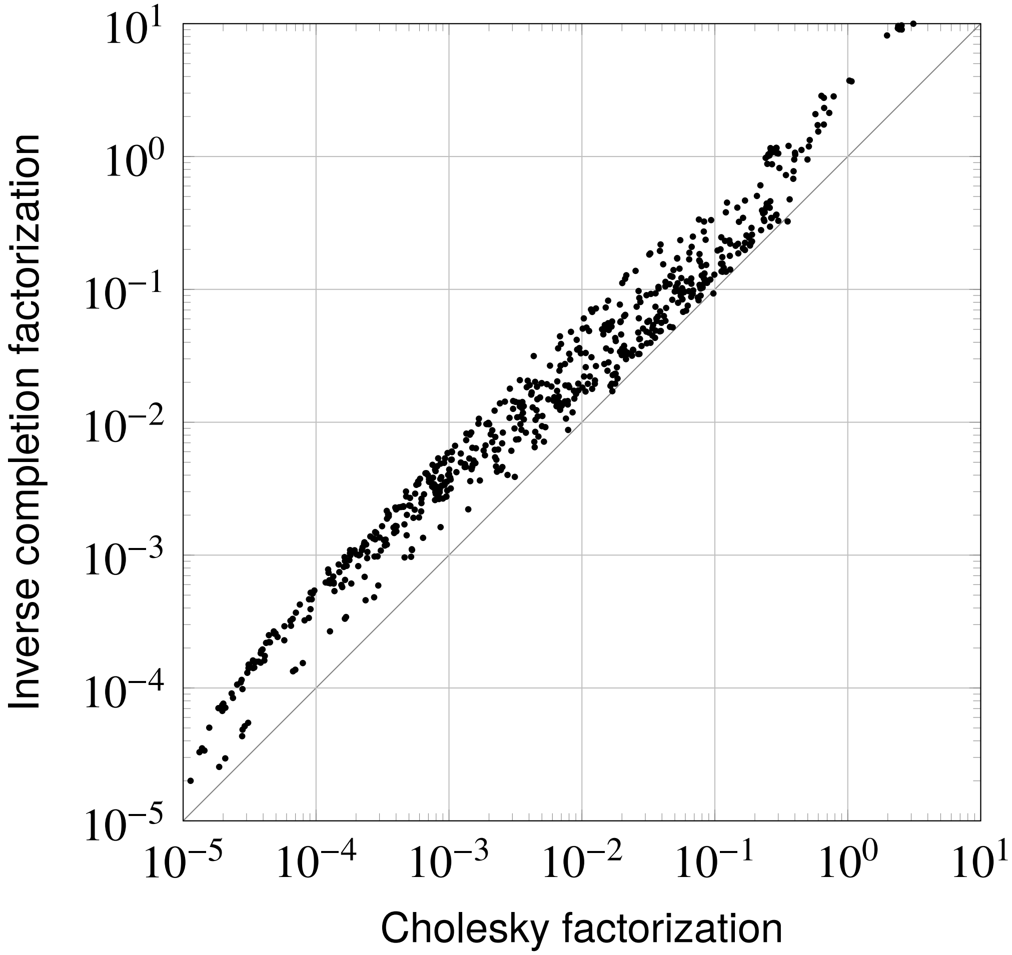
$$\begin{array}{ll} \text{maximize} & \log\det Z \\ \text{subject to} & \Pi_E(Z) = X \end{array}$$

- gradient and Hessian of ϕ_* at X are

$$\nabla\phi_*(X) = -\widehat{S}, \qquad \nabla^2\phi_*(X) = \nabla^2\phi(\widehat{S})^{-1}$$

for chordal E, efficient 'multifrontal' algorithms for Cholesky factors of $\hat{S}$, given X

Inverse completion versus Cholesky factorization



time for Cholesky factorization of inverse of maximum determinant PD completion

Nonsymmetric interior-point methods

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) \\ \text{subject to} & \mathbf{tr}(A_iX) = b_i, \quad i = 1, \ldots, m \\ & X \in \Pi_E(\mathbf{S}^n_+) \end{array}$$

- can be solved by nonsymmetric primal or dual barrier methods
- logarithmic barriers for cone $\Pi_E(\mathbf{S}^n_+)$ and its dual cone $\mathbf{S}^n_+ \cap \mathbf{S}^n_E$:

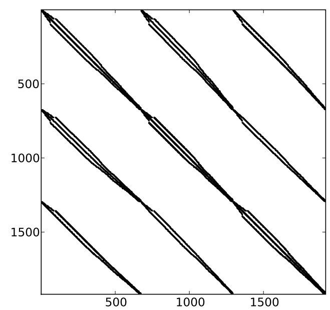
$$\phi_*(X) = \sup_S \left(-\mathbf{tr}(XS) + \log\det S\right), \qquad \phi(S) = -\log\det S$$

- fast evaluation of barrier values and derivatives if pattern is chordal
- examples: linear complexity per iteration for band or arrow pattern
- code and numerical results at `github.com/cvxopt/smcp`

[Fukuda et al. 2000], [Burer 2003], [Srijungtongsiri and Vavasis 2004], [Andersen et al. 2010]

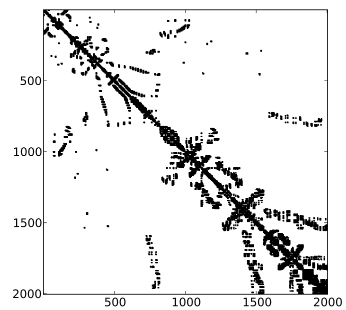
Sparsity patterns

- sparsity patterns from University of Florida Sparse Matrix Collection
- $m = 200$ constraints
- randomly generated data with 0.05% nonzeros in A_i relative to $|E|$



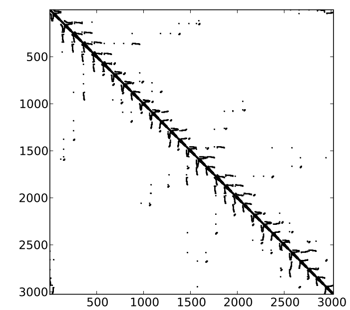
rs228

$n = 1{,}919$



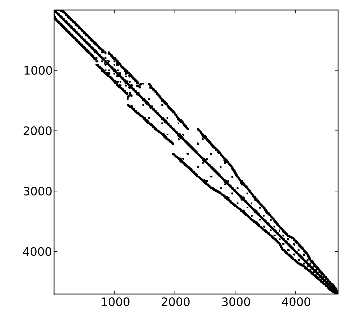
rs35

$n = 2{,}003$



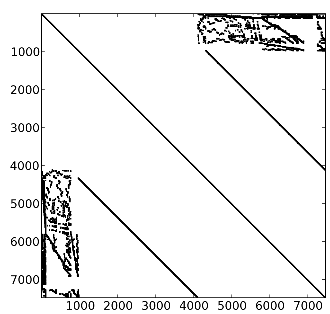
rs200

$n = 3{,}025$



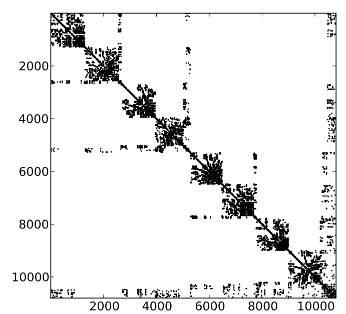
rs365

$n = 4{,}704$



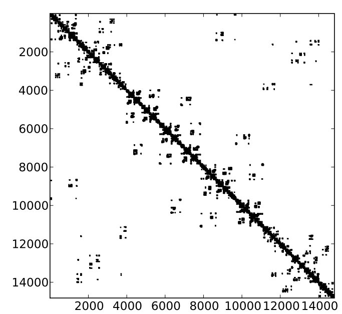
rs1555

$n = 7{,}479$



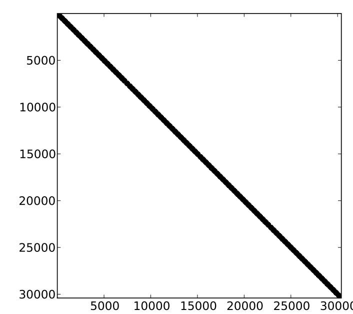
rs828

$n = 10{,}800$



rs1184

$n = 14{,}822$



rs1288

$n = 30{,}401$

Results

n	DSDP	SDPA	SDPA-C	SDPT3	SeDuMi	SMCP
1919	1.4	30.7	5.7	10.7	511.2	2.3
2003	4.0	34.4	41.5	13.0	521.1	15.3
3025	2.9	128.3	6.0	33.0	1856.9	2.2
4704	15.2	407.0	58.8	99.6	4347.0	18.6

n	DSDP	SDPA-C	SMCP
7479	22.1	23.1	9.5
10800	482.1	1812.8	311.2
14822	791.0	2925.4	463.8
30401	mem	2070.2	320.4

- average time per iteration for different solvers
- SMCP uses nonsymmetric matrix cone approach [Andersen *et al.* 2010]

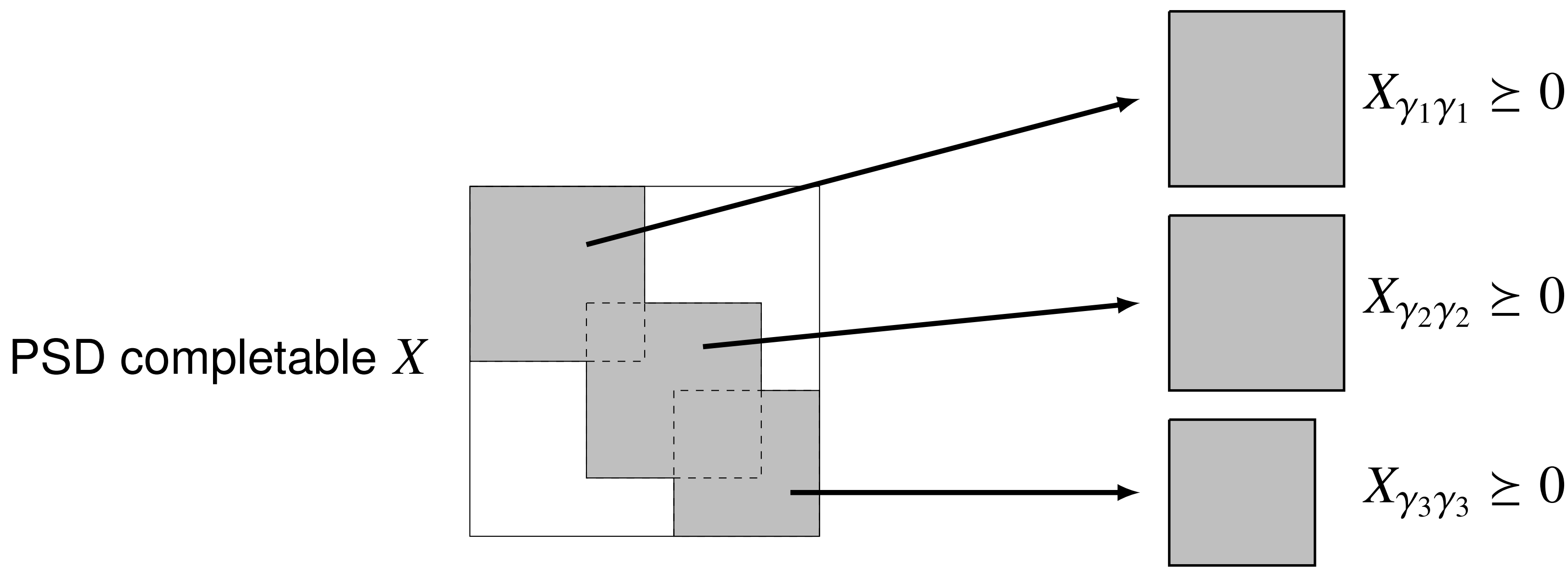
Outline

1. Sparse semidefinite programs
2. Chordal graphs
3. Decomposition of sparse matrix cones
4. Multifrontal algorithms for logarithmic logarithmic barriers
5. **Minimum rank positive semidefinite completion**

Minimum rank PSD completion with chordal sparsity

recall that $X \in \mathbf{S}^n_E$ has a positive semidefinite completion if and only if

$$X_{\gamma_i\gamma_i} \succeq 0 \quad \text{for all cliques } \gamma_i$$



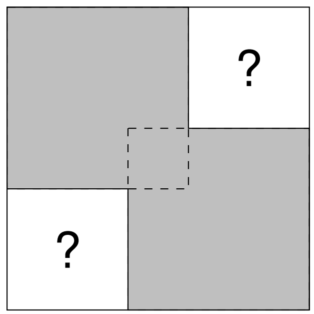
the minimum rank PSD completion has rank equal to

$$\max_{\text{cliques } \gamma_i} \operatorname{rank}(X_{\gamma_i\gamma_i})$$

[Dancis 1992]

Two-block completion problem

we consider the simple two-block completion problem



$$X = \begin{bmatrix} X_{11} & X_{12} & 0 \\ X_{21} & X_{22} & X_{23} \\ 0 & X_{32} & X_{33} \end{bmatrix}$$

- a completion exists if and only if

$$C_1 = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0, \qquad C_2 = \begin{bmatrix} X_{22} & X_{23} \\ X_{32} & X_{33} \end{bmatrix} \succeq 0$$

- we construct a positive semidefinite completion of rank

$$r = \max\{\mathrm{rank}(C_1), \mathrm{rank}(C_2)\}$$

Two-block completion algorithm

- compute matrices U, V, $\tilde{V}$, W of column dimension r such that

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}^T, \qquad \begin{bmatrix} X_{22} & X_{23} \\ X_{32} & X_{33} \end{bmatrix} = \begin{bmatrix} \tilde{V} \\ W \end{bmatrix} \begin{bmatrix} \tilde{V} \\ W \end{bmatrix}^T$$

- since $VV^T = \tilde{V}\tilde{V}^T$, there exists an orthogonal $r \times r$ matrix Q such that

$$V = \tilde{V}Q$$

(computed from SVDs: take $Q = Q_2 Q_1^T$ where $V = P\Sigma Q_1^T$ and $\tilde{V} = P\Sigma Q_2^T$)

- a completion of rank r is given by

$$\begin{bmatrix} UQ^T \\ \tilde{V} \\ W \end{bmatrix} \begin{bmatrix} UQ^T \\ \tilde{V} \\ W \end{bmatrix}^T = \begin{bmatrix} X_{11} & X_{12} & UQ^T W^T \\ X_{21} & X_{22} & X_{23} \\ WQU^T & X_{32} & X_{33} \end{bmatrix}$$

Sparse semidefinite optimization

$$
\begin{array}{ll}
\text{minimize} & \mathbf{tr}(CX) \\
\text{subject to} & \mathbf{tr}(A_i X) = b_i, \quad i = 1, \ldots, m \\
& X \succeq 0
\end{array}
$$

- any feasible X can be replaced by a PSD completion of $\Pi_E(X)$:

$$
\tilde{X} \succeq 0, \qquad \Pi_E(\tilde{X}) = \Pi_E(X)
$$

- for chordal E, can take $\tilde{X} = YY^T$ with rank bounded by largest clique size

$$
\begin{array}{ll}
\text{minimize} & \mathbf{tr}(Y^T C Y) \\
\text{subject to} & \mathbf{tr}(Y^T A_i Y) = b_i, \quad i = 1, \ldots, m
\end{array}
$$

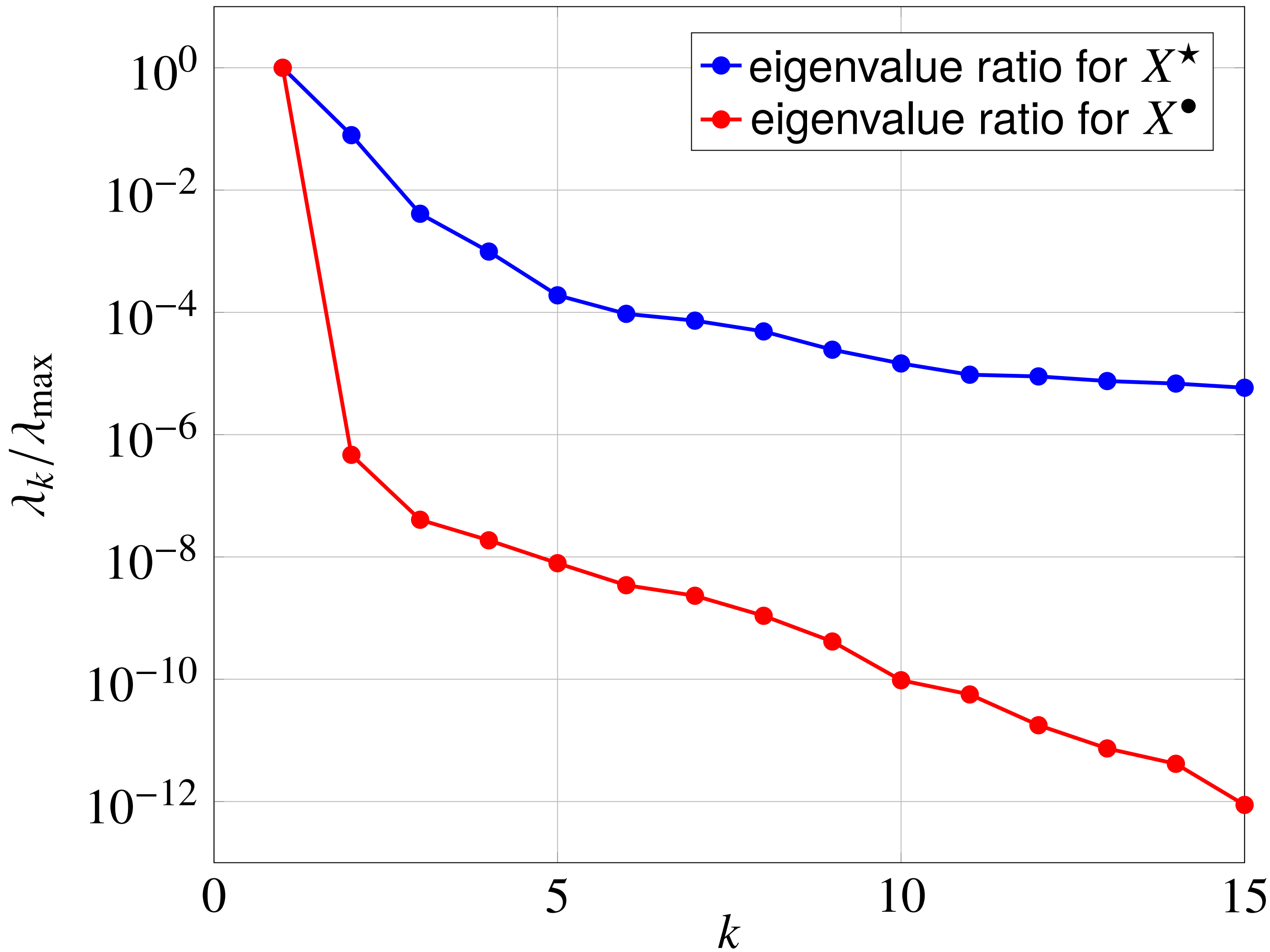
- proves exactness of some simple SDP relaxations
- useful for rounding solution of SDP relaxations to minimum rank solution

SDP relaxation of optimal power flow problem

	n	max. clique	MOSEK 8 rank($X^\star$)	MOSEK 8 rank($X^\bullet$)	SeDuMi v1.05 rank($X^\star$)	SeDuMi v1.05 rank($X^\bullet$)	SDPT3 v4.0 rank($X^\star$)	SDPT3 v4.0 rank($X^\bullet$)
IEEE-118	118	20	1	1	1	1	1	1
IEEE-300	300	17	5	1	5	1	5	1
2383wp	2383	31	17	1	17	1	17	1
2736sp	2736	30	1	1	1	1	1	1
2737sop	2737	29	44	1	43	1	43	1
2746wop	2746	30	32	1	32	1	32	1
2746wp	2746	31	1	1	1	1	1	1
3012wp	3012	32	346	13	346	13	337	17
3120sp	3120	32	514	27	572	32	519	27
3375wp	3375	33	451	19	451	19	454	21

- benchmark problems from MATPOWER package
- rank is number of eigenvalues greater than $10^{-5}\sqrt{n}\lambda_{\max}$
- $X^\star$ is the (Hermitian) solution of the relaxation computed by SDP solver
- $X^\bullet$ is minimum rank PSD completion of $\Pi_E(X^\star)$

IEEE-300 solution



$X^\star$ is computed by SeDuMi; $X^\bullet$ is minimum rank completion of $\Pi_E(X^\star)$

Summary

Sparse matrix theory: PSD and PSD-completable matrices with chordal pattern

- decomposition of sparse matrix cones as sum or intersection of simple cones
- fast algorithms for evaluating barrier functions and derivatives
- simple algorithms for maximum determinant and minimum rank completion

Applications in SDP algorithms

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) \\ \text{subject to} & \mathbf{tr}(A_i X) = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0 \end{array}$$

- decomposition and splitting methods
- nonsymmetric interior-point methods