The Concept of Tensorization – Applications in Blind Signal Separation

TDA 2016

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Bigger picture of the talk

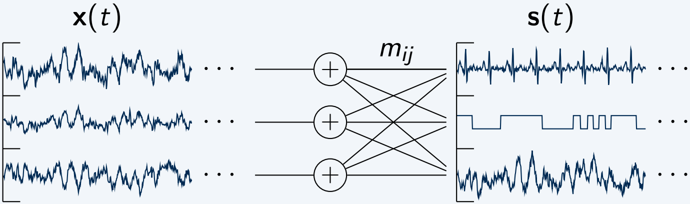
1 | Introduction of BSS

2 | Core elements of multilinear algebra

3 | *Tensorization* as such

4 | Examples of tensorization techniques

What is blind signal separation?



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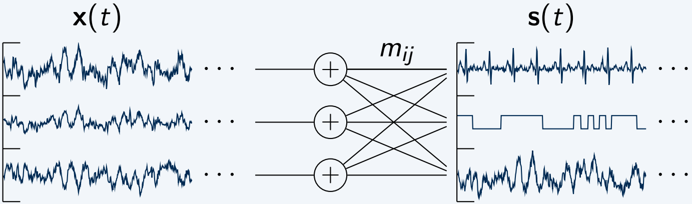

$$\underbrace{\boxed{\mathbf{X}}}_{\text{Known!}}$$

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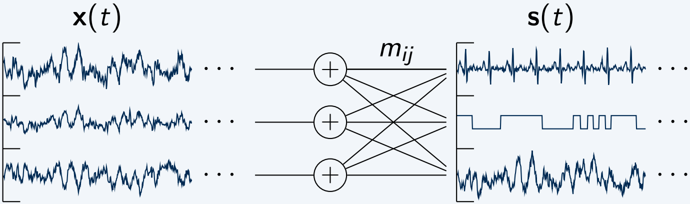

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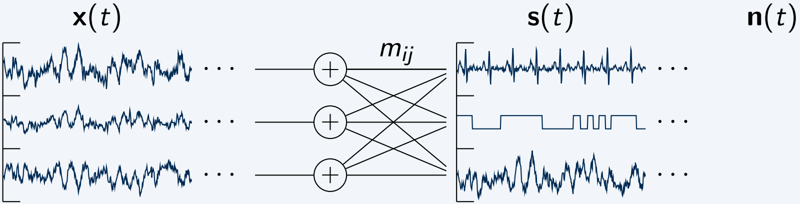

$$\underbrace{\mathbf{X}}_{\text{Known!}} = \underbrace{\mathbf{M}}_{\text{Unknown...}} \times \underbrace{\mathbf{S}}_{\text{Unknown...}} \quad (+\mathbf{N})$$

There are many possible solutions ...

- BSS in general does not lead to a unique solution!

$$\begin{aligned}\mathbf{X} &= \mathbf{M} \cdot \mathbf{S} \\ &= \mathbf{M} \cdot \left(\mathbf{A}^{-1}\mathbf{A}\right) \cdot \mathbf{S} \\ &= \left(\mathbf{M}\mathbf{A}^{-1}\right) \cdot (\mathbf{A}\mathbf{S}) \\ &= \tilde{\mathbf{M}} \cdot \tilde{\mathbf{S}}\end{aligned}$$

- Add constraints/assumptions!

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\end{aligned}
$$

- Add constraints/assumptions!
- Well-known factorizations are not suitable for BSS:
  - SVD: $\boxed{\mathbf{X}} = \boxed{\underset{\perp}{\mathbf{M}}}\ \boxed{\underset{\perp}{\mathbf{S}}}$
  - LQ: $\boxed{\mathbf{X}} = \boxed{\underset{\triangle}{\mathbf{M}}}\ \boxed{\underset{\perp}{\mathbf{S}}}$
  - ...

Specifically tailored BSS techniques

- Independency - Independent Component Analysis (ICA)

$$\mathbf{X} = \mathbf{M} \; \underset{\text{Ind}}{\mathbf{S}}$$

- Nonnegativity - Nonnegative Matrix Factorization (NMF)

$$\mathbf{X} = \underset{+}{\mathbf{M}} \; \underset{+}{\mathbf{S}}$$

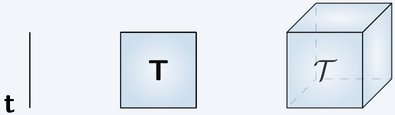
- Sparsity - Sparse Component Analysis (SCA)

$$\mathbf{X} = \mathbf{M} \; \underset{\text{Max0}}{\mathbf{S}}$$

- ...

Multilinear algebra

- Tensor $\mathcal{T}$ of size $I_1 \times I_2 \times \ldots \times I_N$ is a generalization of a vector $\mathbf{t}$ or matrix $\mathbf{T}$:



Multilinear algebra

- We have matrix decompositions ...

$$= \sum_{r=1}^{R} \mathbf{a}_r \otimes \mathbf{b}_r$$

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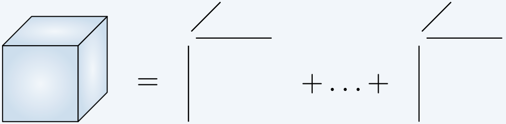

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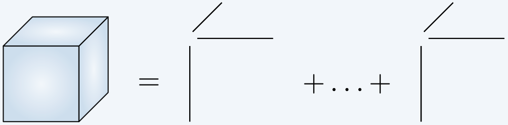
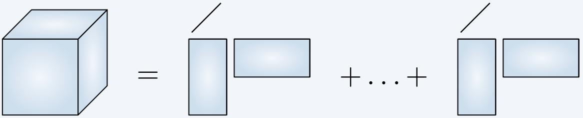


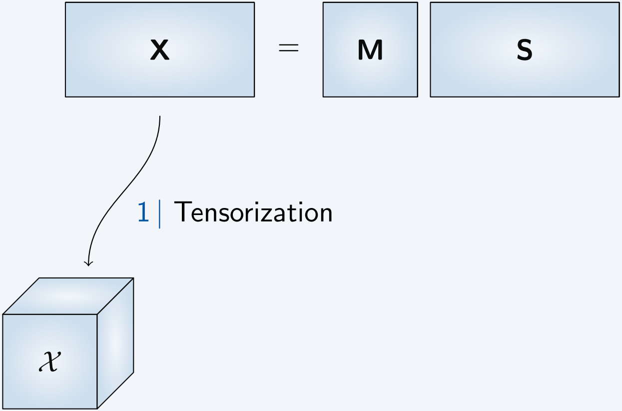
$$= \sum_{r=1}^{R} \mathbf{a}_r \otimes \mathbf{b}_r \otimes \mathbf{c}_r$$

- Or the block term decomposition ... [De Lathauwer, 2008]



$$= \sum_{r=1}^{R} (\mathbf{A}_r \mathbf{B}_r^{\mathrm{T}}) \otimes \mathbf{c}_r$$

Tensorization is often the key!



- Tensorization $\sim$ assumptions

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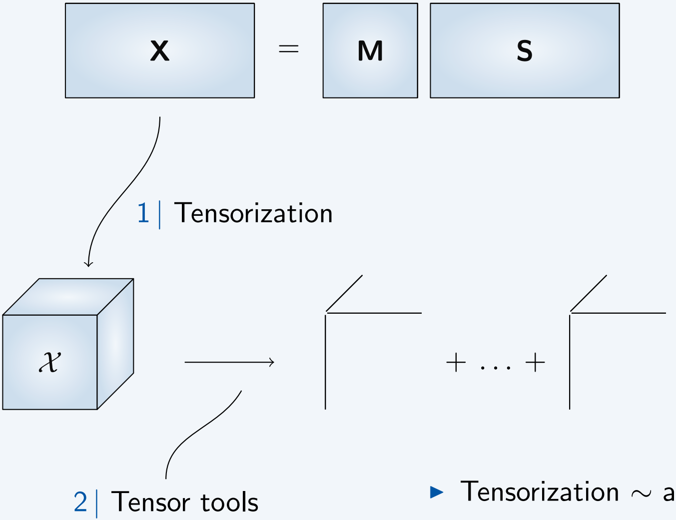

- Tensorization ∼ assumptions
- Tensor tools ∼ uniqueness

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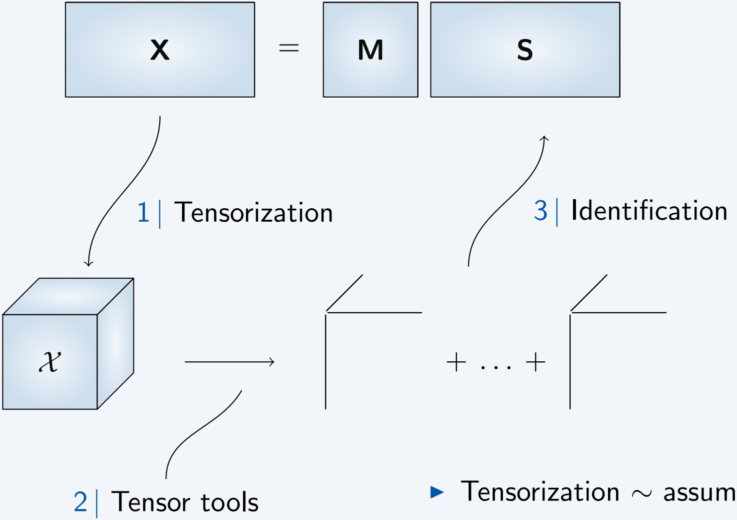

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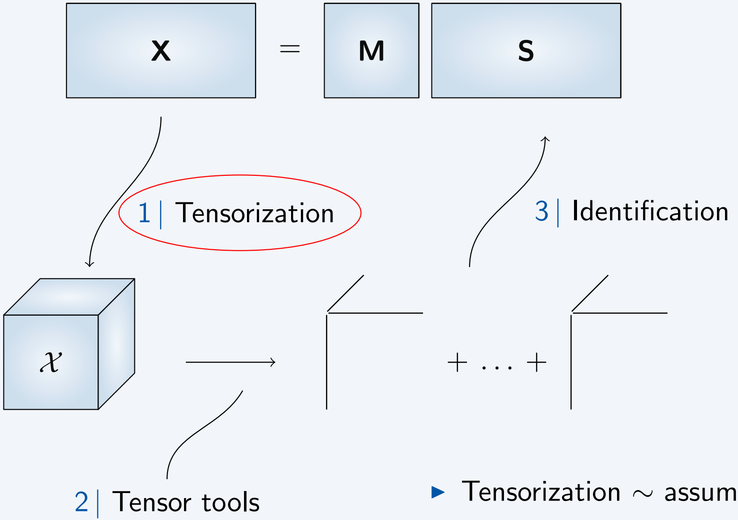

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Overview of the illustrations

Signal model $\longleftrightarrow$ Low-rank constraints

	Working hypothesis on sources	Tensorization
T1:	Exponential polynomials	Hankelization
T2:	Rational functions	Löwnerization
T3:	Sum of Kronecker products	Segmentation
T4:	Independent sources (ICA)	Higher-order statistics
T5:	E.g., independent sources (ICA)	Parameter variation

- Talk of Borbala Hunyadi
- Talk of Takaaki Nara
- Poster of Robert Luce
- Talk of Vladimir Kazeev
- Talk of Namgil Lee
- Talk of Gabriel Hollander
- Poster of Philippe Dreesen
- Poster of Esin Karahan

T1: Hankelization: Intermediary

Hankel matrix:

- Given a vector

$$\begin{bmatrix} h_1 & h_2 & h_3 & h_4 & \dots \end{bmatrix}$$

- We have the Hankel matrix

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ h_3 & h_4 & h_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

T1: Hankelization

- Consider an exponential signal $f(k) = az^k$, being sampled:

$$\begin{bmatrix} a & az & az^2 & az^3 & \cdots \end{bmatrix}$$

- Let's arrange it in a Hankel matrix $\mathbf{H}$

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- $\mathbf{H}$ has rank 1 !

T1: Hankelization

- More general:
  - Consider sinusoids: $f(k) = \cos(2\pi k) = \frac{1}{2}\left(e^{2\pi i}\right)^k + \frac{1}{2}\left(e^{-2\pi i}\right)^k$

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  - Consider sum-of-exponentials: $f(k) = \sum_{r=1}^{R} a_r z_r^k$
  - Consider exponential polynomials: $f(k) = \sum_{r=1}^{R} p_r(k) z_r^k$

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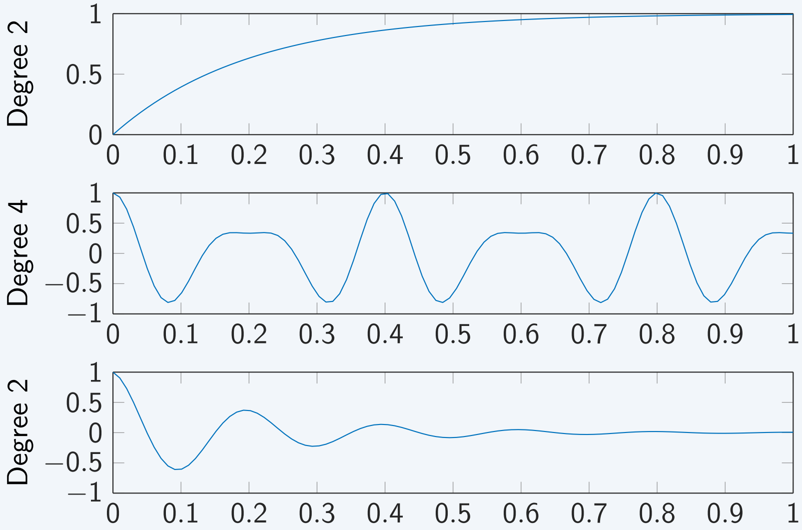
$\sim$ Work of Borbala Hunyadi

T1: Hankelization: some sum-of-exponential functions



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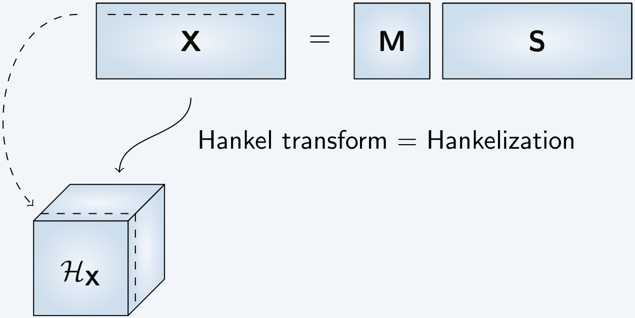

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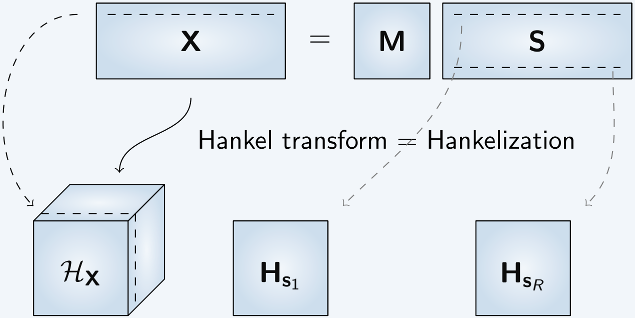

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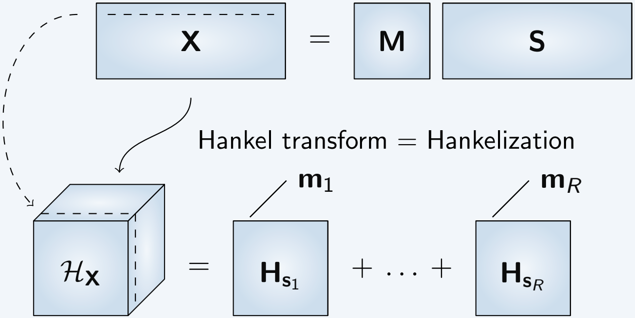

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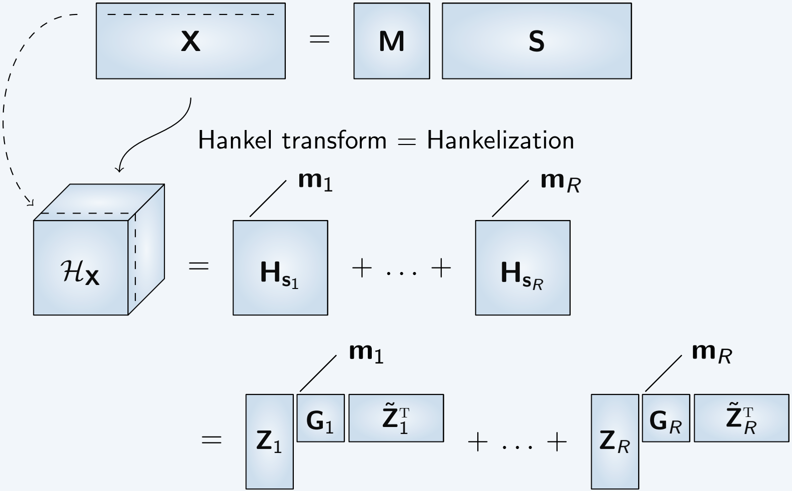

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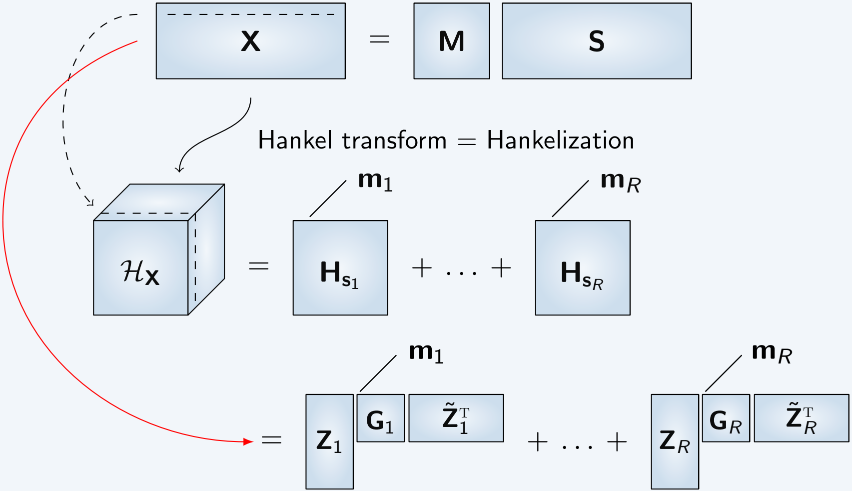


T2: Löwnerization

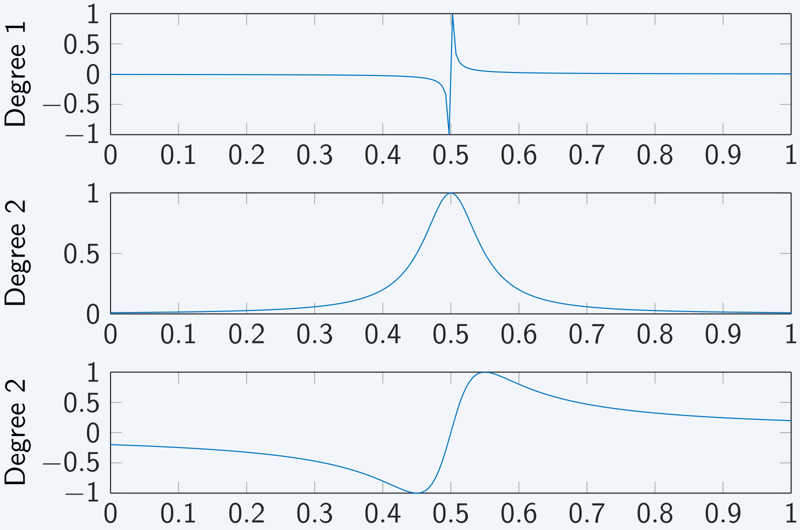
- Sources: not exponential polynomials but rational functions

$$f(t) = \frac{u(t)}{v(t)} = a(t) + \sum_{r=1}^{R} \frac{1}{t - p_r}$$

$\sim$ Work of Takaaki Nara
$\sim$ Work of Robert Luce

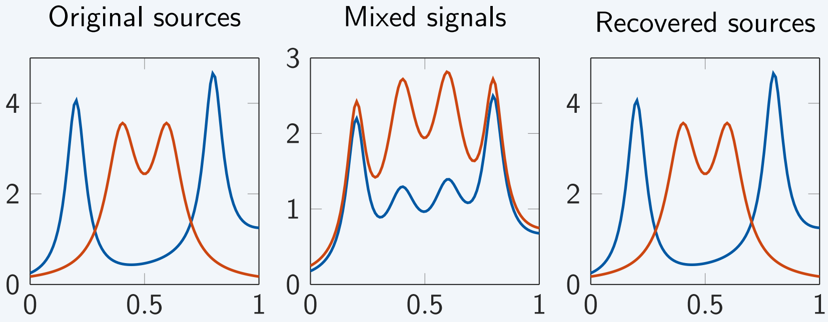
Van Barel, M.,
Debals, O., De Lathauwer, L. "Löwner-based Blind Signal Separation of Rational Functions with Applications". IEEE Trans. Signal Processing (2015)

T2: Löwnerization: some rational functions



Separating rational functions: Example

- Given a mixture of two rational functions, we want to recover the original signals.



T2: Löwnerization

Löwner matrix

- Given a function $f(t)$ sampled in point set T
- Partition point set T in two different point sets X and Y
- Define the Löwner matrix $\mathbf{L}$:

$$(\mathbf{L})_{i,j} = \frac{f(x_i) - f(y_j)}{x_i - y_j}$$

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$$\begin{array}{ccc} \text{Exponential polynomials} & \longleftrightarrow & \text{Hankel matrix} \\ & \Updownarrow & \\ \text{Rational functions} & \longleftrightarrow & \text{Löwner matrix} \end{array}$$

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$$f(t) = \frac{1}{t + 0.5}$$

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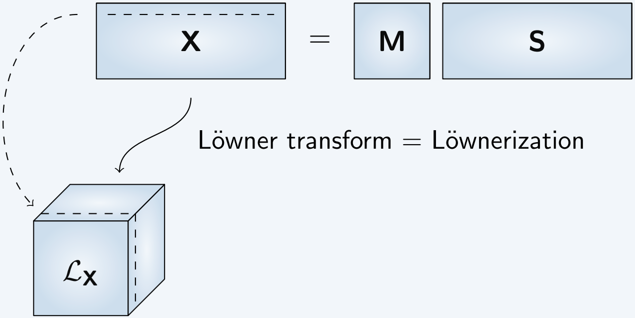
- The Löwner matrix will have rank 1:

$$\mathbf{L} = -\begin{bmatrix} \frac{1}{x_1+0.5} \\ \frac{1}{x_2+0.5} \\ \vdots \end{bmatrix} \begin{bmatrix} \frac{1}{y_1+0.5} & \frac{1}{y_2+0.5} & \cdots \end{bmatrix}$$

- Rational function with degree $D \rightarrow$ Löwner matrix has rank D

T2: Löwnerization



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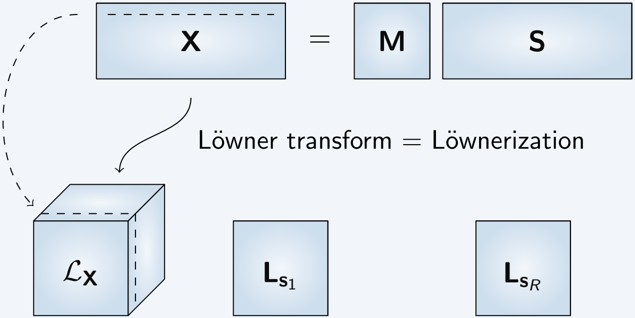

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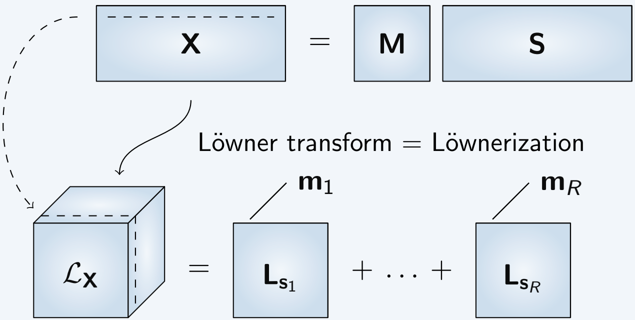

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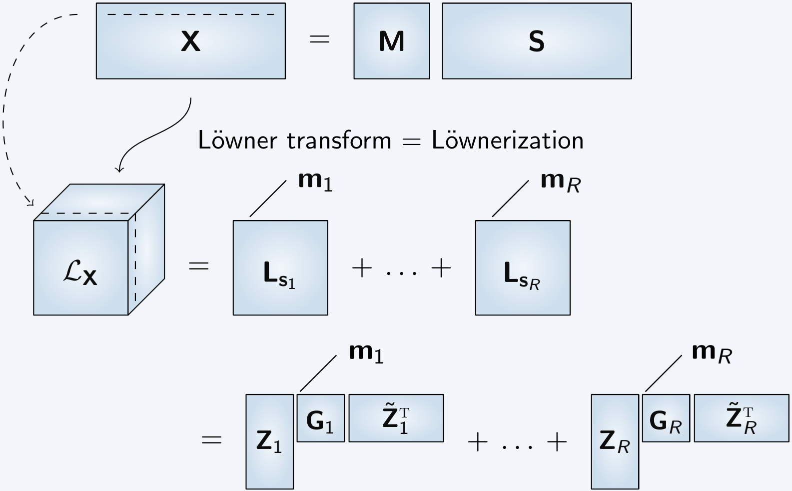

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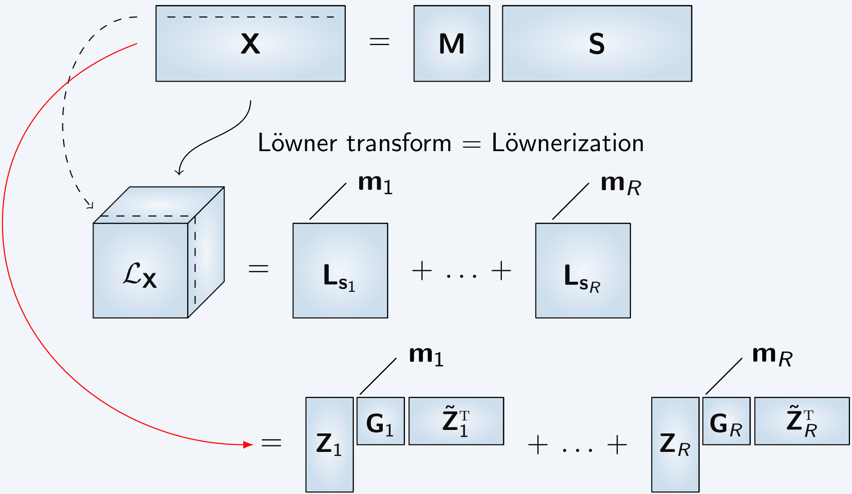

T3a: Segmentation

- Consider a sampled exponential signal $f(k) = z^k$:

$$\begin{bmatrix} 1 & z & z^2 & z^3 & z^4 & z^5 \end{bmatrix}$$

$\sim$ Work of Vladimir Kazeev on quantization
$\sim$ Work of Namgil Lee on solving linear systems

Boussé, M., Debals, O., De Lathauwer, L. "A novel deterministic method for large-scale blind source separation". Proceedings of EUSIPCO (2015)

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- The segments are stacked in the columns of $\mathbf{E}$:

$$\mathbf{E} = \begin{bmatrix} 1 & z^3 \\ z & z^4 \\ z^2 & z^5 \end{bmatrix}$$

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- Assumption: segmented source signal has rank 1 $\rightarrow$ CPD

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possibly fewer parameters! -> large-scale signal separation

- Assumption: segmented source signal has rank 1 $\rightarrow$ CPD
- More realistic: segmented source signal has low rank $\rightarrow$ BTD

T3b: Decimation

- Consider a sampled exponential signal $f(k) = z^k$:

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T4: Higher-order statistics

Let us consider the fourth-order cumulant

$$\left(\mathcal{C}_{\mathbf{s}}^{(4)}\right)_{i_1 i_2 i_3 i_4} \triangleq \mathrm{E}\left\{s_{i_1} s_{i_2} s_{i_3} s_{i_4}\right\} - \mathrm{E}\left\{s_{i_1} s_{i_2}\right\} \mathrm{E}\left\{s_{i_3} s_{i_4}\right\} - \mathrm{E}\left\{s_{i_1} s_{i_3}\right\} \mathrm{E}\left\{s_{i_2} s_{i_4}\right\} - \mathrm{E}\left\{s_{i_1} s_{i_4}\right\} \mathrm{E}\left\{s_{i_2} s_{i_3}\right\}.$$

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- If $\mathbf{S}$ contains independent signals ...
  ... $\mathcal{C}_{\mathbf{s}}^{(4)}$ is diagonal, with diagonal elements $\kappa_{\mathbf{s}_r}$.

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- It is quadrilinear:

$$\text{If } \mathbf{X} = \mathbf{M}\mathbf{S} \quad \rightarrow \quad \mathcal{C}_{\mathbf{x}}^{(4)} = \mathcal{C}_{\mathbf{s}}^{(4)} \cdot_1 \mathbf{M} \cdot_2 \mathbf{M} \cdot_3 \mathbf{M} \cdot_4 \mathbf{M}$$

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- It is quadrilinear:

$$\text{If } \mathbf{X} = \mathbf{MS} \quad \rightarrow \quad \mathcal{C}_{\mathbf{x}}^{(4)} = \mathcal{C}_{\mathbf{s}}^{(4)} \cdot_1 \mathbf{M} \cdot_2 \mathbf{M} \cdot_3 \mathbf{M} \cdot_4 \mathbf{M}$$

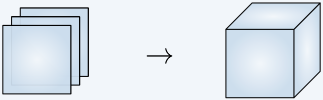
- This can be written as a CPD:

$$\mathcal{C}_{\mathbf{x}}^{(4)} = \sum_{r=1}^{R} \kappa_{s_r} \mathbf{m}_r \otimes \mathbf{m}_r \otimes \mathbf{m}_r \otimes \mathbf{m}_r$$

T5: Parameter Variation

Procedure

1. We generate a set of matrices from $\mathbf{X}$, for diff. param values
2. Then we stack the matrices to a tensor



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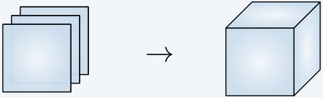

Examples:

- Using lagged covariance matrices
  $\sim$ ICA with second-order blind identification (SOBI)
  $\sim$ Work of Esin Karahan
- Stacking Jacobian matrices
  $\sim$ Work of Gabriel Hollander and Philippe Dreesen (VUB)

T5: Parameter Variation

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- Then, (1) is a CPD:

$$\mathcal{C}_{\mathbf{x}} = \sum_{r=1}^{R} \mathbf{m}_r \otimes \mathbf{m}_r \otimes \mathbf{c}_{\mathbf{s}_r}$$

Not every tensorization technique will work

We are searching for some specific mappings ...

Checklist

$\checkmark$ The mapping is multilinear

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- √ A clear and meaningful assumption on the sources exists

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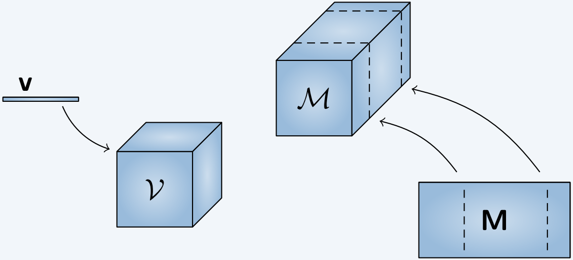
- √ The mapping is multilinear
- √ A clear and meaningful assumption on the sources exists
- √ A tensor decomposition with uniqueness results exists
- √ Recovery of mixing vectors and source signals is possible

Take-home message

- Multilinear algebra has a high influence in BSS
- Tensorization translates signal models/assumptions into low-rank constraints
- The tensor decomposition step tackles the identifiability question
- Many more tensorization techniques: time-frequency, wavelets, space-frequency, constant modulus, ...

Debals, O., De Lathauwer, L. "Stochastic and Deterministic Tensorization for Blind Signal Separation". Latent Variable Analysis and Signal Separation, Springer Berlin/Heidelberg (2015).

Tensorization



Tensorization methods:

- Hankel-based mapping
- Löwner-based mapping
- Segmentation and decimation
- Higher-order and lagged second-order statistics

Also corresponding detensorization methods

Efficient representations

- Consider an Nth-order tensor $\mathcal{T}$ of size $I \times I \times \cdots \times I$
- If $\mathcal{T}$ adheres to, e.g., one of the following structures, efficient storage methods, operations and decompositions are possible!

Structure	#variables, compared to I^N
Hankel	$NI - N + 1$
Löwner	NI
CPD	NIR
LL1	$NILR$
LMLRA	$NIL + L^N$
BTD	$NILR + RL^N$
TT	$NIR + (N-2)IR^2$

- E.g., Hankel tensor of size $334 \times 334 \times 334$: 298 MB

$\Leftrightarrow$ Efficient representation: 10 kB

Tensorlab 3.0 March 2016 (expand/collapse all)

Tensorlab 3.0 features dedicated algorithms for the decomposition in multilinear rank-$(L_r, L_r, 1)$ terms, various tensorization techniques, a more flexible and expanded modeling language for structured data fusion problems, support for efficient representations of structured tensors in most optimization-based decomposition algorithms, and new algorithms for dealing with sparse, incomplete and/or large-scale datasets. A new visualization tool is introduced, many existing algorithms have received performance and flexibility updates, e.g., by using more lenient option parsing, and a number of bugs have been fixed. Finally, the user guide has been extended significantly and illustrated with practical demos.

In these release notes, all new features and updates are discussed in detail. Structured tensors, algorithms for the decomposition in multilinear rank-$(L_r, L_r, 1)$ terms and tensorization are completely new. For the other topics, it is indicated which specific commands are new and which have been updated. Each topic consists of a short overview and a number of key items. The full description of an item can be uncovered by clicking it. Use the show/hide buttons next to the topic titles to (un)cover all items at once.

Structured tensors (expand/collapse topic)

Most optimization-based algorithms are able to exploit the efficient representation of structured tensors. To accommodate this, a large number of new methods have been implemented. This section gives an overview of the different formats supported and of the new algorithms.

- **Supported formats** *CPD, BTD, LMLRA, TT, Hankel and Loewner.*
- **Core computational routines** `frob`, `inprod`, `mtkrprod` *and* `mtkronprod`.
- **Structure detection and validation** `getstructure`, `isvalidtensor` *and* `detectstructure`.
- **Extended `ful` method** *Expand incomplete, sparse and structured tensors and create subtensors.*
- **Auxiliary functions** `getsize` *and* `getorder`.

Canonical polyadic decomposition (expand/collapse topic)

The high-level algorithm `cpd` has an improved initialization and computation strategy. A new large-scale algorithm `cpd_rbs`, an improved algorithm for incomplete tensors, and a method for computing the Cramér-Rao bound (`cpd_crb`) are introduced. Structured tensors are supported in most optimization-based algorithms and a number of new options are added.

- **Improved high-level strategy for `cpd`** *New compression strategy, support for structured tensors and complex decompositions.*
- **`cpd_rbs`** *Large-scale algorithm using randomized block sampling.*
- **Improved performance for incomplete tensors** `UseCPDI` *option for* `cpd_nls`.
- **Cramér-Rao bound** *Cramér-Rao bound for a CPD and additive i.i.d. Gaussian noise.*
- **Structured tensor support** *Most optimization-based algorithms, line and plane search methods exploit the efficient representation of structured tensors.*