A linear operator-theoretic approach to nonlinear systems

Alexandre Mauroy

University of Namur

UNIVERSITÉ DE NAMUR

Namur Institute for Complex Systems

You have probably already used
an operator-theoretic approach to nonlinear systems

You have probably already used an operator-theoretic approach to nonlinear systems

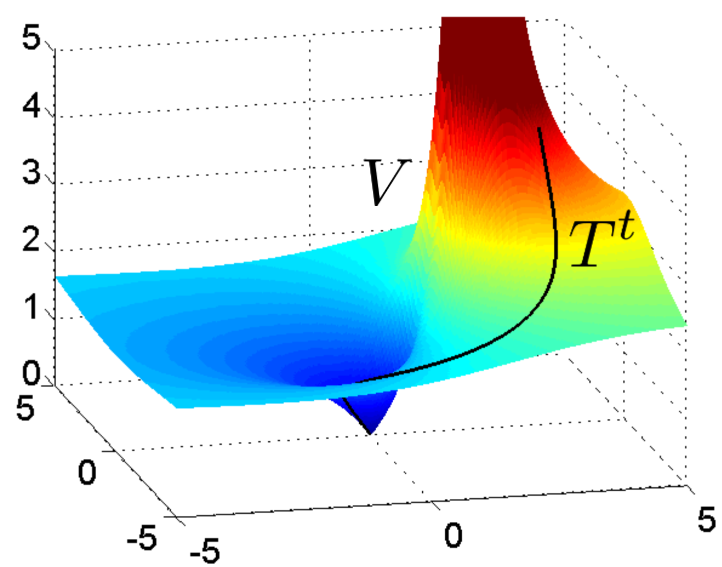
$$x(t) = T^t(x(0))$$

Globally stable equilibrium?

Positive Lyapunov function:

$$\boxed{V \circ T^t}(x) < V(x)$$

$$\forall t > 0, \forall x$$



You have probably already used an operator-theoretic approach to nonlinear systems

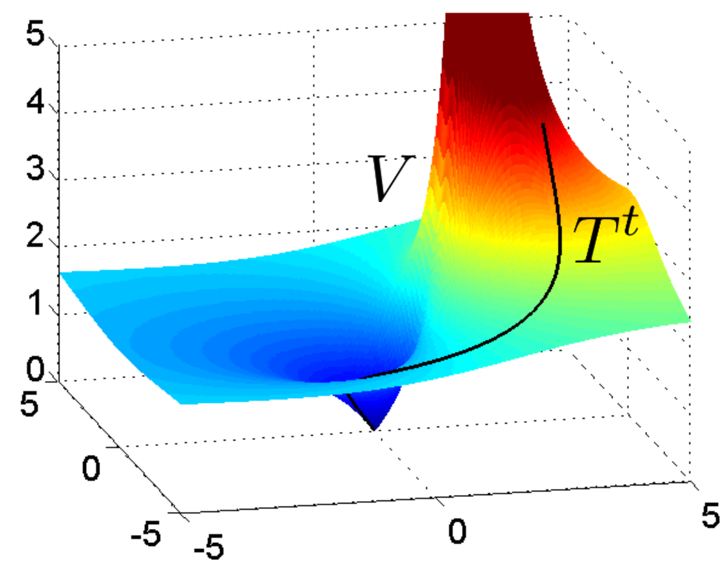
$$x(t) = T^t(x(0))$$

Globally stable equilibrium?

Positive Lyapunov function:

$$V \circ T^t(x) < V(x)$$

$$\forall t > 0, \forall x$$



Operator-theoretic approach:

$$U^t V < V \qquad \forall t > 0$$

Koopman operator $U^t f = f \circ T^t$ acting on the « observable » $f = V$

However, this operator-theoretic approach has been overlooked in nonlinear systems theory

It is surprising to find that Lyapunov's theorem has a close relative (…) that has been **neglected until present date**.
A. Rantzer, A dual to Lyapunov stability theorem, Systems & Control Letters, 42 (2001)

Stability analysis

Lyapunov function V
c. 1890

>100 years

Lyapunov density ρ
[Rantzer, 2001]

However, this operator-theoretic approach has been overlooked in nonlinear systems theory

> It is surprising to find that Lyapunov's theorem has a close relative (...)
> that has been **neglected until present date**.
> *A. Rantzer, A dual to Lyapunov stability theorem, Systems & Control Letters, 42 (2001)*

Stability analysis

Lyapunov function V
c. 1890

>100 years

Lyapunov density ρ
[Rantzer, 2001]

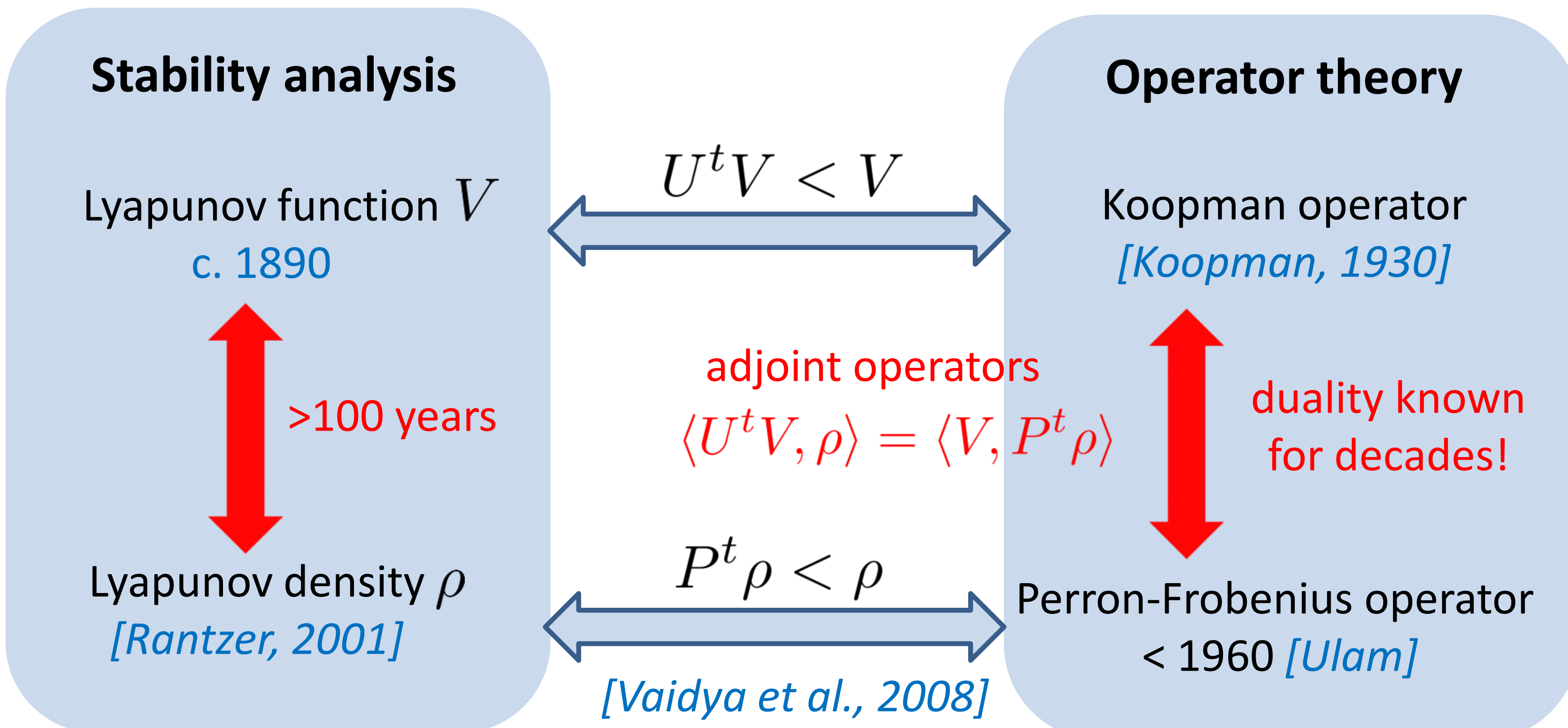
$$U^t V < V$$

Operator theory

Koopman operator
[Koopman, 1930]

However, this operator-theoretic approach has been overlooked in nonlinear systems theory

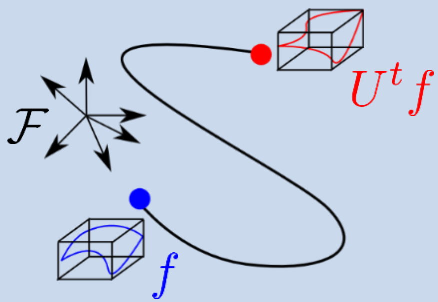
It is surprising to find that Lyapunov's theorem has a close relative (...) that has been **neglected until present date**.
A. Rantzer, A dual to Lyapunov stability theorem, Systems & Control Letters, 42 (2001)



The operator-theoretic approach provides general and systematic linear methods for nonlinear systems

Koopman operator-based description

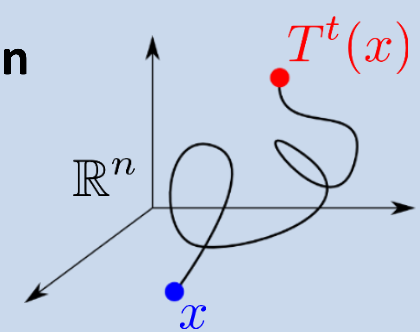
Operator $f \mapsto U^t f = f \circ T^t$
acting on a functional space $\mathcal{F}$



LIFTING

Trajectory-oriented description

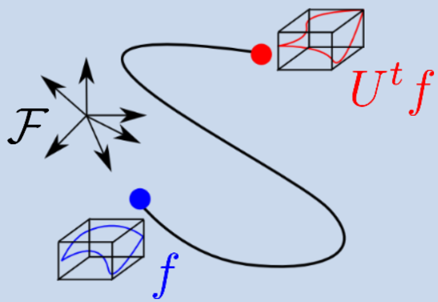
Flow $x \mapsto T^t(x)$
acting on the state space $\mathbb{R}^n$



The operator-theoretic approach provides general and systematic linear methods for nonlinear systems

Koopman operator-based description

Operator $f \mapsto U^t f = f \circ T^t$
acting on a functional space $\mathcal{F}$



☹ **infinite-dimensional**

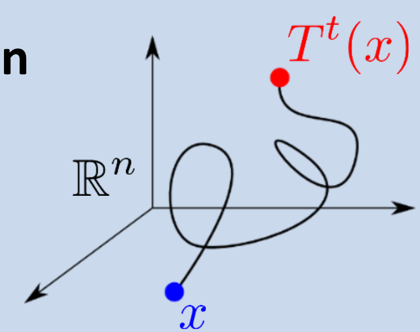
☺ **linear**

Systematic, general
linear methods

LIFTING

Trajectory-oriented description

Flow $x \mapsto T^t(x)$
acting on the state space $\mathbb{R}^n$



☺ **finite-dimensional**

☹ **nonlinear**

Outline

Stability analysis: a systematic method

Joint work with I. Mezic, University of California Santa Barbara

Nonlinear identification: a lifting method

Joint work with J. Goncalves, University of Luxembourg

Control: recent works and perspectives

Global stability is characterized in terms of spectral properties of the Koopman operator

Continuous-time nonlinear system $\dot{x} = F(x)$ $\Leftrightarrow$ $T^t : \mathbb{R}^n \to \mathbb{R}^n$

Koopman eigenfunction $\phi_\lambda \in \mathcal{F}$

Koopman eigenvalue $\lambda \in \sigma(U)$

$\Leftrightarrow$ $U^t \phi_\lambda = \phi_\lambda \circ T^t = e^{\lambda t} \phi_\lambda$

Global stability is characterized in terms of spectral properties of the Koopman operator

Continuous-time nonlinear system $\dot{x} = F(x)$ $\Longleftrightarrow$ $T^t : \mathbb{R}^n \to \mathbb{R}^n$

Koopman eigenfunction $\phi_\lambda \in \mathcal{F}$

Koopman eigenvalue $\lambda \in \sigma(U)$

$\Longleftrightarrow$ $U^t \phi_\lambda = \phi_\lambda \circ T^t = e^{\lambda t} \phi_\lambda$

Theorem: If there exist eigenfunctions $\phi_{\lambda_i} \in C^0(X)$ with eigenvalues $\lambda_i \in \sigma(U)$ such that $\Re\{\lambda_i\} < 0$, $i = 1, \ldots, m$, then the set

$$M = \bigcap_{i=1}^{m} \{x \in X | \phi_{\lambda_i}(x) = 0\}$$

is globally asymptotically stable in X.

[AM and Mezic, IEEE Trans. on Aut. Control 2016]

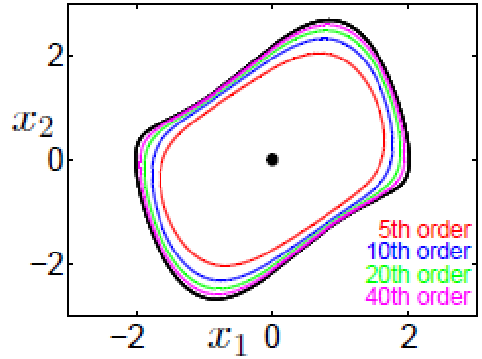
We obtain a systematic approach to global stability, which mirrors linear stability analysis

Hyperbolic equilibrium x^*

Jacobian matrix $\dfrac{\partial F}{\partial x}(x^*)$ has eigenvalues λ_i

Assume that $X \subset \mathbb{R}^n$ is a forward invariant connected set.
The equilibrium x^* is globally asymptotically stable in X iff
(i) the eigenvalues $\lambda_i \in \sigma(U)$ are such that $\Re\{\lambda_i\} < 0$ (local stability)
(ii) there exist n eigenfunctions $\phi_{\lambda_i} \in C^1(X)$ with $\nabla \phi_{\lambda_i}(x^*) \neq 0$

<u>Example:</u>
approximation of the basin of attraction

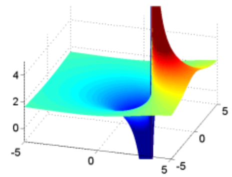


[AM and Mezic, IEEE Trans. on Aut. Control 2016]

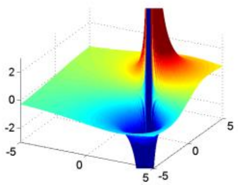
The spectral approach is related to classic and (new) concepts in control theory

Lyapunov function $V(x) = \sqrt{\sum_{j=1}^{N} |\phi_{\lambda_j}(x)|^2}$

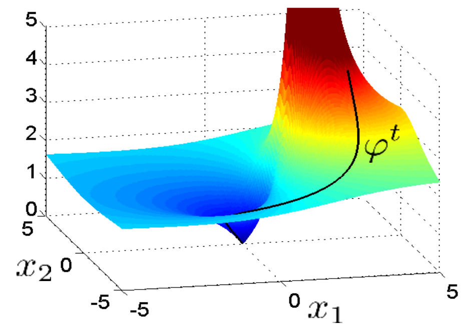
ϕ_{λ_1}



ϕ_{λ_2}

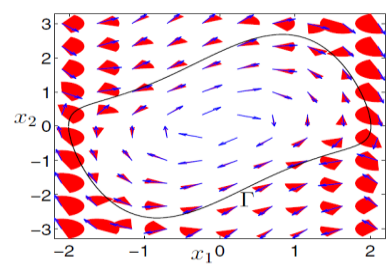


$$V = \sqrt{\phi_{\lambda_1}^2 + \phi_{\lambda_2}^2}$$



Contracting metric $\mathcal{M}(x,y) = \sqrt{\sum_{j=1}^{N} |\phi_{\lambda_j}(x) - \phi_{\lambda_j}(y)|^2}$

Differential positivity (contracting cone field)
[AM, Forni and Sepulchre, CDC 2015]



Eventual monotonicity
[Sootla and AM, arXiv 1510.01149]

Outline

Stability analysis: a systematic method

Joint work with I. Mezic, University of California Santa Barbara

Nonlinear identification: a lifting method

Joint work with J. Goncalves, University of Luxembourg

Control: recent works and perspectives

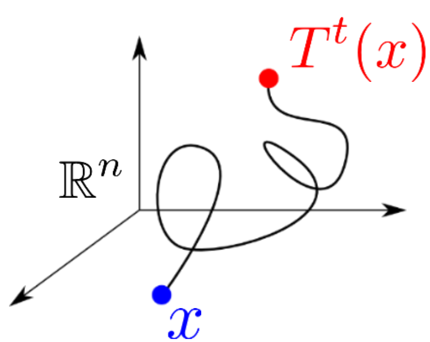
We propose to “identify” the Koopman operator

$$(x_k, y_k) = (x_k, T^{t_s}(x_k))$$

$$k = 1, \ldots, K \qquad t_s \not\ll 1$$

Find c_j such that

$$\dot{x} = F(x) = \sum_{j=1}^{N_F} c_j \, g_j(x)$$



$$\dot{x} = F(x)$$

Nonlinear identification /parameter estimation

[AM and Goncalves, CDC2016]

[AM and Goncalves, arXiv 1709.02003]

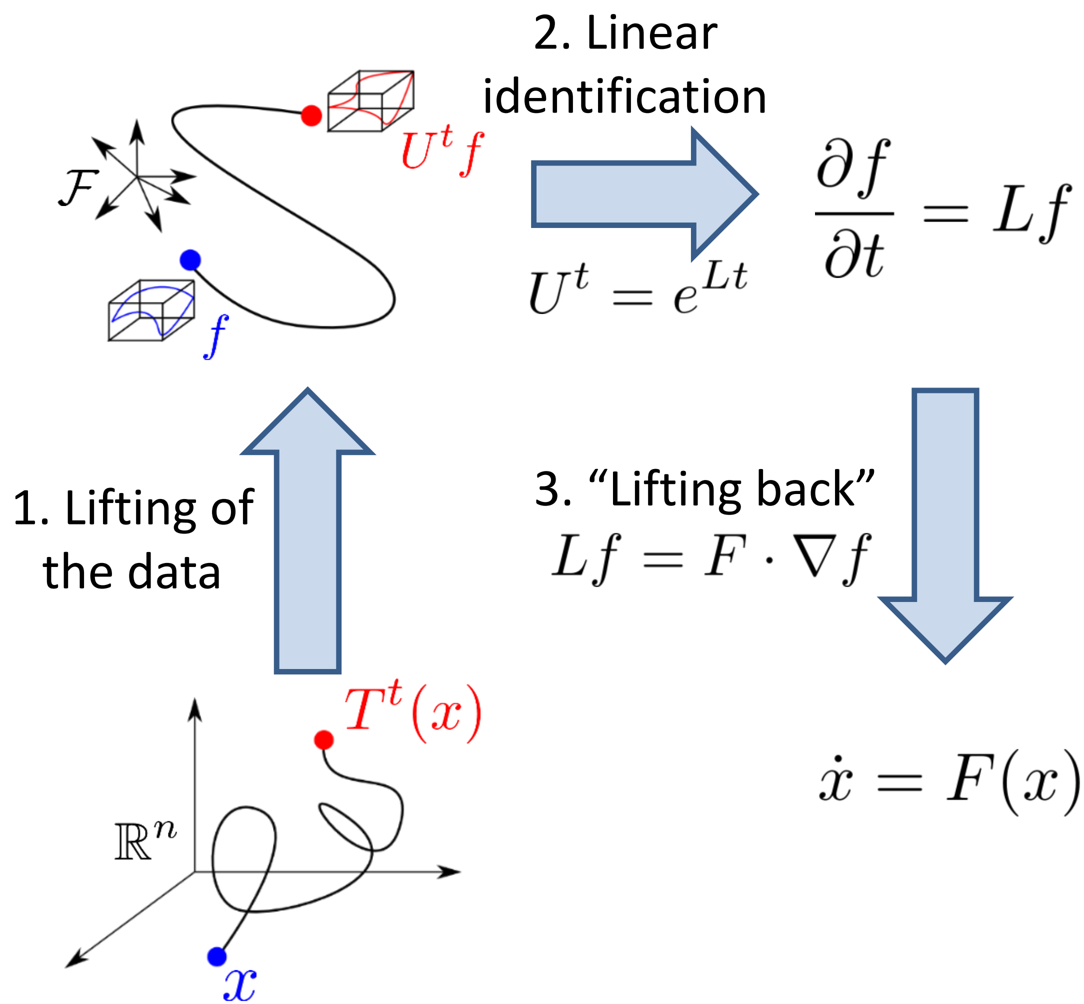
We propose to “identify” the Koopman operator

$$(x_k, y_k) = (x_k, T^{t_s}(x_k))$$

$$k = 1, \ldots, K \qquad t_s \lll 1$$

Find c_j such that

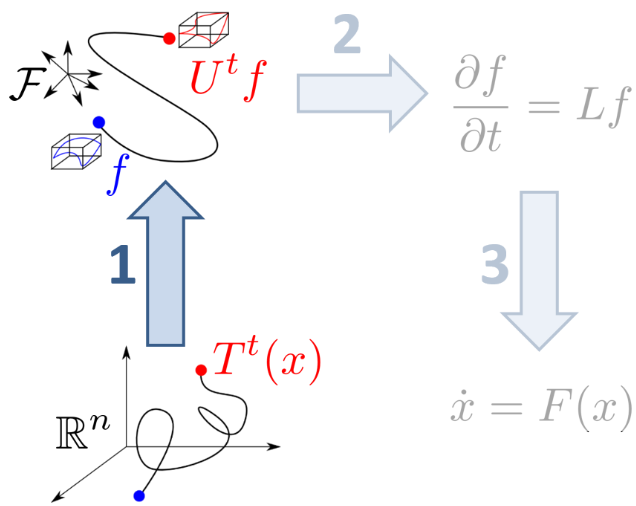
$$\dot{x} = F(x) = \sum_{j=1}^{N_F} c_j \, g_j(x)$$

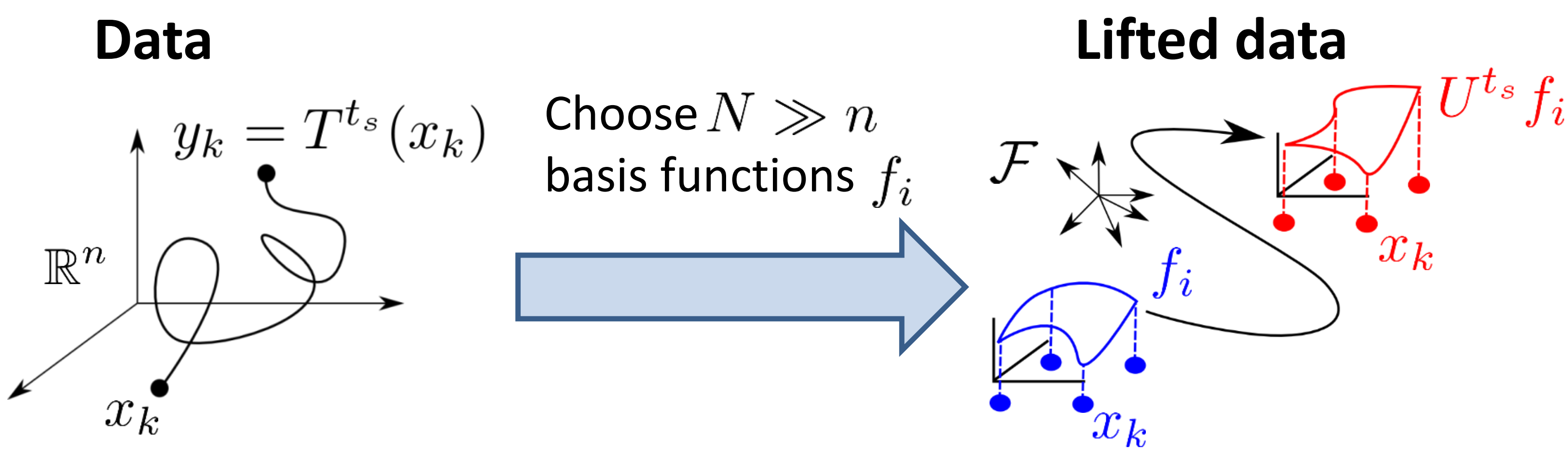


[AM and Goncalves, CDC2016]

[AM and Goncalves, arXiv 1709.02003]

Step 1: Data are lifted to a higher dimensional space

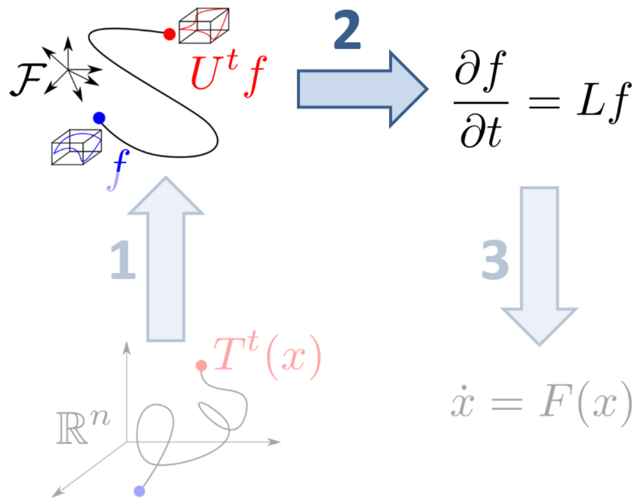




$$(x_k, y_k) = (x_k, T^{t_s}(x_k))$$

$$(f_i(x_k), f_i(y_k)) = (f_i(x_k), U^{t_s} f_i(x_k))$$

Step 2: The Koopman operator is « identified » in the lifted space



Lifted data

linear least squares

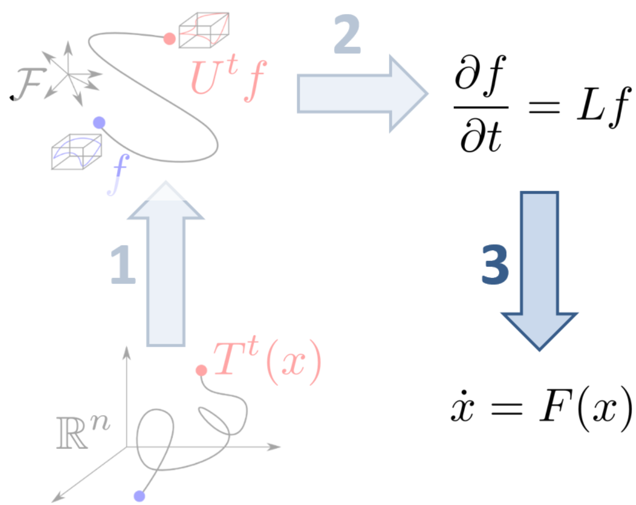
Realization $\mathbf{U} \in \mathbb{R}^{N \times N}$ of $U^{t_s} = e^{L t_s}$ in the basis

matrix logarithm

Realization $\mathbf{L} = 1/t_s \log(\mathbf{U})$ of the infinitesimal generator L

Remark: Dual method for high-dimensional systems $(N > K)$

Step 3: The nonlinear system is finally identified



Realization $\mathbf{L}$ of the infinitesimal generator L

$$Lf = F \cdot \nabla f$$

$$F(x) = \sum_{j=1}^{N_F} c_j \, g_j(x)$$

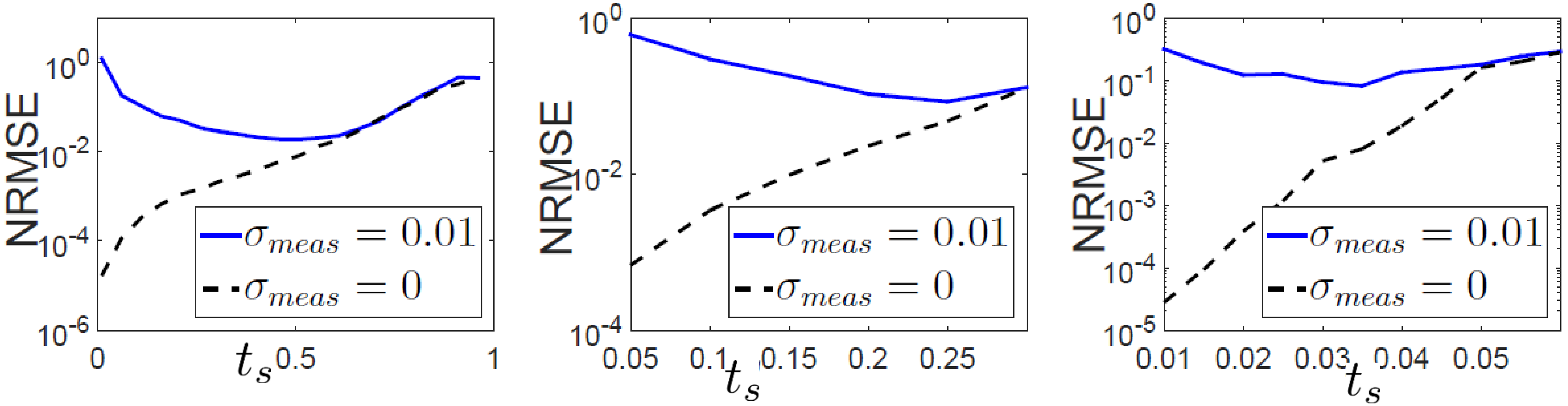
Theoretical and numerical results suggest that the method is efficient

Theoretical convergence results

The error tends to 0 as $N \to \infty$ (in "optimal" conditions)

[AM and Goncalves, arXiv 1709.02003]

Numerical results

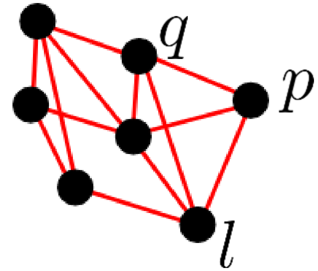


Van der Pol oscillator *Unstable system* *Chaotic Lorenz system*

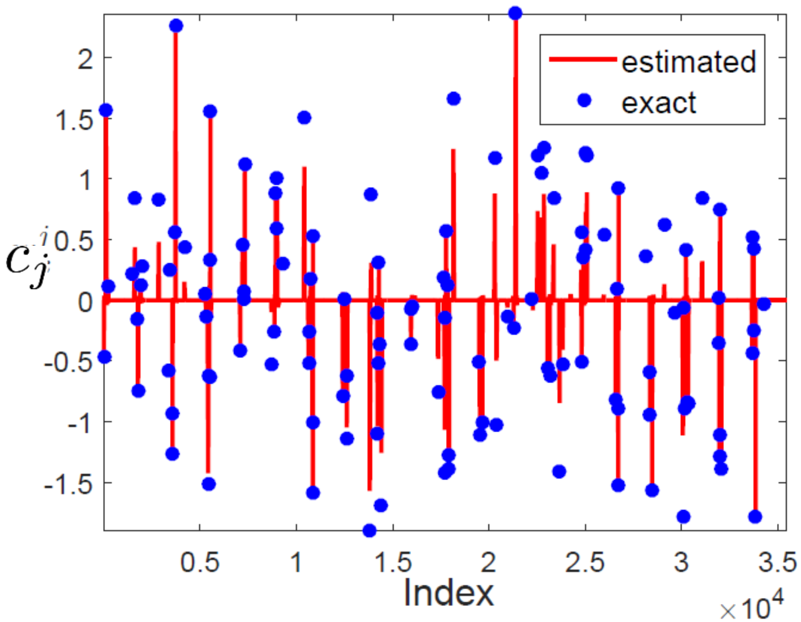
The lifting method is efficient to reconstruct networks with low-sampled data

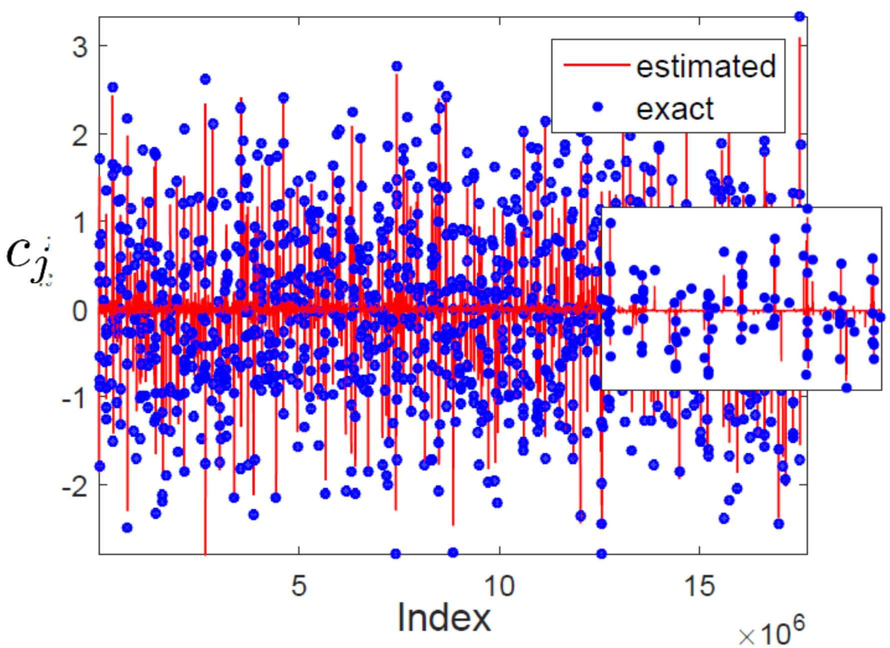
$$\dot{x}_l = \sum_{j=1}^{N_F} c_j \, g_j(x)$$

$$g_j \equiv x_p^a x_q^b$$
$$a + b \in \{1, 2, 3\}$$



Sampling period: $t_s = 0.5$





$n = 20$ states (nodes)

$N_F \approx 3.10^4$ coefficients

$K = 200$ data points

$n = 100$ states (nodes)

$N_F \approx 2.10^7$ coefficients

$K = 1000$ data points

Outline

Stability analysis: a systematic method

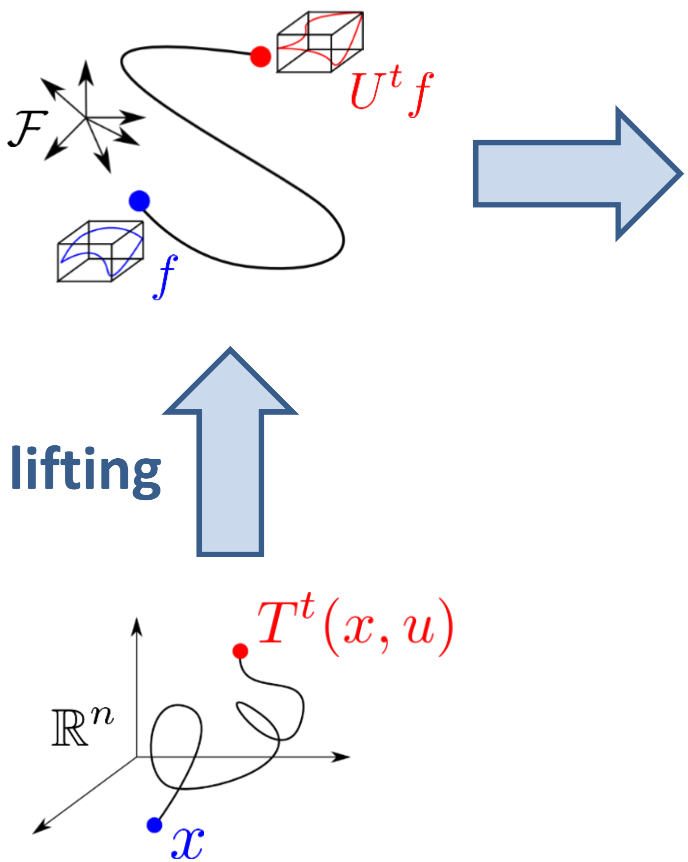
Joint work with I. Mezic, University of California Santa Barbara

Nonlinear identification: a lifting method

Joint work with J. Goncalves, University of Luxembourg

Control: recent works and perspectives

The Koopman operator-theoretic framework has been recently applied to control



linear controller/observer design

Observer synthesis *[Surana, CDC 2016]*

Model predictive control *[Korda and Mezic 2016, arXiv 1611.03537]*

Optimal control *[Kaiser et al. 2016 , arXiv 1707.01146]*

Controllability *[Goswami and Paley, CDC 2017]*

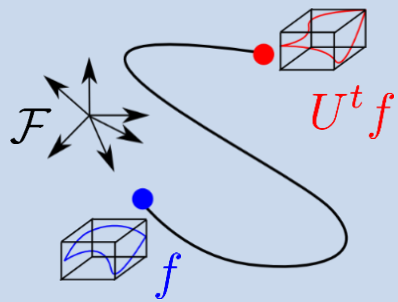
Only numerical results

No theoretical framework

The operator-theoretic approach provides general and systematic linear methods for nonlinear systems

Koopman operator-based description

Operator $f \mapsto U^t f = f \circ T^t$
acting on a functional space $\mathcal{F}$



☹ **infinite-dimensional**

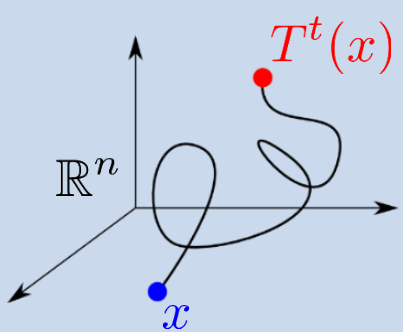
☺ **linear**

Systematic, general linear methods

LIFTING

Trajectory-oriented approach

Flow $x \mapsto T^t(x)$
acting on the state space $\mathbb{R}^n$



- Global stability
- Identification
- Control

What you do with linear systems

can (technically) be done with nonlinear systems

analysis
identification
control…

A linear operator-theoretic approach to nonlinear systems

Alexandre Mauroy
(alexandre.mauroy@unamur.be)

University of Namur