Small dispersion asymptotics for the Korteweg-de Vries equation

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Outline

- Lecture 1: The Korteweg-de Vries equation and the inviscid Burgers' equation
- Lecture 2: Inverse scattering and the Riemann-Hilbert method
- Lecture 3: Relation with orthogonal polynomials and random matrix theory

KdV equation

- Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0, \qquad \epsilon > 0$$

  - model for shallow water waves
  - prototype example for more general class of equations
- initial data $u(x, t = 0, \epsilon) = u_0(x)$ (independent of ϵ)
  - time evolution $u(x, t, \epsilon)$?
- small dispersion limit $\epsilon \to 0$
  - different types of asymptotics for $u(x, t, \epsilon)$ in various regions in (x, t)-plane

Hopf equation

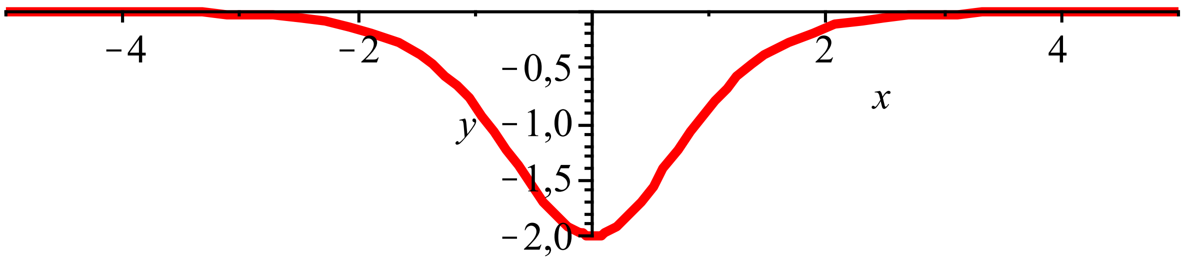
- $\epsilon = 0$: Hopf/inviscid Burgers' equation

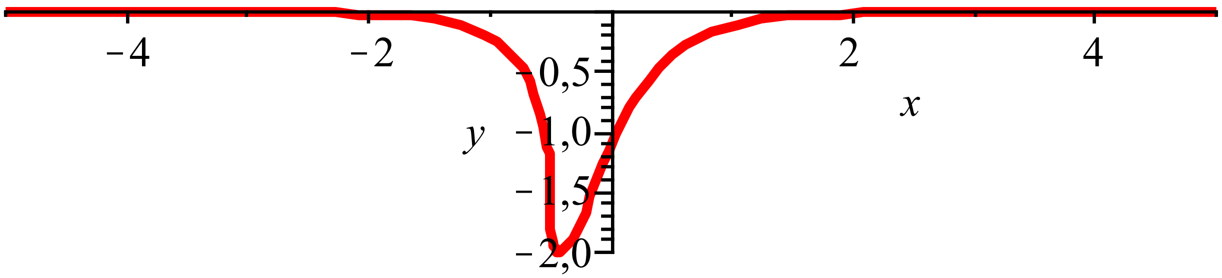
$$u_t + 6uu_x = 0, \qquad u(x,0) = u_0(x)$$

  - can be solved by method of characteristics:
    $\rightarrow$ $u(x,t) = u_0(\xi(x,t))$
    $\rightarrow$ $x = 6tu_0(\xi) + \xi$
  - change of variable $x \mapsto \xi(x,t)$
  - consider initial data $u_0(x)$:
    $\rightarrow$ negative
    $\rightarrow$ smooth
    $\rightarrow$ one local minimum
    $\rightarrow$ converging to 0 rapidly as $x \rightarrow \pm\infty$

Hopf equation

- $$u_t + 6uu_x = 0$$
- initial data $u(x, t = 0, \epsilon) = u_0(x)$





- solution moves to the left, minimum moves faster

Hopf equation

- Solution

$$u(x,t) = u_0(\xi), \quad x = 6tu_0(\xi) + \xi$$

- x-derivative of $u(x,t)$ is equal to $u_0'(\xi).\xi_x$
  - blows up if ξ_x blows up
  - $\xi_x = \frac{1}{1+6tu_0'(\xi)}$
- blow-up at

$$t_c = \min_{\xi\in\mathbb{R}} \frac{1}{-6u_0'(\xi)} = \frac{1}{-6\min_{\xi\in\mathbb{R}} u_0'(\xi)}$$

  - time of gradient catastrophe
- after gradient catastrophe, solution doesn't exist in classical sense

Hopf equation

- singular perturbations can regularize Hopf equation
  - Burgers':
$$u_t + 6uu_x - \epsilon^2 u_{xx} = 0$$
  - KdV:
$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0$$
- heuristics
  - for ϵ small, the perturbative term can be neglected as long as the Hopf solution and derivatives remain *small*
  - as $t \approx t_c$, x-derivatives of u become large, and the perturbative terms $\epsilon^2 u_{xx}$ or $\epsilon^2 u_{xxx}$ will start to contribute, even if ϵ is small
  - what happens for $t > t_c$?

Perturbations

- Burgers': $u_t + 6uu_x - \epsilon^2 u_{xx} = 0$
  - Hopf-Cole transformation $u = -\epsilon^2 \frac{\psi_x}{3\psi}$
  - Burgers' reduces to $\psi_t = \epsilon^2 \psi_{xx}$

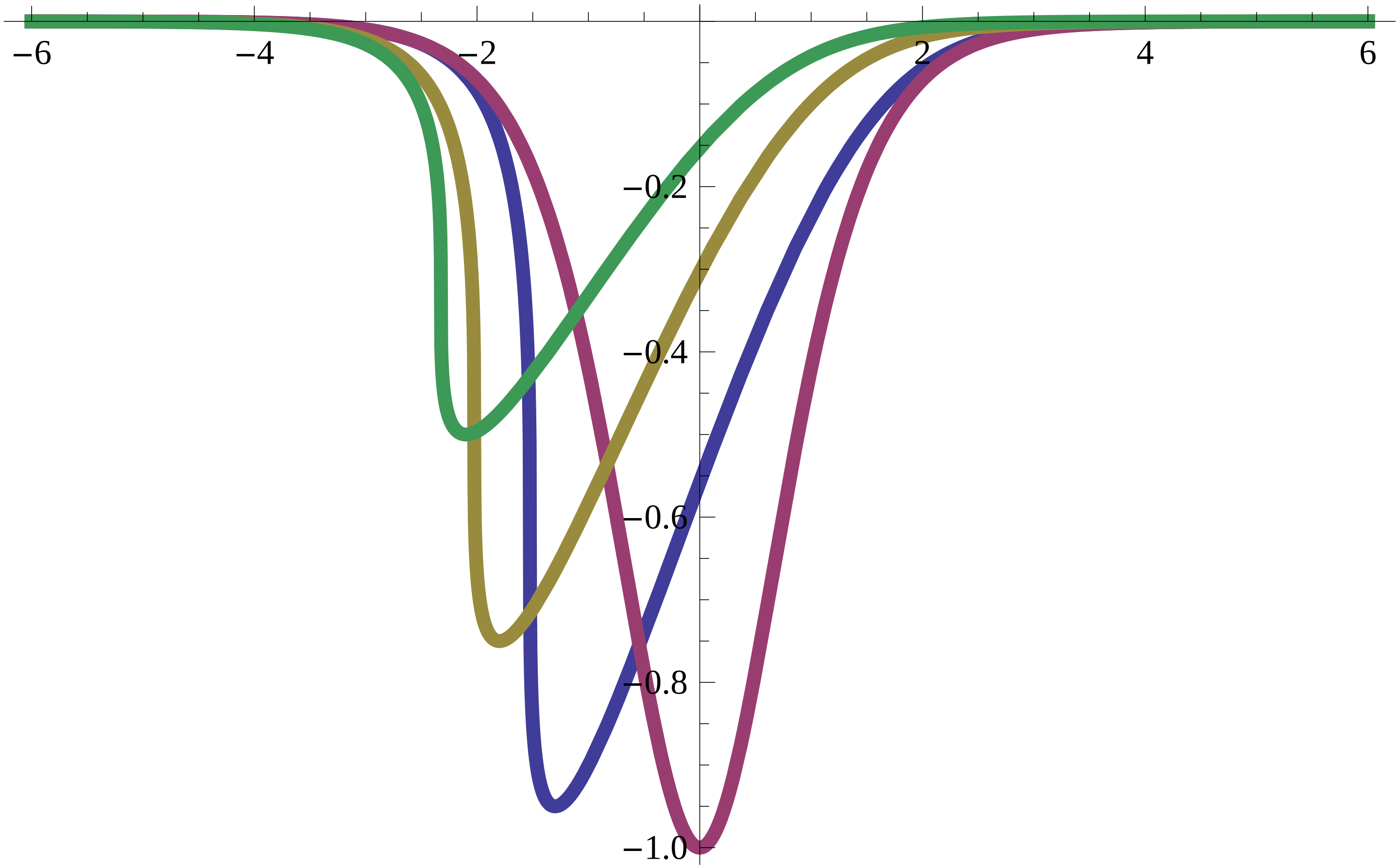
    $\to$ non-linear heat equation

    $\to$ Fourier transform $\hat{\psi}$ solves equation that can be solved by separation of variables

    $\to$ leads to solution ψ of heat equation and u of Burgers' equation

Solution to Burgers' equation

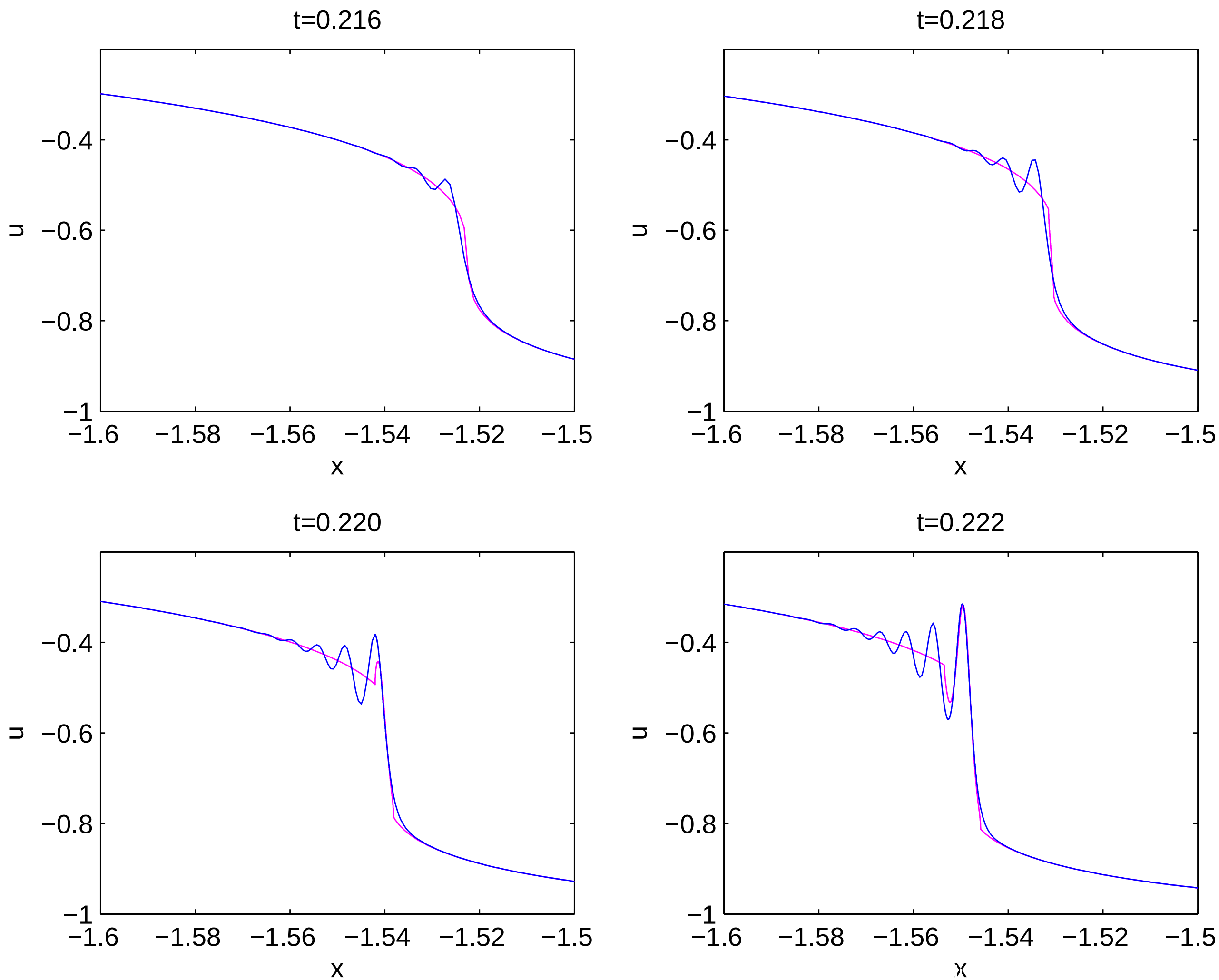
- solution of Burgers' solution near time of gradient catastrophe



  - dissipation regularizes gradient catastrophe

KdV equation

- KdV equation $u_t + 6uu_x + \epsilon^2 u_{xxx} = 0$
  - different type of regularization (pictures Grava-Klein)



KdV equation

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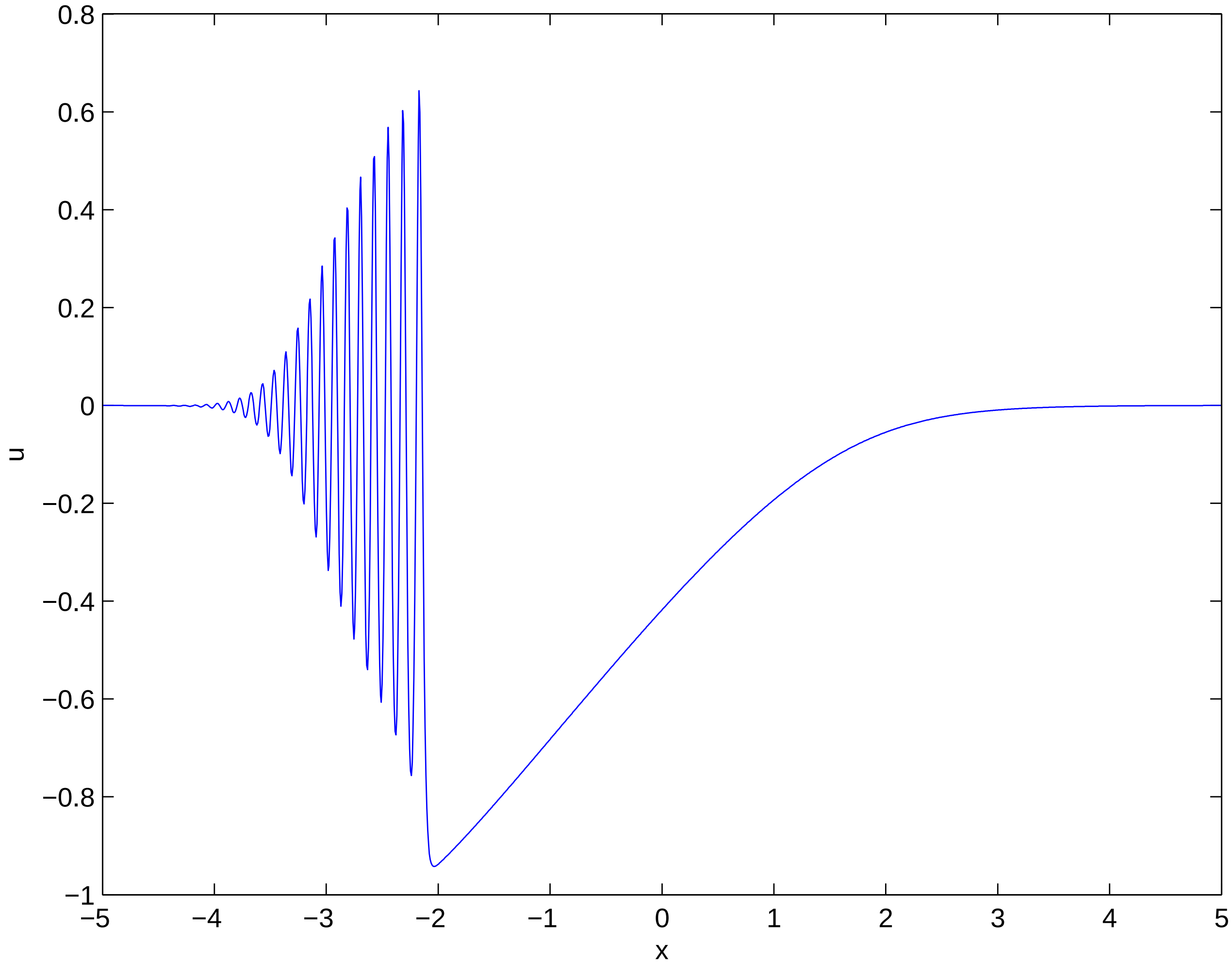
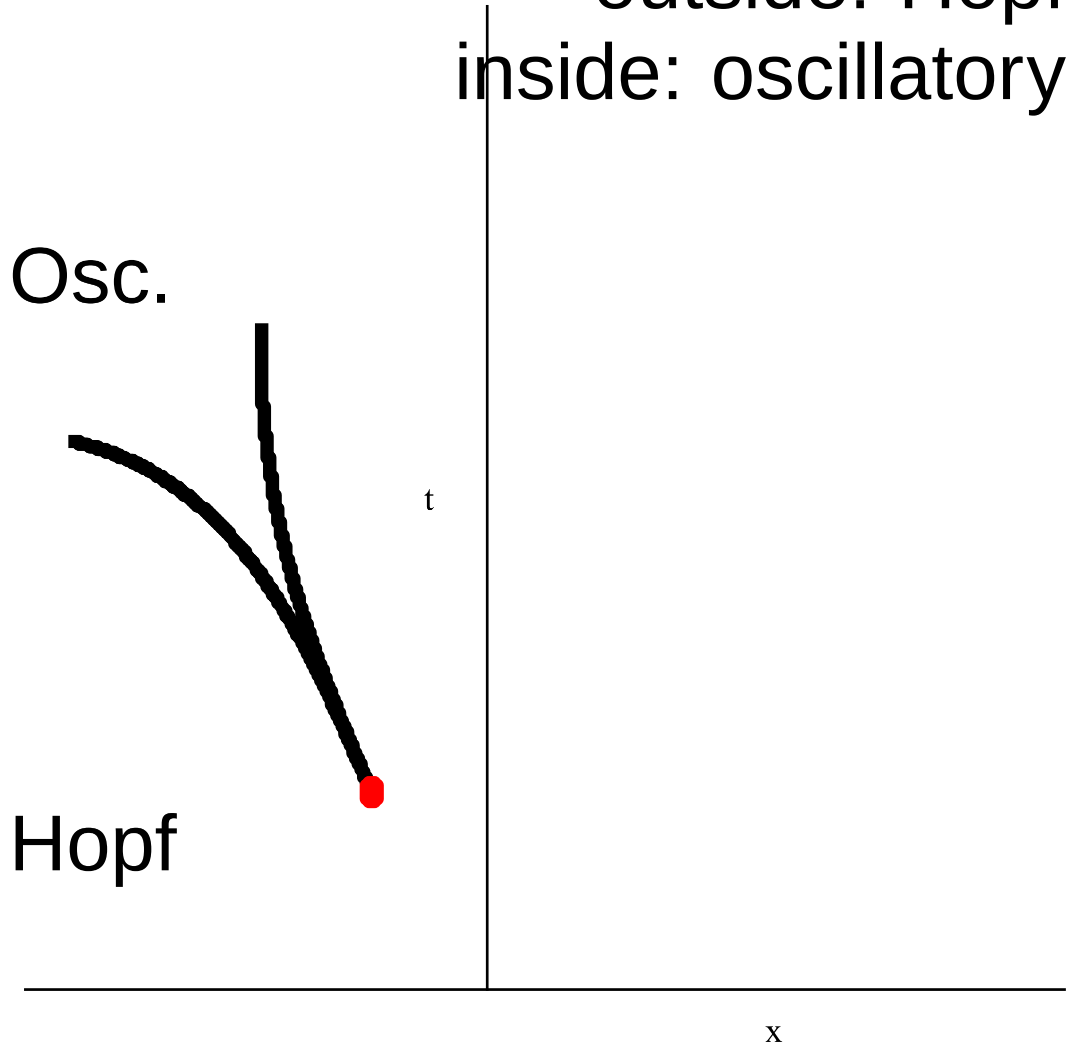


KdV equation

- formation of rapid oscillations after t_c for small ϵ
- In the (x, t)-plane

outside: Hopf asymptotics
inside: oscillatory asymptotics



KdV equation

- oscillatory zone characterized in terms of Whitham equations
  - for $t < t_c < T$, interval $[x^-(t), x^+(t)]$ where oscillations take place
  - x^- characterized by system of equations in u, v, x^-, $u > v$

$$x^- = 6tu + f(u)$$

$$6t + \frac{1}{2\sqrt{u-v}} \int_v^u \frac{f'(\xi)}{\sqrt{\xi - v}} d\xi = 0$$

$$\frac{\partial}{\partial v} \left(\frac{1}{2\sqrt{u-v}} \int_v^u \frac{f'(\xi)}{\sqrt{\xi - v}} d\xi \right) = 0$$

$\rightarrow$ f is inverse of decreasing part of initial data u_0

KdV equation

- oscillatory zone characterized in terms of Whitham equations
  - for $t < t_c < T$, interval $[x^-(t), x^+(t)]$ where oscillations take place
  - x^+ characterized by system of equations in u, v, x^+, $u < v$

$$x^- = 6tu + f(u)$$

$$6t + \frac{1}{2\sqrt{v-u}} \int_u^v \frac{f'(\xi)}{\sqrt{v-\xi}} d\xi = 0$$

$$\int_u^v \left(6t + \frac{1}{2\sqrt{\lambda-u}} \int_u^\lambda \sqrt{\lambda-u} \frac{f'(\xi)}{\sqrt{\lambda-\xi}} d\xi \right) \sqrt{\lambda-u} d\lambda = 0$$

$\rightarrow$ f is inverse of decreasing part of initial data u_0

KdV equation

- Special solutions
  - solitons: $u(x,t,\epsilon) = a\mathrm{sech}^2(bx - ct)$
    $\rightarrow$ substitute into equation to find a, b, c
    $\rightarrow$ shape preserved, travelling to the right at constant speed
  - solutions in terms of elliptic θ-functions
  - reductions to certain ODEs of Painlevé type
- our problem: study KdV with ϵ-independent initial data $u_0(x)$
  - at $t = 0$, special solutions described above depend on ϵ
  - no explicit example to verify behavior

Scattering theory

- Solving the KdV equation is much harder than Burgers'
  - scattering transform - inverse scattering transform
  - Idea:

Initial data $\quad\quad\quad\quad$ reflection coefficien

$$u_0(x) \xrightarrow{\text{scattering transform}} r_0(z)$$

$$u_0(x) \downarrow \textbf{?} \qquad\qquad \text{Explicit time evolution} \downarrow r_0(z)$$

$$u(x,t,\epsilon) \xleftarrow{\text{inverse scattering}} r(z;t)$$

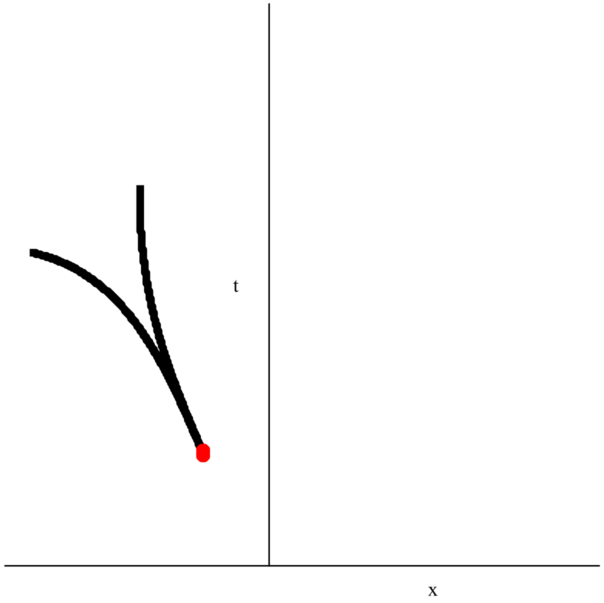
Some asymptotic results

- KdV with initial condition $u_0(x)$ (negative, smooth, one local minimum)
  - Hopf region *(Lax-Levermore, Deift-Venakides-Zhou)*

$$u(x,t,\epsilon) = u(x,t) + \mathcal{O}(\epsilon^2)$$

  - oscillatory region *(Lax-Levermore, Deift-Venakides-Zhou, Venakides, Tian)*
    $\rightarrow$ $u(x,t,\epsilon)$ can be expressed in terms of Jacobi elliptic θ-functions up to $\mathcal{O}(\epsilon)$
  - approximations only valid away from boundaries of the cusp region

Some asymptotic results



- critical asymptotics near boundaries
  - leading edge
  - trailing edge
  - point of gradient catastrophe (cusp singularity)

Painlevé equations

- Fourth order ODE

$$x = tU - \left(\frac{1}{6}U^3 + \frac{1}{24}(U_x^2 + 2UU_{xx}) + \frac{1}{240}U_{xxxx}\right).$$

  - unique real solution $U(x,t)$ satisfying the following conditions:

    $\rightarrow$ $U(x,t)$ has no poles for $x, t \in \mathbb{R}$,

    $\rightarrow$ U has the following asymptotic behavior

$$U(x,t) = \mp 6^{1/3}|x|^{1/3} + \mathcal{O}(x^{-1/3}) \qquad \text{as } x \to \pm\infty.$$

(Brézin-Marinari-Parisi '90, Dubrovin '06, TC-Vanlessen '07)

- U solves KdV $U_t + UU_x + \frac{1}{12}U_{xxx} = 0$

Gradient catastrophe

Theorem *(TC-Grava)*

- Take a double scaling limit where we let $\epsilon \to 0$ and at the same time $x \to x_c$ and $t \to t_c$ in such a way that

$$x - x_c - 6u_c(t - t_c) = \mathcal{O}(\epsilon^{6/7}), \qquad t - t_c = \mathcal{O}(\epsilon^{4/7}).$$

Then we have

$$u(x,t,\epsilon) = u_c + \left(\frac{2\epsilon^2}{k^2}\right)^{1/7} U\left(\frac{x - x_c - 6u_c(t-t_c)}{(8k\epsilon^6)^{\frac{1}{7}}}, \frac{6(t-t_c)}{(4k^3\epsilon^4)^{\frac{1}{7}}}\right) + O\left(\epsilon^{4/7}\right).$$

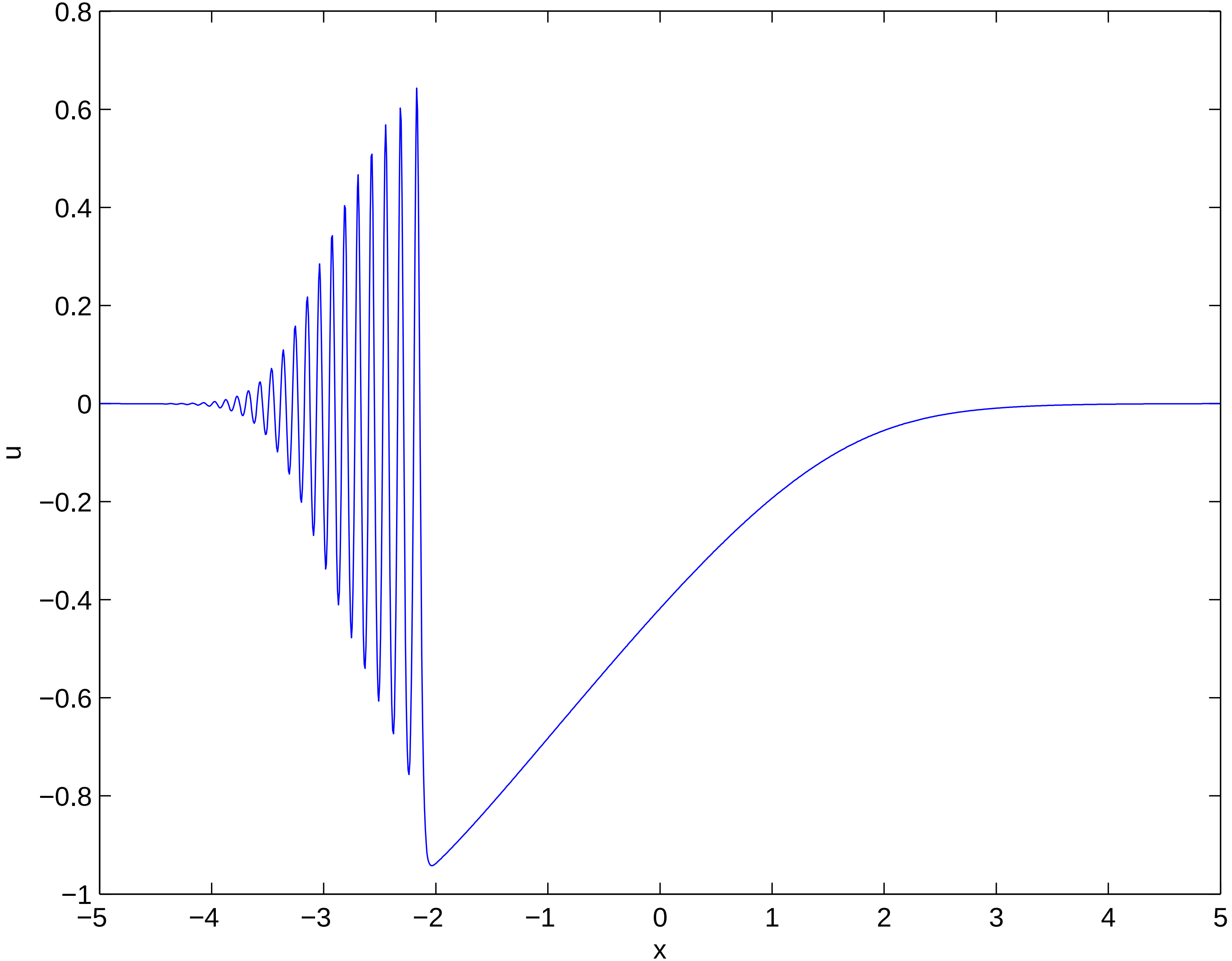
(conjectured by Dubrovin in much more general settings)

Critical asymptotics

- similar results near leading edge and trailing edge
  - leading edge: $u(x,t,\epsilon) = u(x,t) + \mathcal{O}(\epsilon^{2/3})$, sub-leading order expressed in terms of Hastings-McLeod solution to Painlevé II
  - trailing edge: $\mathcal{O}(1)$ correction to $u(x,t)$, expressed in terms of soliton solutions
- proofs using direct and inverse scattering transform

Leading edge

- Oscillatory zone



Scattering theory

- KdV equation $u_t + 6uu_x + \epsilon^2 u_{xxx} = 0$ can be written in the Lax form

$$L_t = [L, P],$$

  - L is the Schrödinger operator

$$L = \epsilon^2 D^2 + u, \quad D = \frac{d}{dx},$$

$$P = 4\epsilon^2 D^3 + 3\left(uD + Du\right)$$

  - Schrödinger equation $Lf = -z^2 f$
    $\to$ eigenvalue equation for L
    $\to$ no L^2 solutions if $u < 0$ (no point spectrum)

Scattering theory

- if $u(x,t) \to 0$ as $x \to \pm\infty$, fundamental solution to Schrödinger equation $Lf = -z^2 f$
  - as $x \to +\infty$: $\psi_\pm(z;x) \sim e^{\mp \frac{i}{\epsilon} xz}$
  - as $x \to -\infty$: $\phi_\pm(z;x) \sim e^{\mp \frac{i}{\epsilon} xz}$
- connection matrix ($z \in \mathbb{R}$)

$$\begin{pmatrix} \psi_+ & \psi_- \end{pmatrix} = \begin{pmatrix} \phi_+ & \phi_- \end{pmatrix} \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}.$$

- a, b depend on z, t, ϵ but not on x

Scattering theory

- connection between $\psi_\pm$ and $\phi_\pm$

$$\frac{\psi_+}{a} \sim \begin{cases} e^{-\frac{i}{\epsilon}z} + \frac{b}{a}e^{\frac{i}{\epsilon}zx}, & x \to -\infty, \\ \frac{1}{a}e^{-\frac{i}{\epsilon}zx}, & x \to +\infty. \end{cases}$$

- reflection coefficient $r(z;t) = b(z;t)/a(z;t)$
- if u evolves according to KdV, then

$$r(z;t) = r_0(z)e^{\frac{8i}{\epsilon}z^3 t}$$

  - follows from Lax pair: if u solves KdV $\frac{d}{dt}\psi_+ + P\psi_+$ is also a solution to the Schrödinger equation. Asymptotics as $x \to +\infty$ imply

$$\frac{d}{dt}\psi_+ = -P\psi_+ + \frac{4i}{\epsilon}z^3\psi_+.$$

Example

- Example $u_0(x) = -\frac{1}{\cosh(x)^2}$
  - $\psi_\pm$ and $\phi_\pm$ can be expressed in terms of hypergeometric function
  - reflection coefficient can be computed using asymptotics for hypergeometric function:

$$r_0(z;\epsilon) = \frac{\Gamma(iz/\epsilon - s)\Gamma(iz/\epsilon + s + 1)\Gamma(-iz/\epsilon)}{\Gamma(s+1)\Gamma(-s)\Gamma(iz/\epsilon)}$$

  - $s = \frac{1}{2} + \frac{i}{\epsilon}\sqrt{1 - \epsilon^2/4}$
- in general, no explicit expressions for $\psi_\pm$, $\phi_\pm$, and r

Small dispersion asymptotics

- What do we know about $r_0(z;\epsilon)$ as $\epsilon \to 0$? *(Ramond)*
  - exponentially small for $-z^2 < \min_{x\in\mathbb{R}} u_0(x)$
  - oscillatory for $\min_{x\in\mathbb{R}} u_0(x) < -z^2 < 0$:
    $r(z;\epsilon) \sim e^{\frac{i}{\epsilon}\rho(z)}$

Scattering theory

- How to recover $u(x,t,\epsilon)$ from reflection coefficient $r(z;t,\epsilon)$?
  - inverse scattering transform

Initial data $\qquad\qquad\qquad\qquad$ reflection coefficient

$$\begin{tikzcd}
u_0(x) \arrow[r, "\text{scattering transform}"'] \arrow[d, "?"] & r_0(z) \arrow[d, "\text{Explicit time evolution}"'] \\
u(x,t,\epsilon) & r(z;t) \arrow[l, "\text{inverse scattering}"']
\end{tikzcd}$$

Inverse scattering

- direct scattering transform
  - KdV $\to$ Lax pair $\to$ Schrödinger equation $\to$ fundamental solutions $\psi_\pm$, $\phi_\pm$
  - fundamental solutions $\psi_\pm$, $\phi_\pm$ can be used to construct a solution to a Riemann-Hilbert (RH) problem

    $\to$ search 2×2 matrix-valued function
    - analytic in $\mathbb{C} \setminus \mathbb{R}$
    - boundary values on $\mathbb{R}$ satisfy a jump relation
    - asymptotic condition

Inverse scattering

- Define

$$M(\lambda;x,t,\epsilon)=\begin{cases}\begin{pmatrix}\phi_-(z)e^{\frac{4i}{\epsilon}tz^3} & \frac{1}{a}\psi_+(z)e^{-\frac{4i}{\epsilon}tz^3}\\ \epsilon\frac{d}{dx}\phi_-(z)e^{\frac{4i}{\epsilon}tz^3} & \frac{\epsilon}{a}\frac{d}{dx}\psi_+(z)e^{-\frac{4i}{\epsilon}tz^3}\end{pmatrix} & \lambda\in\mathbb{C}^+\\ \begin{pmatrix}\frac{1}{a^*}\psi_-(z)e^{\frac{4i}{\epsilon}tz^3} & \phi_+(z)e^{-\frac{4i}{\epsilon}tz^3}\\ \frac{\epsilon}{a^*}\frac{d}{dx}\psi_-(z)e^{\frac{4i}{\epsilon}tz^3} & \epsilon\frac{d}{dx}\phi_+(z)e^{-\frac{4i}{\epsilon}tz^3}\end{pmatrix} & \lambda\in\mathbb{C}^-\end{cases}$$

  where $z=\sqrt{-\lambda}$

  - analyticity and asymptotic properties of the fundamental solutions $\psi_\pm$, $\phi_\pm$ and a,b $\longrightarrow$ analyticity and asymptotic properties for M
  - connection between $\psi_\pm$ and $\phi_\pm$ $\longrightarrow$ jump property for M on the real line in terms of reflection coefficient

Inverse scattering

Riemann-Hilbert problem for M

(a) $M : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic

(b)

$$M_+(\lambda) = M_-(\lambda) \begin{pmatrix} 1 & r_0(z;\epsilon) \\ -\bar{r}_0(z;\epsilon) & 1 - |r_0(z;\epsilon)|^2 \end{pmatrix}, \qquad \text{for } \lambda < 0,$$

$$M_+(\lambda) = M_-(\lambda) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{for } \lambda > 0,$$

(c) as $\lambda \to \infty$,

$$M(\lambda) = (I + \mathcal{O}(\lambda^{-1})) \begin{pmatrix} e^{\frac{i}{\epsilon}xz} e^{\frac{4i}{\epsilon}tz^3} & e^{\frac{-i}{\epsilon}xz} e^{\frac{-4i}{\epsilon}tz^3} \\ ize^{\frac{i}{\epsilon}xz} e^{\frac{4i}{\epsilon}tz^3} & -ize^{\frac{-i}{\epsilon}xz} e^{\frac{-4i}{\epsilon}tz^3} \end{pmatrix}$$

where $z = \sqrt{-\lambda}$,

Riemann-Hilbert problem

- Write $M_{11}(\lambda) = 1 + M_{1,11}/z + \mathcal{O}(z^{-2})$, $z \to \infty$
  - KdV solution can be recovered from RH problem:

$$u(x,t,\epsilon) = -2i\epsilon \frac{\partial}{\partial x} M_{1,11}(x,t,\epsilon)$$

- Remarks on the RH problem
  - RH solution is unique
  - jump matrix depends on reflection coefficient
- Strategy to solve Cauchy problem for KdV with initial data $u_0(x)$
  - compute reflection coefficient $r_0(\lambda;\epsilon)$
  - solve the RH problem at values x, t, ϵ
  - $u(x,t,\epsilon)$ can be recovered from the RH solution

Riemann-Hilbert problem

- This works also backwards: if M solves a RH problem of this form, for some function r_0, then $u(x,t,\epsilon) = -2i\epsilon\frac{\partial}{\partial x}M_{1,11}(x,t,\epsilon)$ solves the KdV equation
- Idea of the proof
  - Consider $A := M_x M^{-1}$ and $B := M_t M^{-1}$

    $\rightarrow$ since the jump matrix for M is independent of x and t, A and B are entire functions in λ

    $\rightarrow$ from the asymptotics of the RH problem, it follows that

$$A(\lambda; x, t, \epsilon) = -\frac{1}{\epsilon}\begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix},$$

Riemann-Hilbert problem

- Idea of the proof
  - Consider $A := M_x M^{-1}$ and $B := M_t M^{-1}$

    $\rightarrow$ from the asymptotics of the RH problem, it follows that

$$A(\lambda; x, t, \epsilon) = -\frac{1}{\epsilon}\begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix},$$

$$B(\lambda; x, t, \epsilon) = \frac{4}{\epsilon}\begin{pmatrix} -f & \lambda + \frac{u}{2} \\ \lambda^2 - \frac{u}{2}\lambda + s & f \end{pmatrix}.$$

Derivation of KdV

- - We have $M_x = AM$, and $M_t = BM$
  - Compatibility $M_{xt} = M_{tx}$ implies the identity

$$A_t - B_x + [A, B] = 0$$

  $\rightarrow$ this is a system of equations in u, s, f from which it follows that

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0$$

- so, for any (sufficiently smooth and decaying at $\pm\infty$) r, if the RH problem has a solution, the RH solution generates a solution to KdV (but not necessarily with ϵ-independent initial data)
- same argument works if $\mathbb{R}$ is replaced by other contour, as long as jumps are independent of x, t

Asymptotic analysis of RH problem

- small ϵ asymptotics for RH problem?
- Idea
  - transform the RH problem to one of the following form, by a series of invertible transformations $M \mapsto T \mapsto S \mapsto R$:

    (a) R analytic except on a contour γ

    (b) $R_+(z) = R_-(z)J(z)$ for $z \in \gamma$, and

    $$J(z) = I + o(1), \qquad \epsilon \to 0$$

    (c) $R(z) \to I$ as $z \to \infty$
  - then the solution $R(z) = I + o(1)$ as $\epsilon \to 0$
  - inverting the transformations $M \mapsto T \mapsto S \mapsto R$ leads to small ϵ asymptotics for M and for u

Asymptotic analysis of RH problem

- Deift/Zhou steepest descent method provides powerful tools to obtain asymptotics for RH solution M, for example in the small dispersion limit $\epsilon \to 0$ *(Deift-Venakides-Zhou '97)*
  - deformations of the jump contour - creates jump matrices that are close to I for ϵ small, except near special points
  - construction of local parametrices - deals with jumps near those special points
  - different transformations depending on x, t
- this procedure leads to asymptotics for M as $\epsilon \to 0$, and asymptotics for u follow from this

Asymptotic analysis of RH problem

- only possible if we have sufficiently accurate information about $r(z;t,\epsilon)$ for small ϵ!
- small ϵ asymptotics for the jump matrix
  - replacing r by its small ϵ approximation, jump matrix for M becomes

$$
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \lambda < \min_{x\in\mathbb{R}} u_0(x)
$$

$$
\begin{pmatrix} 1 & e^{\frac{i}{\epsilon}\rho(\lambda)} \\ -e^{-\frac{i}{\epsilon}\rho(\lambda)} & 0 \end{pmatrix}, \qquad \min_{x\in\mathbb{R}} u_0(x) < \lambda < 0
$$

$$
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \lambda > 0
$$

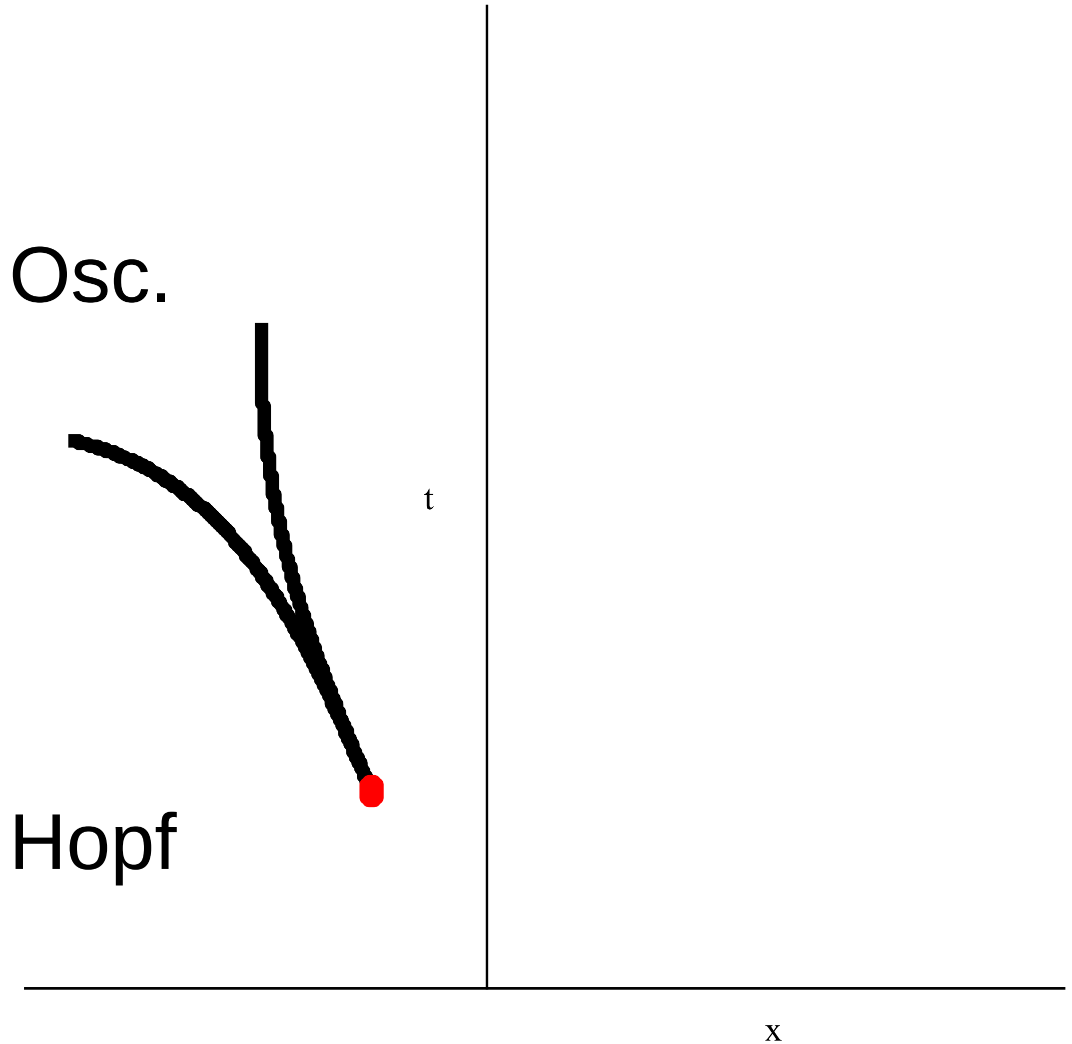
Asymptotic analysis of RH problem

Rough overview of transformations

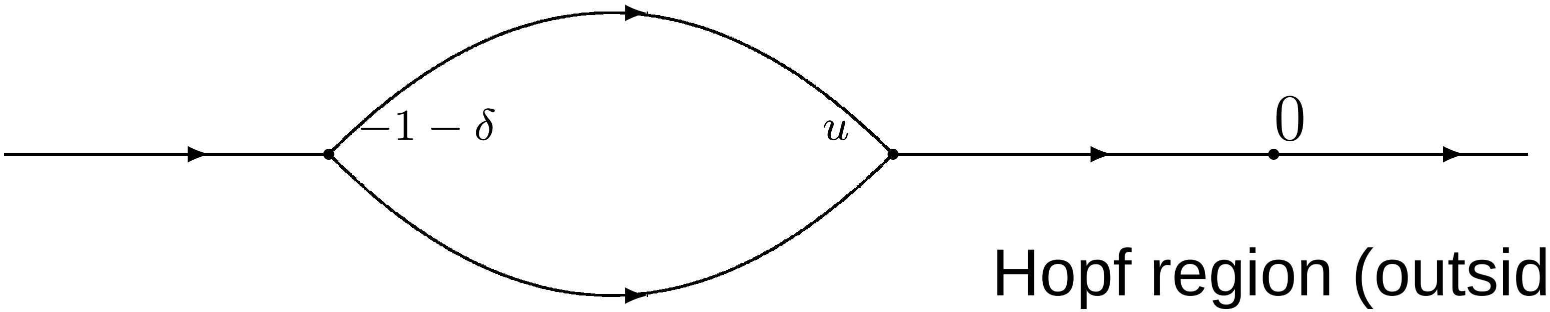
- Transformation $M \mapsto T$ modifies the jumps in a convenient way
  - construction of a suitable g-*function*
- Transformation $T \mapsto S$
  - contour deformation using factorization of jump matrix
- Transformation $S \mapsto R$
  - requires construction of *parametrices*: approximations of the RH solution in different regions of complex plane

Asymptotic analysis of RH problem

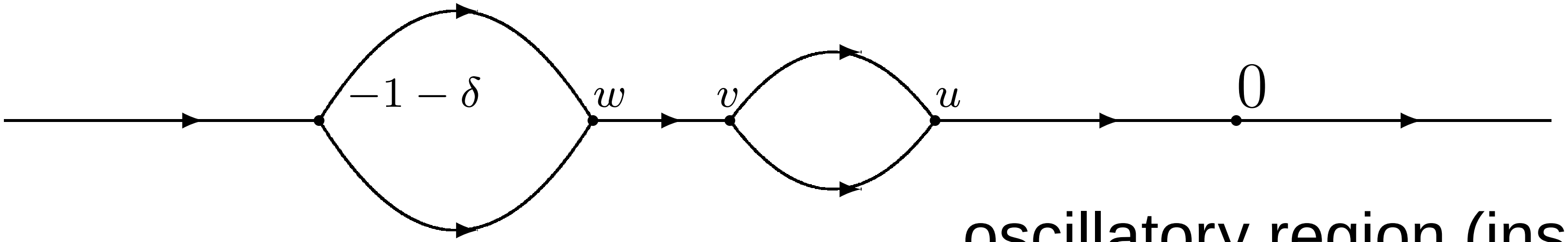
- Remember cusp-shaped region
  - transformations of the RH problem are different inside and outside cusp



Riemann-Hilbert problem - opening of lens

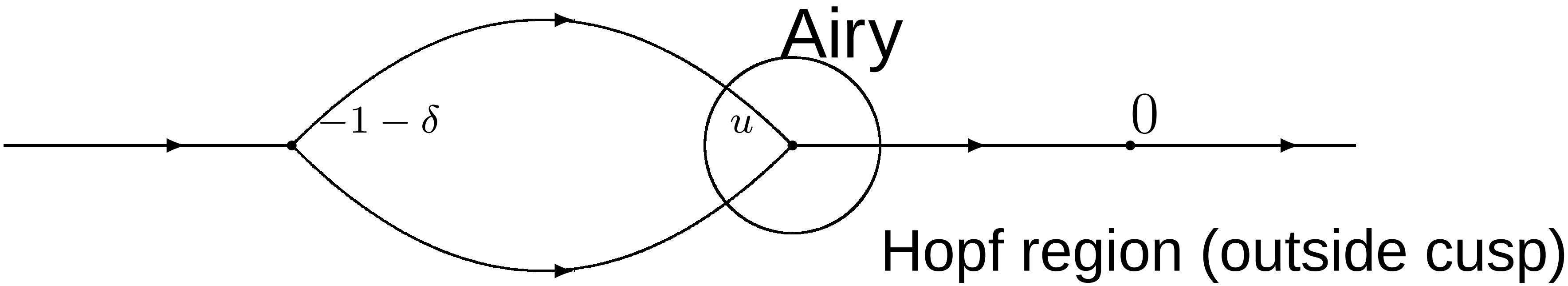


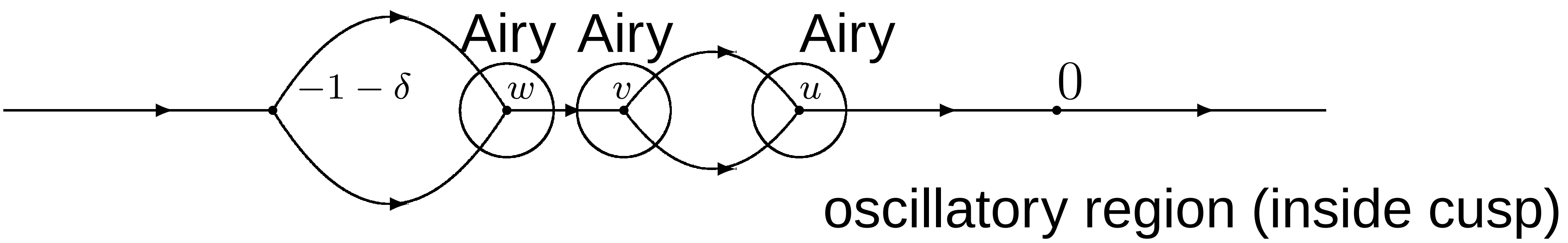
Hopf region (outside cusp)



oscillatory region (inside cusp)

Riemann-Hilbert problem - parametrices





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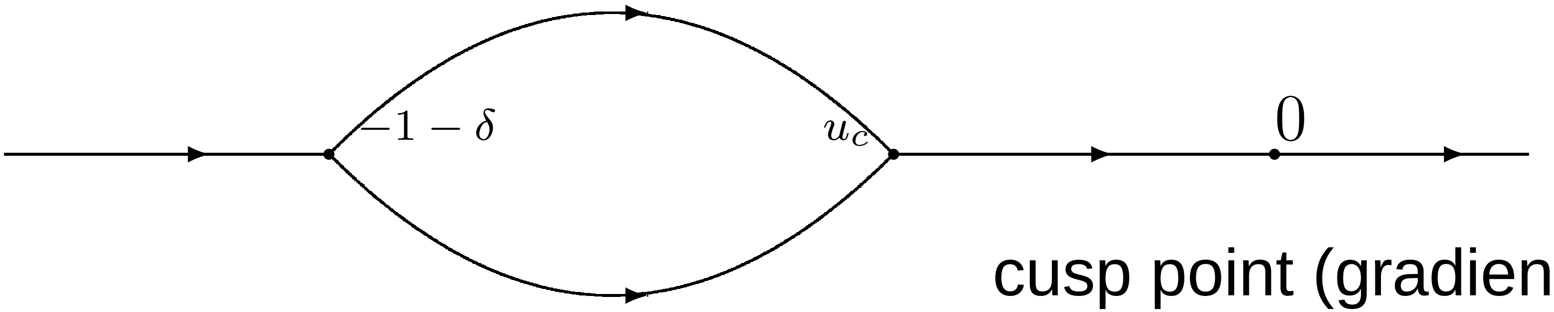
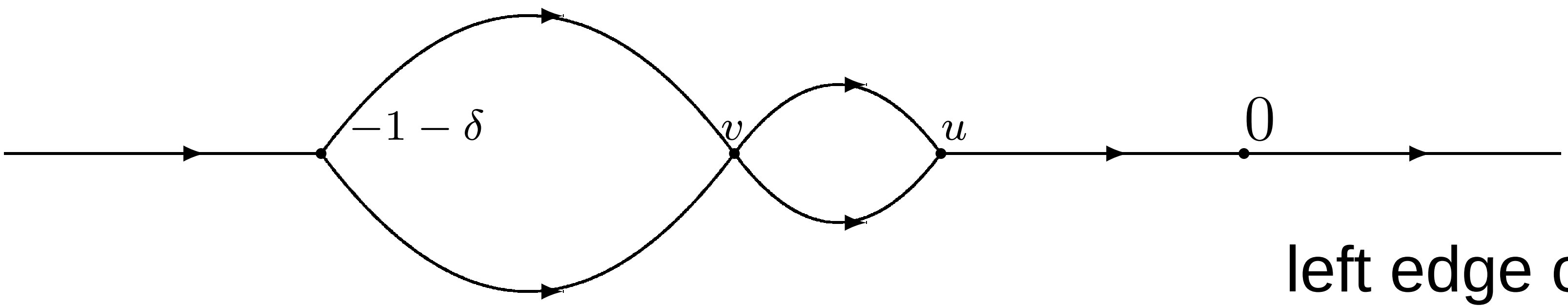
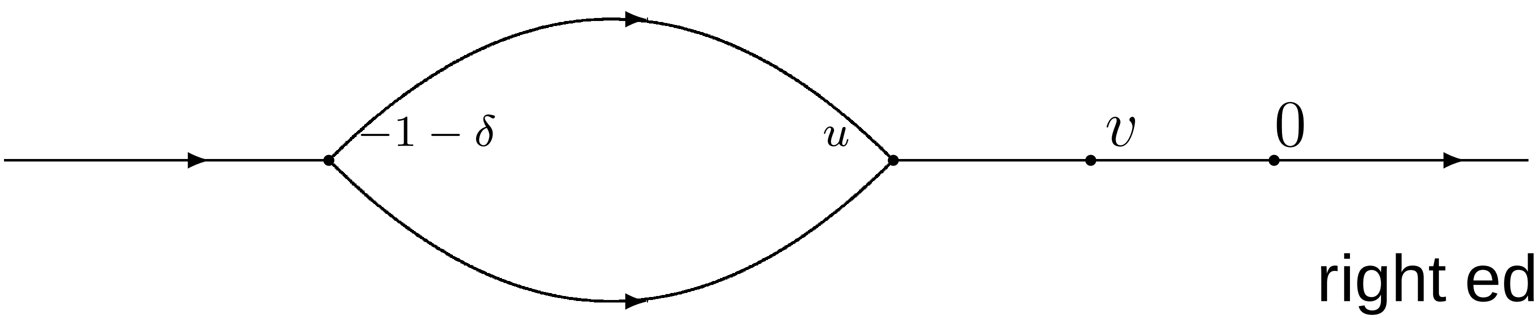


cusp point (gradient catastrophe)

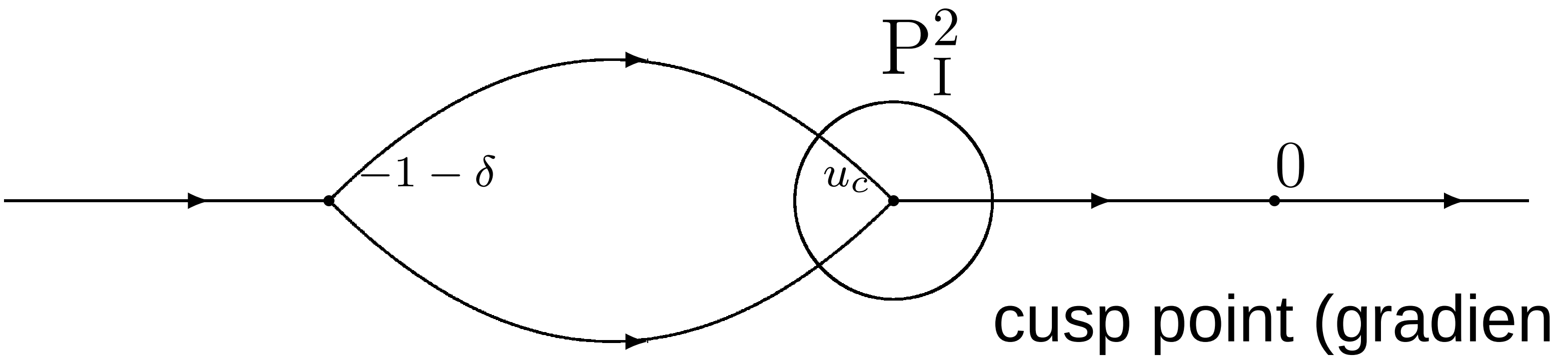


left edge of the cusp



right edge of the cusp

Riemann-Hilbert problem - local parametrices



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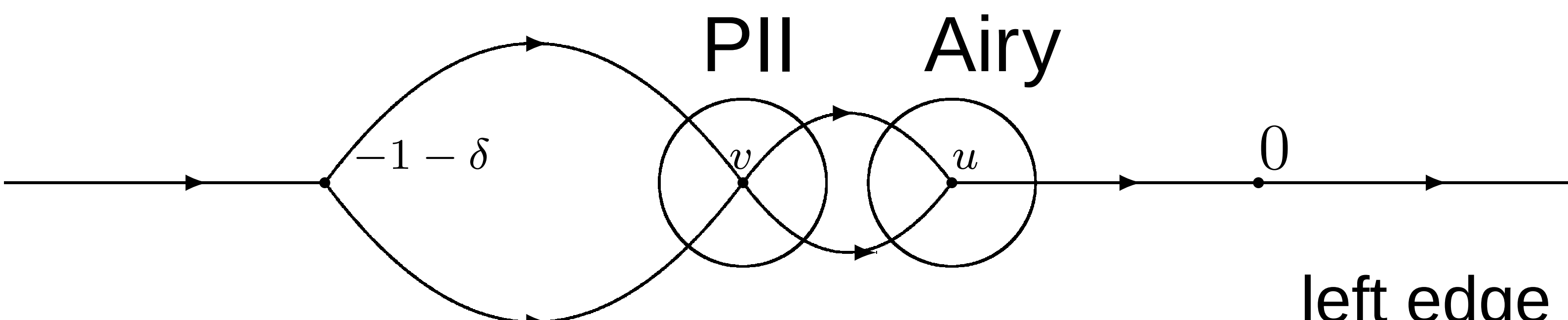

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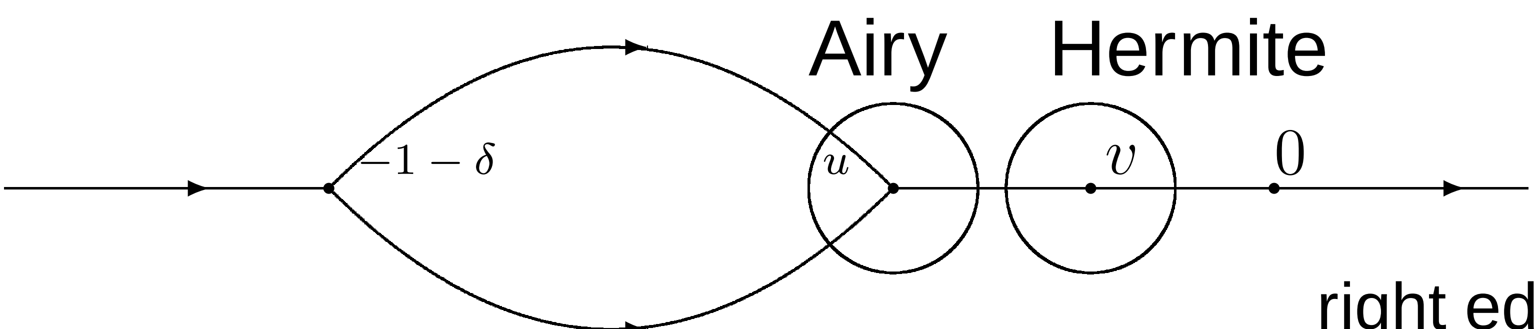


right edge of the cusp

Perturbations of the KdV equation

- much less can be said about other perturbations of the Hopf equation

$$u_t + 6uu_x + \epsilon^2 P(x, u, u_x, u_{xx}, \ldots) = 0,$$

where P depends on $x, u, u_x, \ldots$, for example $P = u_{xxx} + \epsilon^2 u_{xxxxx}$ (Kawahara equation)

  - existence of a solution (even for small t)?
  - is the solution close to the Hopf solution for small t, as $\epsilon \to 0$?
    (Masoero-Raimondo)

Universality conjecture

- class of equations considered by Dubrovin

$$
\begin{aligned}
u_t + 6uu_x + \frac{\epsilon^2}{24}\left[2c\, u_{xxx} + 4c' u_x u_{xx} + c'' u_x^3\right] \\
= \epsilon^4 \left[2p\, u_{xxxxx} + 2p'(5u_{xx}u_{xxx} + 3u_x u_{xxxx}) \right. \\
\left. + p''(7u_x u_{xx}^2 + 6u_x^2 u_{xxx}) + 2p''' u_x^3 u_{xx}\right],
\end{aligned}
$$

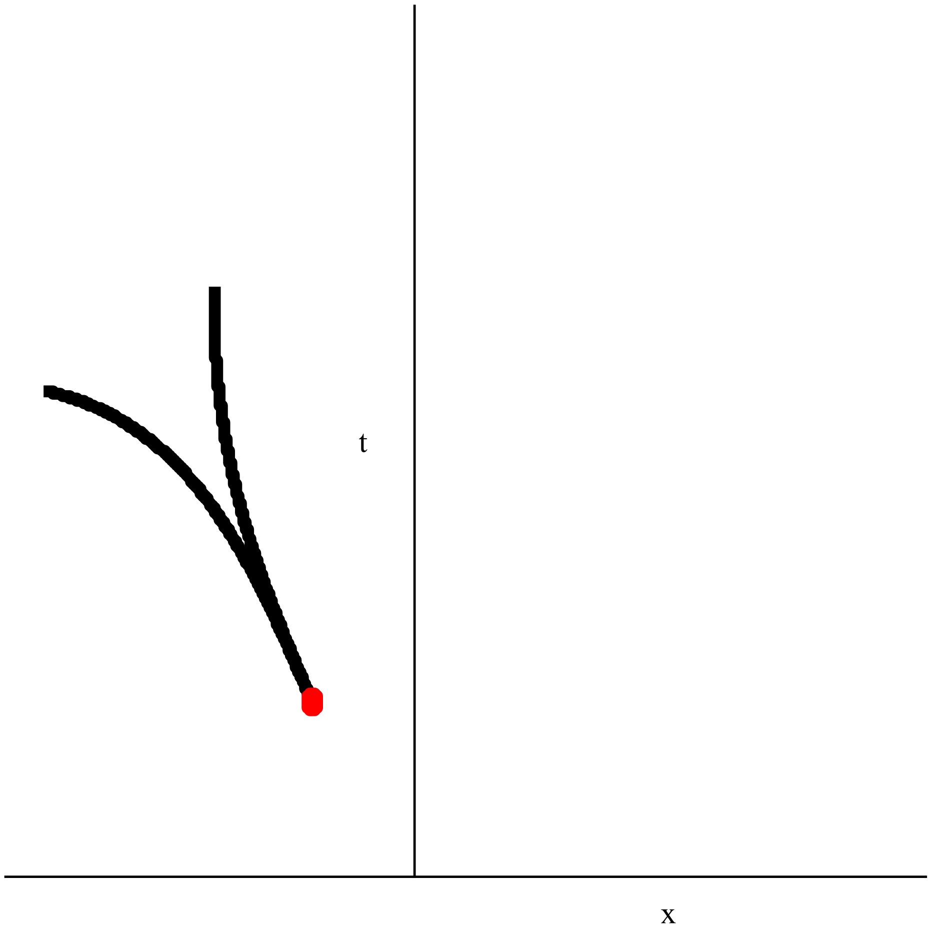
  - universality conjecture: near point of gradient catastrophe of Hopf equation $u_t + 6uu_x = 0$, solutions are generically described in terms of the same Painlevé transcendent U
  - even for Hamiltonian perturbations to hyperbolic systems of equations
  - equations are not integrable in general: no Lax pair

Universality conjecture

- equations in this universality class
  - KdV hierarchy, de-focusing NLS, Camassa-Holm equation, Kawahara equation
- other universality class for elliptic equations
  - focusing NLS, ...
  - regularization asymptotically described in terms of tritronquée solution to Painlevé I equation
- to what extent are asymptotics universal?
  - Hopf asymptotics?
  - critical asymptotics?
  - elliptic asymptotics?

Phase diagram for KdV equation

- small dispersion KdV
  - different behavior inside and outside cusp
  - transitions near leading edge (PII), trailing edge (soliton), point of gradient catastrophe(P_I^2)



Unitary random matrix ensembles

- space of $n \times n$ Hermitian matrices with probability measure
$$\frac{1}{Z_n} \exp(-n \text{Tr}\, V(M)) dM,$$
where
  - V is a polynomial of even degree with positive leading coefficient,
  - $dM = \prod_{j=1}^{n} dM_{jj} \prod_{i<j} d\text{Re}\, M_{ij} d\text{Im}\, M_{ij}$
- Simplest case $V(M) = M^2/2 \longrightarrow$ GUE
  - independent Gaussian distributed matrix entries
- invariant under unitary conjugation
- behavior of eigenvalues?

Unitary random matrix ensembles

- Joint probability distribution of eigenvalues

$$\prod_{1\leq i<j\leq n}(\lambda_i-\lambda_j)^2\prod_{j=1}^{n}e^{-nV(\lambda_j)}d\lambda_j$$

- Eigenvalues follow a determinantal point process with correlation kernel

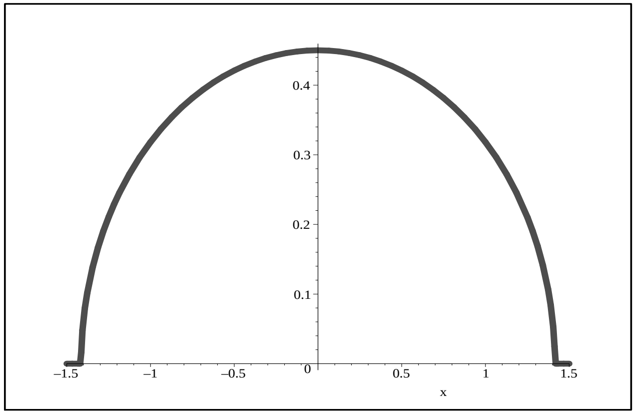
$$K_n(x,y)=e^{-\frac{n}{2}(V(x)+V(y))}\sum_{k=0}^{n-1}p_k^{(n)}(x)p_k^{(n)}(y)$$

  - $p_k^{(n)}$ OPs characterized by

$$\int_{\mathbb{R}}p_k^{(n)}(x)p_\ell^{(n)}(x)e^{-nV(x)}dx=\delta_{k\ell}$$

Limiting mean eigenvalue density

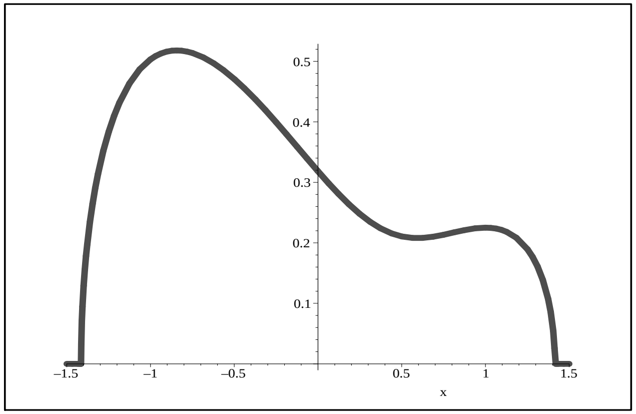
- limiting mean density of eigenvalues
  - for GUE:
    Wigner semi-circle law



  - in general: limiting mean eigenvalue density depends on V

    $\rightarrow$ smooth density on finite union of intervals

    $\rightarrow$ typically square root behavior near endpoints



Limiting distribution of largest eigenvalue

- Limiting mean eigenvalue distribution
  - equilibrium measure minimizing

$$I_V(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x),$$

among all probability measures μ on $\mathbb{R}$

- μ supported on finite number of intervals $\cup_{j=1}^{k} [a_j, b_j]$
  - V convex: 1-cut
  - V of degree $2m$: at most m cuts

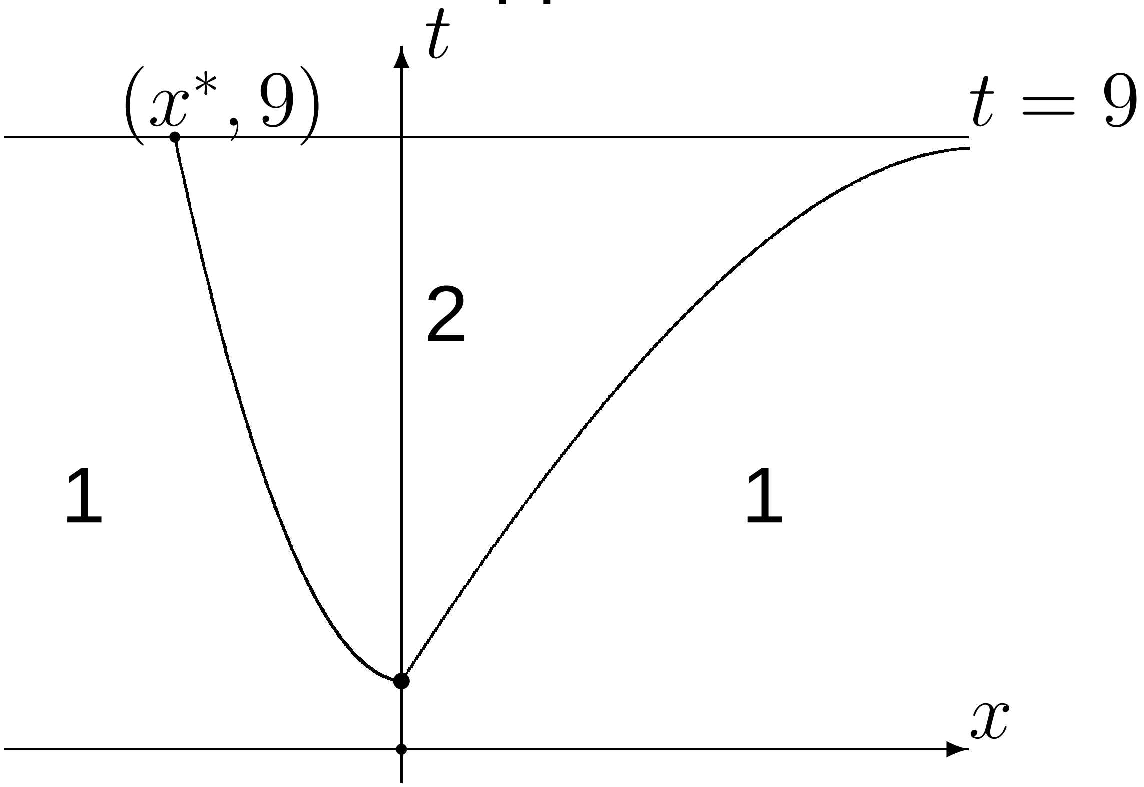
Quartic polynomial

- Consider

$$V_{x,t}(s) = e^x \left[(1-t)\frac{s^2}{2} + t\left(\frac{s^4}{20} - \frac{4s^3}{15} + \frac{s^2}{5} + \frac{8}{5}s \right) \right]$$

with $x \in \mathbb{R}$, $t \geq 0$

- number of intervals in the support



Orthogonal polynomials

- orthogonal polynomials defined by

$$\int_{\mathbb{R}} p_k(s) p_\ell(s) e^{-nV(s)} ds = \delta_{k\ell}$$

  - satisfy three-term recurrence relation

$$s p_n(s) = a_{n+1} p_{n+1}(s) + b_n p_n(s) + a_n p_{n-1}(s)$$

  - asymptotics for recurrence coefficients a_n, b_n as $n \to \infty$?
    $\to$ depend on equilibrium measure μ_V

Orthogonal polynomials

- one-cut case with support $[\alpha, \beta]$:

$$a_n = \frac{\beta - \alpha}{4} + \mathcal{O}(n^{-2}),\ b_n = \frac{\beta + \alpha}{2} + \mathcal{O}(n^{-2})$$

- two-cut case: oscillatory, expressed in terms of elliptic θ-functions
- boundary between one- and two-cut: critical behavior

Orthogonal polynomials

- boundary between one- and two-cut: critical behavior
  - cusp point: in terms of $U(x,t)$

$$a_n = \frac{\beta-\alpha}{4} + c_1 n^{-2/7} U\left(c_2 n^{6/7}(e^x-1), c_2 n^{4/7} e^x (t-1)\right) + \mathcal{O}(n^{-4/}$$

  - Compare to

$$u(x,t,\epsilon) = u_c$$

$$+\left(\frac{2\epsilon^2}{k^2}\right)^{1/7} U\left(\frac{x-x_c-6u_c(t-t_c)}{(8k\epsilon^6)^{\frac{1}{7}}}, \frac{6(t-t_c)}{(4k^3\epsilon^4)^{\frac{1}{7}}}\right) + O\left(\epsilon^{4/7}\right).$$

Painlevé equations

- Fourth order ODE

$$x = tU - \left(\frac{1}{6}U^3 + \frac{1}{24}(U_x^2 + 2UU_{xx}) + \frac{1}{240}U_{xxxx}\right).$$

  - unique real solution $U(x,t)$ satisfying the following conditions:
    - $\rightarrow$ $U(x,t)$ has no poles for $x, t \in \mathbb{R}$,
    - $\rightarrow$ U has the following asymptotic behavior

$$U(x,t) = \mp 6^{1/3}|x|^{1/3} + \mathcal{O}(x^{-1/3}) \qquad \text{as } x \to \pm\infty.$$

Orthogonal polynomials

- boundary between one- and two-cut: critical behavior
  - leading edge: Hastings-McLeod solution to Painlevé II

$$a_n(x,t^*) = \frac{\beta-\alpha}{4} - \frac{1}{2c} q\left(s_{x,n}\right)\cos(2\pi n\omega(x))n^{-1/3} + \mathcal{O}(n^{-2/3}),$$
$$b_n(x,t^*) = \frac{\beta+\alpha}{2} + \frac{1}{c} q\left(s_{x,n}\right)\sin(2\pi n\omega(x)+\theta)n^{-1/3} + \mathcal{O}(n^{-2/3}),$$

  - Compare to

$$u(x,t,\epsilon) = u - \frac{4\epsilon^{1/3}}{c^{1/3}} q\left[s(x,t,\epsilon)\right]\cos\left(\frac{\Theta(x,t)}{\epsilon}\right) + O(\epsilon^{\frac{2}{3}}),$$

Orthogonal polynomials

- let $V(s) = t_{2m}s^{2m} + t_{2m-1}s^{2m-1} + \ldots + t_2 s^2 + t_1 s$,

$$Q = \begin{pmatrix} b_0 & a_1 & 0 & 0 & 0 & \ldots \\ a_1 & b_1 & a_2 & 0 & 0 & \ldots \\ 0 & a_2 & b_2 & a_3 & 0 & \ldots \\ 0 & 0 & a_3 & b_3 & a_4 & \ldots \\ 0 & 0 & 0 & a_4 & b_4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$\epsilon \frac{\partial a_n}{\partial t_k} = \frac{a_n}{2} \left([Q^k]_{n-1,n-1} - [Q^k]_{nn} \right),$$

$$\epsilon \frac{\partial b_n}{\partial t_k} = a_n [Q^k]_{n,n-1} - a_{n+1} [Q^k]_{n+1,n},$$

- Toda hierarchy in Flaschka variables

(Adler-van Moerbeke, Bessis-Itzykson-Zuber)

Toda

- first time flow

$$\epsilon \frac{\partial a_n}{\partial t_1} = \frac{a_n}{2} \left(b_{n-1} - b_n \right),$$
$$\epsilon \frac{\partial b_n}{\partial t_1} = a_n^2 - a_{n+1}^2.$$

- $v_n = -b_n$ and $u_n = \log a_n^2$ solve Toda lattice

$$\frac{du_n}{dt_1} = v_n - v_{n-1}, \qquad \frac{dv_n}{dt_1} = e^{u_{n+1}} - e^{u_n}, \qquad n \in \mathbb{Z}.$$

Toda

- continuum limit: assume $u(x), v(x)$ are functions of x such that

$$u(\epsilon n) = u_n,\ v(\epsilon n) = v_n$$

- then, formally, Toda lattice reduces to

$$u_t = \frac{1}{\epsilon}\left[v(x) - v(x-\epsilon)\right] = v_x - \frac{1}{2}\epsilon v_{xx} + O(\epsilon^2)$$
$$v_t = \frac{1}{\epsilon}\left[e^{u(x+\epsilon)} - e^{u(x)}\right] = e^u u_x + \frac{1}{2}\epsilon(e^u)_{xx} + O(\epsilon^2)$$

Toda

- system of equations in Dubrovins universality class: perturbation of hyperbolic system

$$u_t = v_x$$
$$v_t = e^u u_x$$

  - analogue of Hopf equation
  - can be solved by a generalization of method of characteristics

Open problems

- analogies between multi-matrix models and systems of equations?
  - equations corresponding to Lax operator of higher order than 2
    $\rightarrow$ KP hierarchy, Gelfand-Dikii hierarchy
  - analogues of Pearcey kernel, tacnode kernel ?