Block-Toeplitz Reductions of the 2D-Toda Hierachy

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The semi-infinite 2D-Toda lattice

- The phase space:

$$L_1 = \Lambda + \sum_{k\geq 0} u_k^{(1)} \Lambda^{-k} \qquad L_2 = u_{-1}^{(2)} \Lambda^{-1} + \sum_{k\geq 0} u_k^{(2)} \Lambda^{k}$$

(Lax operators), where

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & & & & \ddots \end{pmatrix} \qquad \Lambda^{-1} \equiv \Lambda^t \qquad u_k^{(i)} = \mathrm{diag}(u_k^{(i)}(n))$$

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- The flows:

$$\frac{\partial}{\partial t_k} L_i = \left[\left(L_1^k \right)_+ , L_i \right] \qquad \frac{\partial}{\partial s_k} L_i = \left[\left(L_2^k \right)_- , L_i \right], \quad i = 1, 2 \quad k \geq 1$$

are pairwise compatible and have bi-hamiltonian structure.

Reductions

The 2D-Toda hierarchy has infinitely many dependent variables

$$u_k^{(i)} = u_k^{(i)}(n; t, s)$$

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Examples:

- Standard Toda:

$$L_1 = L_2 = \Lambda + u_0 + u_1 \Lambda^{-1}$$

- Bi-graded Toda [Carlet, 2006]:

$$L_1^a = L_2^b = \Lambda^a + u_1 \Lambda^{a-1} + \cdots + u_{a+b} \Lambda^{-b}$$

A glimpse of GW-IS correspondence

- For a projective manifold X, one can define GW invariants
$$\langle \tau_{\alpha_1, p_1} \dots \tau_{\alpha_k, p_k} \rangle_{g,\beta}^X \in \mathbb{Q}$$
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- These number are subject to a mysterious web of relations: i.e. the generating function
$$\mathcal{F}^X(t;\epsilon) = \sum_g \epsilon^{2g-2} \mathcal{F}_g^X(t)$$
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satisfies a number of "topological" PDEs.

Example. $F^X(v^1,\dots,v^n) = \mathcal{F}^X_0(t)\big|_{t^{\alpha,0}=v^\alpha,\, t^{\alpha,>0}=0}$ satisfies

$$\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\mu} \eta^{\mu\nu} \frac{\partial^3 F}{\partial v^\nu \partial v^\gamma \partial v^\delta} = (\alpha \leftrightarrow \gamma) \qquad (WDVV)$$

where $\eta_{\alpha\beta} = \partial_1 \partial_\alpha \partial_\beta F$ is a constant nondegenerate matrix.

A glimpse of GW-IS correspondence

- Dispersionless limits of integrable hierarchies are also described by solutions of WDVV (Frobenius manifolds) [Dubrovin, 1994].
- In some simple cases, this correspondence is known to hold at all orders in the genus/dispersion expansion: the full set of topological PDEs for $\mathcal{F}^X$ coincides with an integrable hierarchy.

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Examples

- $X = \mathbb{P}^1 \quad \to \quad$ Standard Toda

  [Getzler, 2000]

- $X = \mathbb{P}^1_{a,b} \quad \to \quad$ Bi-graded Toda

  [Milanov-Tseng, 2007]

where $\mathbb{P}^1_{a,b}$ is the projective line with two orbifold points of weights a, b.

Solutions of 2D-Toda from orthogonal polynomials

Let $\mu_\infty = \mu_\infty(t, s)$ be a semi-infinite solution of the linear system

$$\frac{\partial}{\partial t_k}\mu_\infty = \Lambda^k \mu_\infty \qquad \frac{\partial}{\partial s^k}\mu_\infty = -\mu_\infty \Lambda^{-k} \qquad \text{(I2DT)}$$

If $\det\left((\mu_{ij})_{i,j\leq n}\right) \neq 0$ for all n, μ_∞ has a unique Borel factorization

$$\mu_\infty = S_1^{-1} S_2 \qquad S_1 = I + \sum_{k\geq 1} s_k^{(1)}\Lambda^{-k} \qquad S_2 = h + \sum_{k\geq 1} s_k^{(2)}\Lambda^{k}$$

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μ_∞ defines a bi-moment functional $\mathcal{L} : \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C}$ by $\mathcal{L}(z^i, z^j) = \mu_{ij}$. Then

$$S_1 \chi(z) = \mathbf{p}^{(1)}(z) \qquad h(S_2^t)^{-1}\chi(z) = \mathbf{p}^{(2)}(z)$$

where $\chi = (1, z, z^2, \dots)^t$, $\mathbf{p}^{(i)} = (p_0^{(i)}, p_1^{(i)}, p_2^{(i)}, \dots)^t$ and $p_k^{(i)}$ are monic bi-orthogonal polynomials, $\mathcal{L}(p_k^{(1)}(z), p^{(2)}(z)) = h_k \delta_{kl}$.

Solutions of 2D-Toda from orthogonal polynomials

Recurrence matrices:

$$L_1(t,s) \doteq S_1(t,s)\Lambda S_1(t,s)^{-1} \qquad L_2(t,s) \doteq S_2(t,s)\Lambda^{-1} S_2(t,s)^{-1}$$

$$L_1 \mathbf{p}^{(1)}(z) = z\mathbf{p}^{(1)}(z) \qquad h(L_2^t)^{-1} h^{-1} \mathbf{p}^{(2)}(z) = z\mathbf{p}^{(2)}(z)$$

- $L_1(t,s), L_2(t,s)$ satisfy the 2D-Toda hierarchy.
- For special symmetries of the moment matrix (invariant under the flows), we obtain solutions of reductions of 2D-Toda.

[Adler - Van Moerbeke, 2000]

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Example (Toeplitz reduction):

$$\mathcal{L}(f(z), g(z)) = \frac{1}{2\pi i}\oint_{S^1} f(z)g(z^{-1})\rho(z)\frac{dz}{z}$$

$$\mathcal{L}(zf(z), g(z)) = \mathcal{L}(f(z), z^{-1}g(z)) \qquad \mu_{ij} = \mu_{i-j}$$

Solutions of 2D-Toda from orthogonal polynomials

For the time-dependent weight

$$\rho(z) = \rho_0(z) e^{\sum_{k\geq 1} t_k z^k - s_k z^{-k}}$$

the moment matrix $\mu(t, s)$ satisfies the linearized 2D-Toda lattice.

Solutions of 2D-Toda from orthogonal polynomials

For the time-dependent weight

$$\rho(z) = \rho_0(z) e^{\sum_{k\geq 1} t_k z^k - s_k z^{-k}}$$

the moment matrix $\mu(t, s)$ satisfies the linearized 2D-Toda lattice. The corresponding Lax matrices have the "rank 2" structure

$$L_1 = h \begin{pmatrix} -x_1 y_0 & 1 - x_1 y_1 & 0 & 0 & \dots \\ -x_2 y_0 & -x_2 y_1 & 1 - x_2 y_2 & 0 & \dots \\ -x_3 y_0 & -x_3 y_1 & -x_3 y_2 & 1 - x_3 y_3 & \dots \\ \vdots & & & & \ddots \end{pmatrix} h^{-1}$$

$$L_2 = \begin{pmatrix} -x_0 y_1 & -x_0 y_2 & -x_0 y_3 & -x_0 y_4 & \dots \\ 1 - x_1 y_1 & -x_1 y_2 & -x_1 y_3 & -x_1 y_4 & \dots \\ 0 & 1 - x_2 y_2 & -x_2 y_3 & -x_2 y_4 & \dots \\ \vdots & & & & \ddots \end{pmatrix}$$

where $x_k = p_k^{(1)}(0), y_k = p_k^{(2)}(0), h_{k+1}/h_k = 1 - x_{k+1} y_{k+1}$.

(Toeplitz lattice/Ablowitz-Ladik hierarchy)

Block-Toeplitz and rational reductions

More generally, consider

$$\mathcal{L}_{a,b}(f(z), g(z)) = \frac{1}{2\pi i} \oint_{S^1} f(z^b) g(z^{-a}) \rho(z) \frac{dz}{z}$$

$$\mathcal{L}_{a,b}(z^a f(z), g(z)) = \mathcal{L}(f(z), z^{-b} g(z)) \qquad \mu_{ij} = \mu_{bi-aj}$$

$$\rho(z) = \rho_0(z) \exp\left(\sum_k t_k z^{kb} - s_k z^{-ka}\right)$$

Then $\mu_\infty(t,s)$ is a $b \times a$ block-Toeplitz solution of I2DT, i.e.

$$\frac{\partial}{\partial t_k}\mu_\infty = \Lambda^k \mu_\infty \qquad \frac{\partial}{\partial s^k}\mu_\infty = -\mu_\infty \Lambda^{-k}$$

$$\mu_\infty = \Lambda^a \mu_\infty \Lambda^{-b}$$

In particular

$$L_1^a L_2^b = S_1 \Lambda^a S_1^{-1} S_2 \Lambda^{-b} S_2^{-1} = S_1 \mu_\infty S_2^{-1} = I$$

Block-Toeplitz and rational reductions

Lemma (recurrence relations). For each n, there exists coefficients $C_1^{(n)}, \dots, C_a^{(n)}$, $D_1^{(n)}, \dots, D_b^{(n)}$ s.t.

$$p^{(2)}_{n+b} + D_1 p^{(2)}_{n+b-1} + \cdots + D_b p^{(2)}_n =$$
$$= z^b \left(p^{(2)}_n + C_1 p^{(2)}_{n-1} + \cdots + C_a p^{(2)}_{n-a} \right)$$

Corollary. The recurrence matrices have the form

$$L_1 = (AB^{-1})^{1/a} \qquad L_2 = (BA^{-1})^{1/b}$$

$$A = \Lambda^a + \alpha_1 \Lambda^{a-1} + \cdots + \alpha_a$$
$$B = I + \beta_1 \Lambda^{-1} + \cdots + \beta_b \Lambda^{-b}$$

where the matrices α_j, β_j depend only on the a lowest degree coefficients of $\mathbf{p}^{(1)}$ and b lowest degree coefficients of $\mathbf{p}^{(2)}$.

Bi-graded Ablowitz-Ladik

The induced 2D-Toda flows on the pair (A, B) take the form

$$\frac{\partial}{\partial t_k} A = ((AB^{-1})^{k/a})_+ A - A((B^{-1}A)^{k/a})_+$$

$$\frac{\partial}{\partial t_k} B = ((AB^{-1})^{k/a})_+ B - B((B^{-1}A)^{k/a})_+$$

$$\frac{\partial}{\partial s_k} A = ((BA^{-1})^{k/b})_- A - A((BA^{-1})^{k/b})_-$$

$$\frac{\partial}{\partial s_k} B = ((BA^{-1})^{k/b})_- B - B((BA^{-1})^{k/b})_-$$

(Bi-graded Ablowitz-Ladik hierarchy)

- $a + b$ components rational reduction of 2D-Toda.
- Formulated in the same way in the bi-infinite setting.
- It has a natural Hamiltonian structure.
- Bi-hamiltonianity?

Dispersionless limit and local $\mathbb{P}^1$ orbifolds

- Thanks to the explicit the representation $L_1^a = AB^{-1} = L_2^{-b}$ of the Lax operators, we are able to obtain a complete description of the dispersionless limit [S. R., 2012].

[1] The total space of the bundle $\mathcal{O}(-0) \oplus \mathcal{O}(-\infty) \to \mathbb{P}^1_{a,b}$, where 0 and ∞ are orbifold points of weights a, b, with anti-diagonal Calabi-Yau action on the fibers.

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- The relevant solution of WDVV can be expressed as

  $$F(u, v_{-b+1}, \dots, v_{a-1}) = \text{cubic} + \sum_{-b<k\leq l<a} \epsilon_{k,l} \mathrm{Li}_3 \left(e^{v_k+\dots+v_l}\right)$$

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- This prepotential coincides with the $g = 0$ primary GW potential of the "local football" $(\mathbb{P}^{loc}_{a,b})^1$ [M. Liu, ...].

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Conjecture (GW-IS correspondence for local $\mathbb{P}^1$ orbifolds). The full GW theory of the local football is governed by the bi-graded Ablowitz-Ladik hierarchy.

(checked by Brini [2010] for $a = b = 1$ up to genus 2).

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