The deformation quantizations of the Hermitian symmetric space $SU(1,n)/U(n)$

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Introduction

Start point : P. Bieliavsky, S. Detournay, Ph. Spindel,
The Deformation Quantizations of the Hyperbolic Plane
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$\rightsquigarrow$ Description of a class of associative deformations of $\left(\mathcal{C}^{\infty}\left(SU\left(1,1\right)/U\left(1\right)\right),\ .\ \right)$ from the evolution of a hyperbolic differential operator of order two.

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$\rightsquigarrow$ Description of a class of associative deformations of $(\mathcal{C}^\infty(SU(1,1)/U(1)), .)$ from the evolution of a hyperbolic differential operator of order two.

Goal of the present work : Extension of this approach in the case of the Hermitian symmetric space $SU(1,n)/U(n)$ for $n \in \mathbb{N}\backslash\{0\}$.

I. Notion of deformation quantization

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such that its $\mathbb{C}[[\nu]]$-linear extension to $\mathcal{C}^\infty(M)[[\nu]]$ is associative

+ hypothesis on $C_k : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ for each $k \geq 1$

+ compatibity with $\{\ ,\ \}$

$\Rightarrow$ $*$ is a formal deformation quantization (or star-product)

$*$ is non formal if it is the asymptotic expansion of a non formal expression.

I. Notion of deformation quantization

Example $\left(\nu = \frac{i\hbar}{2}\right)$

For $M = \mathbb{R}^{2n}$ endowed with the standard symplectic structure Ω, the following expression defines a star-product on $\left(\mathbb{R}^{2n}, \Omega\right)$ called Moyal star-product :

$$u *_\nu^0 v := uv + \sum_{k=1}^{\infty} \frac{(i\hbar)^k}{2^k k!} \left(\sum_{\substack{1 \leq i_1, \ldots, i_k \leq 2n \\ 1 \leq j_1, \ldots, j_k \leq 2n}} \Lambda_{i_1 j_1} \ldots \Lambda_{i_k j_k} \; \partial^k_{i_1, \ldots, i_k} u \;\; \partial^k_{j_1, \ldots, j_k} v \right)$$

where $-\Omega^{-1} =: \Lambda = (\Lambda_{ij})$ and $\{ u, v \} := \sum_{i,j} \Lambda_{ij} \, \partial_i u \, \partial_j v$.

Its non formal version is the Weyl product :

$$\left(u *^W v\right)(x) := \left(\frac{1}{\pi\hbar}\right)^{2n} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} u(y)\, v(z)\, e^{\frac{2i}{\hbar}(\Lambda(x,y) + \Lambda(y,z) + \Lambda(z,x))} dy\, dz.$$

II. Basic structure of $SU(1,n)/U(n)$

Let's consider

- $n \in \mathbb{N}\backslash\{0\}$;
- $G := SU(1,n)$ and $\mathfrak{g} := Lie(G)$;
- $K := U(n)$ as Lie subgroup of G.

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On $\mathbb{S}$, we have
- a left-invariant symplectic form $\omega \in \Omega^2(\mathbb{S})$
- a global Darboux chart $\phi : \mathcal{S} := Lie(\mathbb{S}) \to \mathbb{S}$
- an action $G \times \mathbb{S} \xrightarrow{\tau} \mathbb{S}$ (G-equivariant transport of the action of G on G/K $(g, g'K) \mapsto (gg')K$ by π)

III. From a G-invariant star-product on $\mathbb{S}$ to a system of PDEs

Objective : To obtain an explicit description of some/all deformation quantizations $*$ both formal and non formal on $(\mathbb{S} \simeq G/K, \omega)$ which are G-invariant, i.e. such that for each $g \in G$ and $u, v \in \mathcal{C}^{\infty}(\mathbb{S}) : \tau_g^{\star}(u * v) = \tau_g^{\star}(u) * \tau_g^{\star}(v)$.

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Starting idea : Considering $*_{\nu}^{0}$ the Moyal star-product on $(\mathbb{S}, \omega) \simeq (\mathcal{S}, \phi^{\star}\omega)$, find $T \in \mathbb{E} := \mathrm{End}(\mathcal{C}^{\infty}(\mathbb{S})[[\nu]])$ such that $* := T(*_{\nu}^{0})$ defined for each $u, v \in \mathcal{C}^{\infty}(\mathbb{S})$ by $u * v := T\left(T^{-1}(u) *_{\nu}^{0} T^{-1}(v)\right)$ is a G-invariant star-product on $\mathbb{S}$.

Remark

G-invariance of $*$ can be translated at infinitesimal level by

$$X^{\star}(u * v) = X^{\star}(u) * v + u * X^{\star}(v)$$

for each $X \in \mathfrak{g}$ and $u, v \in \mathcal{C}^{\infty}(\mathbb{S})$, where $X^{\star}$ is the fundamental vector field associated to X defined at $s \in \mathbb{S}$ by $X^{\star}_{s} := \frac{d}{dt}_{|_0} \tau_{\exp(-tX)}(s)$.

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In our framework, it is possible to consider a quantum deformation of the action of these fundamental vector fields

$$\rho_{\nu} : \mathfrak{g} \to \mathbb{E} : X \mapsto \rho_{\nu}(X)$$

which is a representation of the Lie algebra $\mathfrak{g}$ such that for each $T \in \mathbb{E}$ and $X \in \mathfrak{g}$:

$$T \circ \rho_{\nu}(X) \circ T^{-1} \in \mathcal{D}er\left(T\left(*^{0}_{\nu}\right)\right).$$

Reformulation of the starting idea : Try to find $T \in \mathbb{E}$ such that

$$T \circ \rho_\nu(X) \circ T^{-1} = X^\star \qquad \mathrm{P}(X)$$

for each $X \in \mathfrak{g}$.

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$\rightsquigarrow$ Two steps : (1) A result from P. Bieliavsky (2002) provides us with an explicit operator $T_\nu \in \mathbb{E}$ satisfying $\mathsf{P}(X)$ for each $X \in \mathcal{S}$ $\rightsquigarrow$ $T_\nu\left(*_\nu^0\right)$ is a $\mathbb{S}$-invariant star-product.

(2) Produce explicitly $U_\nu \in \mathbb{E}$ in such a way that $(U_\nu \circ T_\nu)\left(*_\nu^0\right)$ is G-invariant star-product.

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$\rightsquigarrow$ Looking for U_ν being a convolution operator such that for all $f \in \mathcal{D}(\mathbb{S})$ and $x \in \mathbb{S}$:

$$\left(U_\nu^{-1}(f)\right)(x) = \int_{\mathbb{S}} v\left(y^{-1}x\right) f(y)\, dy,$$

for dy the left-invariant Haar measure on $\mathbb{S}$ and $v \in \mathcal{D}'(\mathbb{S})[[\nu]]$ a (formal) distribution.

Proposition

In this framework :

$$P(X) \text{ for each } X \in \mathfrak{g}$$

$$\Updownarrow$$

$$\left(T_\nu \circ \rho_\nu(X) \circ T_\nu^{-1}\right)(v) = [X]^\star_{\mathcal{S}}(v) \quad \text{for each } X \in \mathfrak{g}.$$

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- $\rightsquigarrow$ Explicit caracterisation of v, U_ν and $*$.

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Thanks for listening !
