Domino-tilings of Aztec diamonds,

Random Surfaces

& the Gaussian Unitary Ensemble (GUE)

Pierre van Moerbeke*

UCLouvain & Brandeis U.

DYGEST Conference at Corfou (May 2013)

1. **Domino tilings of Aztec Diamonds & Random Surfaces**

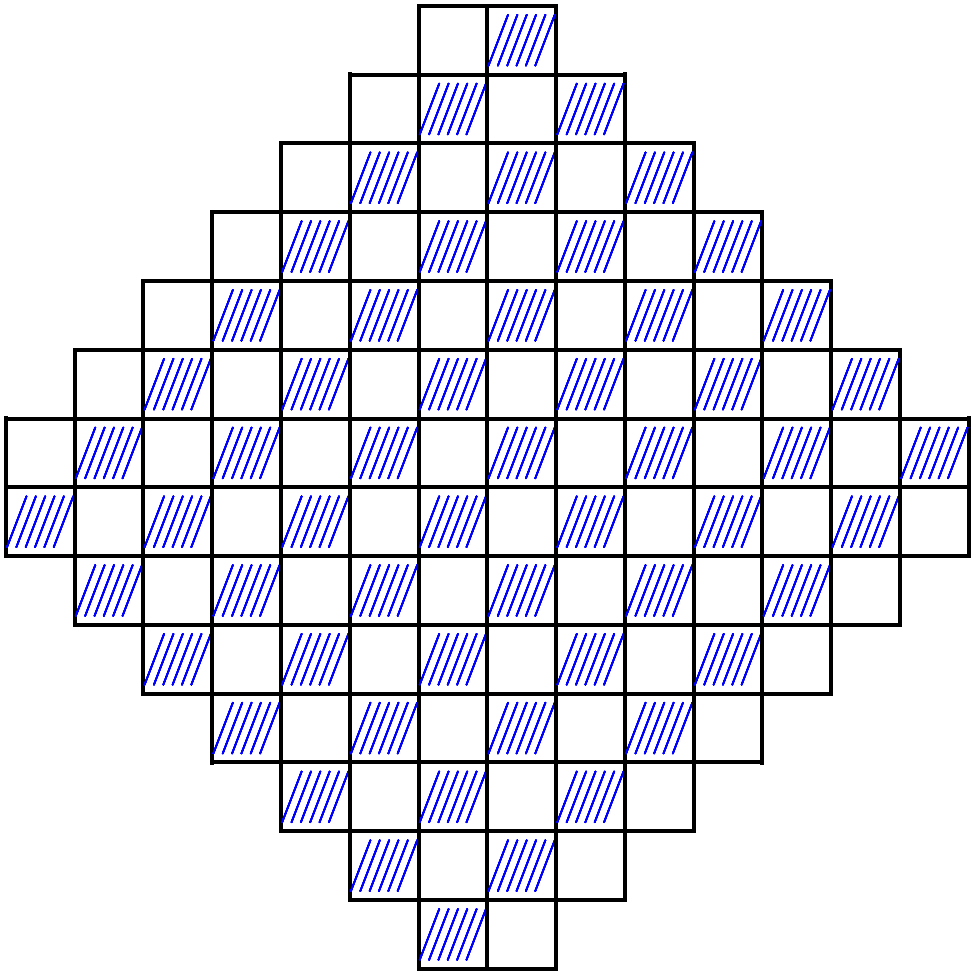
2. **Domino tilings of double Aztec Diamonds**

3. **Letting the size $n \to \infty$**

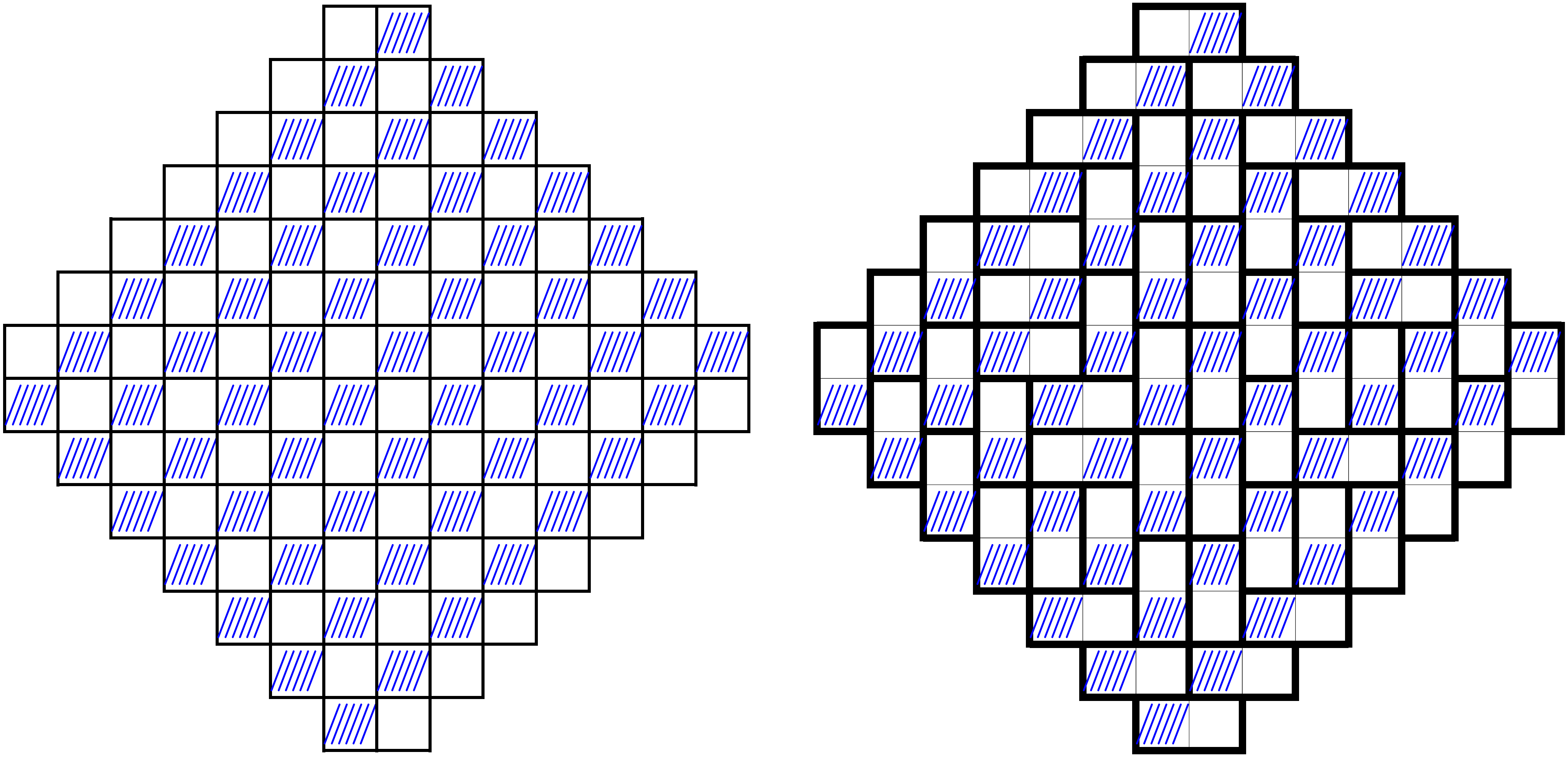
1. Domino tilings of Aztec Diamonds

& Random Surfaces

An Aztec diamond (size n):



Domino tilings of an Aztec diamond :

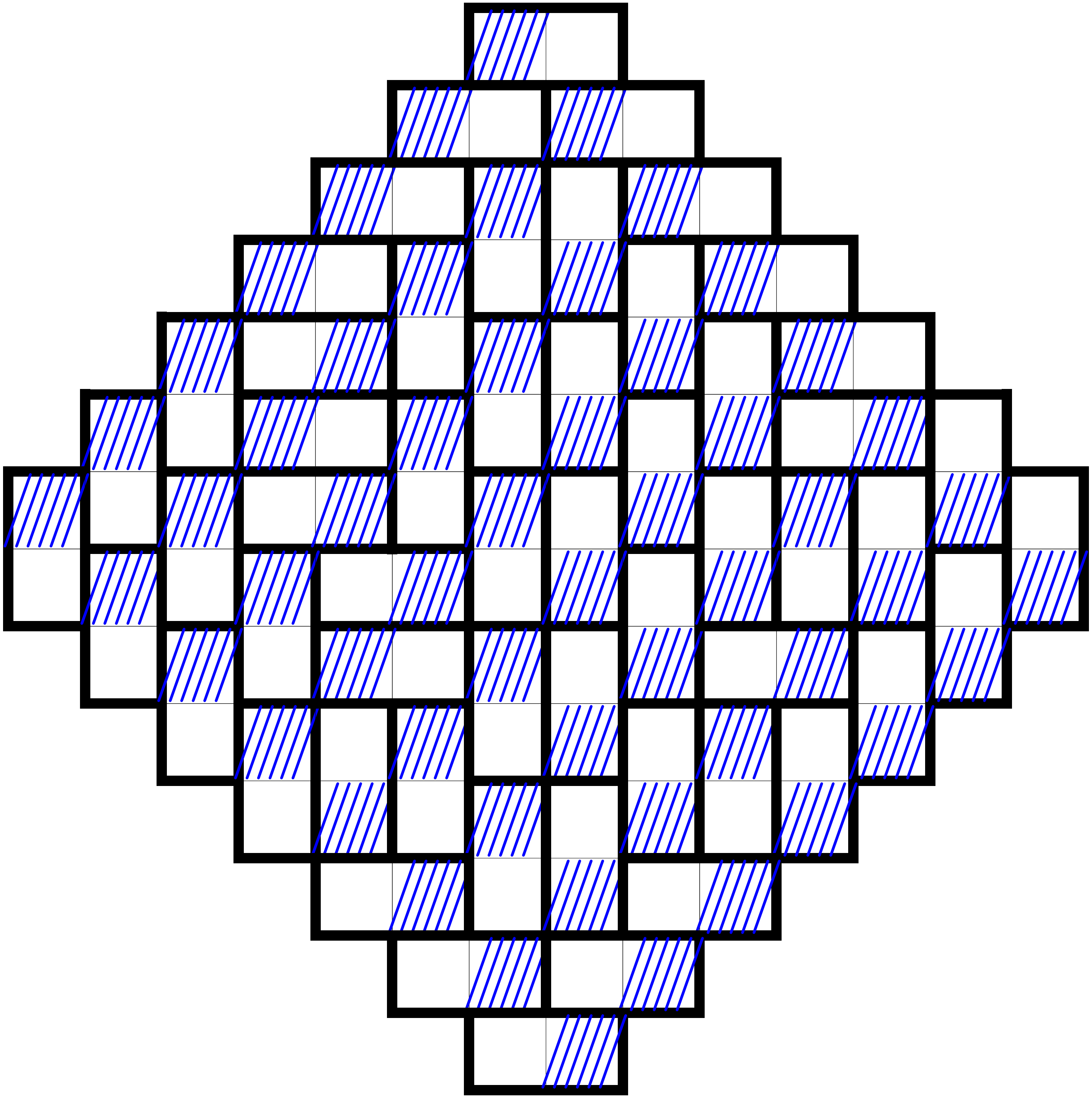


4 types of situations for the domino's!

Kasteleyn 1961, Elkies, Kuperberg, Larsen, Propp '92

Cohn, Elkies, Propp '96, Johansson '00, '03, '05

M. Fulmek, C. Krattenthaler '00, Johansson-Nordenstam '06



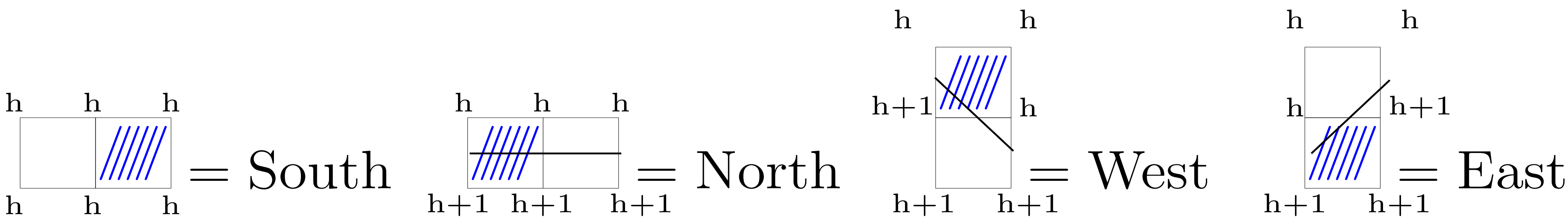
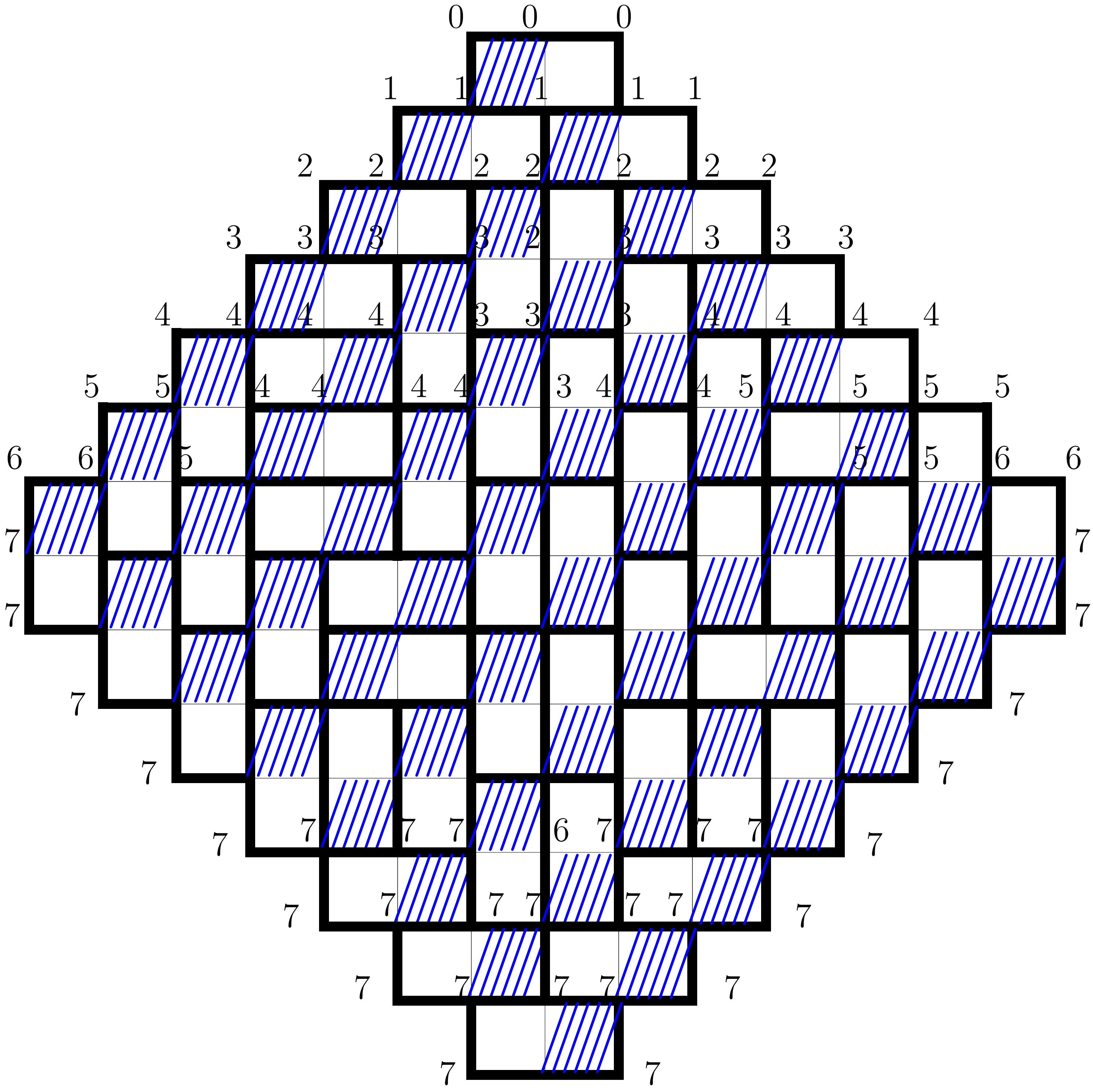
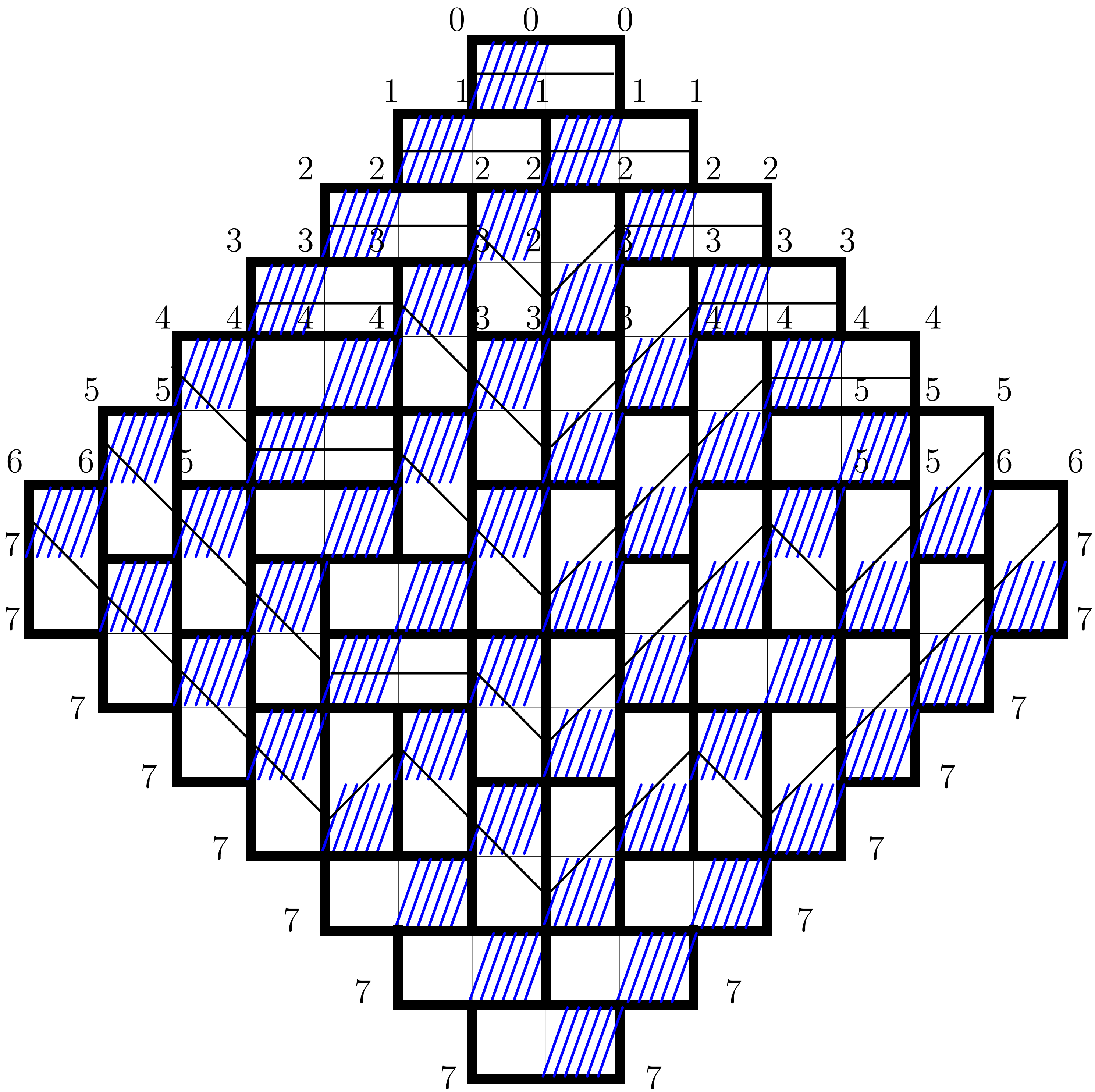
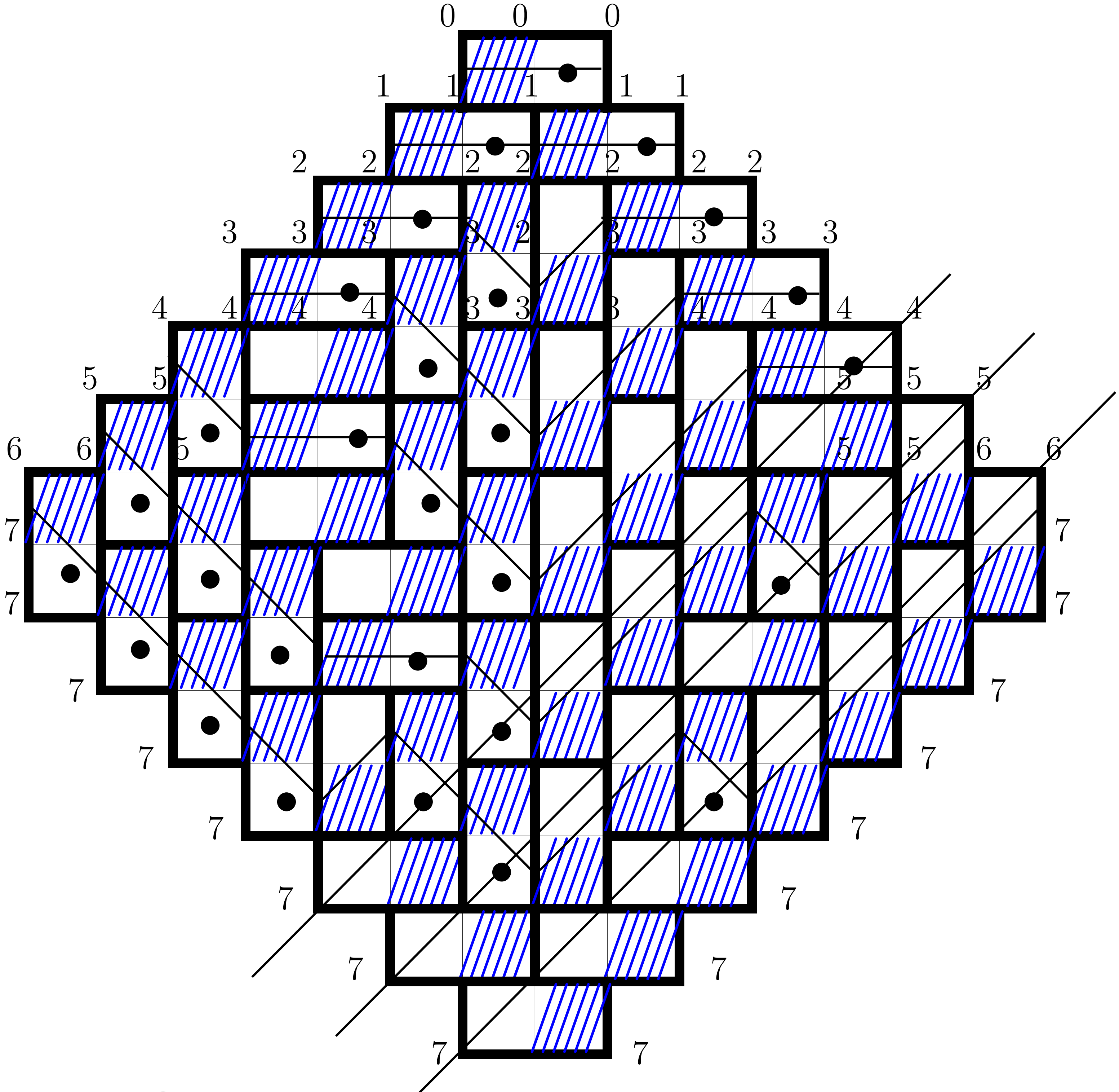


Figure 4. Height function on domino's and level-lines.





Level lines of height $\{\frac{1}{2}, \frac{3}{2}, \ldots, n - \frac{1}{2}\}$



Point process of black dots.

A weight on domino's and a probability on domino tilings:

- put the weight $0 < a \leq 1$ on vertical dominoes
- put the weight 1 on horizontal dominoes,

Define:

$$\mathbb{P}(\text{domino tiling } T) = \frac{a^{\#\text{vertical domino's in } T}}{\sum_{\text{all possible tilings } T} a^{\#\text{vertical domino's in } T}}$$

Which probability are we computing?

$$\mathbb{P}\{\text{The interval } [k,\ell] \subset \{\xi = 2s\} \text{ contains no dot-particles}\}$$

$$= \mathbb{P}\{\text{The random surface is flat along the interval } [k,\ell] \subset \{\xi = 2s\}\}$$

$$= \det\left[\mathbb{1} - \chi_{(k,\ell)}(\eta_1)\ \mathbb{L}_n^{\text{oneAzt}}(2s,\eta_1; 2s,\eta_2)\ \chi_{(k,\ell)}(\eta_2)\right]$$

Which probability are we computing?

$$\mathbb{P}\{\text{The interval } [k,\ell] \subset \{\xi = 2s\} \text{ contains no dot-particles}\}$$

$$= \mathbb{P}\{\text{The random surface is flat along the interval } [k,\ell] \subset \{\xi = 2s\}\}$$

$$= \det\left(\mathbb{1} - \chi_{(k,\ell)}(\eta_1)\mathbb{L}_n^{\text{oneAzt}}(2s,\eta_1;2s,\eta_2)\chi_{(k,\ell)}(\eta_2)\right)$$

with

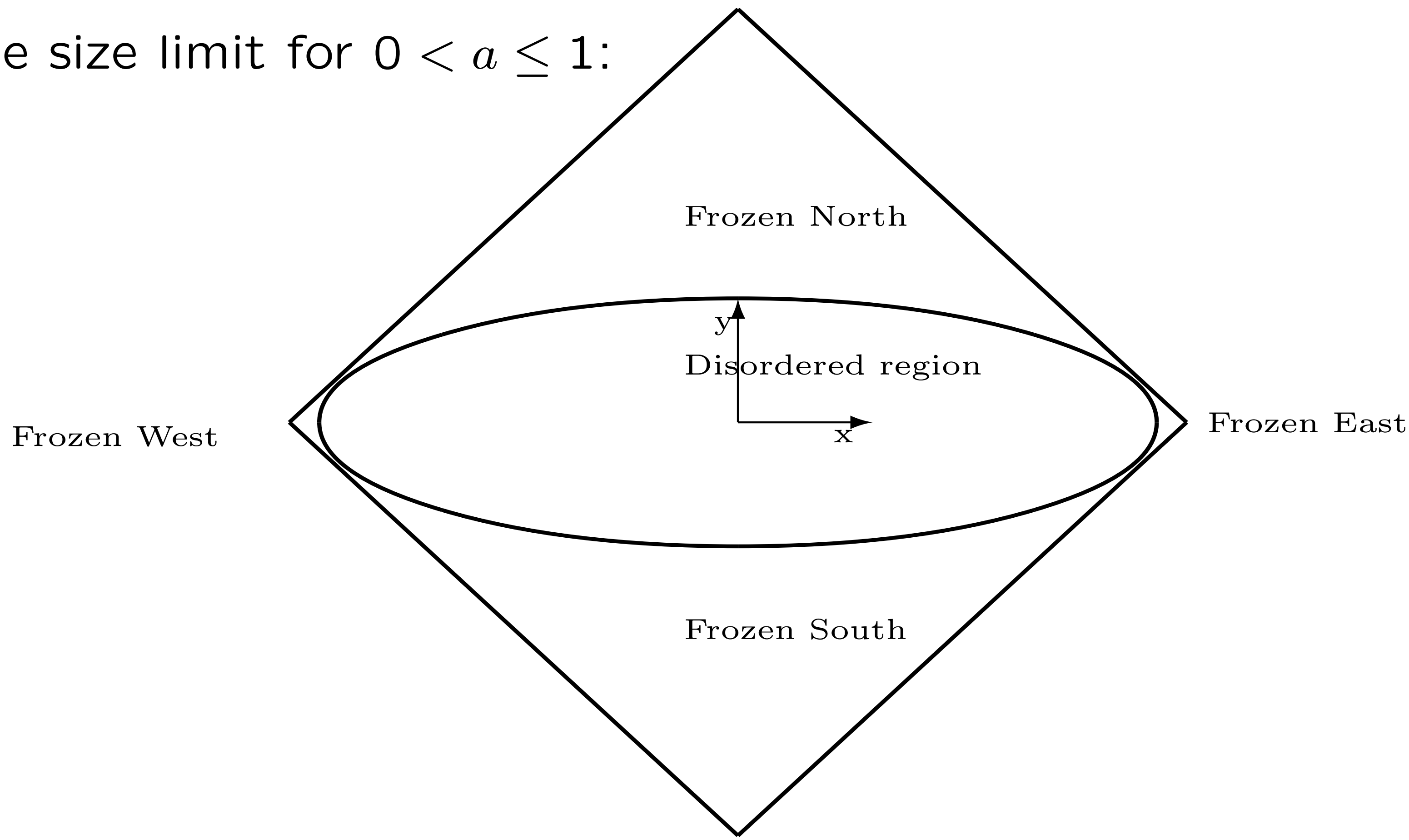
$$\mathbb{L}_n^{\text{oneAzt}}(\xi_1,\eta_1;\xi_2,\eta_2) = -\mathbb{1}_{(\xi_1<\xi_2)}\int_{\Gamma_{0,a}}\frac{dw}{2\pi \mathrm{i}}\;\frac{(1+aw)^{\frac{\eta_1-\eta_2}{2}-1}}{(w-a)^{\frac{\eta_1-\eta_2}{2}+1}}w^{\frac{\xi_1-\xi_2}{2}}$$

$$+\int_{\Gamma_{0,a}}\frac{dz}{(2\pi \mathrm{i})^2}\int_{\Gamma_{0,a,z}}\frac{dw}{w-z}\;\;\frac{(1+az)^{\frac{\eta_1-1}{2}}(z-a)^{n-\frac{\eta_1+1}{2}}}{(1+aw)^{\frac{\eta_2+1}{2}}(w-a)^{n-\frac{\eta_2-1}{2}}}\;\;\frac{w^{n-\frac{\xi_2}{2}}}{z^{n-\frac{\xi_1}{2}}}$$

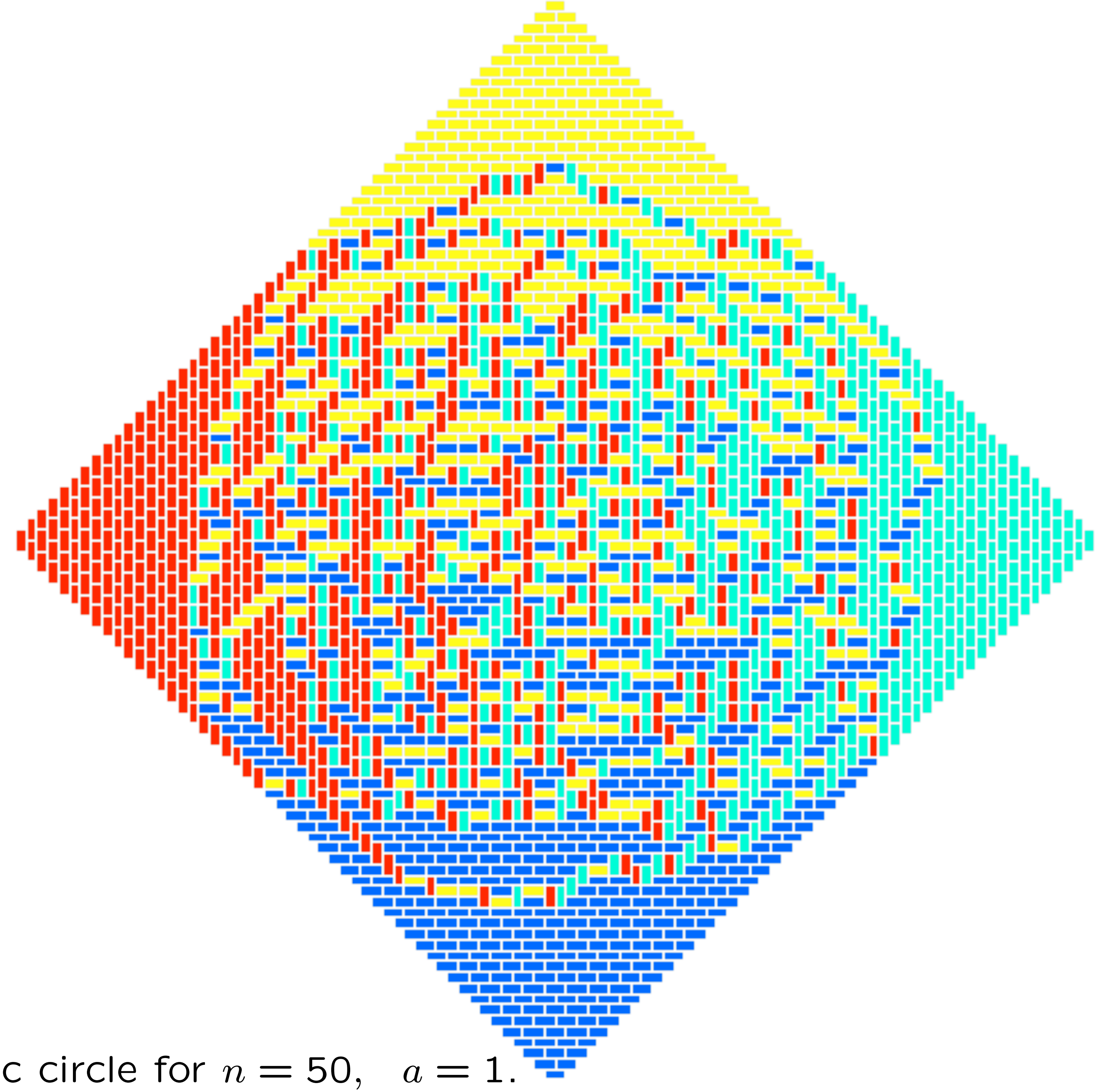
Johansson 03, 05

Letting the size $n \to \infty$:

In the large size limit for $0 < a \leq 1$:



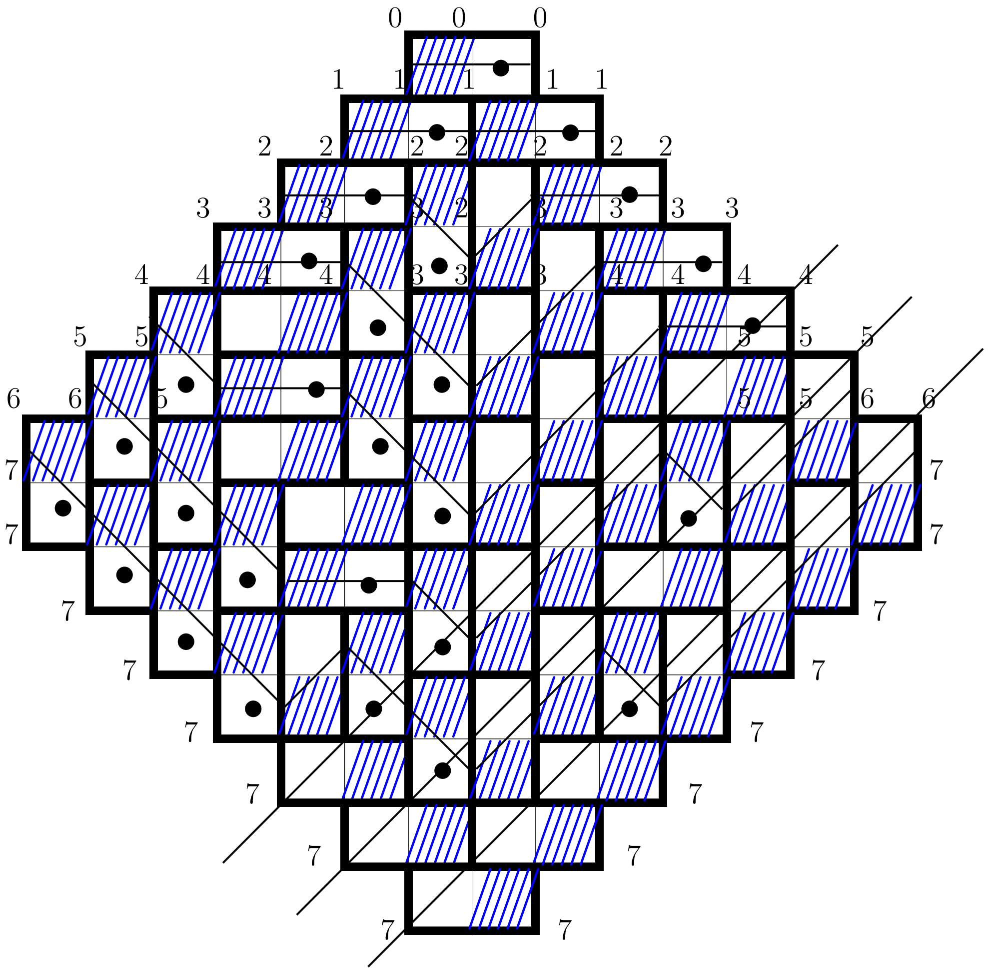
(horizontal domino's are more likely than vertical ones for $0 < a < 1$)



Arctic circle for $n = 50, \quad a = 1.$

Determinantal point process for single Aztec diamond:

$$\left\{\begin{matrix}\text{Interlacing dots on the}\\ \text{oblique lines for } n\to\infty\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{good model for the interlacing spectra}\\ \text{of consecutive minors of GUE-matrices}\end{matrix}\right\}$$



"GUE-minor kernel" $(n \to \infty)$ (Johansson-Nordenstam '06)

Scaling of the coordinates $\eta_i \in \mathbb{Z}$, $\Longrightarrow$ coordinates $x,\ y \in \mathbb{R}$: (Set $a = 1$)

$$\eta_1 = n/2 + x\sqrt{n/2}$$
$$\eta_2 = n/2 + y\sqrt{n/2}$$

LIMITING KERNEL: GUE-minor-Kernel

$$\lim_{n\to\infty} (\sqrt{n/2})\mathbb{L}_n^{\text{oneAztec}}(2r,\eta_1;2s,\eta_2) \ = \mathbb{K}^{\text{GUE-minor}}(r,x;s,y).$$

with

$$\tfrac{1}{2}\text{GUE-minor kernel} = \tfrac{1}{2}\mathbb{K}^{\text{GUE-minor}}(r,x;s,x')$$

$$:= -\mathbb{1}_{r>s}\frac{(2(x-x'))^{r-s-1}}{(r-s-1)!}\mathbb{1}_{x\geq x'} + \int_\Gamma \frac{dz}{(2\pi \mathrm{i})^2} \quad \int_L \frac{dw}{w-z} \quad \frac{e^{-z^2+2zx}}{e^{-w^2+2wx'}}\frac{w^s}{z^r}$$

where $\begin{cases} L := 0^+ + i\mathbb{R}\uparrow \text{ to the right of} \\ \Gamma := \text{circle about } 0 \end{cases}$

Scaling of the coordinates $\eta_i \in \mathbb{Z}$, $\Longrightarrow$ coordinates $x,\ y \in \mathbb{R}$: (Set $a = 1$)

$$\eta_1 = n/2 + x\sqrt{n/2}$$
$$\eta_2 = n/2 + y\sqrt{n/2}$$

LIMITING KERNEL: GUE-minor-Kernel

$$\lim_{n\to\infty} (\sqrt{n/2})\mathbb{L}_n^{\text{oneAztec}}(2r, \eta_1; 2s, \eta_2) = \mathbb{K}^{\text{GUE-minor}}(r, x; s, y).$$

with

$$\tfrac{1}{2}\text{GUE-minor kernel} = \tfrac{1}{2}\mathbb{K}^{\text{GUE-minor}}(r, x; s, x')$$

$$:= -\mathbb{1}_{r>s}\frac{(2(x-x'))^{r-s-1}}{(r-s-1)!}\mathbb{1}_{x\geq x'} + \int_\Gamma \frac{dz}{(2\pi \mathrm{i})^2}\int_L \frac{dw}{w-z}\frac{e^{-z^2+2zx}}{e^{-w^2+2wx'}}\frac{w^s}{z^r}$$

Note:

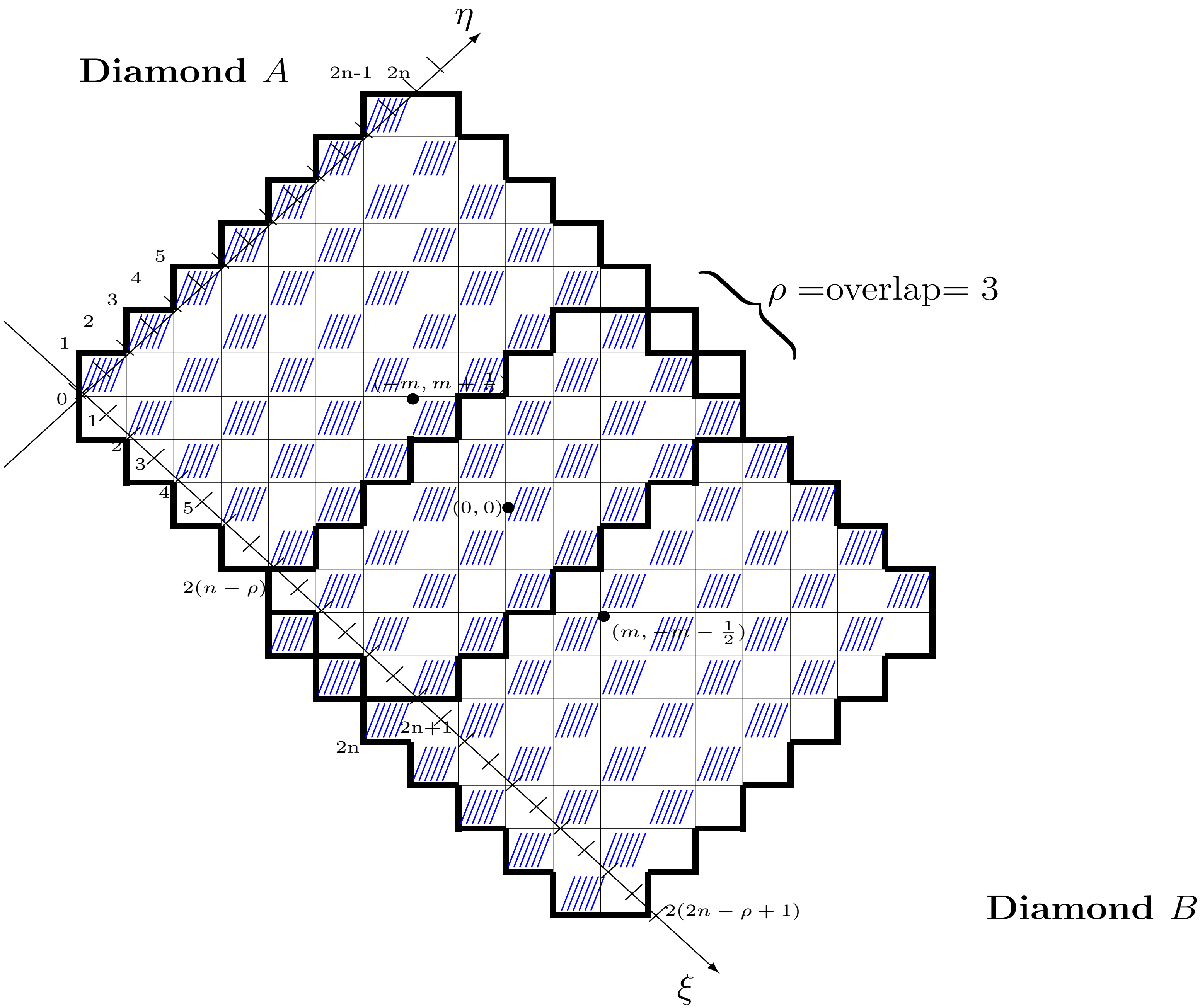
$$\tfrac{1}{2}\mathbb{K}^{\text{GUE-minor}}(r, x; r, x') = \int_\Gamma \frac{dz}{(2\pi \mathrm{i})^2}\int_L \frac{dw}{w-z}\frac{e^{-z^2+2zx}}{e^{-w^2+2wx'}}\frac{w^r}{z^r} = \begin{cases} \text{kernel for} \\ r \times r\text{-GUE} \end{cases}$$

II. Domino tiling of a double Aztec diamond

Adler, Johansson & PvM: *Double Aztec Diamonds and the Tacnode Process, 2011.* (arXiv:1112.5532)

Mark Adler, Sunil Chhita, Kurt Johansson & PvM: *Tacnode GUE-minor Processes and Double Aztec Diamonds*, 2013. (arXiv:1303.5279)

Mark Adler & PvM: *The Double GUE-minor process*, 2013.

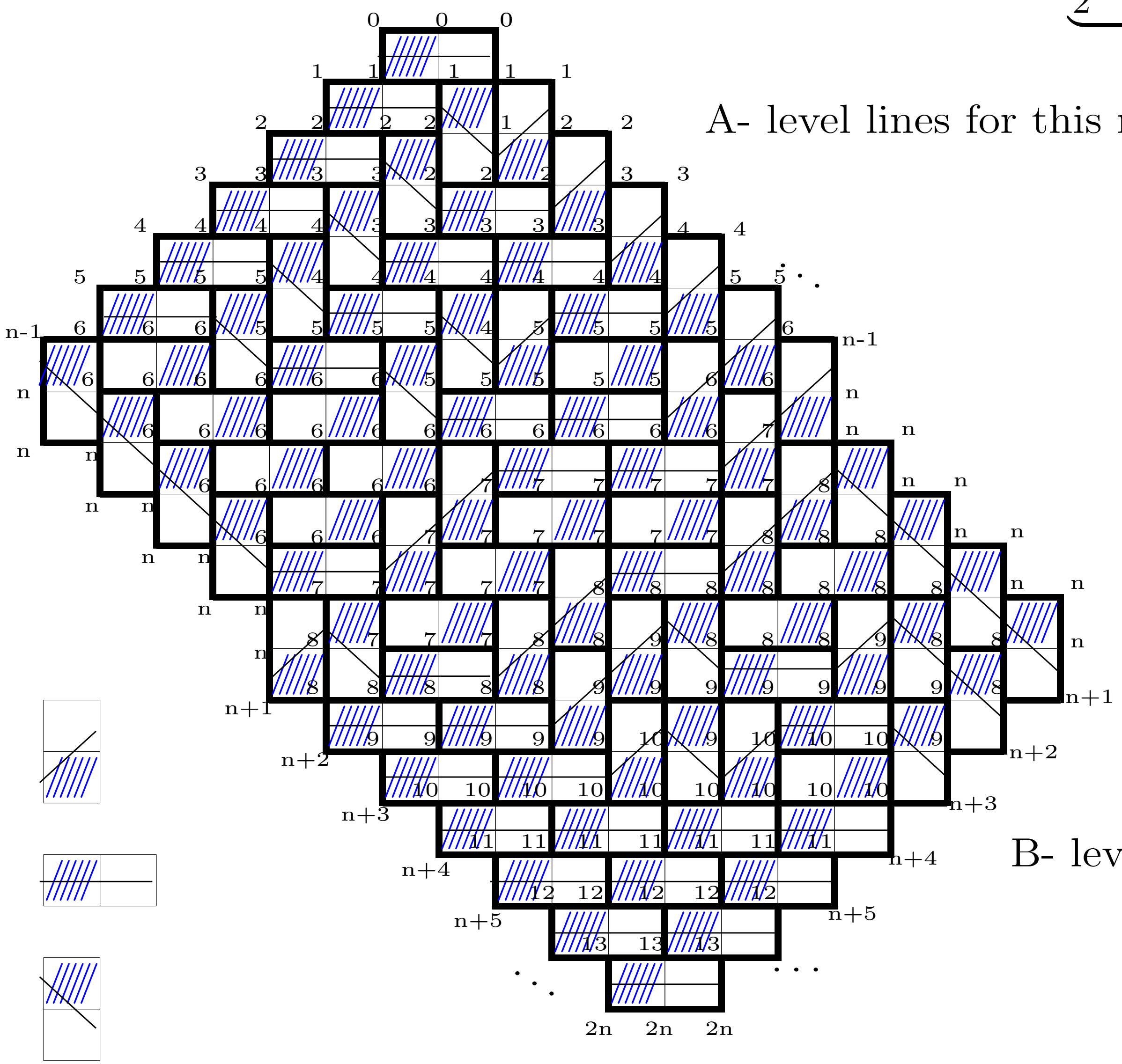


Double Aztec diamond of type $n = 7$, with overlap $=\rho = 3$
Coordinate system: (ξ, η)

Random cover with domino's: two groups of non-intersecting paths

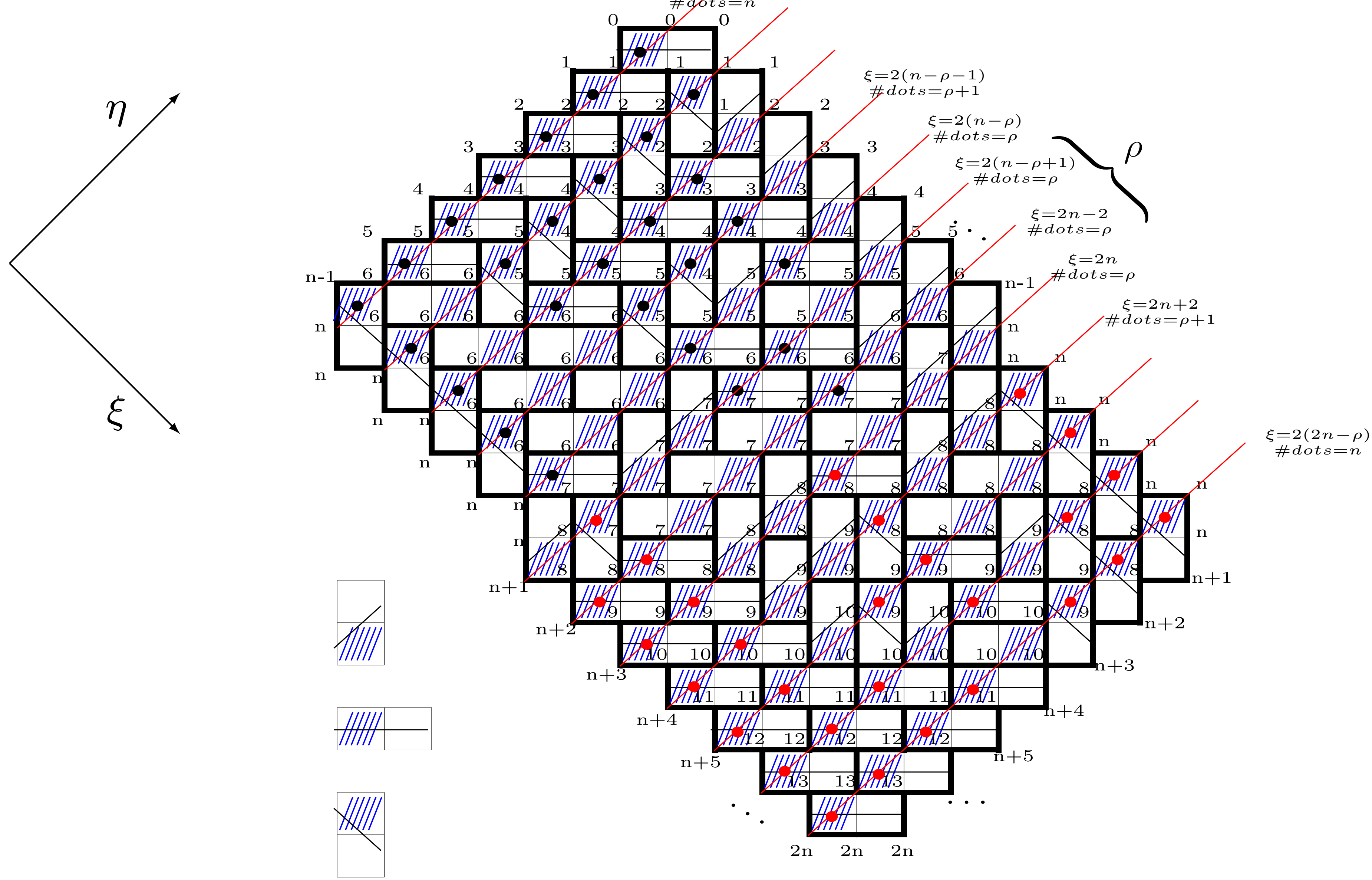
level lines corresponding to height $\underbrace{\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ldots,n-\frac{1}{2}}_{A-\text{level lines}},\ \underbrace{n+\frac{1}{2},\ldots,\ 2n-\frac{1}{2}}_{B-\text{level lines}}$

A- level lines for this random surface



B- level lines for this random surface

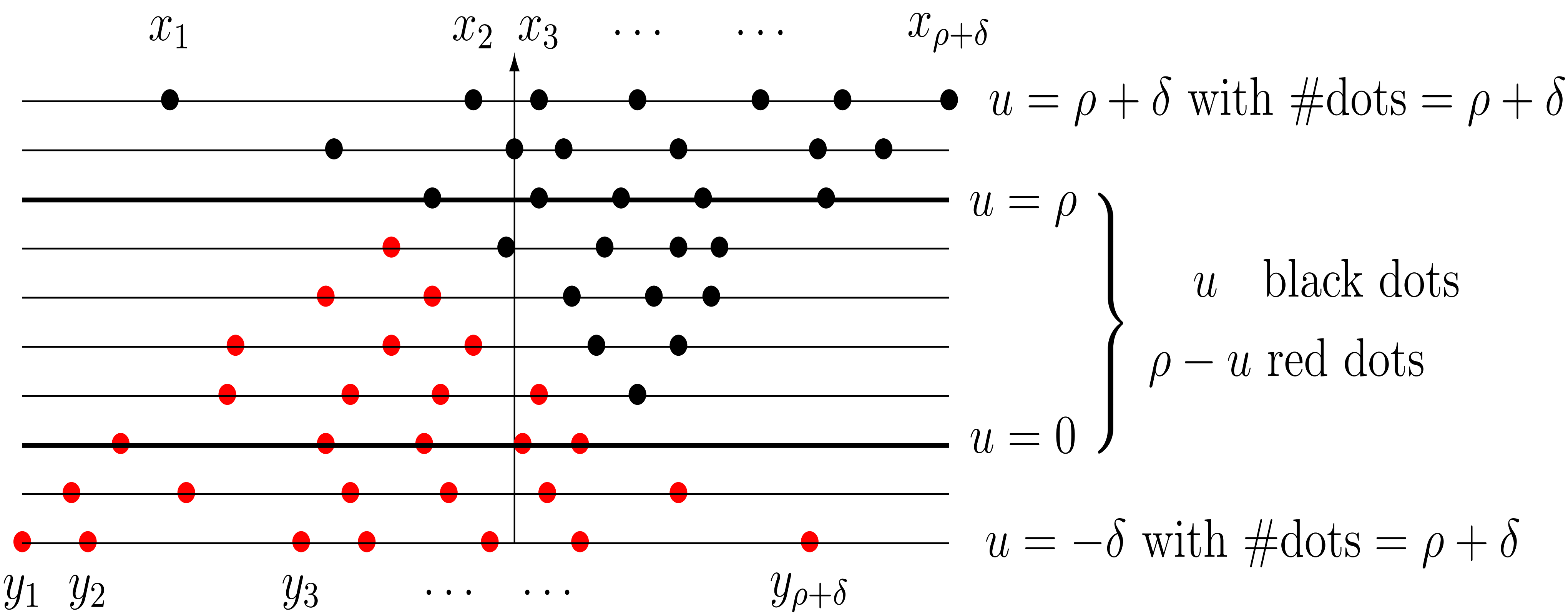
Determinantal Processes



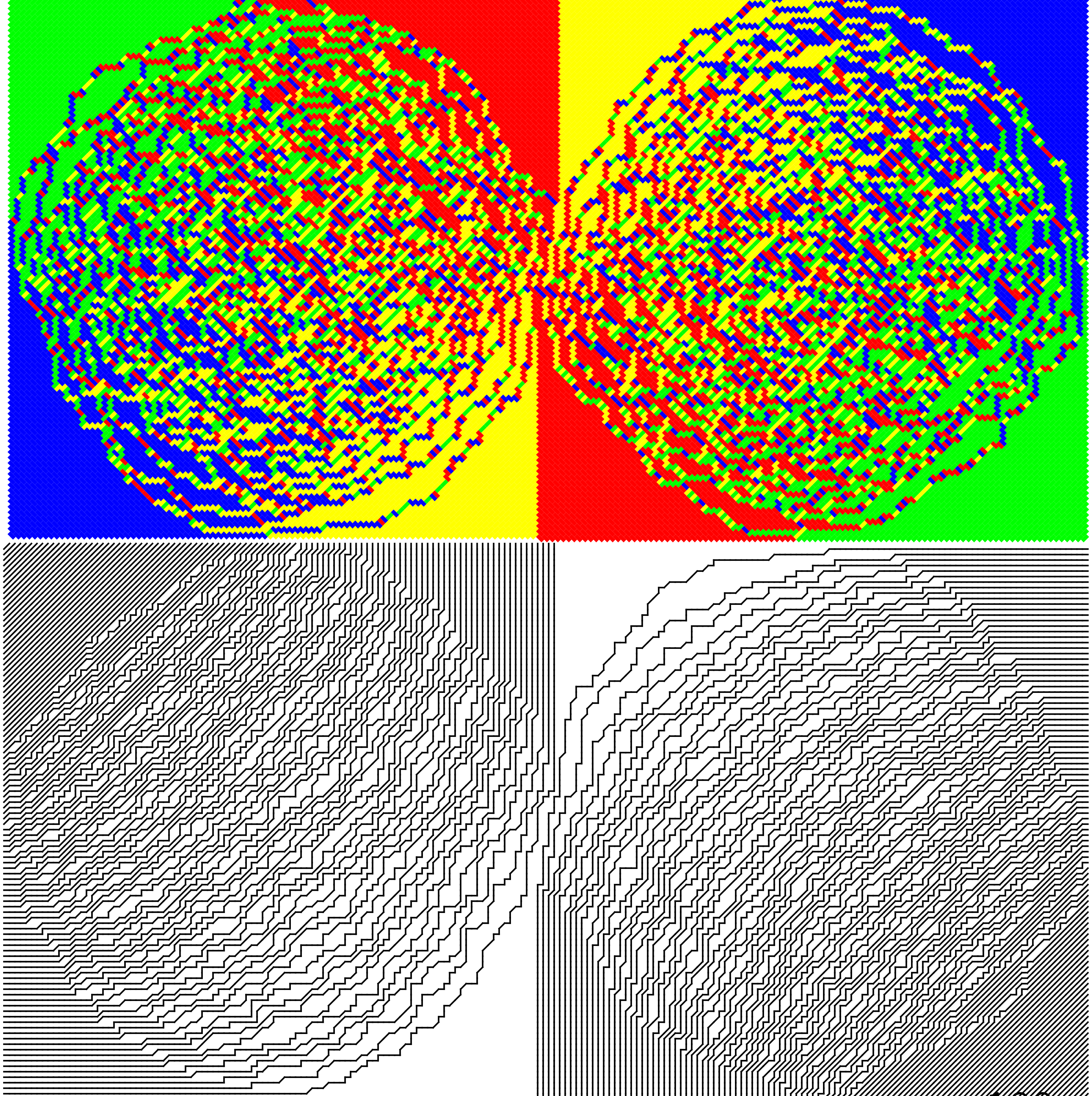
Each time a line $\xi = 2k$ intersects an

$$\left.\begin{array}{l}\text{(1) } A\text{-level line: put a black dot in that square}\\ \text{(2) } B\text{-level line: put a red dot in that square}\end{array}\right\}\ \textit{interlacing}!$$

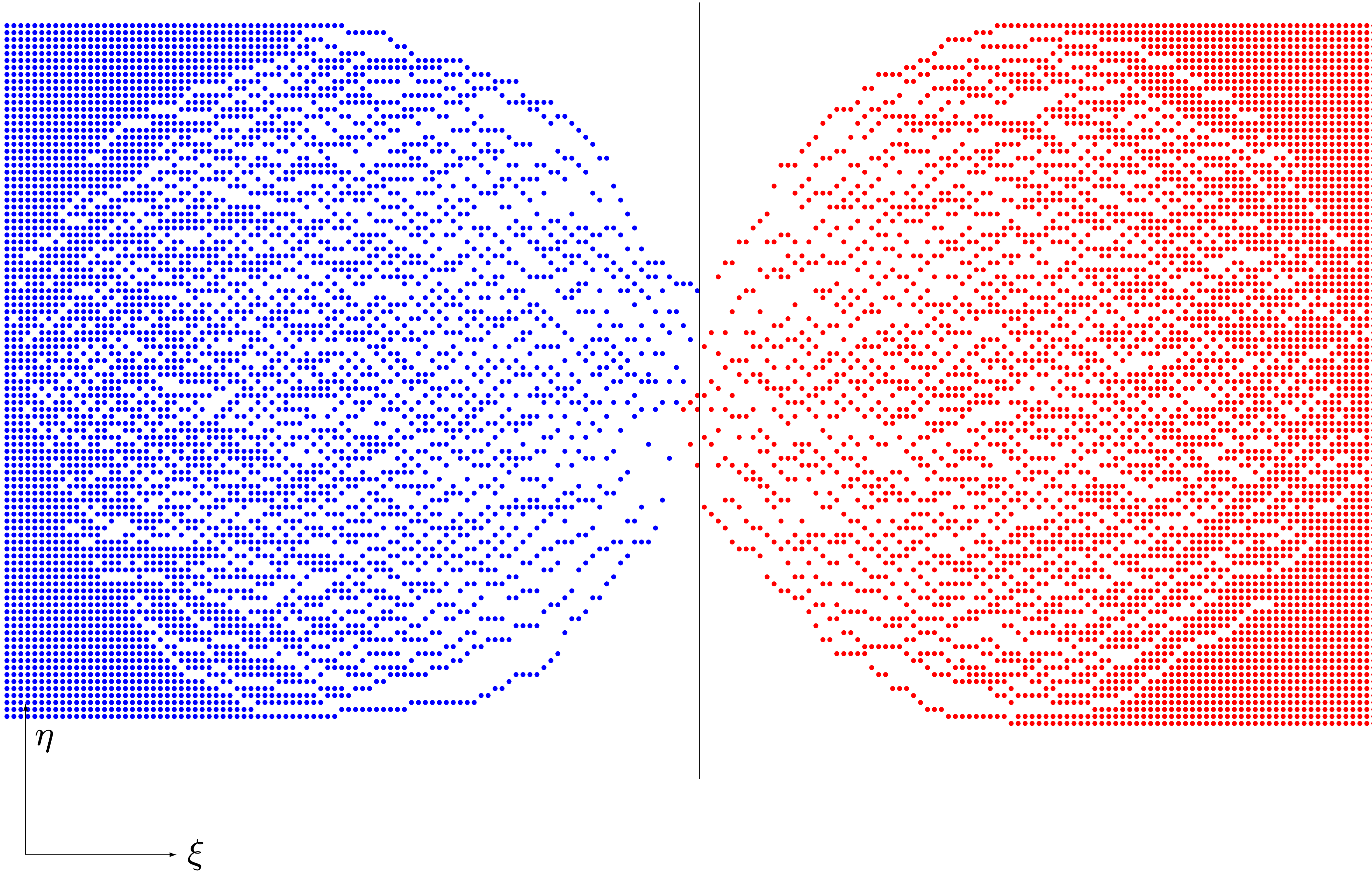
The (discrete) interlacing can schematically be represented by the following picture: (coordinates $2u = 2n - \xi$)



Any such set of (discrete) interlacing black and red dots, respecting the boundary, corresponds to a tiling with domino's !



$n = 100\ ,\ \rho = 4,\ a = 1$



$n = 100$, $\rho = 4$, $a = 1$

Determinantal point process with kernel $\mathbb{L}_{n,\rho}^{\text{doubleAzt}}(\xi_1,\eta_1;\xi_2,\eta_2)$:

$$\frac{1}{1+a^2}\mathbb{L}_{n,\rho}^{\text{doubleAzt}}(\xi_1,\eta_1;\xi_2,\eta_2) = \mathbb{L}_{n}^{\text{oneAzt}}(\xi_1,\eta_1;\xi_2,\eta_2)$$

$$-\left\langle ((I-K_0)^{-1}A_{\xi_1,\eta_1})(k), B_{\xi_2,\eta_2}(k)\right\rangle_{\geq n-\rho+1},$$

Via the Kasteleyn matrix !

Determinantal point process with kernel $\mathbb{L}_{n,\rho}^{\mathrm{doubleAzt}}(\xi_1,\eta_1;\xi_2,\eta_2)$:

$$\begin{aligned}\frac{1}{1+a^2}\mathbb{L}_{n,\rho}^{\mathrm{doubleAzt}}(\xi_1,\eta_1;\xi_2,\eta_2) =&\mathbb{L}_{n}^{\mathrm{oneAzt}}(\xi_1,\eta_1;\xi_2,\eta_2)\\ &-\left\langle ((I-K_0)^{-1}A_{\xi_1,\eta_1})(k), B_{\xi_2,\eta_2}(k)\right\rangle_{\geq n-\rho+1},\end{aligned}$$

with

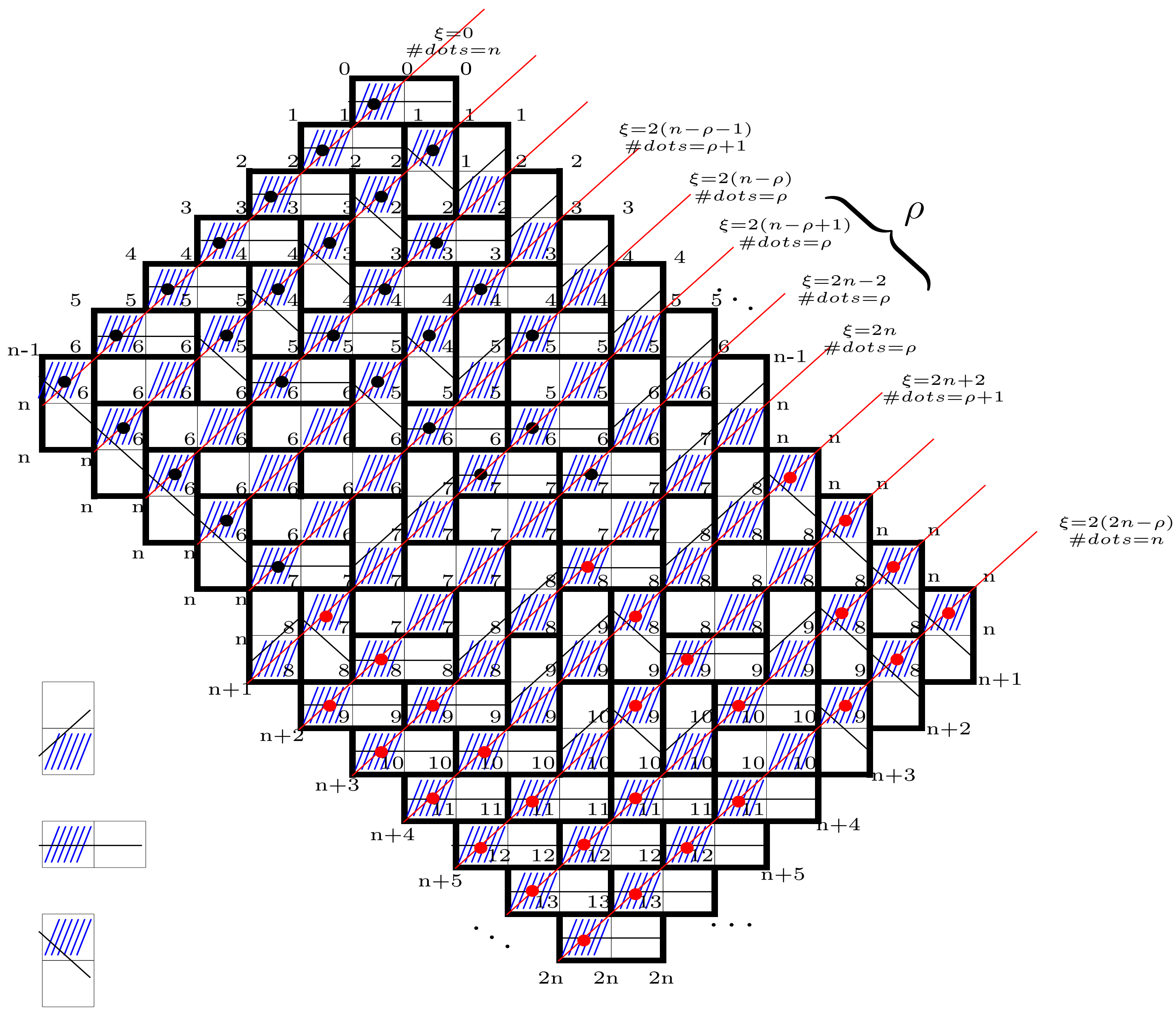
$$K_0(j,k)=\int_{\Gamma_{0,a}}\frac{dw}{(2\pi \mathrm{i})^2}\int_{\Gamma_{0,a,w}}\frac{dz}{z-w}\ \frac{z^{n-j}}{w^{n+1-k}}\frac{\psi_n(w)}{\psi_n(z)},$$

$$A_{\xi_1,\eta_1}(j)=\int_{\Gamma_{0,a}}\frac{dw}{(2\pi \mathrm{i})^2}\int_{\Gamma_{0,a,w}}\frac{dz}{w-z}\ \frac{w^{n-j}}{z^{n-\frac{\xi_1}{2}}}\ \frac{(1+az)^{\frac{\eta_1-1}{2}}(z-a)^{n-\frac{\eta_1+1}{2}}}{\psi_n(w)}$$

$$B_{\xi_2,\eta_2}(k)=\int_{\Gamma_{0,a}}\frac{dz}{(2\pi \mathrm{i})^2}\int_{\Gamma_{0,a,z}}\frac{dw}{w-z}\ \frac{z^{n-\xi_2/2}}{w^{n+1-k}}\ \frac{\psi_n(w)}{(1+az)^{\frac{\eta_2+1}{2}}(z-a)^{n-\frac{\eta_2-1}{2}}}$$

$$\psi_n(w)=(1+aw)^n(w-a)^{n+1}$$

III. LETTING THE SIZE $n \to \infty$, keeping the overlap finite



$\xi=0$
$\#dots=n$
$\xi=2(n-\rho-1)$
$\#dots=\rho+1$
$\xi=2(n-\rho)$
$\#dots=\rho$
$\xi=2(n-\rho+1)$
$\#dots=\rho$
$\xi=2n-2$
$\#dots=\rho$
$\xi=2n$
$\#dots=\rho$
$\xi=2n+2$
$\#dots=\rho+1$
$\xi=2(2n-\rho)$
$\#dots=n$
ρ

Double GUE-minor kernel, as a scaling limit of the double-Aztec kernel

Take scaling limit:

$$\begin{cases} \text{size } n \to \infty, \\ \rho = \text{fixed overlap of two diamonds}, \\ a = 1 - \beta\sqrt{\dfrac{2}{n}} = \text{weight of vertical domino's}, \quad \text{with } \beta \geq 0. \end{cases}$$

Scaling of the coordinates (ξ, η), $\Longrightarrow$ coordinates $(u, y) \in \mathbb{Z} \times \mathbb{R}$:

$$\xi_i = 2n - 2u_i, \qquad \eta = n + [y\sqrt{2n}] - 1.$$

LIMITING KERNEL: Double GUE-Kernel

$$\lim_{n\to\infty} (-a)^{\frac{1}{2}(\eta_1-\eta_2)}(-\sqrt{n/2})^{\frac{1}{2}(\xi_1-\xi_2)}\mathbb{L}^{\text{doubleAzt}}_{n,\rho}(\xi_1,\eta_1;\xi_2,\eta_2)\frac{\Delta\eta_2}{2}$$
$$= \mathbb{K}^{\text{dGUE}}_{\beta,\rho}(u_1,y_1;u_2,y_2)dy_2.$$

Double GUE-minor kernel, two parameters: $\begin{cases} \beta = \text{speed at which } \; a \to 1 \\ \rho = \text{overlap of two diamonds} \end{cases}$

$$\begin{aligned} \tfrac{1}{2}\mathbb{K}^{\mathrm{dGUE}}_{\beta,\rho}(u_1,y_1;u_2,y_2) &= \tfrac{1}{2}\mathbb{K}^{\mathrm{GUEminor}}(u_1,\beta-y_1;\; u_2,\beta-y_2) \\ &\quad + \left\langle (\mathbb{1}-\mathcal{K}^{\beta}(\lambda,\kappa))^{-1}\; \mathcal{A}^{\beta,y_1-\beta}_{u_1}(\kappa), \mathcal{B}^{\beta,y_2-\beta}_{u_2}(\lambda) \right\rangle_{\geq -\rho} \end{aligned}$$

with $u_i \in \mathbb{Z}, \qquad y_i \in \mathbb{R}, \qquad \lambda, \mu \in \mathbb{Z}$

Double GUE-minor kernel, two parameters: $\begin{cases} \beta = \text{speed at which } \; a \to 1 \\ \rho = \text{overlap of two diamonds} \end{cases}$

$$\tfrac{1}{2}\mathbb{K}^{\text{dGUE}}_{\beta,\rho}(u_1,y_1;u_2,y_2) = \tfrac{1}{2}\mathbb{K}^{\text{GUEminor}}(u_1,\beta-y_1;\; u_2,\beta-y_2)$$

$$+\left\langle (\mathbb{1}-\mathcal{K}^{\beta})^{-1}\; \mathcal{A}^{\beta,y_1-\beta}_{u_1}(\lambda), \mathcal{B}^{\beta,y_2-\beta}_{u_2}(\lambda)\right\rangle_{\geq -\rho}$$

$$\uparrow$$

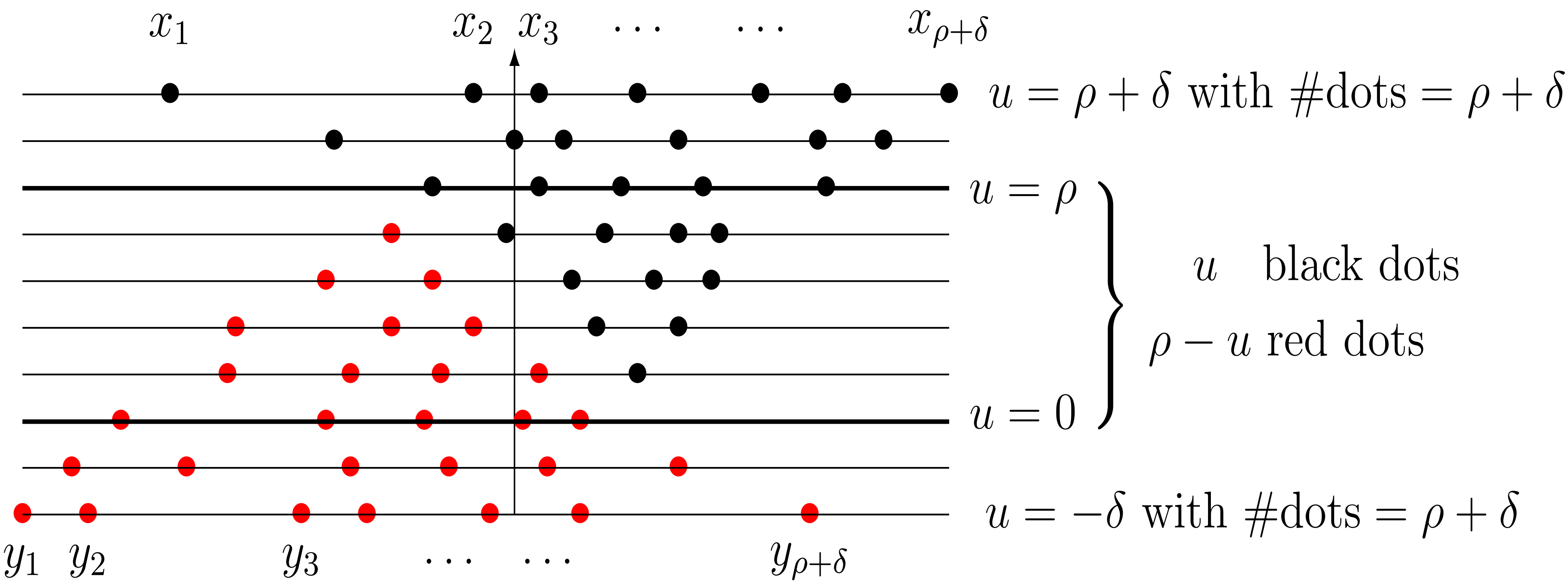
(Finite rank perturbation of GUE-minor kernel!)

$$\mathcal{K}^{(\beta)}(\lambda,\kappa) := \oint_{\Gamma}\frac{d\zeta}{(2\pi \mathrm{i})^2}\int_L \frac{d\omega}{\omega-\zeta}\;\frac{e^{-2\zeta^2+4\beta\zeta}}{e^{-2\omega^2+4\beta\omega}}\frac{\zeta^{\kappa}}{\omega^{\lambda+1}}, \qquad \lambda,\kappa\in\mathbb{Z}$$

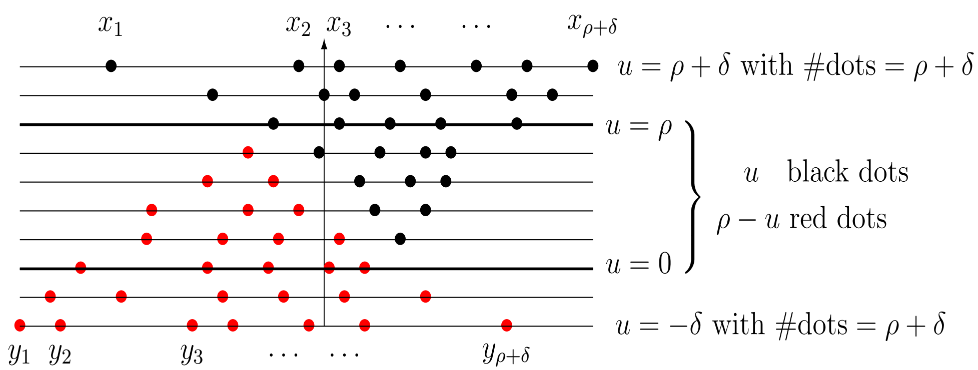
$$\mathcal{A}^{\beta,y}_{v}(\kappa) := \oint_{\Gamma}\frac{d\zeta}{(2\pi \mathrm{i})^2}\int_L \frac{d\omega}{\zeta-\omega}\;\frac{e^{-\zeta^2-2y\zeta}}{e^{-2\omega^2+4\beta\omega}}\frac{\zeta^{-v}}{\omega^{\kappa+1}} + \int_L \frac{d\zeta}{2\pi \mathrm{i}}\frac{e^{\zeta^2-2\zeta(y+2\beta)}}{\zeta^{v+\kappa+1}}$$

$$\mathcal{B}^{\beta,y}_{u}(\lambda) := \oint_{\Gamma}\frac{d\zeta}{(2\pi \mathrm{i})^2}\int_L \frac{d\omega}{\zeta-\omega}\;\frac{e^{-2\zeta^2+4\zeta\beta}}{e^{-\omega^2-2\omega y}}\frac{\zeta^{\lambda}}{\omega^{-u}} + \oint_{\Gamma}\frac{d\omega}{2\pi \mathrm{i}}\;\frac{\omega^{u+\lambda}}{e^{\omega^2-2\omega(y+2\beta)}}.$$

where $\begin{cases} L := 0^+ + i\mathbb{R}\uparrow \\ \Gamma := \text{circle about } 0 \end{cases}$



In the limit, when size $\to \infty$: *Continuous interlacing* of red and blue dots for $\rho = 5$ and $\delta = 2$.



Level $u = -\delta$ at a distance $-\delta$ below the overlap:

density of particles is given by: $(\mathbf{y} = (y_1, \dots, y_{\rho+\delta}))$

$$\det\left(\mathbb{K}^{\mathrm{dGUE}}_{\beta,\rho}(-\delta, y_i; -\delta, y_j)\right)_{1\le i,j\le \rho+\delta} = C_{\rho,\delta}\Delta_{\rho+\delta}(\mathbf{y})\widetilde{\Delta}_{\rho+\delta}(\mathbf{y})\prod_{1}^{\rho+\delta}\frac{e^{-(y_i+\beta)^2}}{\sqrt{\pi}}$$

where

$$\Delta_{\rho+\delta}(\mathbf{y}) = \det\begin{pmatrix} 1 \\ y \\ \vdots \\ y^{\delta-1} \\ y^{\delta} \\ \vdots \\ y^{\delta+\rho-1} \end{pmatrix}_{y=y_1,\dots,y_{\rho+\delta}} \qquad \widetilde{\Delta}_{\rho+\delta}(\mathbf{y}) := \det\begin{pmatrix} 1 \\ y \\ \vdots \\ y^{\delta-1} \\ \Phi_{\delta}(\beta - y) \\ \vdots \\ \Phi_{\delta+\rho-1}(\beta - y) \end{pmatrix}_{y=y_1,\dots,y_{\rho+\delta}}$$

Define for $n \geq 0$

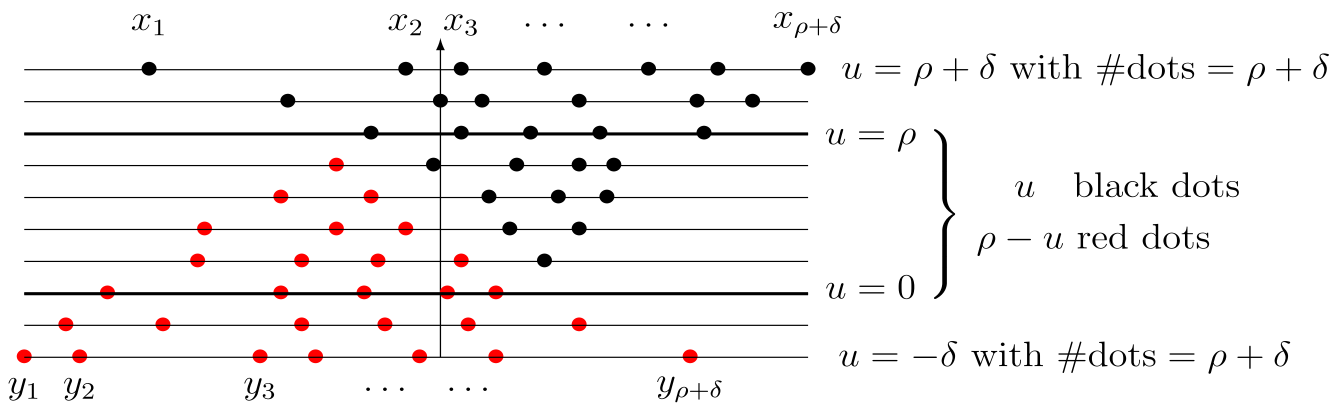
$$\Phi_n(y) := P_n(y) \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{y} e^{-\xi^2} d\xi \right) + Q_{n-1}(y) \left(\frac{1}{\sqrt{\pi}} e^{-y^2} \right)$$

with

$$P_n(y) := \frac{1}{n! \; \mathrm{i}^n} H_n(\mathrm{i}y),$$

$$Q_n(y) := \text{polynomials, defined by recursion relation}$$

$$\tfrac{1}{2}(n+1) Q_n(y) = y Q_{n-1}(y) + Q_{n-2}(y) \text{ with } Q_0 = 1 \text{ and } Q_{-1} = 0$$

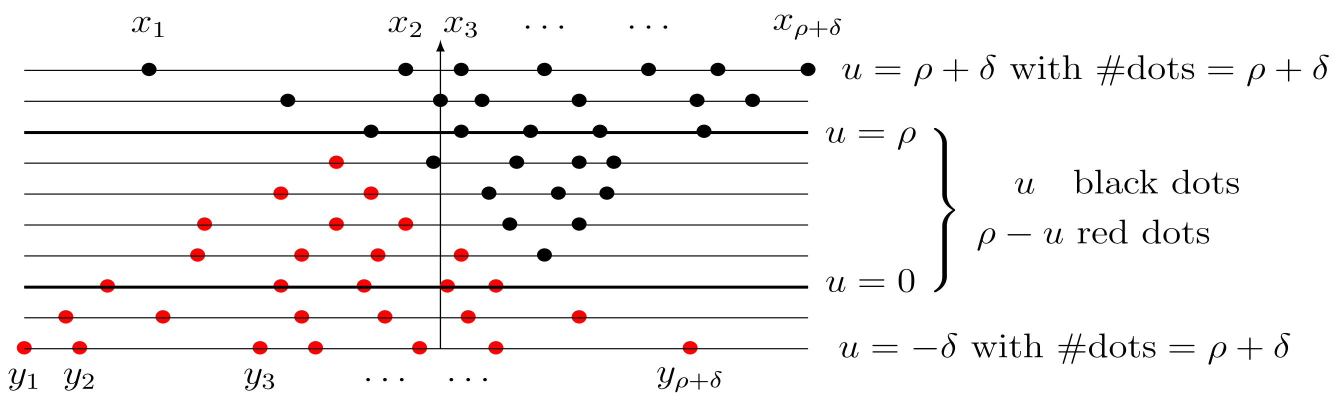


At levels $-\delta$ and $\rho+\delta$ (below and above the overlap) :

joint density of particles at levels is given by:

$$\mathbb{P}\left(\bigcap_{1}^{\rho+\delta}\{x_i \in dx_i,\ \ y_i \in dy_i\}\right) := dxdy$$

$$\times \det\left(\begin{array}{c|c} \left(\frac{1}{2}\mathbb{K}^{\mathrm{dGUE}}_{\beta,\rho}(-\delta, y_i; -\delta, y_j)\right)_{1\le i,j\le \rho+\delta} & \left(\frac{1}{2}\mathbb{K}^{\mathrm{dGUE}}_{\beta,\rho}(-\delta, y_i; \rho+\delta, x_j)\right)_{1\le i,j\le \rho+\delta} \\ \hline \left(\frac{1}{2}\mathbb{K}^{\mathrm{dGUE}}_{\beta,\rho}(\rho+\delta, x_i; -\delta, y_j)\right)_{1\le i,j\le \rho+\delta} & \left(\frac{1}{2}\mathbb{K}^{\mathrm{dGUE}}_{\beta,\rho}(\rho+\delta, x_i; \rho+\delta, x_j)\right)_{1\le i,j\le \rho+\delta} \end{array}\right)$$



•Two level density at levels $-\delta$ and $\rho + \delta$

$$\mathbb{P}\left(\bigcap_1^{\rho+\delta} \{x_i \in dx_i,\ y_i \in dy_i\}\right) = C\left(\Delta^2(\mathbf{x}) \prod_{i=1}^{\rho+\delta} e^{-(x_i-\beta)^2}\right)\left(\Delta^2(\mathbf{y}) \prod_{i=1}^{\rho+\delta} e^{-(y_i+\beta)^2}\right)$$

$$\times \frac{\mathrm{Vol}(\mathcal{C}(\mathbf{x},\mathbf{y}))}{\mathrm{Vol}(\mathcal{C}(\mathbf{x}))\mathrm{Vol}(\mathcal{C}(\mathbf{y}))} \prod_1^{\rho+\delta} dx_i dy_i$$

where $\mathrm{Vol}(\mathcal{C}(\mathbf{x})) = \text{Volume}(\text{cone about } \mathbf{x} = (x_1, \dots, x_{\rho+\delta}))$

$$\mathrm{Vol}(\mathcal{C}(\mathbf{x},\mathbf{y})) = \text{Volume}\begin{pmatrix} \text{“double cone”} \\ \text{about } (\mathbf{x},\mathbf{y}) \end{pmatrix}$$

$$= \int_{\mathbb{R}^{(\rho+\delta)(\rho+\delta-1)}} \mathbb{1}_{\mathbf{x} \prec \mathbf{z}^{(-\delta+1)} \prec \dots \prec \mathbf{z}^{\rho+\delta-1} \prec \mathbf{y}} (\mathbf{z})\ d\mathbf{z}$$

For real physics application of the generic points of the boundary between the frozen and the stochastic region, see:

K. A. Takeuchi & M. Sano: *Evidence for geometry-dependent universal fluctuations of the Kardar-Parisi-Zhang interfaces in liquid-crystal turbulence*, J. Stat. Phys. 147, 853-890 (2012).

http://dx.doi.org/10.1007/s10955-012-0503-0