W^*-superrigidity for left-right wreath products

Mihai Berbec

KU Leuven

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- Then $\{\lambda_g \mid g \in \Gamma\}$ is a self-adjoint subalgebra of $B(\ell^2(\Gamma))$.
- Define the group von Neumann algebra $\mathrm{L}\,\Gamma$ as the weak closure of $\{\lambda_g \mid g \in \Gamma\}$ in $B(\ell^2(\Gamma))$.

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$\rightsquigarrow$ The structure of group II_1 factors is largely non-understood.

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- Rigidity: $\mathrm{L}\,\Gamma$ remembers the group Γ.
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Soft groups - examples

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$\rightsquigarrow$Recall: Γ is amenable if the left regular representation of Γ has almost inviariant vectors. (finite groups, abelian groups, solvable groups, etc.)

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$\rightsquigarrow$ Recall: the wreath product $H \wr \Gamma$ is defined as $H \wr \Gamma = H^{(\Gamma)} \rtimes \Gamma$.

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- Ioana-Popa-Vaes '10: There exist infinitely many non-isomorphic groups Λ such that $\mathrm{L}\,\Lambda \cong \mathrm{L}(\mathbb{Z}/2\mathbb{Z} \wr \mathrm{PSL}(n, \mathbb{Z}))$ $(n \geq 2)$.

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$\rightsquigarrow$ Recall: Γ has Kazhdan's property (T) if the following holds: if π is a unitary representation of Γ that has almost invariant vectors, then π must have a nonzero invariant vector - opposite to amenability ! (e.g. finite groups, $\mathrm{SL}(n, \mathbb{Z})$ $(n \geq 3)$, (lattices in) $\mathrm{Sp}(n, 1)$ $(n \geq 2)$, simple Lie groups with higher rank)

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$\rightsquigarrow$ It suffices to consider icc and property (T) only for the group Γ. (Connes-Jones '85)

Definition (Ioana- Popa-Vaes '10)

The group Γ is called W*-superrigid if the following holds: whenever $\pi : \mathrm{L}\,\Lambda \to \mathrm{L}\,\Gamma$ is an isomorphism, for some group Λ, there exists a group isomorphism $\Lambda \to \Gamma$ that implements π.

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$\rightsquigarrow$ Ioana-Popa-Vaes '10: First examples of W*-superrigid groups !

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Theorem (IPV '10)

Let Γ_0 be a non-amenable group. Consider

$$\Gamma = \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{(I)} \rtimes (\Gamma_0 \wr \mathbb{Z}), \text{ where } I := (\Gamma_0 \wr \mathbb{Z})/\mathbb{Z} = \Gamma_0^{(\mathbb{Z})}.$$

Then Γ is W*-superrigid.

W*-superrigid groups

Theorem (B-Vaes '12)

Let Γ be one of the following groups:

- an icc hyperbolic group,
- a lattice in $SO(n, 1)$, $SU(n, 1)$ or $Sp(n, 1)$,
- a free product $\Gamma_1 * \Gamma_2$, with $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$.

Consider the left-right multiplication action of $\Gamma \times \Gamma$ on Γ and define the wreath product $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$.
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$\rightsquigarrow$ but it is not true, in general, for any non-amenable group (e.g. for $\Gamma = \mathbb{F}_2 \times S(\infty)$, $L\,\mathcal{G}$ is a McDuff factor).

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$\rightarrow$ define $\mathrm{L}^\infty(X, \mu) \rtimes \Gamma$ to be the weak closure of the linear span of

$$\{F u_g \mid g \in \Gamma, F \in \mathrm{L}^\infty(X, \mu)\}$$

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Assume now that $M \cong \mathrm{L}\,\Lambda$, for an arbitrary countable group Λ, and define the comultiplication associated to Λ by

$$\Delta_\Lambda : \mathrm{L}\,\Lambda \to \mathrm{L}\,\Lambda \overline{\otimes} \mathrm{L}\,\Lambda\,,\ \Delta(v_s) = v_s \otimes v_s,\ \text{for all } s \in \Lambda.$$

(here $(v_s)_{s\in\Lambda}$ denote the generating unitaries of $\mathrm{L}\,\Lambda$)

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Recall our setting:
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$\rightsquigarrow$ $\mathcal{G} = H^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$, where $H = \mathbb{Z}/2\mathbb{Z}$, $M := \mathrm{L}\,\mathcal{G}$
$\rightsquigarrow$ $M \cong A \rtimes (\Gamma \times \Gamma)$, where $A = \mathrm{L}^\infty(\widehat{H}^{(\Gamma)})$ is abelian

Assume now that $M \cong \mathrm{L}\,\Lambda$, for an arbitrary countable group Λ, and define the comultiplication associated to Λ by

$$\Delta_\Lambda : \mathrm{L}\,\Lambda \to \mathrm{L}\,\Lambda \overline{\otimes} \mathrm{L}\,\Lambda\,,\ \Delta(v_s) = v_s \otimes v_s,\ \text{for all } s \in \Lambda.$$

(here $(v_s)_{s\in\Lambda}$ denote the generating unitaries of $\mathrm{L}\,\Lambda$)

Similarly, we can define the comultiplication $\Delta_{\mathcal{G}} : M \to M \overline{\otimes} M$ associated to $\mathcal{G}$:

$$\Delta_{\mathcal{G}}(u_g) = u_g \otimes u_g,\ \text{for all } g \in \mathcal{G}.$$

Strategy of the proof

- We start with $M = \mathrm{L}\,\mathcal{G}$ and we associate an embedding $\Delta_\Lambda : M \to M \overline{\otimes} M$ to this "mysterious" group von Neumann algebra decomposition $M \cong \mathrm{L}\,\Lambda$.

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- We analyze how Δ_Λ relates with the original structure of $M = \mathrm{L}\,\mathcal{G}$ and with the similar embedding $\Delta_\mathcal{G}$. Ideally, we prove that Δ_Λ and $\Delta_\mathcal{G}$ are unitarily conjugate:

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$$\sigma(\Omega) = \Omega \ \ (\sigma = \text{ the flip }).$$

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which exactly says that $\Lambda \cong \mathcal{G}$ and this isomorphism implements the isomorphism $\mathrm{L}\,\Lambda \cong \mathrm{L}\,\mathcal{G}$.