L^2-Betti numbers for locally compact groups

David Kyed
KU Leuven

based on joint work with Stefaan Vaes & Henrik D. Petersen

DYGEST Corfu meeting 2013

May 2013

Aim of the talk

To discuss a notion of L^2-Betti numbers for locally compact groups and explain

- how it relates to the classical notion for discrete groups.
- some of the computational results obtained in this setting.
- why this theory is relevant even if you only care about discrete groups.

Aim of the talk

To discuss a notion of L^2-Betti numbers for locally compact groups and explain

- how it relates to the classical notion for discrete groups.
- some of the computational results obtained in this setting.
- why this theory is relevant even if you only care about discrete groups.

Aim of the talk

To discuss a notion of L^2-Betti numbers for locally compact groups and explain

- how it relates to the classical notion for discrete groups.
- some of the computational results obtained in this setting.
- why this theory is relevant even if you only care about discrete groups.

Aim of the talk

To discuss a notion of L^2-Betti numbers for locally compact groups and explain

- how it relates to the classical notion for discrete groups.
- some of the computational results obtained in this setting.
- why this theory is relevant even if you only care about discrete groups.

L^2-Betti numbers

Fix a discrete countable group $\Gamma \rightsquigarrow \mathbb{Z}, \mathrm{SL}(n, \mathbb{Z}), \mathbb{F}_2, \ldots.$

Associated with Γ we have:

- The Hilbert space $\ell^2(\Gamma) = \{x \colon \Gamma \to \mathbb{C} \mid \sum_\gamma |x(\gamma)|^2 < \infty\}$.
- The left regular representation $\lambda \colon \Gamma \to \mathbb{U}(\ell^2(\Gamma))$ which simply shifts the variable: $\lambda(\gamma)\delta_\mu = \delta_{\gamma\mu}$.
- The complex group algebra $\mathbb{C}[\Gamma] := \operatorname{span}_{\mathbb{C}}\{\lambda(\gamma) \mid \gamma \in \Gamma\}$.
- The group von Neumann algebra $L\Gamma = \mathbb{C}[\Gamma]''$; the closure of $\mathbb{C}[\Gamma]$ in the strong operator topology (a.k.a. the topology of pointwise convergence).
- The group cohomology $H^p(\Gamma, \ell^2(\Gamma))$.

The latter is dubbed the L^2-cohomology of Γ — what is this thing?

L^2-BETTI NUMBERS

Fix a discrete countable group $\Gamma \rightsquigarrow \mathbb{Z}, \mathrm{SL}(n,\mathbb{Z}), \mathbb{F}_2, \ldots$.
Associated with Γ we have:

- The Hilbert space $\ell^2(\Gamma) = \{x \colon \Gamma \to \mathbb{C} \mid \sum_\gamma |x(\gamma)|^2 < \infty\}$.
- The left regular representation $\lambda \colon \Gamma \to \mathbb{U}(\ell^2(\Gamma))$ which simply shifts the variable: $\lambda(\gamma)\delta_\mu = \delta_{\gamma\mu}$.
- The complex group algebra $\mathbb{C}[\Gamma] := \mathrm{span}_{\mathbb{C}}\{\lambda(\gamma) \mid \gamma \in \Gamma\}$.
- The group von Neumann algebra $L\Gamma = \mathbb{C}[\Gamma]''$; the closure of $\mathbb{C}[\Gamma]$ in the strong operator topology (a.k.a. the topology of pointwise convergence).
- The group cohomology $H^p(\Gamma, \ell^2(\Gamma))$.

The latter is dubbed the L^2-cohomology of Γ — what is this thing?

L^2-Betti numbers

Fix a discrete countable group $\Gamma \rightsquigarrow \mathbb{Z}, \mathrm{SL}(n,\mathbb{Z}), \mathbb{F}_2, \ldots.$
Associated with Γ we have:

- The Hilbert space $\ell^2(\Gamma) = \{x \colon \Gamma \to \mathbb{C} \mid \sum_\gamma |x(\gamma)|^2 < \infty\}$.
- The left regular representation $\lambda \colon \Gamma \to \mathbb{U}(\ell^2(\Gamma))$ which simply shifts the variable: $\lambda(\gamma)\delta_\mu = \delta_{\gamma\mu}$.
- The complex group algebra $\mathbb{C}[\Gamma] := \mathrm{span}_{\mathbb{C}}\{\lambda(\gamma) \mid \gamma \in \Gamma\}$.
- The group von Neumann algebra $L\Gamma = \mathbb{C}[\Gamma]''$; the closure of $\mathbb{C}[\Gamma]$ in the strong operator topology (a.k.a. the topology of pointwise convergence).
- The group cohomology $H^p(\Gamma, \ell^2(\Gamma))$.

The latter is dubbed the L^2-cohomology of Γ — what is this thing?

L^2-BETTI NUMBERS

Fix a discrete countable group $\Gamma \rightsquigarrow \mathbb{Z}, \mathrm{SL}(n,\mathbb{Z}), \mathbb{F}_2, \ldots$.
Associated with Γ we have:

- The Hilbert space $\ell^2(\Gamma) = \{x \colon \Gamma \to \mathbb{C} \mid \sum_\gamma |x(\gamma)|^2 < \infty\}$.
- The left regular representation $\lambda \colon \Gamma \to \mathbb{U}(\ell^2(\Gamma))$ which simply shifts the variable: $\lambda(\gamma)\delta_\mu = \delta_{\gamma\mu}$.
- The complex group algebra $\mathbb{C}[\Gamma] := \mathrm{span}_{\mathbb{C}}\{\lambda(\gamma) \mid \gamma \in \Gamma\}$.
- The group von Neumann algebra $L\Gamma = \mathbb{C}[\Gamma]''$; the closure of $\mathbb{C}[\Gamma]$ in the strong operator topology (a.k.a. the topology of pointwise convergence).
- The group cohomology $H^p(\Gamma, \ell^2(\Gamma))$.

The latter is dubbed the L^2-cohomology of Γ — what is this thing?

L^2-BETTI NUMBERS

Fix a discrete countable group $\Gamma \rightsquigarrow \mathbb{Z}, \mathrm{SL}(n,\mathbb{Z}), \mathbb{F}_2, \ldots.$
Associated with Γ we have:

- The Hilbert space $\ell^2(\Gamma) = \{x \colon \Gamma \to \mathbb{C} \mid \sum_\gamma |x(\gamma)|^2 < \infty\}$.
- The left regular representation $\lambda \colon \Gamma \to \mathbb{U}(\ell^2(\Gamma))$ which simply shifts the variable: $\lambda(\gamma)\delta_\mu = \delta_{\gamma\mu}$.
- The complex group algebra $\mathbb{C}[\Gamma] := \mathrm{span}_{\mathbb{C}}\{\lambda(\gamma) \mid \gamma \in \Gamma\}$.
- The group von Neumann algebra $L\Gamma = \mathbb{C}[\Gamma]''$; the closure of $\mathbb{C}[\Gamma]$ in the strong operator topology (a.k.a. the topology of pointwise convergence).
- The group cohomology $H^p(\Gamma, \ell^2(\Gamma))$.

The latter is dubbed the L^2-cohomology of Γ — what is this thing?

L^2-Betti numbers

Fix a discrete countable group $\Gamma \rightsquigarrow \mathbb{Z}, \mathrm{SL}(n, \mathbb{Z}), \mathbb{F}_2, \ldots$.
Associated with Γ we have:

- The Hilbert space $\ell^2(\Gamma) = \{x \colon \Gamma \to \mathbb{C} \mid \sum_\gamma |x(\gamma)|^2 < \infty\}$.
- The left regular representation $\lambda \colon \Gamma \to \mathbb{U}(\ell^2(\Gamma))$ which simply shifts the variable: $\lambda(\gamma)\delta_\mu = \delta_{\gamma\mu}$.
- The complex group algebra $\mathbb{C}[\Gamma] := \mathrm{span}_{\mathbb{C}}\{\lambda(\gamma) \mid \gamma \in \Gamma\}$.
- The group von Neumann algebra $L\Gamma = \mathbb{C}[\Gamma]''$; the closure of $\mathbb{C}[\Gamma]$ in the strong operator topology (a.k.a. the topology of pointwise convergence).
- The group cohomology $H^p(\Gamma, \ell^2(\Gamma))$.

The latter is dubbed the L^2-cohomology of Γ — what is this thing?

A quick cohomology reminder in degree one

- A 1-cocycle is a map $c\colon \Gamma \to \ell^2(\Gamma)$ satisfying $c(gh) = \lambda_g c(h) + c(g)$.

There is a trivial source of such cocycles, namely the so-called

- 1-coboundaries; i.e. the maps $g \mapsto \lambda_g \xi - \xi$ for some vector $\xi \in \ell^2(\Gamma)$.

The 1-cohomology is by definition the difference:

$$H^1(\Gamma, \ell^2(\Gamma)) := \frac{\{\text{1-cocycles}\}}{\{\text{1-coboundaries}\}}$$

As usual there are also higher cohomology groups

$$H^2(\Gamma, \ell^2(\Gamma)),\ H^3(\Gamma, \ell^2(\Gamma)),\ H^4(\Gamma, \ell^2(\Gamma)).....$$

We want to measure "the size" of $H^p(\Gamma, \ell^2(\Gamma))$; alas, this is in general infinite dimensional as a complex vector space.

A quick cohomology reminder in degree one

- A 1-cocycle is a map $c\colon \Gamma \to \ell^2(\Gamma)$ satisfying $c(gh) = \lambda_g c(h) + c(g)$.

There is a trivial source of such cocycles, namely the so-called

- 1-coboundaries; i.e. the maps $g \mapsto \lambda_g \xi - \xi$ for some vector $\xi \in \ell^2(\Gamma)$.

The 1-cohomology is by definition the difference:

$$H^1(\Gamma, \ell^2(\Gamma)) := \frac{\{\text{1-cocycles}\}}{\{\text{1-coboundaries}\}}$$

As usual there are also higher cohomology groups

$$H^2(\Gamma, \ell^2(\Gamma)),\ H^3(\Gamma, \ell^2(\Gamma)),\ H^4(\Gamma, \ell^2(\Gamma)).....$$

We want to measure "the size" of $H^p(\Gamma, \ell^2(\Gamma))$; alas, this is in general infinite dimensional as a complex vector space.

A quick cohomology reminder in degree one

- A 1-cocycle is a map $c\colon \Gamma \to \ell^2(\Gamma)$ satisfying $c(gh) = \lambda_g c(h) + c(g)$.

There is a trivial source of such cocycles, namely the so-called

- 1-coboundaries; i.e. the maps $g \mapsto \lambda_g \xi - \xi$ for some vector $\xi \in \ell^2(\Gamma)$.

The 1-cohomology is by definition the difference:

$$H^1(\Gamma, \ell^2(\Gamma)) := \frac{\{\text{1-cocycles}\}}{\{\text{1-coboundaries}\}}$$

As usual there are also higher cohomology groups

$$H^2(\Gamma, \ell^2(\Gamma)),\ H^3(\Gamma, \ell^2(\Gamma)),\ H^4(\Gamma, \ell^2(\Gamma)).....$$

We want to measure "the size" of $H^p(\Gamma, \ell^2(\Gamma))$; alas, this is in general infinite dimensional as a complex vector space.

A quick cohomology reminder in degree one

- A 1-cocycle is a map $c\colon \Gamma \to \ell^2(\Gamma)$ satisfying $c(gh) = \lambda_g c(h) + c(g)$.

There is a trivial source of such cocycles, namely the so-called

- 1-coboundaries; i.e. the maps $g \mapsto \lambda_g \xi - \xi$ for some vector $\xi \in \ell^2(\Gamma)$.

The 1-cohomology is by definition the difference:

$$H^1(\Gamma, \ell^2(\Gamma)) := \frac{\{\text{1-cocycles}\}}{\{\text{1-coboundaries}\}}$$

As usual there are also higher cohomology groups

$$H^2(\Gamma, \ell^2(\Gamma)),\ H^3(\Gamma, \ell^2(\Gamma)),\ H^4(\Gamma, \ell^2(\Gamma)).....$$

We want to measure “the size” of $H^p(\Gamma, \ell^2(\Gamma))$; alas, this is in general infinite dimensional as a complex vector space.

A quick cohomology reminder in degree one

- A 1-cocycle is a map $c\colon \Gamma \to \ell^2(\Gamma)$ satisfying $c(gh) = \lambda_g c(h) + c(g)$.

There is a trivial source of such cocycles, namely the so-called

- 1-coboundaries; i.e. the maps $g \mapsto \lambda_g\xi - \xi$ for some vector $\xi \in \ell^2(\Gamma)$.

The 1-cohomology is by definition the difference:

$$H^1(\Gamma, \ell^2(\Gamma)) := \frac{\{\text{1-cocycles}\}}{\{\text{1-coboundaries}\}}$$

As usual there are also higher cohomology groups

$$H^2(\Gamma, \ell^2(\Gamma)),\ H^3(\Gamma, \ell^2(\Gamma)),\ H^4(\Gamma, \ell^2(\Gamma)).....$$

We want to measure "the size" of $H^p(\Gamma, \ell^2(\Gamma))$; alas, this is in general infinite dimensional as a complex vector space.

A quick cohomology reminder in degree one

- A 1-cocycle is a map $c\colon \Gamma \to \ell^2(\Gamma)$ satisfying $c(gh) = \lambda_g c(h) + c(g)$.

There is a trivial source of such cocycles, namely the so-called

- 1-coboundaries; i.e. the maps $g \mapsto \lambda_g \xi - \xi$ for some vector $\xi \in \ell^2(\Gamma)$.

The 1-cohomology is by definition the difference:

$$H^1(\Gamma, \ell^2(\Gamma)) := \frac{\{\text{1-cocycles}\}}{\{\text{1-coboundaries}\}}$$

As usual there are also higher cohomology groups

$$H^2(\Gamma, \ell^2(\Gamma)),\ H^3(\Gamma, \ell^2(\Gamma)),\ H^4(\Gamma, \ell^2(\Gamma)).....$$

We want to measure “the size” of $H^p(\Gamma, \ell^2(\Gamma))$; alas, this is in general infinite dimensional as a complex vector space.

The von Neumann dimension

But, $H^p(\Gamma, \ell^2(\Gamma))$ is more than a vector space — it is also a right module for the von Neumann algebra $L\Gamma$ and as such is has a von Neumann dimension $\dim_{L\Gamma} H^p(\Gamma, \ell^2(\Gamma)) =: \beta_p^{(2)}(\Gamma)$. This is the p-th L^2-Betti number of Γ.

- Defining $\dim_{L\Gamma}(-)$ properly would lead us too far astray.
- But, if $K \subset \ell^2(\Gamma)$ is closed and right Γ-invariant then

$$\dim_{L\Gamma}(K) := \tau(P_K) := \langle P_K \delta_e | \delta_e \rangle \in [0, 1].$$

- L^2-Betti numbers turn out to be of great importance in geometry, topology, algebra, operator algebras....

The von Neumann dimension

But, $H^p(\Gamma, \ell^2(\Gamma))$ is more than a vector space — it is also a right module for the von Neumann algebra $L\Gamma$ and as such is has a von Neumann dimension $\dim_{L\Gamma} H^p(\Gamma, \ell^2(\Gamma)) =: \beta_p^{(2)}(\Gamma)$. This is the p-th L^2-Betti number of Γ.

- Defining $\dim_{L\Gamma}(-)$ properly would lead us too far astray.
- But, if $K \subset \ell^2(\Gamma)$ is closed and right Γ-invariant then

$$\dim_{L\Gamma}(K) := \tau(P_K) := \langle P_K \delta_e | \delta_e \rangle \in [0, 1].$$

- L^2-Betti numbers turn out to be of great importance in geometry, topology, algebra, operator algebras....

The von Neumann dimension

But, $H^p(\Gamma, \ell^2(\Gamma))$ is more than a vector space — it is also a right module for the von Neumann algebra $L\Gamma$ and as such is has a von Neumann dimension $\dim_{L\Gamma} H^p(\Gamma, \ell^2(\Gamma)) =: \beta_p^{(2)}(\Gamma)$. This is the p-th L^2-Betti number of Γ.

- Defining $\dim_{L\Gamma}(-)$ properly would lead us too far astray.
- But, if $K \subset \ell^2(\Gamma)$ is closed and right Γ-invariant then

$$\dim_{L\Gamma}(K) := \tau(P_K) := \langle P_K \delta_e | \delta_e \rangle \in [0,1].$$

- L^2-Betti numbers turn out to be of great importance in geometry, topology, algebra, operator algebras....

The von Neumann dimension

But, $H^p(\Gamma, \ell^2(\Gamma))$ is more than a vector space — it is also a right module for the von Neumann algebra $L\Gamma$ and as such is has a von Neumann dimension $\dim_{L\Gamma} H^p(\Gamma, \ell^2(\Gamma)) =: \beta_p^{(2)}(\Gamma)$. This is the p-th L^2-Betti number of Γ.

- Defining $\dim_{L\Gamma}(-)$ properly would lead us too far astray.
- But, if $K \subset \ell^2(\Gamma)$ is closed and right Γ-invariant then

$$\dim_{L\Gamma}(K) := \tau(P_K) := \langle P_K \delta_e | \delta_e \rangle \in [0, 1].$$

- L^2-Betti numbers turn out to be of great importance in geometry, topology, algebra, operator algebras....

An important theorem by one of the heroes of the field:

THEOREM (GABORIAU, 2000)

If Λ and Γ are lattices in a common locally compact group G then $\beta_p^{(2)}(\Gamma) = \frac{\text{covol}(\Gamma)}{\text{covol}(\Lambda)}\beta_p^{(2)}(\Lambda)$.

This leads to the following

- Question (Petersen, 2010): Is there a notion of $\beta_p^{(2)}(G)$ such that one has

$$\beta_p^{(2)}(\Gamma) = \text{covol}(\Gamma)\beta_p^{(2)}(G) \qquad (\dagger)$$

  that for any lattice $\Gamma \leqslant G$??

A partial answer for G locally compact, second countable and unimodular with Haar measure λ is given by

THEOREM (PETERSEN 2011)

Yes, $\beta_p^{(2)}(G, \lambda)$ makes sense and $(\dagger)$ holds for cocompact lattices or in general if G is totally disconnected.

An important theorem by one of the heroes of the field:

THEOREM (GABORIAU, 2000)

If Λ and Γ are lattices in a common locally compact group G then $\beta_p^{(2)}(\Gamma) = \frac{\mathrm{covol}(\Gamma)}{\mathrm{covol}(\Lambda)}\beta_p^{(2)}(\Lambda)$.

This leads to the following

- Question (Petersen, 2010): Is there a notion of $\beta_p^{(2)}(G)$ such that one has

$$\beta_p^{(2)}(\Gamma) = \mathrm{covol}(\Gamma)\beta_p^{(2)}(G) \tag{†}$$

  that for any lattice $\Gamma \leqslant G$??

A partial answer for G locally compact, second countable and unimodular with Haar measure λ is given by

THEOREM (PETERSEN 2011)

Yes, $\beta_p^{(2)}(G, \lambda)$ makes sense and (†) holds for cocompact lattices or in general if G is totally disconnected.

An important theorem by one of the heroes of the field:

THEOREM (GABORIAU, 2000)

If Λ and Γ are lattices in a common locally compact group G then $\beta_p^{(2)}(\Gamma) = \frac{\text{covol}(\Gamma)}{\text{covol}(\Lambda)}\beta_p^{(2)}(\Lambda)$.

This leads to the following

- Question (Petersen, 2010): Is there a notion of $\beta_p^{(2)}(G)$ such that one has

$$\beta_p^{(2)}(\Gamma) = \text{covol}(\Gamma)\beta_p^{(2)}(G) \qquad (\dagger)$$

  that for any lattice $\Gamma \leqslant G$??

A partial answer for G locally compact, second countable and unimodular with Haar measure λ is given by

THEOREM (PETERSEN 2011)

Yes, $\beta_p^{(2)}(G, \lambda)$ makes sense and $(\dagger)$ holds for cocompact lattices or in general if G is totally disconnected.

A poor man's cocompact lattice

Unfortunately, very few lc. groups have cocompact lattices. But, when such a group G acts on a measure space the action allows for a so-called cocompact cross section (Forrest, 1974) which in turn gives rise to a natural countable equivalence relation $\mathcal{R}$ which can 'substitute' for the cocompact lattice to yield:

Theorem (K-Petersen-Vaes, 2012)

The scaling formula (†) *holds in general; i.e. if* $\Gamma \leqslant G$ *is a lattice then* $\beta_p^{(2)}(\Gamma) = \mathrm{covol}_\lambda(\Gamma)\beta_p^{(2)}(G,\lambda)$.

- Note that this gives a one-line conceptual proof of Gaboriau's result: $\Lambda, \Gamma \leqslant G \rightsquigarrow \beta_p^{(2)}(\Gamma) = \frac{\mathrm{covol}(\Gamma)}{\mathrm{covol}(\Lambda)}\beta_p^{(2)}(\Lambda)$

A poor man's cocompact lattice

Unfortunately, very few lc. groups have cocompact lattices. But, when such a group G acts on a measure space the action allows for a so-called cocompact cross section (Forrest, 1974) which in turn gives rise to a natural countable equivalence relation $\mathcal{R}$ which can 'substitute' for the cocompact lattice to yield:

Theorem (K-Petersen-Vaes, 2012)

The scaling formula (†) *holds in general; i.e. if* $\Gamma \leqslant G$ *is a lattice then* $\beta_p^{(2)}(\Gamma) = \mathrm{covol}_\lambda(\Gamma)\beta_p^{(2)}(G,\lambda)$.

- Note that this gives a one-line conceptual proof of Gaboriau's result: $\Lambda, \Gamma \leqslant G \rightsquigarrow \beta_p^{(2)}(\Gamma) = \frac{\mathrm{covol}(\Gamma)}{\mathrm{covol}(\Lambda)}\beta_p^{(2)}(\Lambda)$

A poor man's cocompact lattice

Unfortunately, very few lc. groups have cocompact lattices. But, when such a group G acts on a measure space the action allows for a so-called cocompact cross section (Forrest, 1974) which in turn gives rise to a natural countable equivalence relation $\mathcal{R}$ which can 'substitute' for the cocompact lattice to yield:

Theorem (K-Petersen-Vaes, 2012)

The scaling formula (†) *holds in general; i.e. if* $\Gamma \leqslant G$ *is a lattice then* $\beta_p^{(2)}(\Gamma) = \mathrm{covol}_\lambda(\Gamma)\beta_p^{(2)}(G,\lambda)$.

- Note that this gives a one-line conceptual proof of Gaboriau's result: $\Lambda, \Gamma \leqslant G \rightsquigarrow \beta_p^{(2)}(\Gamma) = \frac{\mathrm{covol}(\Gamma)}{\mathrm{covol}(\Lambda)}\beta_p^{(2)}(\Lambda)$

A POOR MAN'S COCOMPACT LATTICE

Unfortunately, very few lc. groups have cocompact lattices. But, when such a group G acts on a measure space the action allows for a so-called cocompact cross section (Forrest, 1974) which in turn gives rise to a natural countable equivalence relation $\mathcal{R}$ which can 'substitute' for the cocompact lattice to yield:

THEOREM (K-PETERSEN-VAES, 2012)

The scaling formula $(\dagger)$ *holds in general; i.e. if* $\Gamma \leqslant G$ *is a lattice then* $\beta_p^{(2)}(\Gamma) = \mathrm{covol}_\lambda(\Gamma)\beta_p^{(2)}(G, \lambda)$.

- Note that this gives a one-line conceptual proof of Gaboriau's result: $\Lambda, \Gamma \leqslant G \rightsquigarrow \beta_p^{(2)}(\Gamma) = \frac{\mathrm{covol}(\Gamma)}{\mathrm{covol}(\Lambda)}\beta_p^{(2)}(\Lambda)$

VANISHING RESULTS

Our techniques also allow for l.c. versions of the classical computational results. For instance the following:

- If G contains a non-compact, normal, amenable subgroup $\beta_p^{(2)}(G) = 0$ for all p. In particular this holds when G itself is amenable.

So, even if you *only* care about discrete groups you for instance get the following:

COROLLARY (KPV)

If G is a locally compact group which contains some non-compact, closed, amenable, normal subgroup then $\beta_p^{(2)}(\Gamma) = 0$ for every lattice $\Gamma \leqslant G$.

Vanishing results

Our techniques also allow for l.c. versions of the classical computational results. For instance the following:

- If G contains a non-compact, normal, amenable subgroup $\beta_p^{(2)}(G) = 0$ for all p. In particular this holds when G itself is amenable.

So, even if you *only* care about discrete groups you for instance get the following:

Corollary (KPV)

If G is a locally compact group which contains some non-compact, closed, amenable, normal subgroup then $\beta_p^{(2)}(\Gamma) = 0$ for every lattice $\Gamma \leqslant G$.

Vanishing results

Our techniques also allow for l.c. versions of the classical computational results. For instance the following:

- If G contains a non-compact, normal, amenable subgroup $\beta_p^{(2)}(G) = 0$ for all p. In particular this holds when G itself is amenable.

So, even if you *only* care about discrete groups you for instance get the following:

Corollary (KPV)

If G is a locally compact group which contains some non-compact, closed, amenable, normal subgroup then $\beta_p^{(2)}(\Gamma) = 0$ for every lattice $\Gamma \leqslant G$.