Random Matrices

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Part 7) Biorthogonal Ensembles

OP ensemble

- **Vandermonde determinant**

$$\Delta(\lambda) = \prod_{j<k} (\lambda_j - \lambda_k)^2$$

- **Eigenvalue density of invariant ensemble**

$$\frac{1}{Z_n} \Delta(\lambda)^2 \prod_{j=1}^{n} e^{-V(\lambda_j)} = \frac{1}{Z_n} \left(\det \left[f_j(\lambda_k) \right]_{j,k=1,\dots,n} \right)^2$$

with

$$f_j(x) = x^{j-1} e^{-\frac{1}{2} V(x)}$$

- **This is an OP ensemble**

Biorthogonal ensemble

- **Joint probability density function on $\mathbb{R}^n$ of the form**

$$\frac{1}{Z_n} \det\left[f_j(x_k)\right]_{j,k} \det\left[g_j(x_k)\right]_{j,k}$$

is a biorthogonal ensemble

- **It is always determinantal** **Borodin (1998)**
- **Correlation kernel is**

$$K_n(x,y) = \sum_{j=1}^{n} \sum_{k=1}^{n} f_j(x) \left[A^{-1}\right]_{k,j} g_k(y)$$

where A^{-1} is the inverse of the matrix

$$A = (A_{j,k}), \qquad A_{j,k} = \int f_j(x) g_k(x)\, dx$$

Biorthogonal functions

- **Change basis functions**

$$f_1, \ldots, f_n \mapsto \phi_1, \ldots, \phi_n$$
$$g_1, \ldots, g_n \mapsto \psi_1, \ldots, \psi_n$$

- **New basis is biorthogonal if**

$$\int_{-\infty}^{\infty} \phi_j(x)\psi_k(x)\,dx = \delta_{j,k}$$

- **Then correlation kernel is single sum**

$$K_n(x,y) = \sum_{j=1}^{n} \phi_j(x)\psi_j(y)$$

Non-intersecting paths

Example: Assume $X_1(t), \ldots, X_n(t)$ are independent copies of one-dimensional diffusion process conditioned such that

- $X_j(0) = a_j$ **with** $a_1 < a_2 < \cdots < a_n$
- $X_j(T) = b_j$ **with** $b_1 < b_2 < \cdots < b_n$**,**
- **the paths do not intersect**

Then positions of the paths at any time $t \in (0, T)$ are biorthogonal ensemble Karlin-McGregor (1959)

Biorthogonal ensemble

- **Let $p_t(x, y)$ be the transition probability density**

  For Brownian motion

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$$

- **Biorthogonal ensemble at time t with functions**

$$f_j(x) = p_t(a_j, x), \qquad g_j(x) = p_{T-t}(x, b_j)$$

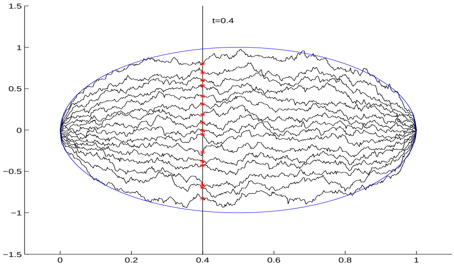
- **This is not an OP ensemble**

Confluent limit

- **Take Brownian motion, and confluent limit**

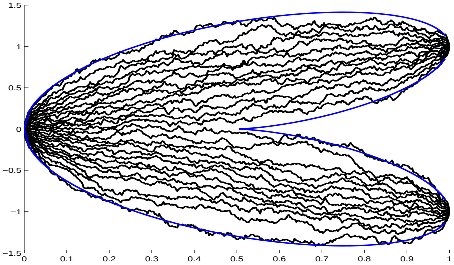
$$a_j \to 0, \qquad b_j \to 0$$

- **Positions at time t are eigenvalues of GUE (up to scaling)**



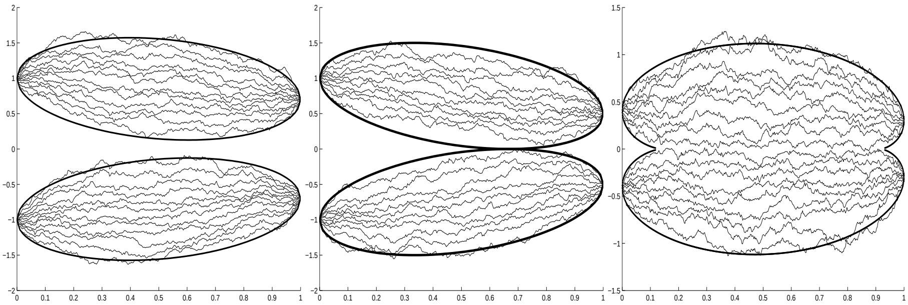
Cusp point

- **All $a_j \to 0$, half of $b_j \to 1$, other half of $b_j \to -1$**



Tacnode problem

- **Two starting points, two ending points**



Part 8) Multiple Orthogonal Ensembles

MOP ensemble

- **Biorthogonal ensemble of the form**

$$\frac{1}{Z_n}\Delta(x)\,\det\left[f_j(x_k)\right]$$

with $\Delta(x)$ the Vandermonde determinant

- **Suppose**

$$f_j\textbf{s are: } x^{j-1}w_k(x) \textbf{ for } j=1,\ldots,n_k,\ k=1,\ldots,r$$

with small number r.

- **Biorthogonal functions take the form**

$$\phi_j(x)=P_{j-1}(x), \qquad \psi_j(x)=\sum_{k=1}^{r} A_{j,k}(x)w_k(x)$$

with polynomials

$$\deg P_{j-1}=j-1, \qquad \sum_{k=1}^{r}\deg A_{j,k}=j-1.$$

Type II multiple orthogonality

- P_n **satisfies**

$$\int_{-\infty}^{\infty} P_n(x) x^{j-1} w_k(x)\, dx = 0, \qquad j = 1, \dots, n_k,\ k = 1, \dots, r.$$

- **This is multiple orthogonality of type II**
- **Multi-index notation** $P_n = P_{n_1,\dots,n_r}$

- **Polynomials** $A_{n,k}$ **are type I MOPs**

Examples of MOP ensembles

- **Non-intersecting Brownian motions, with one starting point and r ending points**
- **Eigenvalues of external source model**

$$\frac{1}{Z_n} e^{-\operatorname{Tr}(V(M) - AM)} dM$$

with deterministic A having r distinct eigenvalues

- **Eigenvalues of matrix M_1 in two matrix model**

$$\frac{1}{Z_n} e^{-\operatorname{Tr}(V(M_1) + W(M_2) - \tau M_1 M_2)} dM_1 dM_2$$

if W has degree $r+1$

More examples

- **Non-intersecting squared Bessel paths**
- **Eigenvalues in normal matrix model (complex matrices)**

$$\frac{1}{Z_n} e^{-\operatorname{Tr}(M^* M - V(M) - V(M^*))} dM$$

if V **has degree** $r+1$

- **Squared singular values of product**

$$X_r \dots X_2 X_1$$

of independent Ginibre random matrices

Tools

Useful tools for asymptotic analysis

- **Riemann-Hilbert problem** **YES**
- **Equilibrium problem** **SOMETIMES**

Part 9) RH problem for MOPs

Riemann-Hilbert problem (case $r = 2$)

RH problem of size 3×3 (if $r = 2$)

RH-Y1 $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{3\times 3}$ **is analytic.**

RH-Y2 Y **has boundary values for** $x \in \mathbb{R}$**, denoted by** $Y_\pm(x)$**, and**

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{for } x \in \mathbb{R}.$$

RH-Y3 **As** $z \to \infty$**,**

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^{n_1+n_2} & 0 & 0 \\ 0 & z^{-n_1} & 0 \\ 0 & 0 & z^{-n_2} \end{pmatrix}$$

Solution in terms of type II MOPs

Theorem (Van Assche, Geronimo, K (2001))

RH problem has a unique solution if and only if the type II MOP P_{n_1,n_2} uniquely exists.
In that case the first row of Y is given by

$$\begin{pmatrix} P_{n_1,n_2}(z) & \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{P_{n_1,n_2}(s)w_1(s)}{s-z}\,ds & \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{P_{n_1,n_2}(s)w_2(s)}{s-z}\,ds \\ * & * & * \\ * & * & * \end{pmatrix}$$

- **Other rows contain P_{n_1-1,n_2} and P_{n_1,n_2-1}**

Inverse of Y

- **The type I MOPs are in the inverse of** Y

$$Y^{-1}(z) = \begin{pmatrix} -\displaystyle\int_{-\infty}^{\infty} \frac{\psi(s)}{s-z}\, ds & * & * \\ 2\pi i A_1(z) & * & * \\ 2\pi i A_2(z) & * & * \end{pmatrix}$$

where $\psi(s) = A_1(s)w_1(s) + A_2(s)w_2(s)$

Christoffel Darboux formula

Theorem (Bleher-K (2004) for $r = 2$)

The correlation kernel K_n for the MOP ensemble is given by

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Extension to $r \geq 3$: Daems-K (2004)

Part 10) Equilibrium problem for MOPs (incomplete)

Vector equilibrium problem

- **For a RH problem of size $(r+1)\times(r+1)$ we would need *r* measures $\mu_1, \ldots, \mu_r$ and associated g-functions**

$$g_j(z) = \int \log(z-s)\,d\mu_j(s)$$

- **Equilibrium problem for vector of measures**

General form of VEP

Minimize

$$\sum_{j=1}^{r}\sum_{k=1}^{r} c_{j,k} \iint \log \frac{1}{|x-y|} d\mu_j(x) d\mu_k(y) + \sum_{j=1}^{r} \int V_j(x) d\mu_j(x)$$

among vector of measures $(\mu_1, \ldots, \mu_r)$ **where**

$$\mu_j \text{ is supported on } \Sigma_j \qquad \text{with } \mu_j(\Sigma_j) = p_j$$

and maybe extra conditions

- **General existence and uniqueness result in case the interaction matrix** $(c_{j,k})_{j,k=1,\ldots r}$ **is positive definite**

Hardy-K (2012)