Random Matrices

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Part 4) Orthogonal polynomials

Reminder: GUE and Hermite polynomials

- **Eigenvalues in GUE**

$$\frac{1}{Z_n} e^{-\frac{1}{2}\operatorname{Tr} M^2} dM$$

are determinantal point process with kernel

$$K_n(x,y) = e^{-\frac{1}{4}(x^2+y^2)} \sum_{k=0}^{n-1} H_k(x) H_k(y)$$

- H_n **is the Hermite polynomial**

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-\frac{1}{2}x^2} dx = \delta_{n,m}$$

Invariant ensembles

- **Immediate extension to invariant ensembles**

$$\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM$$

with arbitrary V,

$$\lim_{|x| \to +\infty} \frac{V(x)}{\log(1 + |x|)} = +\infty.$$

- **Joint eigenvalue density**

$$\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{n} e^{-nV(\lambda_j)}$$

Determinantal point process

- **Orthonormal polynomials w.r.t.** $e^{-nV(x)}$

$$\int_{-\infty}^{\infty} p_k(x) p_j(x) e^{-nV(x)} dx = \delta_{k,j}$$

- **Determinantal point process with kernel**

$$K_n(x,y) = e^{-\frac{n}{2}(V(x)+V(y))} \sum_{k=0}^{n-1} p_k(x) p_k(y)$$

Double integral

Correlation kernel has double integral representation in GUE case

$$K_n(x,y) = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} ds \oint_\Gamma dt \; e^{\frac{n}{2}(s-x)^2 - \frac{n}{2}(t-y)^2} \left(\frac{s}{t}\right)^n \frac{1}{s-t}$$

where $c > 0$ **and** Γ **goes around zero such that** $\mathrm{Re}\, t < c$ **for** $t \in \Gamma$**.**

- **Apply classical steepest descent techniques to obtain large** n **behavior.**
- **Matrix Riemann-Hilbert problem is nonlinear/non-commutative analogue of integral representation.**

Matrix Riemann-Hilbert problem for OPs

- **Find 2×2 matrix valued function $Y(z)$ such that**

RH-Y1 $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ **is analytic.**

RH-Y2 Y **has boundary values for $x \in \mathbb{R}$, denoted by $Y_\pm(x)$, and**

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-nV(x)} \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{R}.$$

RH-Y3 **As $z \to \infty$,**

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$$

Fokas, Its, Kitaev RH problem for OP

Theorem (Fokas, Its, Kitaev (1992))

The Riemann-Hilbert problem has the unique solution

$$Y(z) = \begin{pmatrix} \gamma_n^{-1} p_n(z) & \frac{1}{2\pi i}\gamma_n^{-1} \int\limits_{-\infty}^{\infty} \frac{p_n(s)e^{-nV(s)}}{s-z} ds \\ -2\pi i \gamma_{n-1} p_{n-1}(z) & -\gamma_{n-1} \int\limits_{-\infty}^{\infty} \frac{p_{n-1}(s)e^{-nV(s)}}{s-z} ds \end{pmatrix}$$

- p_n **is the orthonormal polynomial w.r.t.** $e^{-nV(x)}dx$
- γ_n **is the leading coefficient of** p_n**.**

Proof

- **We give the proof for the first row. The first entry $Y_{11}(z)$ satisfies**

$$\begin{aligned} (Y_{11})_+(x) &= (Y_{11})_-(x), \quad x \in \mathbb{R}, \\ Y_{11}(z) &= z^n + O(z^{n-1}), \quad \text{as } z \to \infty. \end{aligned}$$

- **Therefore Y_{11} is a monic polynomial of degree n, which we call P_n.**

Proof (cont.)

- **For Y_{12} we have** $Y_{12}(z) = O(z^{-n-1})$ **as** $z \to \infty$**, and**

$$(Y_{12})_+(x) = (Y_{12})_-(x) + P_n(x)w(x)$$

- **By Sokhotskii-Plemelj formula**

$$Y_{12}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{P_n(s)w(s)}{s-z}ds = O(z^{-1})$$

- **To satisfy the asymptotic condition** $O(z^{-n-1})$ **we need**

$$\int s^k P_n(s)w(s)ds = 0, \qquad k = 0, \ldots, n-1,$$

  which means that P_n **is the monic OP.**

- **The proof for the second row is similar.**

OP kernel in terms of the RH problem

- **OP kernel is**

$$K_n(x,y) = \frac{\sqrt{e^{-nV(x)}}\sqrt{e^{-nV(y)}}}{2\pi i(x-y)}\left[Y_+^{-1}(y)Y_+(x)\right]_{2,1}$$

$$= \frac{\sqrt{e^{-nV(x)}}\sqrt{e^{-nV(y)}}}{2\pi i(x-y)}\begin{pmatrix}0 & 1\end{pmatrix} Y_+^{-1}(y)Y_+(x)\begin{pmatrix}1\\0\end{pmatrix}.$$

This is based on the Christoffel-Darboux formula

Deift-Zhou steepest descent analysis

- **Deift-Zhou (1993) steepest descent analysis for RH problems**
- **Sequence of explicit and invertible transformations $Y \mapsto X \mapsto \cdots \mapsto R$ leading to a RH problem for R on contour Σ_R, which takes the form**

RH-R1 $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ **is analytic.**

RH-R2 **On Σ_R we have a jump J_R,**

$$R_+(z) = R_-(z) J_R(z),$$

RH-R3 $R(z) = I + O(1/z)$ **as $z \to \infty$.**

R is close to the identity matrix

- **Goal:** J_R **depends on** n **and**

$$J_R \to I \qquad \textbf{as } n \to \infty.$$

- **Then also**

$$R(z) \to I \qquad \textbf{as } n \to \infty$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$**.**

Part 5) Equilibrium measures

Why varying weight $e^{-nV(x)}$?

- **Joint eigenvalue pdf from unitary matrix model is**

$$\frac{1}{Z_n}\prod_{i<j}(x_j-x_i)^2\prod_{j=1}^{n}e^{-nV(x_j)}$$
$$=\frac{1}{Z_n}\exp\left[-n^2\left(-\frac{1}{n^2}\sum_{i\neq j}\log|x_j-x_i|+\frac{1}{n}\sum_j V(x_j)\right)\right].$$

- **Most likely distributions of eigenvalues are such that**

$$-\frac{1}{n^2}\sum_i\sum_{j\neq i}\log|x_j-x_i|+\frac{1}{n}\sum_j V(x_j)\qquad \textbf{is minimal}$$

- **Continuum limit (mean-field approximation)**

$$-\iint\log|x-y|d\mu(x)d\mu(y)+\int V(x)d\mu(x)\qquad \textbf{is minimal}$$

among probability measures μ on $\mathbb{R}$.

Equilibrium measure

- **Balance between mutual repulsion of eigenvalues and the confining potential V.**
- **To minimize**

$$-\iint \log|x-y|d\mu(x)d\mu(y)+\int V(x)d\mu(x)$$

is a well-studied equilibrium problem in logarithmic potential theory.

Saff-Totik book (1997)

- **There is a unique minimizer $\mu=\mu_V$ which has a density**

$$d\mu_V(x)=\rho_V(x)dx$$

Equilibrium constant

- **There is a constant ℓ_V**

$$2\int \log\frac{1}{|x-y|}\rho_V(y)dy + V(x) = \ell_V \qquad \textbf{on}\ \ \mathrm{supp}(\rho_V)$$

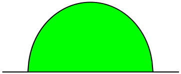
and

$$2\int \log\frac{1}{|x-y|}\rho_V(y)dy + V(x) \geq \ell_V \qquad \textbf{on}\ \mathbb{R}\setminus\mathrm{supp}(\rho_V)$$

Example: Semicircle law

- **In GUE case** $V(x) = 2x^2$ **the equilibrium density can be explicitly calculated**

$$\rho_V(x) = \frac{1}{2\pi}\sqrt{1-x^2}, \qquad -1 \le x \le 1$$



- **This is Wigner semi-circle law**
- **If** V **is real analytic the support is a finite union of intervals**

$$\mathrm{supp}(\rho_V) = \bigcup_{j=1}^{N} [a_j, b_j]$$

Deift-Kriecherbauer-McLaughlin (1997)

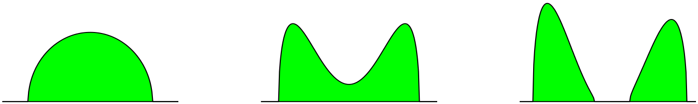
Global eigenvalue behavior

- **Global eigenvalue behavior is determined by the minimizer of the equilibrium problem**

$$\lim_{n\to\infty} \frac{1}{n} K_n(x,x) = \rho_V(x)$$

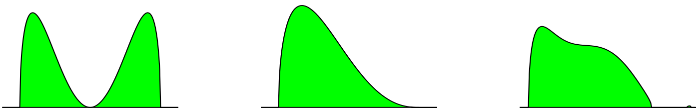
Regular cases

- $\rho_V > 0$ **in the interior of each interval,**
- ρ_V **vanishes like a square root at each endpoint,**
- $2\int \log \frac{1}{|x-y|}\rho_V(y)dy + V(x) > \ell$ **outside** $\mathrm{supp}(\rho_V)$**.**



Singular cases

- **Singular case I: ρ_V vanishes at an interior point**
- **Singular case II: ρ_V vanishes to higher order at an endpoint.**
- **Singular case III: Equality in $2\int \log \frac{1}{|x-y|}\rho_V(y) + V(x) \geq \ell$ at exterior point.**



- **Different local eigenvalue behavior in singular cases near critical points.**

Part 6) RH steepest descent

RH problem for Y

RH-Y1 $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ **is analytic.**

RH-Y2 Y **has jump on real line**

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-nV(x)} \\ 0 & 1 \end{pmatrix}$$

RH-Y3 $Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ **as** $z \to \infty$**.**

Steepest descent analysis of this RH problem is due to Deift, Kriecherbauer, McLaughlin, Venakides, Zhou (1999)

First transformation

- **We use the equilibrium measure in the first transformation of the RH problem**
- **Define the g-function**

$$g(z) = \int \log(z-s)\rho_V(s)ds$$

- **From equilibrium problem with $\ell = \ell_V$:**

$$\begin{aligned} g_+(x) + g_-(x) - V(x) + \ell &= 0 \qquad \textbf{for } x \in \operatorname{supp}(\mu_V), \\ g_+(x) + g_-(x) - V(x) + \ell &\leq 0 \qquad \textbf{for } x \in \mathbb{R}. \end{aligned}$$

- **From definition as a log-transform:**

$$g_+(x) - g_-(x) = 2\pi i \int_x^\infty \rho_V(s)ds$$

is purely imaginary.

First transformation

- **We define**

$$T(z) = \begin{pmatrix} e^{n\ell/2} & 0 \\ 0 & e^{-n\ell/2} \end{pmatrix} Y(z) \begin{pmatrix} e^{-n(g(z)+\ell/2)} & 0 \\ 0 & e^{n(g(z)+\ell/2)} \end{pmatrix}$$

- **Then** $T(z) = I + O(1/z)$ **as** $z \to \infty$**.**

ϕ functions

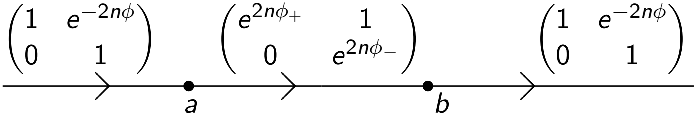
- **Jumps can all be expressed nicely in terms of analytic function ϕ,**

$$2\phi(z) = -2g(z) + V(z) - \ell$$

- **ϕ satisfies**

$$g_+(x) - g_-(x) = -2\phi_+(x) = 2\phi_-(x) \quad \textbf{for } x \in (a, b)$$

Jumps for T in one-interval case



- $\phi(x) > 0$ **for** $x < a$ **and for** $x > b$**,**
- $\phi_+ = -\phi_-$ **is purely imaginary on** (a, b) **and**

$$\frac{d}{dx}\phi_+(x) = \pi i \rho_V(x) \qquad \textbf{with} \qquad \rho_V(x) > 0$$

Correlation kernel in terms of T

- **Recall that**

$$K_n(x,y) = \frac{\sqrt{e^{-nV(x)}}\sqrt{e^{-nV(y)}}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- **Assume** $x, y \in (a, b)$**.**
- **Transformation** $Y \mapsto T$ **gives**

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{-n\phi_+(y)} \end{pmatrix} T_+^{-1}(y) T_+(x) \begin{pmatrix} e^{-n\phi_+(x)} \\ 0 \end{pmatrix}.$$

  - **This is based on** $2g_+ - V + \ell = -2\phi_+$ **on** (a, b)**.**

Second transformation $T \mapsto S$

- **Factorization of jump matrix for T on (a, b),**

$$\begin{pmatrix} e^{2n\phi_+} & 1 \\ 0 & e^{2n\phi_-} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{2n\phi_-} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{2n\phi_+} & 1 \end{pmatrix}.$$

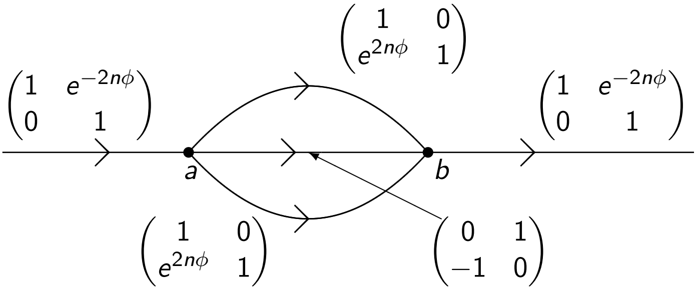
- **Open a lens around $[a, b]$ and define**

$$S = T \begin{pmatrix} 1 & 0 \\ -e^{2n\phi} & 1 \end{pmatrix} \qquad \text{in upper part of the lens}$$

$$S = T \begin{pmatrix} 1 & 0 \\ e^{2n\phi} & 1 \end{pmatrix} \qquad \text{in lower part of the lens}$$

$$S = T \qquad \text{outside the lens}$$

RH problem for S in one-interval case



- **We have $\phi > 0$ on $(-\infty, a)$ and on (b, ∞)**
- **From Cauchy-Riemann equations:**

$$\operatorname{Re} \phi < 0 \qquad \textbf{on the lips of the lens}$$

provided that $\rho_V(x) > 0$ on (a, b)

Correlation kernel in terms of S

- **We have for** $x, y \in (a, b)$**,**

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{-n\phi_+(y)} \end{pmatrix} T_+^{-1}(y) T_+(x) \begin{pmatrix} e^{-n\phi_+(x)} \\ 0 \end{pmatrix}.$$

- **Transformation** $T \mapsto S$ **gives**

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \times$$
$$\begin{pmatrix} -e^{n\phi_+(y)} & e^{-n\phi_+(y)} \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-n\phi_+(x)} \\ e^{n\phi_+(x)} \end{pmatrix}$$

Sine kernel in the bulk

- **The outcome of the steepest descent analysis will be that for $x, y \in (a+\delta, b-\delta)$,**

$$S_+^{-1}(y) S_+(x) = I + O(x-y) \qquad \textbf{as } y \to x$$

- **Then for x and y with $x-y$ small**

$$K_n(x,y) \approx \frac{1}{2\pi i (x-y)} \begin{pmatrix} -e^{n\phi_+(y)} & e^{-n\phi_+(y)} \end{pmatrix} \begin{pmatrix} e^{-n\phi_+(x)} \\ e^{n\phi_+(x)} \end{pmatrix}$$

- **Replacing**

$$x \mapsto x^* + \frac{x}{n\rho_V(x^*)}, \qquad y \mapsto x^* + \frac{y}{n\rho_V(x^*)}$$

and taking limit $n \to \infty$ we arrive at the sine kernel

$$\frac{\sin \pi(x-y)}{\pi(x-y)}.$$