Random Matrices

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Part 1) Random Matrix Theory Overview

Random matrices

What is a a GUE random matrix?

- **Take $n \times n$ matrix X with independent complex Gaussian entries.** **Ginibre random matrix**
- **Then**

$$M = X + X^*$$

is a GUE matrix **Gaussian Unitary Ensemble**

- **Same construction with real matrices leads to GOE** **Gaussian Orthogonal Ensemble**

Main results in RMT

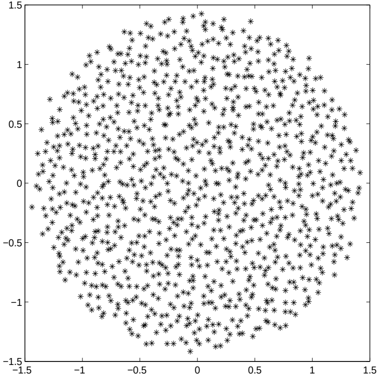
Five main results from Random Matrix Theory

1) **Ginibre eigenvalues:** **circular law**
2) **GUE eigenvalues:** **semi-circle law**
3) **Local repulsion of GUE eigenvalues:** **sine kernel**
4) **Largest GUE eigenvalue:** **Tracy-Widom distribution**
5) **Law of addition:** **free convolution**

1) Ginibre eigenvalues

Ginibre random matrix X

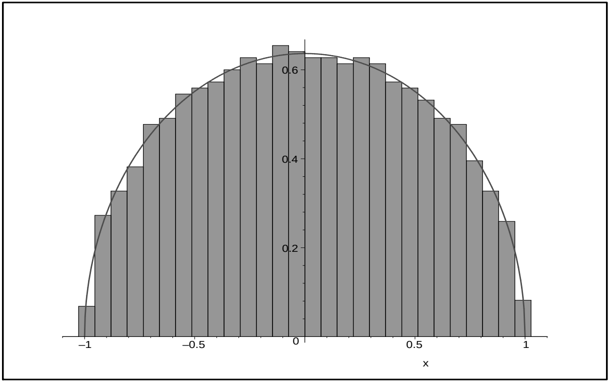
- **Eigenvalues follow the circular law**



2) GUE eigenvalues

GUE matrix: $M = X + X^*$

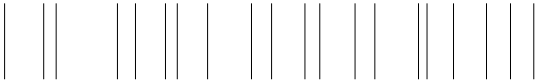
- **Eigenvalues follow the semicircle law** **Wigner 1950s**



- **Histogram of** 1000 **rescaled eigenvalues of a GUE matrix.**

3) Local repulsion, Universality

- **Eigenvalues on a local scale show universal repelling behavior**



- **This is called Universality in RMT**

Correlation functions

- **Joint probability density function for eigenvalues of** M

$$p_n(x_1, \ldots, x_n)$$

- k **point correlation function is**

$$R_{n,k}(x_1, \ldots, x_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} p_n(x_1, \ldots, x_n) dx_{k+1} \cdots dx_n$$

Local scaling limit

- **Take** x^* **with** $\rho(x^*) > 0$
- **Rescale eigenvalues around** x^*

$$x_j \mapsto n\rho(x^*)(x_j - x^*)$$

- k **point correlation function for rescaled eigenvalues is**

$$\frac{1}{(n\rho(x^*))^k} R_{n,k}\left(x^* + \frac{x_1}{n\rho(x^*)}, \ldots, x^* + \frac{x_k}{n\rho(x^*)}\right)$$

- **Universality: This has a limit as** $n \to \infty$

$$\det\left[K^{\sin}(x_i, x_j)\right]_{i,j=1,\ldots,k}$$

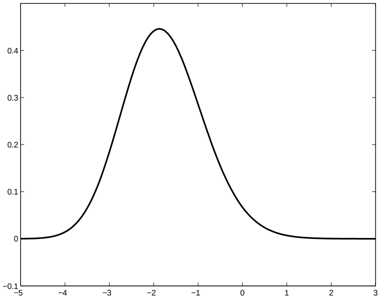
with sine kernel

$$K^{\sin}(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$$

4) Largest eigenvalue

- **Largest eigenvalue** $x_{\max,n} \sim 1$ **has fluctuations on scale** $n^{-2/3}$**.**
- **Limit law is Tracy-Widom distribution**

$$\lim_{n\to\infty} \mathbb{P}\left(cn^{2/3}(x_{n,n}-1) \le t\right) = F_{TW}(t)$$



- **Density** $\frac{d}{dt}F_{TW}(t)$ **of the Tracy-Widom distribution.**

Tracy-Widom distribution

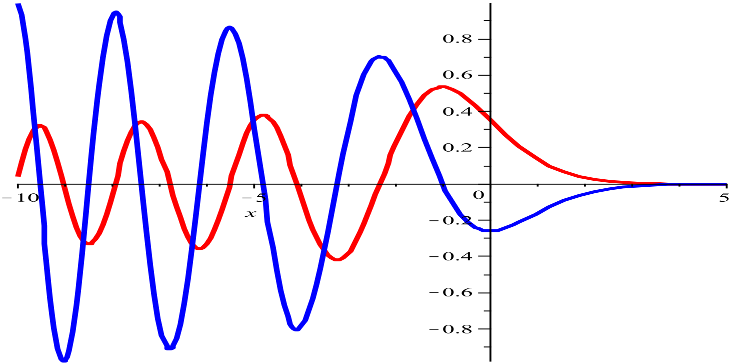
Two formulas for the TW distribution

- **Fredholm determinant with Airy kernel**
- **Formula with a solution of Painlevé II**

Airy functions

- **Airy function Ai solves $y'' = xy$ with**

$$\mathrm{Ai}(x) = \frac{1}{2\sqrt{\pi}x^{1/4}} e^{-\frac{2}{3}x^{3/2}}(1 + O(x^{-3/2})) \qquad \textbf{as } x \to \infty$$



Airy operator

- **Airy kernel**

$$\frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x - y}$$

- **Airy operator** K_t **acts on** $L^2(t, \infty)$

$$(K_t f)(x) = \int_t^\infty \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x - y} f(y)\, dy$$

- **Fredholm determinant for TW distribution**

$$F_{TW}(t) = \det(I - K_t)$$

Painnlevé connection

- **Tracy-Widom (1994):**

$$\frac{d}{dt} \log F_{TW}(t) = u(t), \qquad u' = -q^2$$

where q satisfies Painlevé II equation

$$q''(t) = tq(t) + 2q^3(t)$$

$$q(t) = \mathrm{Ai}(t)(1 + o(1)) \quad \textbf{as } t \to +\infty$$

- **Equivalently**

$$F_{TW}(t) = \exp\left(\int_t^{\infty} (s-t) q^2(s) ds\right)$$

5) Law of addition

- **Suppose A and B are large Hermitian matrices with eigenvalue distributions μ and ν.**
- **Then**

$$A + UBU^* \qquad\qquad U \text{ is Haar distributed}$$

has eigenvalue distribution $\mu \boxplus \nu$. **free convolution**

Free convolution

- **Cauchy transform**

$$F_\mu(z) = \int \frac{1}{z-s} d\mu(s) = \frac{1}{z} + O\left(\frac{1}{z^2}\right) \qquad \textbf{as } z \to \infty$$

- R **transform**

$$R_\mu(w) = (F_\mu)^{-1}(w) - \frac{1}{w}$$

- $\mu \boxplus \nu$ **satisfies**

$$\boxed{R_{\mu \boxplus \nu} = R_\mu + R_\nu}$$

Voiculescu (1986), Speicher (1994)

Part 2) Gaussian Unitary Ensemble

Invariant formulation of GUE

- **Flat Lebesgue measure on $n \times n$ Hermitian matrices**

$$dM = \prod_{j=1} dM_{jj} \prod_{j<k} d\mathrm{Re}\, M_{jk}\, d\mathrm{Im}\, M_{jk}$$

- **Probability measure**

$$\frac{1}{Z_n} e^{-\frac{1}{2} \operatorname{Tr} M^2}\, dM$$

where Z_n is normalization constant

$$Z_n = \int_{\mathcal{H}_n} e^{-\frac{1}{2} \operatorname{Tr} M^2}\, dM$$

GUE

$$\frac{1}{Z_n} e^{-\frac{1}{2}\operatorname{Tr} M^2}\, dM$$

- **Entries of M are independent Gaussian random variables with**

$$\mathbb{E} M_{jk} = 0$$

$$\mathbb{E} M_{jj}^2 = 1, \qquad \mathbb{E}\mathrm{Re}\, M_{jk}^2 = \mathbb{E}\mathrm{Im}\, M_{jk}^2 = \frac{1}{2}$$

- **Invariance under unitary conjugation $M \mapsto U^* M U$**

$$U^* dM U = dM \qquad \operatorname{Tr}(U^* M U) = \operatorname{Tr} M$$

Eigenvalue distribution

- **Joint eigenvalue distribution is explicit**

$$\frac{1}{Z_n} \prod_{1 \le j < k \le n} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{n} e^{-\frac{1}{2}\lambda_j^2}$$

- **Proof: Suppose $M = U \Lambda U^*$.**

$$dM = dU\Lambda U^* + U d\Lambda U^* + U\Lambda dU^*$$

- **By unitary invariance**

$$dM = U^* dMU = U^* dU\Lambda + d\Lambda + \Lambda(U^* dU)^*$$

- $U^* dU$ **is anti-Hermitian**

$$dM = U^* dU\Lambda - \Lambda U^* dU + d\Lambda$$

Proof (cont.)

- **If** $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

$$
\begin{aligned}
dM &= U^* dU\Lambda - \Lambda U^* dU + d\Lambda \\
&= \begin{pmatrix}
d\lambda_1 & (\lambda_1 - \lambda_2)u_1^* du_2 & (\lambda_1 - \lambda_3)u_1^* du_3 & \cdots \\
(\lambda_2 - \lambda_1)u_2^* du_1 & d\lambda_2 & (\lambda_2 - \lambda_3)u_2^* du_3 & \cdots \\
(\lambda_3 - \lambda_1)u_3^* du_1 & (\lambda_3 - \lambda_2)u_3^* du_2 & d\lambda_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\end{aligned}
$$

- **That is,**

$$
\begin{aligned}
dM_{jj} &= d\lambda_j \\
d\mathrm{Re}\, M_{jk} &= (\lambda_j - \lambda_k)\mathrm{Re}\,(u_j^* du_k) \\
d\mathrm{Im}\, M_{jk} &= (\lambda_j - \lambda_k)\mathrm{Im}\,(u_j^* du_k)
\end{aligned}
$$

Proof (cont.)

$$\prod_{j=1}^{n} dM_{jj} \prod_{j<k} d\mathrm{Re}\, M_{jk}\, d\mathrm{Im}\, M_{jk}$$

$$= \prod_{j=1}^{n} d\lambda_j \prod_{j<k} (\lambda_j - \lambda_k)^2 \underbrace{\prod_{j<k} \mathrm{Re}\,(u_j^* du_k) \mathrm{Im}\,(u_j^* du_k)}_{=dU}$$

Hermite polynomials

- **Orthonormal polynomials w.r.t.** $e^{-\frac{1}{2}x^2}$

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-\frac{1}{2}x^2}dx = \delta_{n,m}$$

- **Hermite functions** $\phi_n(x) = e^{-\frac{1}{4}x^2}H_n(x)$ **are orthonormal basis of** $L^2(\mathbb{R})$

$$\int_{-\infty}^{\infty} \phi_n(x)\phi_m(x)\,dx = \delta_{n,m}$$

- **Reproducing kernel**

$$K_n(x,y) = \sum_{k=0}^{n-1} \phi_k(x)\phi_k(y)$$

Correlation kernel

Theorem

$$\frac{1}{Z_n}\prod_{j<k}(\lambda_j-\lambda_k)^2\prod_{j=1}^{n}e^{-\frac{1}{2}\lambda_j^2}=\frac{1}{n!}\det[K_n(\lambda_j,\lambda_k)]_{j,k=1,\dots,n}$$

Also k-point correlation functions are $k\times k$ determinants

$$R_{n,k}(\lambda_1,\dots,\lambda_k)=\det\left[K_n(\lambda_i,\lambda_j)\right]_{i,j,\dots,k}$$

This is the structure of a determinantal point process with correlation kernel K_n.

Part 3) Outlook

Large n limit: semi-circle law

- **First do rescaling $\lambda_j \mapsto \lambda_j/\sqrt{2n}$.**
- **New correlation kernel**

$$\widehat{K}_n(x,y) = \sqrt{2n}K_n(\sqrt{2n}x, \sqrt{2n}y)$$

- **Semi-circle law**

$$\lim_{n\to\infty} \frac{1}{n}\widehat{K}_n(x,x) = \rho(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \textbf{for } x \in [-1,1], \\ 0 & \textbf{elsewhere} \end{cases}$$

Large n limits: sine kernel

- **Universality of local eigenvalue statistics is expressed as**

$$\lim_{n\to\infty} \frac{1}{n\rho(x^*)} \widehat{K}_n \left(x^* + \frac{x}{n\rho(x^*)}, x^* + \frac{y}{n\rho(x^*)} \right) = \frac{\sin \pi(x-y)}{\pi(x-y)}$$

whenever $x^* \in (-1, 1)$