Classification of Cat-Groups

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1 Introduction

Cat-groups first appeared in Mathematical literature in the early seventies. They came out from the independent works of P. Deligne [5] in Algebraic Geometry and A. Fröhlich and C. T. C. Wall [6] in Ring Theory. The latter two used cat-groups in order to study, for a given Dedekind domain, the cohomology of the group of fractional ideals.

The idea of a cat-group is, quite explicitly, to mix the concept of a category and a group. To define it precisely, we will need to introduce monoidal categories, which will be our interest in Chapter 2. There, it will be explained that a monoidal category is a category equipped with a 'weak' structure of monoid on its objects. That is, for each pair of objects X and Y, their product $X \otimes Y$ is a new object of the category. This product has to satisfy the well-known axioms of monoids, but only up to isomorphisms. For example, we will have $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ for all objects X, Y, Z. Moreover, these isomorphisms will be asked to satisfy coherence axioms, like the 'Pentagon Axiom' or the 'Triangle Axiom'. Hence, in Chapter 2, we will study monoidal categories and the suitable functors between them: monoidal functors.

After this introduction to monoidal categories, we will be able to define cat-groups. Briefly, a cat-group is a monoidal category where all morphisms and objects are invertible. Thus, they have a 'weak' group structure on their object class. Therefore, monoidal categories are analogous to monoids in the same way that cat-groups are analogous to groups.

Monoids	Groups
Monoidal Categories	Cat-Groups

In Chapter 3, we will prove basic properties of cat-groups and cat-group functors, which are the suitable functors between them. In view of our analogy described above, we could also think of cat-group functors as we think of group homomorphisms.

Monoid Homomorphisms	Group Homomorphisms
Monoidal Functors	Cat-Group Functors

After these Chapters 2 and 3, since a small cat-group $\mathcal{G}$ is a category with a 'weak' group structure on its objects, it will make sense to introduce the group $\Pi_0(\mathcal{G})$ of classes of isomorphic objects. Moreover, since all arrows are invertible, we will also define $\Pi_1(\mathcal{G}) = \mathcal{G}(I, I)$ as the abelian group of endomorphisms of the unit object I. This group will be endued with an action of $\Pi_0(\mathcal{G})$. Finally, we will see that the associativity isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ induce another invariant: $a \in H^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}))$ in the third cohomology group. It will be called the Postnikov invariant of $\mathcal{G}$. The main aim of this essay will be to present, in Chapter 5, a proof of H. X. Sinh's Theorem, which states that we only need to know the triple $(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}), a)$ to reconstruct, up to cat-group equivalence, the cat-group $\mathcal{G}$. This is the classification of cat-groups.

As an important corollary of this classification, we will prove that the 2-category of small cat-groups is biequivalent to the 2-category whose objects are triples (G, A, a) where G is a group, A is a G-module and $a \in Z^3(G, A)$ is a 3-cocycle. Since there are several definitions of biequivalence for 2-categories in Mathematical literature, we will fix the one we will work with in Chapter 4. We will want such a definition of biequivalence to admit a characterisation as we have for equivalences in categories (i.e. an equivalence is a fully faithful and essentially surjective functor). Furthermore, so as to be coherent with its name, we will also require it to be an actual equivalence relation on 2-categories. This implies that we will have to introduce a definition of biequivalence in terms of pseudo-2-functors, pseudo-2-natural transformations and pseudo-modifications.

Note that we will assume the axiom of choice in this essay, as usual in Category Theory.

2 Monoidal Categories

If A and B are two abelian groups, we can construct their tensor product $A \otimes B$. This induces a binary operation on objects of Ab, the category of abelian groups. Moreover, up to isomorphisms, this tensor product gives to Ab a 'weak' structure of monoid. Indeed, it is wrong to write $A \otimes (B \otimes C) = (A \otimes B) \otimes C$, but we have associativity up to isomorphisms, $A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$, and, also up to isomorphisms, $\mathbb{Z}$ satisfies the axioms of identity.

The aim of this chapter is to generalise this example and give a rigorous definition of such 'monoidal categories'. We are also going to study the suitable functors between them.

Monoidal categories were introduced by J. Bénabou in [2] in 1963 while S. Mac Lane stated the coherence axioms (see definition 2.1) for the first time also in 1963 in [11]. G. M. Kelly and S. Eilenberg also studied monoidal categories in [9]. For recent books dealing with this subject, we refer the reader to Chapter 6 of [4] (Borceux), Chapter 11 of [8] (Kassel) or Chapter 7 of [10] (Mac Lane).

2.1 Definition and Examples

Definition 2.1. A monoidal category $(\mathcal{C}, \otimes, I, l, r, a)$ is the data of

- a category $\mathcal{C}$,
- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- an object $I \in \mathrm{ob}\,\mathcal{C}$,
- three families of natural isomorphisms

  $l = \{l_X :\ I \otimes X \xrightarrow{\sim} X\ \}_{X \in \mathrm{ob}\,\mathcal{C}}$

  $r = \{r_X :\ X \otimes I \xrightarrow{\sim} X\ \}_{X \in \mathrm{ob}\,\mathcal{C}}$

  $a = \{a_{X,Y,Z} :\ (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)\ \}_{X,Y,Z \in \mathrm{ob}\,\mathcal{C}}$

satisfying the following coherence axioms:

- **Pentagon Axiom :** For all $X, Y, Z, W \in \mathrm{ob}\,\mathcal{C}$,

$$a_{X,Y,Z\otimes W}\ a_{X\otimes Y,Z,W} = (1_X \otimes a_{Y,Z,W})\ a_{X,Y\otimes Z,W}\ (a_{X,Y,Z} \otimes 1_W),$$

$$\begin{tikzcd}
 & ((X \otimes Y) \otimes Z) \otimes W \arrow[dl, "a_{X,Y,Z}\otimes 1_W"'] \arrow[dr, "a_{X\otimes Y,Z,W}"] & \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[d, "a_{X,Y\otimes Z,W}"'] & \circlearrowleft & (X \otimes Y) \otimes (Z \otimes W) \arrow[d, "a_{X,Y,Z\otimes W}"] \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "1_X\otimes a_{Y,Z,W}"'] & & X \otimes (Y \otimes (Z \otimes W))
\end{tikzcd}$$

- **Triangle Axiom :** For all $X, Y \in \operatorname{ob}\mathcal{C}$, $(1_X \otimes l_Y)\, a_{X,I,Y} = r_X \otimes 1_Y$.

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "a_{X,I,Y}"] \arrow[dr, "r_X \otimes 1_Y"'] & \circlearrowleft & X \otimes (I \otimes Y) \arrow[dl, "1_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

Remark that the naturality of families l, r and a means that the following diagrams commute for all suitable $f, f_1, f_2, f_3 \in \operatorname{mor}\mathcal{C}$:

$$\begin{tikzcd}
I \otimes X \arrow[r, "l_X"] \arrow[d, "1_I \otimes f"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\
I \otimes Y \arrow[r, "l_Y"'] & Y
\end{tikzcd}
\quad
\begin{tikzcd}
X \otimes I \arrow[r, "r_X"] \arrow[d, "f \otimes 1_I"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\
Y \otimes I \arrow[r, "r_Y"'] & Y
\end{tikzcd}
\quad
\begin{tikzcd}
(X_1 \otimes X_2) \otimes X_3 \arrow[r, "a_{X_1,X_2,X_3}"] \arrow[d, "(f_1 \otimes f_2) \otimes f_3"'] \arrow[dr, phantom, "\circlearrowleft"] & X_1 \otimes (X_2 \otimes X_3) \arrow[d, "f_1 \otimes (f_2 \otimes f_3)"] \\
(Y_1 \otimes Y_2) \otimes Y_3 \arrow[r, "a_{Y_1,Y_2,Y_3}"'] & Y_1 \otimes (Y_2 \otimes Y_3)
\end{tikzcd}$$

Remark also that the functoriality of $\otimes$ means that the following diagrams commute for all $X, Y \in \operatorname{ob}\mathcal{C}$ and for all suitable $f, f', g, g' \in \operatorname{mor}\mathcal{C}$:

$$\begin{tikzcd}
X \otimes Y \arrow[r, shift left, "1_{X \otimes Y}"] \arrow[r, shift right, "1_X \otimes 1_Y"'] \arrow[r, phantom, "\circlearrowleft"] & X \otimes Y
\end{tikzcd}
\quad
\begin{tikzcd}
X \otimes X' \arrow[rr, "gf \otimes g'f'"] \arrow[dr, "f \otimes f'"'] & \circlearrowleft & Z \otimes Z' \\
& Y \otimes Y' \arrow[ur, "g \otimes g'"'] &
\end{tikzcd}$$

Remark 2.2. By abuse of notation, we will often write $\mathcal{C}$ to mean the monoidal category $(\mathcal{C}, \otimes, I, l, r, a)$.

We are now going to give a few examples of monoidal categories.

Example 2.3. If Ab is the category of abelian groups and $\otimes$ their usual tensor product, then $(\mathrm{Ab}, \otimes, \mathbb{Z}, l, r, a)$ is a monoidal category where l, r and a are the obvious isomorphisms.

Example 2.4. More generally, if R is a commutative ring, R-Mod the category of R-modules and $\otimes$ their usual tensor product, then, $(R\text{-Mod}, \otimes, R, l, r, a)$ is a monoidal category where l, r and a are the obvious isomorphisms. Note that $\mathrm{Ab} = \mathbb{Z}\text{-Mod}$.

Let us fix here some notations: If $\mathcal{C}$ and $\mathcal{D}$ are any categories, we write $[\mathcal{C}, \mathcal{D}]$ for the category of functors $F : \mathcal{C} \to \mathcal{D}$ and natural transformations. Moreover, if in the following diagram,

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}$$

$\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories, F, F', G, G' are functors and α, β are natural transformations, we define a new natural transformation $\beta \star \alpha : GF \Rightarrow G'F'$ by

$$(\beta \star \alpha)_X = \beta_{F'X} G(\alpha_X) = G'(\alpha_X)\beta_{FX}$$

for all $X \in \operatorname{ob}\mathcal{C}$.

Example 2.5. If $\mathcal{C}$ is any category, then $([\mathcal{C}, \mathcal{C}], \circ, 1_{\mathcal{C}}, 1, 1, 1)$ is a monoidal category where $\circ$ is the composition of functors and $\alpha \otimes \beta = \alpha \star \beta$.

Example 2.6. If $\mathcal{C}$ is a category with finite products, then $(\mathcal{C}, \times, 1, l, r, a)$ is a monoidal category where 1 is the terminal object and l, r and a are defined by the universal property of the product. Note that this construction use the axiom of choice to select, for all $X, Y \in \operatorname{ob}\mathcal{C}$, a product $X \times Y$ among all isomorphic possibilities.

And here is our last example.

Example 2.7. Let $A \xrightarrow{f} B$ be a morphism of abelian groups. Then, we define $\underline{\operatorname{Coker}} f$ to be the category with elements of B as objects and $\underline{\operatorname{Coker}} f(b, b') = \{a \in A \mid f(a) = b' - b\}$ with composition being the addition in A. Let $b \otimes b' = b + b'$ for all $b, b' \in B$ and $a \otimes a' = a + a'$ for all $a, a' \in A$. We have just construct the monoidal category $(\underline{\operatorname{Coker}} f, \otimes, 0_B, 0_A, 0_A, 0_A)$. We will see in Chapter 5 (example 5.9) why this is denoted $\underline{\operatorname{Coker}} f$.

2.2 Basic Properties

As written in the title, we are going to prove in this section the first properties of monoidal categories.

Lemma 2.8. If f and g are isomorphisms in a monoidal category, then $f \otimes g$ is also a isomorphism and $(f \otimes g)^{-1} = f^{-1} \otimes g^{-1}$.

Proof. Straightforward consequence of the functoriality of $\otimes$.

□

We present in the next lemma a useful equivalence between a monoidal category and itself.

Lemma 2.9. Let $\mathcal{C}$ be a monoidal category. We define two functors

$$\begin{array}{rrcl} I \otimes - : & \mathcal{C} & \longrightarrow & \mathcal{C} \\ & X & \mapsto & I \otimes X \\ & f & \mapsto & 1_I \otimes f \end{array} \qquad \text{and} \qquad \begin{array}{rrcl} - \otimes I : & \mathcal{C} & \longrightarrow & \mathcal{C} \\ & X & \mapsto & X \otimes I \\ & f & \mapsto & f \otimes 1_I . \end{array}$$

They form an equivalence $\mathcal{C} \underset{- \otimes I}{\overset{I \otimes -}{\rightleftarrows}} \mathcal{C}$. In particular, both functors are faithful.

Proof. They are functors since $\otimes$ is also a functor. We have to find two natural isomorphisms $\alpha : (- \otimes I) \circ (I \otimes -) \overset{\sim}{\Longrightarrow} 1_{\mathcal{C}}$ and $\beta : (I \otimes -) \circ (- \otimes I) \overset{\sim}{\Longrightarrow} 1_{\mathcal{C}}$. For all $X \in \operatorname{ob}\mathcal{C}$, we define $\alpha_X = r_X(l_X \otimes 1_I)$ and $\beta_X = l_X(1_I \otimes r_X)$. They are isomorphisms by lemma 2.8. It remains to show their naturality. We will do it for α; β is similar. Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{C}$. We can compute

$$\begin{aligned} \alpha_Y \ ((1_I \otimes f) \otimes 1_I) &= r_Y \ (l_Y \otimes 1_I) \ ((1_I \otimes f) \otimes 1_I) \\ &= r_Y \ (f \otimes 1_I) \ (l_X \otimes 1_I) \\ &= f \ r_X \ (l_X \otimes 1_I) \\ &= f \ \alpha_X , \end{aligned}$$

which proves the naturality of α.

□

The following lemma will be used to make computations easier. We will use it without referring to it.

Lemma 2.10. In a monoidal category $\mathcal{C}$, we have, for all $X, Y \in \operatorname{ob}\mathcal{C}$, the following equalities:

1. $l_{X\otimes Y}\ a_{I,X,Y} = l_X \otimes 1_Y$,
2. $(1_X \otimes r_Y)\ a_{X,Y,I} = r_{X\otimes Y}$,
3. $l_{I\otimes Y} = 1_I \otimes l_Y$,
4. $r_{Y\otimes I} = r_Y \otimes 1_I$,
5. $r_I = l_I$.

Proof. 1. To prove the first point, we have to do some computations. By the Pentagon Axiom, we know that

$$(1_I \otimes l_{X\otimes Y})\ (1_I \otimes a_{I,X,Y})\ a_{I,I\otimes X,Y}\ (a_{I,I,X} \otimes 1_Y) = (1_I \otimes l_{X\otimes Y})\ a_{I,I,X\otimes Y}\ a_{I\otimes I,X,Y}.$$

But we also have

$$\begin{aligned}
(1_I \otimes l_{X\otimes Y}) a_{I,I,X\otimes Y} a_{I\otimes I,X,Y} &= (r_I \otimes 1_{X\otimes Y}) a_{I\otimes I,X,Y} && \text{Triangle Axiom}\\
&= a_{I,X,Y}((r_I \otimes 1_X) \otimes 1_Y) && \text{Naturality of } a\\
&= a_{I,X,Y}((1_I \otimes l_X) \otimes 1_Y)(a_{I,I,X} \otimes 1_Y) && \text{Triangle Axiom}\\
&= (1_I \otimes (l_X \otimes 1_Y)) a_{I,I\otimes X,Y}(a_{I,I,X} \otimes 1_Y) && \text{Naturality of } a.
\end{aligned}$$

Thus, we can write $1_I \otimes (l_{X\otimes Y} a_{I,X,Y}) = 1_I \otimes (l_X \otimes 1_Y)$. We conclude by lemma 2.9.

$$\begin{tikzcd}
((I\otimes I)\otimes X)\otimes Y \arrow[rrrr, "a_{I,I,X}\otimes 1_Y"] \arrow[drr, "(r_I\otimes 1_X)\otimes 1_Y"'] \arrow[ddd, "a_{I\otimes I,X,Y}"'] & & \arrow[d, phantom, "\circlearrowleft"] & & (I\otimes(I\otimes X))\otimes Y \arrow[dll, "(1_I\otimes l_X)\otimes 1_Y"] \arrow[ddd, "a_{I,I\otimes X,Y}"] \\
& & (I\otimes X)\otimes Y \arrow[d, "a_{I,X,Y}"] & & \\
& \circlearrowleft & I\otimes(X\otimes Y) & \circlearrowleft & \\
(I\otimes I)\otimes(X\otimes Y) \arrow[urr, "r_I\otimes 1_{X\otimes Y}"] \arrow[drr, "a_{I,I,X\otimes Y}"'] & \circlearrowleft & & & I\otimes((I\otimes X)\otimes Y) \arrow[ull, "1_I\otimes(l_X\otimes 1_Y)"'] \arrow[dll, "1_I\otimes a_{I,X,Y}"] \\
& & I\otimes(I\otimes(X\otimes Y)) \arrow[uu, "1_I\otimes l_{X\otimes Y}"'] & &
\end{tikzcd}$$

2. Similar to 1.
3. By naturality of l, $l_Y\ l_{I\otimes Y} = l_Y\ (1_I \otimes l_Y)$, which gives $l_{I\otimes Y} = 1_I \otimes l_Y$ since l_Y is an isomorphism.
4. Similar to 3.

5. By lemma 2.9, it is enough to prove $r_I \otimes 1_I = l_I \otimes 1_I$:

$$\begin{aligned} r_I \otimes 1_I &= (1_I \otimes l_I)\, a_{I,I,I} && \text{Triangle Axiom} \\ &= l_{I\otimes I}\, a_{I,I,I} && 2.10.3 \\ &= l_I \otimes 1_I && 2.10.1. \end{aligned}$$

□

In any category $\mathcal{C}$, we know that $\mathcal{C}(X, X)$ is a monoid for all $X \in \mathrm{ob}\,\mathcal{C}$. If $\mathcal{C}$ is monoidal and $X = I$, we can show that this monoid is commutative.

Proposition 2.11. If $\mathcal{C}$ is a monoidal category, then $\mathcal{C}(I, I)$ is a commutative monoid.

Proof. Let $f, g \in \mathcal{C}(I, I)$. Using the fact that $r_I = l_I$ is an isomorphism, we can compute

$$\begin{aligned} gf &= r_I\, l_I^{-1}\, g\, f\, r_I\, l_I^{-1} \\ &= r_I\, (1_I \otimes g)\, l_I^{-1}\, r_I\, (f \otimes 1_I)\, l_I^{-1} \\ &= r_I\, (1_I \otimes g)\, (f \otimes 1_I)\, l_I^{-1} \\ &= r_I\, (f \otimes g)\, l_I^{-1} \\ &= r_I\, (f \otimes 1_I)\, (1_I \otimes g)\, l_I^{-1} \\ &= f\, r_I\, l_I^{-1}\, g \\ &= fg. \end{aligned}$$

□

2.3 Monoidal Functors

A morphism between two monoids is a function preserving the monoidal structure, i.e. a function f such that $f(1) = 1$ and $f(xy) = f(x)f(y)$. Keeping this in mind, we will define a monoidal functor between two monoidal categories to be a functor F such that $F(I) \simeq I$ and $F(X \otimes Y) \simeq F(X) \otimes F(Y)$ where the isomorphisms satisfy some coherence axioms. In this section, we are also going to define monoidal natural transformations and monoidal equivalences.

Definition 2.12. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, l, r, a)$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, l', r', a')$ be two monoidal categories. A monoidal functor

$$(F, F_I, \widetilde{F}) : (\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, l, r, a) \to (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, l', r', a')$$

is the data of:

- a functor $F : \mathcal{C} \to \mathcal{D}$,
- an isomorphism $F_I :\ I_{\mathcal{D}} \xrightarrow{\sim} F(I_{\mathcal{C}})$,
- a family of natural isomorphisms

$$\widetilde{F} = \{\widetilde{F}_{X,Y} :\ F(X) \otimes_{\mathcal{D}} F(Y) \xrightarrow{\sim} F(X \otimes_{\mathcal{C}} Y)\ \}_{X,Y \in \mathrm{ob}\,\mathcal{C}}$$

such that for all $X, Y, Z \in \operatorname{ob}\mathcal{C}$, the following diagrams commute:

$$\begin{tikzcd}
(F(X) \otimes_{\mathcal{D}} F(Y)) \otimes_{\mathcal{D}} F(Z) \arrow[rr, "a'_{F(X),F(Y),F(Z)}"] \arrow[d, "\widetilde{F}_{X,Y} \otimes_{\mathcal{D}} 1_{F(Z)}"'] & & F(X) \otimes_{\mathcal{D}} (F(Y) \otimes_{\mathcal{D}} F(Z)) \arrow[d, "1_{F(X)} \otimes_{\mathcal{D}} \widetilde{F}_{Y,Z}"] \\
F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} F(Z) \arrow[d, "\widetilde{F}_{X \otimes_{\mathcal{C}} Y, Z}"'] & \circlearrowleft & F(X) \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) \arrow[d, "\widetilde{F}_{X, Y \otimes_{\mathcal{C}} Z}"] \\
F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) \arrow[rr, "F(a_{X,Y,Z})"'] & & F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z))
\end{tikzcd}$$

$$\begin{tikzcd}
F(X) \otimes_{\mathcal{D}} I_{\mathcal{D}} \arrow[r, "1_{F(X)} \otimes_{\mathcal{D}} F_I"] \arrow[d, "r'_{F(X)}"'] & F(X) \otimes_{\mathcal{D}} F(I_{\mathcal{C}}) \arrow[d, "\widetilde{F}_{X, I_{\mathcal{C}}}"] \\
F(X) \arrow[ur, phantom, "\circlearrowleft"] & F(X \otimes_{\mathcal{C}} I_{\mathcal{C}}) \arrow[l, "F(r_X)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
I_{\mathcal{D}} \otimes_{\mathcal{D}} F(X) \arrow[r, "F_I \otimes_{\mathcal{D}} 1_{F(X)}"] \arrow[d, "l'_{F(X)}"'] & F(I_{\mathcal{C}}) \otimes_{\mathcal{D}} F(X) \arrow[d, "\widetilde{F}_{I_{\mathcal{C}}, X}"] \\
F(X) \arrow[ur, phantom, "\circlearrowleft"] & F(I_{\mathcal{C}} \otimes_{\mathcal{C}} X) \arrow[l, "F(l_X)"]
\end{tikzcd}$$

Remark that the naturality of the family $\widetilde{F}$ means that the following diagram commutes for all suitable $f, g \in \operatorname{mor}\mathcal{C}$:

$$\begin{tikzcd}
F(X) \otimes_{\mathcal{D}} F(Y) \arrow[r, "\widetilde{F}_{X,Y}"] \arrow[d, "F(f) \otimes_{\mathcal{D}} F(g)"'] & F(X \otimes_{\mathcal{C}} Y) \arrow[d, "F(f \otimes_{\mathcal{C}} g)"] \\
F(X') \otimes_{\mathcal{D}} F(Y') \arrow[ur, phantom, "\circlearrowleft"] \arrow[r, "\widetilde{F}_{X',Y'}"'] & F(X' \otimes_{\mathcal{C}} Y')
\end{tikzcd}$$

Remark 2.13. By abuse of notation, we will often write $F : \mathcal{C} \to \mathcal{D}$ to mean the monoidal functor $(F, F_I, \widetilde{F})$.

As it happens with functors, we have an identity and a composition law for monoidal functors.

Proposition 2.14. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be three monoidal categories. Then, $(1_{\mathcal{C}}, 1_I, 1) : \mathcal{C} \to \mathcal{C}$ is a monoidal functor. Moreover, if $(F, F_I, \widetilde{F}) : \mathcal{C} \to \mathcal{D}$ and $(G, G_I, \widetilde{G}) : \mathcal{D} \to \mathcal{E}$ are monoidal functors, then $(GF, G(F_I)G_I, \widetilde{GF}) : \mathcal{C} \to \mathcal{E}$ is also a monoidal functor where $\widetilde{GF}_{X,Y} = G(\widetilde{F}_{X,Y})\widetilde{G}_{F(X),F(Y)}$ for all $X, Y \in \operatorname{ob}\mathcal{C}$.
This form the category MC of small monoidal categories and monoidal functors.

Proof. It is trivial to check that $(1_{\mathcal{C}}, 1_I, 1)$ is monoidal. The facts that $(GF, G(F_I)G_I, \widetilde{GF})$ is monoidal and MC is a category follow directly from definition 2.12. □

In the same way, we introduce the notion of monoidal natural transformations.

Definition 2.15. Let $(F, F_I, \widetilde{F}), (G, G_I, \widetilde{G}) : \mathcal{C} \to \mathcal{D}$ be two monoidal functors where $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories. A monoidal natural transformation $\alpha : (F, F_I, \widetilde{F}) \Rightarrow (G, G_I, \widetilde{G})$ is a natural transformation $\alpha : F \Rightarrow G$ such that the following diagrams commute for all $X, Y \in \operatorname{ob}\mathcal{C}$:

$$\begin{tikzcd}
 & I \arrow[dl, "F_I"'] \arrow[dr, "G_I"] \arrow[d, phantom, "\circlearrowleft"] & \\
F(I) \arrow[rr, "\alpha_I"'] & {} & G(I)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\widetilde{F}_{X,Y}"] \arrow[d, "\alpha_X \otimes \alpha_Y"'] & F(X \otimes Y) \arrow[d, "\alpha_{X \otimes Y}"] \\
G(X) \otimes G(Y) \arrow[ur, phantom, "\circlearrowleft"] \arrow[r, "\widetilde{G}_{X,Y}"'] & G(X \otimes Y)
\end{tikzcd}$$

We also have an identity and a composition law for monoidal natural transformation.

Proposition 2.16. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be monoidal categories. Let $F, F', F'' : \mathcal{C} \to \mathcal{D}$ and $G, G' : \mathcal{D} \to \mathcal{E}$ be monoidal functors and let $\alpha : F \Rightarrow F'$, $\alpha' : F' \Rightarrow F''$ and $\beta : G \Rightarrow G'$ be monoidal natural transformations. Then, $1_F : F \Rightarrow F$, $\alpha'\alpha : F \Rightarrow F''$ and $\beta \star \alpha : GF \Rightarrow G'F'$ are monoidal natural transformations.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[r, bend right=90, "F''"'] \arrow[r, phantom, bend right=60, "\Downarrow\alpha'"'] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}$$

Moreover, if α_X is an isomorphism for all $X \in \operatorname{ob}\mathcal{C}$, α^{-1} is also a monoidal natural transformation.

Proof. 1_F and $\alpha'\alpha$ are obviously monoidal. Let us compute it for $\beta \star \alpha$:

$$\begin{aligned}
(\beta \star \alpha)_I \, (GF)_I &= \beta_{F'(I)} \, G(\alpha_I) \, G(F_I) \, G_I = \beta_{F'(I)} \, G(F'_I) \, G_I \\
&= G'(F'_I) \, \beta_I \, G_I = G'(F'_I) \, G'_I = (G'F')_I
\end{aligned}$$

and

$$\begin{aligned}
(\beta \star \alpha)_{X \otimes Y} \, \widetilde{GF}_{X,Y} &= \beta_{F'(X \otimes Y)} \, G(\alpha_{X \otimes Y}) \, G(\widetilde{F}_{X,Y}) \, \widetilde{G}_{F(X),F(Y)} \\
&= \beta_{F'(X \otimes Y)} \, G(\widetilde{F'}_{X,Y}) \, G(\alpha_X \otimes \alpha_Y) \, \widetilde{G}_{F(X),F(Y)} \\
&= G'(\widetilde{F'}_{X,Y}) \, \beta_{F'(X) \otimes F'(Y)} \, \widetilde{G}_{F'(X),F'(Y)} \, (G(\alpha_X) \otimes G(\alpha_Y)) \\
&= G'(\widetilde{F'}_{X,Y}) \, \widetilde{G'}_{F'(X),F'(Y)} \, (\beta_{F'(X)} \otimes \beta_{F'(Y)}) \, (G(\alpha_X) \otimes G(\alpha_Y)) \\
&= \widetilde{G'F'}_{X,Y} \, ((\beta \star \alpha)_X \otimes (\beta \star \alpha)_Y).
\end{aligned}$$

Finally, the fact that α^{-1} is monoidal if α is a monoidal natural isomorphism follows directly from the definition.

$\square$

We are now going to focus on monoidal equivalences.

Definition 2.17. Let $\mathcal{C}$ and $\mathcal{D}$ be two monoidal categories. A monoidal equivalence between $\mathcal{C}$ and $\mathcal{D}$ is the data of two monoidal functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ and two monoidal natural isomorphisms $\alpha : GF \overset{\sim}{\Longrightarrow} 1_{\mathcal{C}}$ and $\beta : FG \overset{\sim}{\Longrightarrow} 1_{\mathcal{D}}$. If such a monoidal equivalence exists, we say that $\mathcal{C}$ and $\mathcal{D}$ are monoidally equivalent.

Remark 2.18. By propositions 2.14 and 2.16, we know that this is an equivalence relation on monoidal categories.

In order to prove that $\mathcal{C}$ and $\mathcal{D}$ are monoidally equivalent, we actually do not need to show that G, α and β are monoidal.

Proposition 2.19. Let $\mathcal{C}$ and $\mathcal{D}$ be two monoidal categories and $(F, F_I, \widetilde{F}) : \mathcal{C} \to \mathcal{D}$ be a monoidal functor. Suppose we have a functor $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms $\gamma : GF \overset{\sim}{\Longrightarrow} 1_{\mathcal{C}}$ and $\beta : FG \overset{\sim}{\Longrightarrow} 1_{\mathcal{D}}$. Then, there exists a monoidal structure $(G, G_I, \widetilde{G})$ on G and another natural isomorphism $\alpha : GF \overset{\sim}{\Longrightarrow} 1_{\mathcal{C}}$ such that α and β are monoidal and such that $F(\alpha_X) = \beta_{F(X)}$ and $G(\beta_Y) = \alpha_{G(Y)}$ for all $X \in \operatorname{ob}\mathcal{C}$ and $Y \in \operatorname{ob}\mathcal{D}$ (i.e. α^{-1} and β satisfy the usual triangular identities).

Proof. Since F and G are part of an equivalence, they are essentially surjective on objects, full and faithful. Let $G_I : I \xrightarrow{\sim} G(I)$ be the unique isomorphism such that $F(G_I) = F(I) \xrightarrow{F_I^{-1}} I \xrightarrow{\beta_I^{-1}} FG(I)$ and let $\widetilde{G}$ be the unique family of isomorphisms $\widetilde{G}_{X,Y} : G(X) \otimes G(Y) \xrightarrow{\sim} G(X \otimes Y)$ such that $F(\widetilde{G}_{X,Y}) =$

$$F(G(X) \otimes G(Y)) \xrightarrow{\widetilde{F}^{-1}_{G(X),G(Y)}} FG(X) \otimes FG(Y) \xrightarrow{\beta_X \otimes \beta_Y} X \otimes Y \xrightarrow{\beta^{-1}_{X \otimes Y}} FG(X \otimes Y)$$

for all $X, Y \in \operatorname{ob} \mathcal{D}$.

We want to check that this family is natural. To do so, let $X \xrightarrow{f} X'$ and $Y \xrightarrow{g} Y'$ be two morphisms in $\mathcal{D}$. We have to prove that $\widetilde{G}_{X',Y'}\ (G(f) \otimes G(g)) = G(f \otimes g)\ \widetilde{G}_{X,Y}$. Using naturalities, we can compute

$$\begin{aligned}
F(\widetilde{G}_{X',Y'})\ F(G(f) \otimes G(g)) &= \beta^{-1}_{X' \otimes Y'}\ (\beta_{X'} \otimes \beta_{Y'})\ \widetilde{F}^{-1}_{G(X'),G(Y')}\ F(G(f) \otimes G(g)) \\
&= \beta^{-1}_{X' \otimes Y'}\ (\beta_{X'} \otimes \beta_{Y'})\ (FG(f) \otimes FG(g))\ \widetilde{F}^{-1}_{G(X),G(Y)} \\
&= \beta^{-1}_{X' \otimes Y'}\ (f \otimes g)\ (\beta_X \otimes \beta_Y)\ \widetilde{F}^{-1}_{G(X),G(Y)} \\
&= FG(f \otimes g)\ \beta^{-1}_{X \otimes Y}\ (\beta_X \otimes \beta_Y)\ \widetilde{F}^{-1}_{G(X),G(Y)} \\
&= FG(f \otimes g)\ F(\widetilde{G}_{X,Y}).
\end{aligned}$$

We conclude using the fact that F is faithful. With the same kind of ideas, we can prove that the three diagrams of definition 2.12 commute, making $(G, G_I, \widetilde{G})$ a monoidal functor.

Now, we prove that β is monoidal:

$$\begin{aligned}
\beta_I\ (FG)_I &= \beta_I\ F(G_I)\ F_I \\
&= \beta_I\ \beta_I^{-1}\ F_I^{-1}\ F_I \\
&= 1_I \\
&= (1_{\mathcal{D}})_I
\end{aligned}$$

and

$$\begin{aligned}
\beta_{X \otimes Y}\ \widetilde{FG}_{X,Y} &= \beta_{X \otimes Y}\ F(\widetilde{G}_{X,Y})\ \widetilde{F}_{G(X),G(Y)} \\
&= \beta_{X \otimes Y}\ \beta^{-1}_{X \otimes Y}\ (\beta_X \otimes \beta_Y)\ \widetilde{F}^{-1}_{G(X),G(Y)}\ \widetilde{F}_{G(X),G(Y)} \\
&= \widetilde{(1_{\mathcal{D}})}_{X,Y}\ (\beta_X \otimes \beta_Y)
\end{aligned}$$

for all $X, Y \in \operatorname{ob} \mathcal{D}$. So β is monoidal.

Now, for all $X \in \operatorname{ob} \mathcal{C}$, we define $\alpha_X : GF(X) \xrightarrow{\sim} X$ to be the unique isomorphism such that $F(\alpha_X) = \beta_{F(X)}$. Suppose $Y \in \operatorname{ob} \mathcal{D}$. Since, $\beta_Y\ \beta_{FG(Y)} = \beta_Y\ FG(\beta_Y)$, we know that $F(\alpha_{G(Y)}) = \beta_{FG(Y)} = FG(\beta_Y)$. Therefore, $\alpha_{G(Y)} = G(\beta_Y)$.

It remains to show that $\alpha : GF \xRightarrow{\sim} 1_{\mathcal{C}}$ is a monoidal natural transformation. For the naturality, let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{C}$:

$$F(\alpha_Y)\ FGF(f) = \beta_{F(Y)}\ FGF(f) = F(f)\ \beta_{F(X)} = F(f)\ F(\alpha_X).$$

Hence α is natural since F is faithful.

Eventually, we prove that α is monoidal, using the fact that F is faithful:

$$\begin{aligned} F(\alpha_I)\; F((GF)_I) &= \beta_{F(I)}\; FG(F_I)\; F(G_I) \\ &= F_I\; \beta_I\; \beta_I^{-1}\; F_I^{-1} \\ &= F(1_I) \end{aligned}$$

and

$$\begin{aligned} F(\alpha_{X\otimes Y})\; F(\widetilde{GF}_{X,Y}) &= \beta_{F(X\otimes Y)}\; FG(\widetilde{F}_{X,Y})\; F(\widetilde{G}_{F(X),F(Y)}) \\ &= \widetilde{F}_{X,Y}\; \beta_{F(X)\otimes F(Y)}\; \beta^{-1}_{F(X)\otimes F(Y)}\; (\beta_{F(X)} \otimes \beta_{F(Y)})\; \widetilde{F}^{-1}_{GF(X),GF(Y)} \\ &= \widetilde{F}_{X,Y}\; (F(\alpha_X) \otimes F(\alpha_Y))\; \widetilde{F}^{-1}_{GF(X),GF(Y)} \\ &= F(\alpha_X \otimes \alpha_Y)\; \widetilde{F}_{GF(X),GF(Y)}\; \widetilde{F}^{-1}_{GF(X),GF(Y)} \\ &= F(\alpha_X \otimes \alpha_Y) \end{aligned}$$

for all $X, Y \in \mathrm{ob}\,\mathcal{C}$.

□

Due to this proposition, we have a characterisation of monoidal equivalences.

Corollary 2.20. Let $\mathcal{C}$ and $\mathcal{D}$ be two monoidal categories. They are monoidally equivalent if and only if there exists a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ which is essentially surjective on objects, full and faithful.

Proof. Follows immediately from proposition 2.19 and the axiom of choice.

□

2.4 Adjunctions

In Category Theory, an adjunction is the data of two functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ linked by two natural transformations $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$ satisfying two triangle identities. The same idea is used in a monoidal category to link two objects with two morphisms satisfying two identities.

We present here some basic lemmas about adjunctions in order to use them in the next chapter. A good reference for this is G. Maltsiniotis'paper [12].

Definition 2.21. Let $\mathcal{C}$ be a monoidal category. An adjunction (or duality) (X, X^*, i_X, e_X) in $\mathcal{C}$ is the data of two objects $X, X^* \in \mathrm{ob}\,\mathcal{C}$, a morphism $i_X : I \longrightarrow X \otimes X^*$ and a morphism $e_X : X^* \otimes X \longrightarrow I$ such that the following diagrams commute.

$$\begin{tikzcd} X \arrow[r, "l_X^{-1}"] \arrow[d, "1_X"'] & I \otimes X \arrow[r, "i_X \otimes 1_X"] \arrow[d, phantom, "\circlearrowleft"] & (X \otimes X^*) \otimes X \arrow[d, "a_{X,X^*,X}"] \\ X & X \otimes I \arrow[l, "r_X"] & X \otimes (X^* \otimes X) \arrow[l, "1_X \otimes e_X"] \end{tikzcd}$$

$$\begin{tikzcd} X^* \arrow[r, "r_{X^*}^{-1}"] \arrow[d, "1_{X^*}"'] & X^* \otimes I \arrow[r, "1_{X^*} \otimes i_X"] \arrow[d, phantom, "\circlearrowleft"] & X^* \otimes (X \otimes X^*) \arrow[d, "a^{-1}_{X^*,X,X^*}"] \\ X^* & I \otimes X^* \arrow[l, "l_{X^*}"] & (X^* \otimes X) \otimes X^* \arrow[l, "e_X \otimes 1_{X^*}"] \end{tikzcd}$$

Example 2.22. If $\mathcal{C}$ is a monoidal category, (I, I, r_I^{-1}, r_I) is an adjunction. Indeed,

$$r_I\ (1_I \otimes r_I)\ a_{I,I,I}\ (r_I^{-1} \otimes 1_I)\ l_I^{-1} \ = \ r_I\ r_{I\otimes I}\ r_{I\otimes I}^{-1}\ r_I^{-1} \ = \ 1_I$$

and

$$l_I\ (r_I \otimes 1_I)\ a_{I,I,I}^{-1}\ (1_I \otimes r_I^{-1})\ r_I^{-1} \ = \ r_I\ r_{I\otimes I}\ r_{I\otimes I}^{-1}\ r_I^{-1} \ = \ 1_I.$$

Example 2.23. If (X, X^*, i_X, e_X) and (Y, Y^*, i_Y, e_Y) are two adjunctions in a monoidal category $\mathcal{C}$, then

$$\begin{aligned}(X \otimes Y, Y^* \otimes X^*,\ & a_{X,Y,Y^*\otimes X^*}^{-1}\ (1_X \otimes a_{Y,Y^*,X^*})\ (1_X \otimes (i_Y \otimes 1_{X^*}))\ (1_X \otimes l_{X^*}^{-1})\ i_X, \\ & e_Y\ (1_{Y^*} \otimes l_Y)\ (1_{Y^*} \otimes (e_X \otimes 1_Y))\ (1_{Y^*} \otimes a_{X^*,X,Y}^{-1})\ a_{Y^*,X^*,X\otimes Y})\end{aligned}$$

is an adjunction. To see it, it suffices to prove the two identities using definitions.

The next lemma says that, given an object $X \in \mathcal{C}$, there exists at most one adjunction (X, X^*, i_X, e_X), up to isomorphism.

Lemma 2.24. Let $\mathcal{C}$ be a monoidal category and (X, X^*, i_X, e_X) and $(X, \overline{X}, \alpha_X, \beta_X)$ be two adjunctions in $\mathcal{C}$. There exists a unique isomorphism $\varphi :\ X^* \xrightarrow{\sim} \overline{X}$ such that

$$\begin{tikzcd}
X^* \otimes X \arrow[dd, "\varphi\otimes 1_X"'] \arrow[drr, "e_X"] & & & & X \otimes X^* \arrow[dd, "1_X\otimes\varphi"] \\
& \circlearrowleft & I \arrow[urr, "i_X"] \arrow[drr, "\alpha_X"'] & \circlearrowleft & \\
\overline{X} \otimes X \arrow[urr, "\beta_X"'] & & & & X \otimes \overline{X}
\end{tikzcd}$$

commutes.

Proof. First, we prove the uniqueness. Let φ and ψ be two such isomorphisms. Then,

$$\begin{aligned}
\psi \ &= \ \psi\ l_{X^*}\ (e_X \otimes 1_{X^*})\ a_{X^*,X,X^*}^{-1}\ (1_{X^*} \otimes i_X)\ r_{X^*}^{-1} \\
&= \ l_{\overline{X}}\ (1_I \otimes \psi)\ (e_X \otimes 1_{X^*})\ a_{X^*,X,X^*}^{-1}\ (1_{X^*} \otimes i_X)\ r_{X^*}^{-1} \\
&= \ l_{\overline{X}}\ (e_X \otimes \psi)\ a_{X^*,X,X^*}^{-1}\ (1_{X^*} \otimes i_X)\ r_{X^*}^{-1} \\
&= \ l_{\overline{X}}\ (\beta_X \otimes 1_{\overline{X}})\ ((\varphi \otimes 1_X) \otimes \psi)\ a_{X^*,X,X^*}^{-1}\ (1_{X^*} \otimes i_X)\ r_{X^*}^{-1} \\
&= \ l_{\overline{X}}\ (\beta_X \otimes 1_{\overline{X}})\ a_{\overline{X},X,\overline{X}}^{-1}\ (\varphi \otimes (1_X \otimes \psi))\ (1_{X^*} \otimes i_X)\ r_{X^*}^{-1} \\
&= \ l_{\overline{X}}\ (\beta_X \otimes 1_{\overline{X}})\ a_{\overline{X},X,\overline{X}}^{-1}\ (\varphi \otimes \alpha_X)\ r_{X^*}^{-1} \\
&= \ l_{\overline{X}}\ (\beta_X \otimes 1_{\overline{X}})\ a_{\overline{X},X,\overline{X}}^{-1}\ (1_{\overline{X}} \otimes \alpha_X)\ (\varphi \otimes 1_I)\ r_{X^*}^{-1} \\
&= \ l_{\overline{X}}\ (\beta_X \otimes 1_{\overline{X}})\ a_{\overline{X},X,\overline{X}}^{-1}\ (1_{\overline{X}} \otimes \alpha_X)\ r_{\overline{X}}^{-1}\ \varphi \\
&= \ \varphi.
\end{aligned}$$

It remains to prove the existence. We set φ to be the morphism

$$\begin{tikzcd}
X^* \arrow[r, "r_{X^*}^{-1}"] & X^* \otimes I \arrow[r, "1_{X^*}\otimes\alpha_X"] & X^* \otimes (X \otimes \overline{X}) \arrow[d, "a_{X^*,X,\overline{X}}^{-1}"] \\
\overline{X} & I \otimes \overline{X} \arrow[l, "l_{\overline{X}}"] & (X^* \otimes X) \otimes \overline{X} \arrow[l, "e_X\otimes 1_{\overline{X}}"]
\end{tikzcd}$$

and φ^{-1} to be the morphism

$$\begin{tikzcd}
\overline{X} \arrow[r, "r_{\overline{X}}^{-1}"] & \overline{X} \otimes I \arrow[r, "1_{\overline{X}} \otimes i_X"] & \overline{X} \otimes (X \otimes X^*) \arrow[d, "a_{\overline{X},X,X^*}^{-1}"] \\
X^* & I \otimes X^* \arrow[l, "l_{X^*}"] & (\overline{X} \otimes X) \otimes X^* \arrow[l, "\beta_X \otimes 1_{X^*}"]
\end{tikzcd}$$

Using the axioms, we can show that $\varphi^{-1}\varphi = 1_{X^*}$, $\varphi\varphi^{-1} = 1_{\overline{X}}$, $\beta_X(\varphi \otimes 1_X) = e_X$ and $(1_X \otimes \varphi)i_X = \alpha_X$. Hence, φ is the isomorphism we were looking for.

□

In the last lemma of this section, we prove that monoidal functors preserve adjunctions.

Lemma 2.25. Let $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor between two monoidal categories $\mathcal{C}$ and $\mathcal{D}$. If (X, X^*, i_X, e_X) is an adjunction in $\mathcal{C}$, then

$$(F(X), F(X^*), \widetilde{F}_{X,X^*}^{-1} F(i_X) F_I, F_I^{-1} F(e_X) \widetilde{F}_{X^*,X})$$

is an adjuction in $\mathcal{D}$.

Proof. It suffices to prove the two identities of definition 2.21. We are going to prove the first one by computations. The second one is similar.

$$\begin{aligned}
& r_{F(X)}\ (1_{F(X)} \otimes F_I^{-1})\ (1_{F(X)} \otimes F(e_X))\ (1_{F(X)} \otimes \widetilde{F}_{X^*,X})\ a_{F(X),F(X^*),F(X)} \\
& \qquad (\widetilde{F}_{X,X^*}^{-1} \otimes 1_{F(X)})\ (F(i_X) \otimes 1_{F(X)})\ (F_I \otimes 1_{F(X)})\ l_{F(X)}^{-1} \\
=\ & F(r_X)\ \widetilde{F}_{X,I}\ (1_{F(X)} \otimes F(e_X))\ (1_{F(X)} \otimes \widetilde{F}_{X^*,X})\ a_{F(X),F(X^*),F(X)} \\
& \qquad (\widetilde{F}_{X,X^*}^{-1} \otimes 1_{F(X)})\ (F(i_X) \otimes 1_{F(X)})\ \widetilde{F}_{I,X}^{-1}\ F(l_X^{-1}) \\
=\ & F(r_X)\ F(1_X \otimes e_X)\ \widetilde{F}_{X,X^* \otimes X}\ (1_{F(X)} \otimes \widetilde{F}_{X^*,X})\ a_{F(X),F(X^*),F(X)} \\
& \qquad (\widetilde{F}_{X,X^*}^{-1} \otimes 1_{F(X)})\ \widetilde{F}_{X \otimes X^*,X}^{-1}\ F(i_X \otimes 1_X)\ F(l_X^{-1}) \\
=\ & F(r_X)\ F(1_X \otimes e_X)\ F(a_{X,X^*,X})\ \widetilde{F}_{X \otimes X^*,X}\ (\widetilde{F}_{X,X^*} \otimes 1_{F(X)}) \\
& \qquad (\widetilde{F}_{X,X^*}^{-1} \otimes 1_{F(X)})\ \widetilde{F}_{X \otimes X^*,X}^{-1}\ F(i_X \otimes 1_X)\ F(l_X^{-1}) \\
=\ & F(r_X)\ F(1_X \otimes e_X)\ F(a_{X,X^*,X})\ F(i_X \otimes 1_X)\ F(l_X^{-1}) \\
=\ & 1_{F(X)}.
\end{aligned}$$

□

3 Cat-Groups

What a cat-group is to a group is what a monoidal category is to a monoid. Indeed, in this chapter, we are going to define a cat-group to be a monoidal category where all morphisms and all objects are invertible. Therefore, a cat-group is a groupoid with a (weak) group structure on its objects.

As we said earlier, cat-groups were introduced by P. Deligne in [5] and A. Fröhlich and C. T. C. Wall in [6]. As more recent references, we can cite [1] (Baez-Lauda), [7] (Joyal-Street) and [15] (Vitale).

3.1 Definition and Characterisation

Definition 3.1. A groupoid is a category where all morphisms are isomorphisms.

Definition 3.2. A cat-group $(\mathcal{G}, \otimes, I, l, r, a, *, i, e)$ is the data of

- a groupoid $\mathcal{G}$,
- a monoidal category $(\mathcal{G}, \otimes, I, l, r, a)$,
- an application $* : \mathrm{ob}\,\mathcal{G} \to \mathrm{ob}\,\mathcal{G} : X \mapsto X^*$,
- a family of morphisms $i = \{i_X : I \xrightarrow{\sim} X \otimes X^* \}_{X \in \mathrm{ob}\,\mathcal{G}}$,
- a family of morphisms $e = \{e_X : X^* \otimes X \xrightarrow{\sim} I \}_{X \in \mathrm{ob}\,\mathcal{G}}$

such that (X, X^*, i_X, e_X) is an adjunction for all $X \in \mathrm{ob}\,\mathcal{G}$.

Remark 3.3. By abuse of notation, we will often write $\mathcal{G}$ to mean the cat-group $(\mathcal{G}, \otimes, I, l, r, a, *, i, e)$.

Before we give some examples, we are going to prove a characterisation of cat-groups, in order to make this concept easier to understand.

Lemma 3.4. Let $(\mathcal{C}, \otimes, I, l, r, a)$ be a monoidal category and $X, X^* \in \mathrm{ob}\,\mathcal{C}$. If we have two isomorphisms $i_X : I \xrightarrow{\sim} X \otimes X^*$ and $\beta_X : X^* \otimes X \xrightarrow{\sim} I$, then,

1. each of the following functors is part of an equivalence,

$$\begin{aligned} X \otimes - : \quad \mathcal{C} &\longrightarrow \mathcal{C} \\ Y &\mapsto X \otimes Y \\ f &\mapsto 1_X \otimes f \end{aligned} \qquad\qquad \begin{aligned} - \otimes X : \quad \mathcal{C} &\longrightarrow \mathcal{C} \\ Y &\mapsto Y \otimes X \\ f &\mapsto f \otimes 1_X \end{aligned}$$

2. there exists an isomorphism $e_X : X^* \otimes X \xrightarrow{\sim} I$ such that (X, X^*, i_X, e_X) is an adjunction.

Proof. 1. Let F and G be respectively, the functors $X \otimes -$ and $X^* \otimes -$. We have natural isomorphisms $\alpha : 1_{\mathcal{C}} \overset{\sim}{\Longrightarrow} GF$ and $\gamma : 1_{\mathcal{C}} \overset{\sim}{\Longrightarrow} FG$ defined by

$$\alpha_Y = Y \xrightarrow{l_Y^{-1}} I \otimes Y \xrightarrow{\beta_X^{-1} \otimes 1_Y} (X^* \otimes X) \otimes Y \xrightarrow{a_{X^*,X,Y}} X^* \otimes (X \otimes Y)$$

and

$$\gamma_Y = Y \xrightarrow{l_Y^{-1}} I \otimes Y \xrightarrow{i_X \otimes 1_Y} (X \otimes X^*) \otimes Y \xrightarrow{a_{X,X^*,Y}} X \otimes (X^* \otimes Y)$$

for all $Y \in \operatorname{ob}\mathcal{C}$. Hence $F = X \otimes -$ is part of an equivalence. The proof is similar for $- \otimes X$.

2. We consider the isomorphism

$$X \otimes (X^* \otimes X) \xrightarrow{a^{-1}_{X,X^*,X}} (X \otimes X^*) \otimes X \xrightarrow{i_X^{-1} \otimes 1_X} I \otimes X \xrightarrow{l_X} X \xrightarrow{r_X^{-1}} X \otimes I \, .$$

By what we did above, we know that the functor $X \otimes -$ is full and faithful. So there exists a unique morphism $e_X : X^* \otimes X \longrightarrow I$ such that

$$1_X \otimes e_X = r_X^{-1}\, l_X\, (i_X^{-1} \otimes 1_X)\, a^{-1}_{X,X^*,X}.$$

Again by the fact that $X \otimes -$ is full and faithful, e_X is actually an isomorphism.

By construction, the first diagram of definition 2.21 commutes. So, it remains to prove that the second one commutes. Let us make some computations:

$$\begin{aligned}
&(1_X \otimes l_{X^*})\,(1_X \otimes (e_X \otimes 1_{X^*}))\,(1_X \otimes a^{-1}_{X^*,X,X^*})\,(1_X \otimes (1_{X^*} \otimes i_X))\,(1_X \otimes r^{-1}_{X^*})\\
&= (1_X \otimes l_{X^*})\, a_{X,I,X^*}\, ((1_X \otimes e_X) \otimes 1_{X^*})\, a^{-1}_{X,X^*\otimes X,X^*}\, (1_X \otimes a^{-1}_{X^*,X,X^*})\\
&\quad (1_X \otimes (1_{X^*} \otimes i_X))\,(1_X \otimes r^{-1}_{X^*})\\
&= (1_X \otimes l_{X^*})\, a_{X,I,X^*}\, (r_X^{-1} \otimes 1_{X^*})\,(l_X \otimes 1_{X^*})\,((i_X^{-1} \otimes 1_X) \otimes 1_{X^*})\,(a^{-1}_{X,X^*,X} \otimes 1_{X^*})\\
&\quad a^{-1}_{X,X^*\otimes X,X^*}\,(1_X \otimes a^{-1}_{X^*,X,X^*})\,(1_X \otimes (1_{X^*} \otimes i_X))\,(1_X \otimes r^{-1}_{X^*})\\
&= (l_X \otimes 1_{X^*})\,((i_X^{-1} \otimes 1_X) \otimes 1_{X^*})\, a^{-1}_{X\otimes X^*,X,X^*}\, a^{-1}_{X,X^*,X\otimes X^*}\\
&\quad (1_X \otimes (1_{X^*} \otimes i_X))\,(1_X \otimes r^{-1}_{X^*})\\
&= (l_X \otimes 1_{X^*})\, a^{-1}_{I,X,X^*}\,(i_X^{-1} \otimes 1_{X\otimes X^*})\,(1_{X\otimes X^*} \otimes i_X)\, a^{-1}_{X,X^*,I}\,(1_X \otimes r^{-1}_{X^*})\\
&= l_{X\otimes X^*}\,(1_I \otimes i_X)\,(i_X^{-1} \otimes 1_I)\, r^{-1}_{X\otimes X^*}\\
&= i_X\, l_I\, r_I^{-1}\, i_X^{-1}\\
&= 1_{X\otimes X^*}.
\end{aligned}$$

This proves the second identity of adjunctions since $X \otimes -$ is faithful.

□

We can now state and prove a characterisation of cat-groups.

Proposition 3.5. Let $(\mathcal{G}, \otimes, I, l, r, a)$ be a monoidal category. There exists a cat-group $(\mathcal{G}, \otimes, I, l, r, a, *, i, e)$ if and only if $\mathcal{G}$ is a groupoid and for all $X \in \operatorname{ob}\mathcal{G}$, there exists a $X^* \in \operatorname{ob}\mathcal{G}$ such that $X \otimes X^* \simeq I$.

Proof. The 'only if part' is trivial since $i_X : I \xrightarrow{\sim} X \otimes X^*$ gives the isomorphism. For the 'if part', let $X \in \operatorname{ob} \mathcal{G}$. We can choose a $X^* \in \operatorname{ob} \mathcal{G}$ and some isomorphisms $i_X : I \xrightarrow{\sim} X \otimes X^*$ and $i_{X^*} : I \xrightarrow{\sim} X^* \otimes X^{**}$. By lemma 3.4, it suffices to find a morphism $\beta_X : X^* \otimes X \longrightarrow I$. We can choose

$$\begin{tikzcd}
X^* \otimes X \arrow[r, "1_{X^*} \otimes r_X"] & X^* \otimes (X \otimes I) \arrow[r, "1_{X^*} \otimes (1_X \otimes i_{X^*})"] & X^* \otimes (X \otimes (X^* \otimes X^{**})) \arrow[d, "1_{X^*} \otimes a^{-1}_{X,X^*,X^{**}}"] \\
X^* \otimes X^{**} \arrow[d, "i^{-1}_{X^*}"] & X^* \otimes (I \otimes X^{**}) \arrow[l, "1_{X^*} \otimes l_{X^{**}}"] & X^* \otimes ((X \otimes X^*) \otimes X^{**}) \arrow[l, "1_{X^*} \otimes (i^{-1}_X \otimes 1_{X^{**}})"] \\
I & &
\end{tikzcd}$$

□

Remark 3.6. Notice that it can happen that several choices of $*, i$ and e are possible. Therefore, the notion of cat-group is richer than the notion of monoidal category satisfying the property of proposition 3.5. However, we will prove later (see corollary 3.28) that, up to an equivalence of cat-groups, we can choose any triple $(*, i, e)$ without changing the associated cat-group.

Remark 3.7. As we have written in the beginning of this chapter, we now see that a cat-group is a monoidal category in which every morphism is invertible (is an isomorphism) and each object has a (weak) inverse for the (weak) monoidal structure on the objects.

Definition 3.8. Let $\mathcal{C}$ be a monoidal category. We say that an object $X \in \operatorname{ob} \mathcal{C}$ is weakly invertible if there exists an object $X^* \in \operatorname{ob} \mathcal{C}$ such that $X \otimes X^* \simeq I \simeq X^* \otimes X$. Such a X^* is called a weak inverse for X.

Now, we give some examples of cat-groups, using proposition 3.5.

Example 3.9. Let G be a group. We construct the cat-group $\underline{\mathrm{D}}(G)$: the objects are the elements of G and there is no more morphisms than the identities. The functor $\otimes$ is defined by the group law of G.

Example 3.10. Let A be an abelian group. We define the category $A!$ as follows: I is the unique object and $A!(I, I) = A$ with addition in A as composition. Therefore, by proposition 3.5, the monoidal category $(A!, \otimes, I, 1, 1, 1)$, where the functor $\otimes$ is also defined by addition in A, can be extended to a cat-group. Note that we need A to be abelian in order to prove that $\otimes$ is a functor.

Example 3.11. The monoidal category Ab (as described in 2.3) can not be extended to a cat-group since Ab is not a groupoid.

Example 3.12. If $A \xrightarrow{f} B$ is a morphism of abelian groups, the monoidal category $\underline{\mathrm{Coker}} f$ (see example 2.7) can be extended to a cat-group. Indeed, it is a groupoid since $b' \xrightarrow{-a} b$ is the inverse of $b \xrightarrow{a} b'$ and $-b \in \operatorname{ob} \underline{\mathrm{Coker}} f$ is a weak inverse of b.

Example 3.13. If $\mathcal{C}$ is a monoidal category, we can construct the Picard cat-group of $\mathcal{C}$, denoted $\mathrm{Pic}(\mathcal{C})$, as follows: the objects are the weakly invertible objects of $\mathcal{C}$ and the morphisms are the isomorphisms of $\mathcal{C}$. Compositions, $\otimes$, I, l, r and a are the restrictions of those in $\mathcal{C}$. By proposition 3.5, it can be extended to a cat-group.

If $\mathcal{G}$ is a groupoid, we know that $\mathcal{G}(X,X)$ is a group by composition for all $X \in \operatorname{ob}\mathcal{G}$. We conclude this section with a proposition saying that, if $\mathcal{G}$ is a cat-group, these groups are isomorphic as X runs in $\operatorname{ob}\mathcal{G}$.

Proposition 3.14. Let $\mathcal{G}$ be a cat-group and $X \in \operatorname{ob}\mathcal{G}$. We have two group isomorphisms:

1.
$$\begin{aligned} \gamma_X &: \mathcal{G}(I,I) \to \mathcal{G}(X,X) : \ f \mapsto l_X \ (f \otimes 1_X)\ l_X^{-1} \\ \gamma_X^{-1} &: \mathcal{G}(X,X) \to \mathcal{G}(I,I) : \ g \mapsto i_X^{-1} \ (g \otimes 1_{X^*})\ i_X \end{aligned}$$

2.
$$\begin{aligned} \delta_X &: \mathcal{G}(I,I) \to \mathcal{G}(X,X) : \ f \mapsto r_X \ (1_X \otimes f)\ r_X^{-1} \\ \delta_X^{-1} &: \mathcal{G}(X,X) \to \mathcal{G}(I,I) : \ g \mapsto e_X \ (1_{X^*} \otimes g)\ e_X^{-1} \end{aligned}$$

Proof. 1. These functions are group morphisms since

$$\gamma_X(f)\gamma_X(f') = l_X \ (f \otimes 1_X)\ l_X^{-1}\ l_X \ (f' \otimes 1_X)\ l_X^{-1} = l_X \ (ff' \otimes 1_X)\ l_X^{-1} = \gamma_X(ff')$$

for all $f, f' \in \mathcal{G}(I,I)$ and

$$\gamma_X^{-1}(g)\gamma_X^{-1}(g') = i_X^{-1} \ (g \otimes 1_{X^*})\ i_X \ i_X^{-1} \ (g' \otimes 1_{X^*})\ i_X = i_X^{-1} \ (gg' \otimes 1_{X^*})\ i_X = \gamma_X^{-1}(gg')$$

for all $g, g' \in \mathcal{G}(X,X)$. Moreover, they are injective since $- \otimes X$ and $- \otimes X^*$ are faithful (lemma 3.4.1). So, it remains to show that $\gamma_X^{-1}(\gamma_X(f)) = f$ for all $f \in \mathcal{G}(I,I)$:

$$\begin{aligned} \gamma_X^{-1}(\gamma_X(f)) &= i_X^{-1} \ (l_X \otimes 1_{X^*})\ ((f \otimes 1_X) \otimes 1_{X^*})\ (l_X^{-1} \otimes 1_{X^*})\ i_X \\ &= i_X^{-1} \ l_{X \otimes X^*}\ a_{I,X,X^*}\ ((f \otimes 1_X) \otimes 1_{X^*})\ a_{I,X,X^*}^{-1}\ l_{X \otimes X^*}^{-1}\ i_X \\ &= l_I \ (1_I \otimes i_X^{-1})\ (f \otimes 1_{X \otimes X^*})\ l_{X \otimes X^*}^{-1}\ i_X \\ &= r_I \ (f \otimes 1_I)\ (1_I \otimes i_X^{-1})\ l_{X \otimes X^*}^{-1}\ i_X \\ &= f \ r_I \ l_I^{-1}\ i_X^{-1}\ i_X \\ &= f. \end{aligned}$$

$$\begin{tikzcd}
I \arrow[r, "i_X"] \arrow[rrrrd, bend left, "\gamma_X^{-1}(\gamma_X(f))"] \arrow[rrrddd, bend right, "1_I"'] & X \otimes X^* \arrow[d, "\wr"] \arrow[rrd, phantom, "\circlearrowleft"] & & & \\
& (I \otimes X) \otimes X^* \arrow[r, "(f \otimes 1_X) \otimes 1_{X^*}"] \arrow[d, "\wr"] \arrow[rd, phantom, "\circlearrowleft"] & (I \otimes X) \otimes X^* \arrow[r, "\sim"] \arrow[d, "\wr"] \arrow[rd, phantom, "\circlearrowleft"] & X \otimes X^* \arrow[r, "i_X^{-1}"] & I \\
& I \otimes (X \otimes X^*) \arrow[r, "f \otimes 1_{X \otimes X^*}"] \arrow[rd, "1_I \otimes i_X^{-1}"'] & I \otimes (X \otimes X^*) \arrow[r, "1_I \otimes i_X^{-1}"] \arrow[d, phantom, "\circlearrowleft"] & I \otimes I \arrow[ru, "\sim"] \arrow[d, phantom, "\circlearrowleft"] & \\
& \arrow[u, phantom, "\circlearrowleft"] & I \otimes I \arrow[ru] \arrow[r, "f \otimes 1_I", "\sim"'] & I \arrow[ruu, bend right, "f"'] &
\end{tikzcd}$$

2. Similar.

□

Corollary 3.15. If $\mathcal{G}$ is a cat-group and $X \in \operatorname{ob}\mathcal{G}$, then $\mathcal{G}(X,X)$ is an abelian group.

Proof. By proposition 3.14, $\mathcal{G}(X,X) \simeq \mathcal{G}(I,I)$. But $\mathcal{G}(I,I)$ is abelian by proposition 2.11.

□

3.2 Cat-Group Functors

Since a cat-group has more structure than a monoidal category, a suitable functor between cat-groups has to satisfy more axioms than a monoidal functor, i.e. it has to preserve the weak inverse X^*. We will see in this section that a cat-group functor is actually nothing but a monoidal functor. The same work will be done for cat-group natural transformations.

Definition 3.16. Let $\mathcal{G}$ and $\mathcal{H}$ be two cat-groups. A cat-group functor

$$(F, F_I, \widetilde{F}, F^*) : \mathcal{G} \to \mathcal{H}$$

is the data of

- a monoidal functor $(F, F_I, \widetilde{F}) : \mathcal{G} \to \mathcal{H}$,
- a family of isomorphisms $F^* = \{F^*_X : \ F(X)^* \xrightarrow{\sim} F(X^*)\ \}_{X \in \operatorname{ob}\mathcal{G}}$

such that, for all $X \in \operatorname{ob}\mathcal{G}$, the following diagrams commute:

$$\begin{tikzcd}
I \arrow[r, "i_{F(X)}"] \arrow[d, "F_I"'] & F(X) \otimes F(X)^* \arrow[r, "1_{F(X)} \otimes F^*_X"] \arrow[d, phantom, "\circlearrowleft"] & F(X) \otimes F(X^*) \arrow[d, "\widetilde{F}_{X,X^*}"] \\
F(I) \arrow[rr, "F(i_X)"'] & & F(X \otimes X^*)
\end{tikzcd}$$

$$\begin{tikzcd}
F(X)^* \otimes F(X) \arrow[r, "F^*_X \otimes 1_{F(X)}"] \arrow[d, "e_{F(X)}"'] & F(X^*) \otimes F(X) \arrow[r, "\widetilde{F}_{X^*,X}"] \arrow[d, phantom, "\circlearrowleft"] & F(X^* \otimes X) \arrow[d, "F(e_X)"] \\
I \arrow[rr, "F_I"'] & & F(I)
\end{tikzcd}$$

Remark 3.17. As usual, by abuse of notation, we will often write F to mean the cat-group functor $(F, F_I, \widetilde{F}, F^*)$.

Definition 3.18. Let $(F, F_I, \widetilde{F}, F^*), (G, G_I, \widetilde{G}, G^*) : \mathcal{G} \to \mathcal{H}$ be two cat-group functors between the two cat-groups $\mathcal{G}$ and $\mathcal{H}$. A cat-group natural transformation

$$\alpha : (F, F_I, \widetilde{F}, F^*) \Rightarrow (G, G_I, \widetilde{G}, G^*)$$

is a monoidal natural transformation $\alpha : (F, F_I, \widetilde{F}) \Rightarrow (G, G_I, \widetilde{G})$ satisfying for all $X \in \operatorname{ob}\mathcal{G}$

$$\begin{aligned}\alpha_{X^*}\ F^*_X = G^*_X\ l_{G(X)^*}\ (e_{F(X)} \otimes 1_{G(X)^*})\ a^{-1}_{F(X)^*,F(X),G(X)^*} \\ (1_{F(X)^*} \otimes (\alpha_X^{-1} \otimes 1_{G(X)^*}))\ (1_{F(X)^*} \otimes i_{G(X)})\ r^{-1}_{F(X)^*}.\end{aligned}$$

Remark 3.19. If $\mathcal{G}$ is a cat-group, we can define the functor

$$\begin{array}{cccc} (-)^* : & \mathcal{G}^{\mathrm{op}} & \longrightarrow & \mathcal{G} \\ & X & \longmapsto & X^* \\ & \left(X \xrightarrow{f} Y \right) & \longmapsto & \left(Y^* \xrightarrow{f^*} X^* \right) \end{array}$$

where f^* is

$$\begin{tikzcd} Y^* \arrow[r, "r_{Y^*}^{-1}"] & Y^* \otimes I \arrow[r, "1_{Y^*} \otimes i_X"] & Y^* \otimes (X \otimes X^*) \arrow[r, "1_{Y^*} \otimes (f \otimes 1_{X^*})"] & Y^* \otimes (Y \otimes X^*) \arrow[d, "a_{Y^*,Y,X^*}^{-1}"] \\ X^* & & I \otimes X^* \arrow[ll, "l_{X^*}"'] & (Y^* \otimes Y) \otimes X^* \arrow[l, "e_Y \otimes 1_{X^*}"'] \end{tikzcd}$$

The fact that $(-)^*$ is a functor follows from some (quite long) computations using definitions.

We can now rewrite the condition in definition 3.18 as the commutation of

$$\begin{tikzcd} F(X)^* \arrow[r, "F_X^*"] \arrow[d, "(\alpha_X^{-1})^*"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X^*) \arrow[d, "\alpha_{X^*}"] \\ G(X)^* \arrow[r, "G_X^*"'] & G(X^*) \end{tikzcd}$$

for all $X \in \mathrm{ob}\,\mathcal{G}$.

As announced earlier, we now prove that a cat-group functor is exactly a monoidal functor.

Proposition 3.20. Let $(F, F_I, \widetilde{F}) : \mathcal{G} \to \mathcal{H}$ be a monoidal functor between the two cat-groups $\mathcal{G}$ and $\mathcal{H}$. There exists a unique family of isomorphisms

$$F^* = \{F_X^* : \ F(X)^* \xrightarrow{\sim} F(X^*) \ \}_{X \in \mathrm{ob}\,\mathcal{G}}$$

such that $(F, F_I, \widetilde{F}, F^*)$ is a cat-group functor.

Proof. By lemma 2.25, we know that $(F(X), F(X^*), \widetilde{F}_{X,X^*}^{-1} F(i_X) F_I, F_I^{-1} F(e_X) \widetilde{F}_{X^*,X})$ is an adjunction for all $X \in \mathcal{G}$. We conclude by lemma 2.24.

□

Remark 3.21. Due to this proposition, we see that, speaking about cat-groups, there is no difference between a monoidal functor and a cat-group functor. So, we will not talk about cat-group functors anymore, but only about monoidal functors.

We would like to prove the same thing for natural transformations, i.e. that every monoidal natural transformation is a cat-group natural transformation. In order to do so, we need some lemmas.

Lemma 3.22. Let $\mathcal{G}$ be a cat-group and $X \xrightarrow{f} Y$ a morphism in $\mathcal{G}$. The following diagrams commute.

$$\begin{tikzcd} X^* \otimes X \arrow[rr, "(f^{-1})^* \otimes f"] \arrow[dr, "e_X"'] & \circlearrowleft & Y^* \otimes Y \arrow[dl, "e_Y"] \\ & I & \end{tikzcd} \qquad \begin{tikzcd} X \otimes X^* \arrow[rr, "f \otimes (f^{-1})^*"] & \circlearrowleft & Y \otimes Y^* \\ & I \arrow[ul, "i_X"] \arrow[ur, "i_Y"'] & \end{tikzcd}$$

Proof. We prove the first one, the second one is similar:

$$\begin{aligned}
e_Y\ ((f^{-1})^* \otimes f) &= e_Y\ (l_{Y^*} \otimes 1_Y)\ ((e_X \otimes 1_{Y^*}) \otimes 1_Y)\ (a_{X^*,X,Y^*} \otimes 1_Y) \\
&\quad ((1_{X^*} \otimes (f^{-1} \otimes 1_{Y^*})) \otimes 1_Y)\ ((1_{X^*} \otimes i_Y) \otimes 1_Y)\ (r_{X^*}^{-1} \otimes 1_Y)\ (1_{X^*} \otimes f) \\
&= e_Y\ l_{Y^* \otimes Y}\ a_{I,Y^*,Y}\ ((e_X \otimes 1_{Y^*}) \otimes 1_Y)\ (((1_{X^*} \otimes f^{-1}) \otimes 1_{Y^*}) \otimes 1_Y) \\
&\quad (a_{X^*,Y,Y^*}^{-1} \otimes 1_Y)\ ((1_{X^*} \otimes i_Y) \otimes 1_Y)\ (r_{X^*}^{-1} \otimes 1_Y)\ (1_{X^*} \otimes f) \\
&= l_I\ (1_I \otimes e_Y)\ (e_X \otimes 1_{Y^* \otimes Y})\ a_{X^* \otimes X,Y^*,Y}\ (((1_{X^*} \otimes f^{-1}) \otimes 1_{Y^*}) \otimes 1_Y) \\
&\quad (a_{X^*,Y,Y^*}^{-1} \otimes 1_Y)\ ((1_{X^*} \otimes i_Y) \otimes 1_Y)\ (r_{X^*}^{-1} \otimes 1_Y)\ (1_{X^*} \otimes f) \\
&= l_I\ (e_X \otimes 1_I)\ (1_{X^* \otimes X} \otimes e_Y)\ ((1_{X^*} \otimes f^{-1}) \otimes 1_{Y^* \otimes Y})\ a_{X^* \otimes Y,Y^*,Y} \\
&\quad (a_{X^*,Y,Y^*}^{-1} \otimes 1_Y)\ ((1_{X^*} \otimes i_Y) \otimes 1_Y)\ (r_{X^*}^{-1} \otimes 1_Y)\ (1_{X^*} \otimes f) \\
&= r_I\ (e_X \otimes 1_I)\ ((1_{X^*} \otimes f^{-1}) \otimes 1_I)\ (1_{X^* \otimes Y} \otimes e_Y)\ a_{X^*,Y,Y^* \otimes Y}^{-1} \\
&\quad (1_{X^*} \otimes a_{Y,Y^*,Y})\ a_{X,Y \otimes Y^*,Y}\ ((1_{X^*} \otimes i_Y) \otimes 1_Y)\ (r_{X^*}^{-1} \otimes 1_Y)\ (1_{X^*} \otimes f) \\
&= e_X\ (1_{X^*} \otimes f^{-1})\ r_{X^* \otimes Y}\ a_{X^*,Y,I}^{-1}\ (1_{X^*} \otimes (1_Y \otimes e_Y))\ (1_{X^*} \otimes a_{Y,Y^*,Y}) \\
&\quad (1_{X^*} \otimes (i_Y \otimes 1_Y))\ a_{X^*,I,Y}\ (r_{X^*}^{-1} \otimes 1_Y)\ (1_{X^*} \otimes f) \\
&= e_X\ (1_{X^*} \otimes f^{-1})\ r_{X^* \otimes Y}\ a_{X^*,Y,I}^{-1}\ (1_{X^*} \otimes r_Y^{-1})\ (1_{X^*} \otimes l_Y)\ a_{X^*,I,Y} \\
&\quad (r_{X^*}^{-1} \otimes 1_Y)\ (1_{X^*} \otimes f) \\
&= e_X.
\end{aligned}$$

□

Lemma 3.23. Let $X \xrightarrow{f} Y$ be a morphism in a cat-group $\mathcal{G}$. We have 1-1 correspondences

$$\mathcal{G}(A,B) \xrightarrow{-\otimes f} \mathcal{G}(A \otimes X, B \otimes Y) \quad \text{and} \quad \mathcal{G}(A,B) \xrightarrow{f \otimes -} \mathcal{G}(X \otimes A, Y \otimes B)$$

for all $A, B \in \operatorname{ob} \mathcal{G}$.

Proof. Let us prove it for the first one, the second one is similar. We show that

$$\mathcal{G}(A \otimes X, B \otimes Y) \to \mathcal{G}(A,B) : g \mapsto r_B\ (1_B \otimes i_Y^{-1})\ a_{B,Y,Y^*}\ (g \otimes (f^{-1})^*)\ a_{A,X,X^*}^{-1}\ (1_A \otimes i_X)\ r_A^{-1}$$

is the inverse of $- \otimes f$. If $h \in \mathcal{G}(A,B)$,

$$\begin{aligned}
&r_B\ (1_B \otimes i_Y^{-1})\ a_{B,Y,Y^*}\ ((h \otimes f) \otimes (f^{-1})^*)\ a_{A,X,X^*}^{-1}\ (1_A \otimes i_X)\ r_A^{-1} \\
&= r_B\ (1_B \otimes i_Y^{-1})\ (h \otimes (f \otimes (f^{-1})^*))\ (1_A \otimes i_X)\ r_A^{-1} \\
&= r_B\ (1_B \otimes i_Y^{-1})\ (h \otimes i_Y)\ r_A^{-1} \\
&= r_B\ (h \otimes 1_I)\ r_A^{-1} \\
&= h
\end{aligned}$$

using lemma 3.22. We can also prove that, if $g \in \mathcal{G}(A \otimes X, B \otimes Y)$,

$$\begin{aligned}
g = (r_B \otimes 1_Y)\ ((1_B \otimes i_Y^{-1}) \otimes 1_Y)\ (a_{B,Y,Y^*} \otimes 1_Y)\ ((g \otimes (f^{-1})^*) \otimes f) \\
(a_{A,X,X^*}^{-1} \otimes 1_X)\ ((1_A \otimes i_X) \otimes 1_X)\ (r_A^{-1} \otimes 1_X).
\end{aligned}$$

This follows from definitions and lemma 3.22.

□

The next lemma says that diagrams of lemma 3.22 uniquely determine $(f^{-1})^*$ for a fixed f and vice-versa.

Lemma 3.24. Let $X \xrightarrow{f} Y$ be a morphism in a cat-group $\mathcal{G}$.

1. If $g \in \mathcal{G}(X, Y)$ is such that $e_Y\ ((f^{-1})^* \otimes g) = e_X$ or $(g \otimes (f^{-1})^*)\ i_X = i_Y$, then $g = f$.

2. If $h \in \mathcal{G}(X^*, Y^*)$ is such that $e_Y\ (h \otimes f) = e_X$ or $(f \otimes h)\ i_X = i_Y$, then $h = (f^{-1})^*$.

Proof. Suppose we have a $g \in \mathcal{G}(X, Y)$ such that $e_Y\ ((f^{-1})^* \otimes g) = e_X$. By lemma 3.22, $e_Y\ ((f^{-1})^* \otimes g) = e_Y\ ((f^{-1})^* \otimes f)$. Thus $(f^{-1})^* \otimes g = (f^{-1})^* \otimes f$. We conclude by lemma 3.23. The other ones are similar.

□

We can now prove that a monoidal natural transformation is actually a cat-group natural transformation.

Proposition 3.25. Let $F, G : \mathcal{G} \to \mathcal{H}$ be two monoidal functors between the cat-groups $\mathcal{G}$ and $\mathcal{H}$. Let $\alpha : F \Rightarrow G$ be a monoidal natural transformation. Then, α is a cat-group natural transformation.

Proof. We have to prove $\alpha_{X^*} F^*_X = G^*_X (\alpha_X^{-1})^*$. By lemma 3.24, it suffices to prove $e_{G(X)}\ ((G^*_X)^{-1} \otimes 1_{G(X)})\ (\alpha_{X^*} \otimes 1_{G(X)})\ (F^*_X \otimes 1_{G(X)})\ (1_{F(X)^*} \otimes \alpha_X) = e_{F(X)}$:

$$
\begin{aligned}
& e_{G(X)}\ ((G^*_X)^{-1} \otimes 1_{G(X)})\ (\alpha_{X^*} \otimes 1_{G(X)})\ (F^*_X \otimes 1_{G(X)})\ (1_{F(X)^*} \otimes \alpha_X) \\
&= G_I^{-1}\ G(e_X)\ \widetilde{G}_{X^*,X}\ (\alpha_{X^*} \otimes 1_G(X))\ (1_{F(X^*)} \otimes \alpha_X)\ (F^*_X \otimes 1_{F(X)}) \\
&= G_I^{-1}\ G(e_X)\ \widetilde{G}_{X^*,X}\ (\alpha_{X^*} \otimes \alpha_X)\ \widetilde{F}^{-1}_{X^*,X}\ F(e_X)^{-1}\ F_I\ e_{F(X)} \\
&= G_I^{-1}\ G(e_X)\ \alpha_{X^* \otimes X}\ F(e_X)^{-1}\ F_I\ e_{F(X)} \\
&= G_I^{-1}\ \alpha_I\ F_I\ e_{F(X)} \\
&= e_{F(X)}.
\end{aligned}
$$

□

Remark 3.26. Due to this proposition, we can give up the notion of cat-group natural transformation and only consider monoidal natural transformations.

Remark 3.27. We could have defined cat-group equivalences between cat-groups, but we now see that it is actually the same as monoidal equivalences.

Corollary 3.28. Let $(\mathcal{G}, \otimes, I, l, r, a)$ be a monoidal category where $\mathcal{G}$ is a groupoid and every object has weak inverse. Consider two cat-group structures on $\mathcal{G}$: $(\mathcal{G}, \otimes, I, l, r, a, *, i, e)$ and $(\mathcal{G}, \otimes, I, l, r, a, *', i', e')$. They are monoidally equivalent.

Proof. It suffices to consider the monoidal functor $1_{\mathcal{G}}$.

□

We can actually do better. Indeed, if we consider cat-groups, a monoidal functor is a pair $(F, \widetilde{F})$ where $\widetilde{F}$ commutes with a. Thus, we do not need to construct the isomorphism F_I anymore, it will come uniquely with $(F, \widetilde{F})$. Moreover, to check that a natural transformation is monoidal, we do not have to check $\alpha_I\ F_I = G_I$, it follows from $\widetilde{G}_{X,Y}\ (\alpha_X \otimes \alpha_Y) = \alpha_{X \otimes Y}\ \widetilde{F}_{X,Y}$.

Proposition 3.29. Let $\mathcal{G}$ and $\mathcal{H}$ be two cat-groups. If $(F, \widetilde{F})$ is pair where $F : \mathcal{G} \to \mathcal{H}$ is a functor and $\widetilde{F}$ is a family of natural isomorphisms

$$\widetilde{F} = \{\widetilde{F}_{X,Y} : \ F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y) \}_{X,Y \in \operatorname{ob} \mathcal{G}}$$

such that

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[rr, "a_{F(X),F(Y),F(Z)}"] \arrow[d, "\widetilde{F}_{X,Y} \otimes 1_{F(Z)}"'] & & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[d, "1_{F(X)} \otimes \widetilde{F}_{Y,Z}"] \\
F(X \otimes Y) \otimes F(Z) \arrow[d, "\widetilde{F}_{X \otimes Y, Z}"'] & \circlearrowleft & F(X) \otimes F(Y \otimes Z) \arrow[d, "\widetilde{F}_{X, Y \otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[rr, "F(a_{X,Y,Z})"'] & & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

commutes for all $X, Y, Z \in \operatorname{ob} \mathcal{G}$, then, there exists a unique isomorphism $F_I : \ I \xrightarrow{\sim} F(I)$ such that $(F, F_I, \widetilde{F})$ is a monoidal functor.

Proof. By lemma 3.4, the functor $F(I) \otimes - : \mathcal{H} \to \mathcal{H}$ is an equivalence. Thus, there exists a unique isomorphism $F_I : \ I \xrightarrow{\sim} F(I)$ such that

$$\begin{tikzcd}
F(I) \otimes I \arrow[r, "1_{F(I)} \otimes F_I"] \arrow[d, "r_{F(I)}"'] \arrow[dr, phantom, "\circlearrowleft"] & F(I) \otimes F(I) \arrow[d, "\widetilde{F}_{I,I}"] \\
F(I) & F(I \otimes I) \arrow[l, "F(r_I)"]
\end{tikzcd}$$

commutes. It remains to prove that, for all $X \in \operatorname{ob} \mathcal{G}$, $F(r_X)\ \widetilde{F}_{X,I}\ (1_{F(X)} \otimes F_I) = r_{F(X)}$ and $F(l_X)\ \widetilde{F}_{I,X}\ (F_I \otimes 1_{F(X)}) = l_{F(X)}$. Let $X \in \operatorname{ob} \mathcal{G}$. Let us do some computations:

$$\begin{aligned}
1_{F(I)} \otimes l_{F(X)} &= (r_{F(I)} \otimes 1_{F(X)})\ a^{-1}_{F(I),I,F(X)} \\
&= (F(r_I) \otimes 1_{F(X)})\ (\widetilde{F}_{I,I} \otimes 1_{F(X)})\ ((1_{F(I)} \otimes F_I) \otimes 1_{F(X)})\ a^{-1}_{F(I),I,F(X)} \\
&= (F(r_I) \otimes 1_{F(X)})\ (\widetilde{F}_{I,I} \otimes 1_{F(X)})\ a^{-1}_{F(I),F(I),F(X)}\ (1_{F(I)} \otimes (F_I \otimes 1_{F(X)})) \\
&= (F(r_I) \otimes 1_{F(X)})\ \widetilde{F}^{-1}_{I \otimes I, X}\ F(a_{I,I,X})^{-1}\ \widetilde{F}_{I, I \otimes X}\ (1_{F(I)} \otimes \widetilde{F}_{I,X}) \\
&\quad\ (1_{F(I)} \otimes (F_I \otimes 1_{F(X)})) \\
&= \widetilde{F}^{-1}_{I,X}\ F(r_I \otimes 1_X)\ F(a^{-1}_{I,I,X})\ \widetilde{F}_{I,I \otimes X}\ (1_{F(I)} \otimes \widetilde{F}_{I,X})\ (1_{F(I)} \otimes (F_I \otimes 1_{F(X)})) \\
&= \widetilde{F}^{-1}_{I,X}\ F(1_I \otimes l_X)\ \widetilde{F}_{I,I \otimes X}\ (1_{F(I)} \otimes \widetilde{F}_{I,X})\ (1_{F(I)} \otimes (F_I \otimes 1_{F(X)})) \\
&= (1_{F(I)} \otimes F(l_X))\ (1_{F(I)} \otimes \widetilde{F}_{I,X})\ (1_{F(I)} \otimes (F_I \otimes 1_{F(X)})).
\end{aligned}$$

Therefore, $l_{F(X)} = F(l_X)\ \widetilde{F}_{I,X}\ (F_I \otimes 1_{F(X)})$ since $F(I) \otimes - : \mathcal{H} \to \mathcal{H}$ is an equivalence. In particular, F_I is the unique morphism such that $l_{F(I)} = F(l_I)\ \widetilde{F}_{I,I}\ (F_I \otimes 1_{F(I)})$. Thus, we can show in a similar way that $r_{F(X)} = F(r_X)\ \widetilde{F}_{X,I}\ (1_{F(X)} \otimes F_I)$.

□

Proposition 3.30. Let $F, G : \mathcal{G} \to \mathcal{H}$ be two monoidal functors between the cat-groups $\mathcal{G}$ and $\mathcal{H}$. If $\alpha : F \Rightarrow G$ is a natural transformation such that

$$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "\widetilde{F}_{X,Y}"] \arrow[d, "\alpha_X \otimes \alpha_Y"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X \otimes Y) \arrow[d, "\alpha_{X \otimes Y}"] \\ G(X) \otimes G(Y) \arrow[r, "\widetilde{G}_{X,Y}"'] & G(X \otimes Y) \end{tikzcd}$$

commutes for all $X, Y \in \operatorname{ob}\mathcal{G}$, then, α is a monoidal natural transformation.

Proof. We have to prove that $\alpha_I \; F_I = G_I$. By lemma 3.23, it suffices to show

$$(\alpha_I \otimes \alpha_I) \; (F_I \otimes 1_{F(I)}) = G_I \otimes \alpha_I.$$

$$\begin{aligned} (\alpha_I \otimes \alpha_I) \; (F_I \otimes 1_{F(I)}) &= (\alpha_I \otimes \alpha_I) \; \widetilde{F}_{I,I}^{-1} \; F(l_I^{-1}) \; l_{F(I)} \\ &= \widetilde{G}_{I,I}^{-1} \; \alpha_{I \otimes I} \; F(l_I^{-1}) \; l_{F(I)} \\ &= \widetilde{G}_{I,I}^{-1} \; G(l_I^{-1}) \; \alpha_I \; l_{F(I)} \\ &= \widetilde{G}_{I,I}^{-1} \; G(l_I^{-1}) \; l_{G(I)} \; (1_I \otimes \alpha_I) \\ &= (G_I \otimes 1_{G(I)}) \; (1_I \otimes \alpha_I) \\ &= G_I \otimes \alpha_I. \end{aligned}$$

□

What we have done in this section can be resumed thanks to our analogy (monoidal categories - monoids) and (cat-groups - groups). Indeed, in one hand, a monoid is a set with a composition law and an identity. Thus, a morphism of monoids has to preserve both of them. On the other hand, a group is a monoid with an inverse for each element. So, a morphism of groups has to preserve the composition law, the identity and inverses. Hence, it seems we need three axioms to define a morphism of group. However, it is well-know that, the fact that a function preserves the composition law is enough to prove it is a morphism of group. It is exactly what we have done in this section for cat-groups.

4 2-Categories

A 2-category is a category $\mathcal{C}$ where all $\mathcal{C}(A,B)$'s are themselves categories. In other words, a 2-category is a category with arrows between morphisms. This chapter is a quick introduction to 2-categories and the different kinds of functors between them. We introduce them in order to define the 2-category of small cat-groups and study it in Chapter 5. We can cite Chapter 7 of Borceux's book [3] as a good reference for 2-categories.

4.1 2-Categories and 2-Functors

Definition 4.1. A 2-category is a category $\mathcal{C}$ such that

- for all objects A and B, $\mathcal{C}(A,B)$ is a small category,
- for all morphisms $\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & C \end{tikzcd}$, we have an application

$$\mathcal{C}(A,B)(f,f') \times \mathcal{C}(B,C)(g,g') \longrightarrow \mathcal{C}(A,C)(gf, g'f') : (\alpha,\beta) \longmapsto \beta \star \alpha,$$

satisfying the axioms:

- $1_g \star 1_f = 1_{gf}$ for all diagrams $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow 1_g"] & C \end{tikzcd}$,
- $\alpha \star 1_{1_A} = \alpha = 1_{1_B} \star \alpha$ for all diagrams $\begin{tikzcd} A \arrow[r, bend left, "1_A"] \arrow[r, bend right, "1_A"'] \arrow[r, phantom, "\Downarrow 1_{1_A}"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow \alpha"] & B \arrow[r, bend left, "1_B"] \arrow[r, bend right, "1_B"'] \arrow[r, phantom, "\Downarrow 1_{1_B}"] & B \end{tikzcd}$,
- $\gamma \star (\beta \star \alpha) = (\gamma \star \beta) \star \alpha$ for all diagrams $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow \alpha"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow \beta"] & C \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow \gamma"] & D \end{tikzcd}$,
- $(\psi \star \beta) \circ (\varphi \star \alpha) = (\psi \circ \varphi) \star (\beta \circ \alpha)$ for all diagrams $\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "f'"] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=3, "\Downarrow \alpha"] \arrow[r, phantom, shift right=3, "\Downarrow \beta"] & B \arrow[r, bend left=60, "g"] \arrow[r, "g'"] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=3, "\Downarrow \varphi"] \arrow[r, phantom, shift right=3, "\Downarrow \psi"] & C \end{tikzcd}$.

Remark 4.2.
- We call a morphism in the category $\mathcal{C}$ a '1-cell'.
- A morphism in any category $\mathcal{C}(A,B)$ is a '2-cell'.
- An object of $\mathcal{C}$ is sometimes called a '0-cell'.
- A 2-isomorphism is an invertible 2-cell.

- We may write $\mathcal{C}(f,g)$ instead of $\mathcal{C}(A,B)(f,g)$ for suitable 1-cells f and g.
- The last axiom of definition 4.1 is called the 'interchange law'.
- A 2-category $\mathcal{C}$ is said to be small if $\mathrm{ob}\,\mathcal{C}$ is a set.

Example 4.3. Any category $\mathcal{C}$ can be turned into a 2-category: it suffices to think $\mathcal{C}(A,B)$ as a discrete category for all $A, B \in \mathrm{ob}\,\mathcal{C}$.

Example 4.4. Let Gp be the category of groups. We can turn Gp into a 2-category in the following way: given two group morphisms $f, g : G \to H$, a 2-cell $\alpha : f \longrightarrow g$ is an element of H such that $g(x) = \alpha \cdot f(x) \cdot \alpha^{-1}$ for all $x \in G$. Composition of $f \xrightarrow{\alpha} g \xrightarrow{\beta} h$ in $\mathrm{Gp}(G,H)$ is given by $f \xrightarrow{\beta\cdot\alpha} h$ and the identity is the unit element. Given 2-cells
$\begin{tikzcd} G \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"] & H \arrow[r, bend left, "g"{name=U2}] \arrow[r, bend right, "g'"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\beta"] & K \end{tikzcd}$, we define $gf \xrightarrow{\beta\star\alpha} g'f'$ as $g'(\alpha)\cdot\beta = \beta\cdot g(\alpha)$. By easy computations, we can check that this defines a 2-category.

Example 4.5. Small categories, functors and natural transformations form the 2-category CAT. In the same way, we have the 2-category MC (respectively CG) of small monoidal categories (respectively of small cat-groups), monoidal functors and monoidal natural transformations.

We are now going to define the suitable notions of 2-functors between 2-categories and of 2-natural transformations between 2-functors. Moreover, we will also define arrows between those 2-natural transformations, called modifications.

Definition 4.6. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories. A 2-functor $F : \mathcal{C} \to \mathcal{D}$ is the data of:

- a functor $F : \mathcal{C} \to \mathcal{D}$,
- for each $A, B \in \mathrm{ob}\,\mathcal{C}$, a functor $F_{A,B} : \mathcal{C}(A,B) \to \mathcal{D}(F(A), F(B))$ such that $F_{A,B}(f) = F(f)$ for all $f \in \mathcal{C}(A,B)$

satisfying $F_{A,C}(\beta \star \alpha) = F_{B,C}(\beta) \star F_{A,B}(\alpha)$ for all diagrams $\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"] & B \arrow[r, bend left, "g"{name=U2}] \arrow[r, bend right, "g'"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\beta"] & C \end{tikzcd}$.

Remark 4.7. By abuse of notation, we will write F to mean $F_{A,B}$.

Lemma 4.8. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be 2-categories and $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be 2-functors. Then, $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ and $\mathcal{C} \xrightarrow{GF} \mathcal{D}$ are 2-functors.

Proof. Follows immediately from the definition.

□

Definition 4.9. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories and $F, G : \mathcal{C} \to \mathcal{D}$ be two 2-functors. A 2-natural transformation $\theta : F \Rightarrow G$ is a natural transformation $\theta : F \Rightarrow G$ between the underlying functors such that $1_{\theta_B} \star F(\alpha) = G(\alpha) \star 1_{\theta_A}$ for all $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$.

$$\begin{tikzcd}[column sep=large, row sep=large]
F(A) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\Downarrow F(\alpha)"] \arrow[d, bend right, "\theta_A"'] \arrow[d, bend left, "\theta_A"] \arrow[d, phantom, "\Leftarrow 1_{\theta_A}"] & F(B) \arrow[d, bend right, "\theta_B"'] \arrow[d, bend left, "\theta_B"] \arrow[d, phantom, "\Leftarrow 1_{\theta_B}"] \\
G(A) \arrow[r, bend left, "G(f)"] \arrow[r, bend right, "G(g)"'] \arrow[r, phantom, "\Downarrow G(\alpha)"] & G(B)
\end{tikzcd}$$

Lemma 4.10. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be 2-categories. Let $F, F', F'' : \mathcal{C} \to \mathcal{D}$ and $G, G' : \mathcal{D} \to \mathcal{E}$ be 2-functors and let $\theta : F \Rightarrow F'$, $\theta' : F' \Rightarrow F''$ and $\psi : G \Rightarrow G'$ be 2-natural transformations. Then, $1_F : F \Rightarrow F$, $\theta'\theta : F \Rightarrow F''$ and $\psi \star \theta : GF \Rightarrow G'F'$ are 2-natural transformations.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left=2, "F"] \arrow[r, shift right=2, "F'"'] \arrow[r, phantom, "\Downarrow\theta"] \arrow[r, bend right=90, looseness=2, "F''"'] \arrow[r, phantom, shift right=6, "\Downarrow\theta'"] & \mathcal{D} \arrow[r, shift left=2, "G"] \arrow[r, shift right=2, "G'"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathcal{E}
\end{tikzcd}$$

Moreover, if θ_A is an isomorphism for all $A \in \operatorname{ob}\mathcal{C}$, then, θ^{-1} is also a 2-natural transformation. In this case, θ is called a 2-natural isomorphism.

Proof. It can be proved by some straightforward computations from definitions 4.1, 4.6 and 4.9. For example, for $\psi \star \theta$:

$$\begin{aligned}
1_{(\psi\star\theta)_B} \star G(F(\alpha)) &= 1_{\psi_{F'(B)}} \star 1_{G(\theta_B)} \star G(F(\alpha)) \\
&= 1_{\psi_{F'(B)}} \star G(1_{\theta_B} \star F(\alpha)) \\
&= 1_{\psi_{F'(B)}} \star G(F'(\alpha)) \star 1_{G(\theta_A)} \\
&= G'(F'(\alpha)) \star 1_{\psi_{F'(A)}} \star 1_{G(\theta_A)} \\
&= G'(F'(\alpha)) \star 1_{(\psi\star\theta)_A}.
\end{aligned}$$

□

Corollary 4.11. Small 2-categories, 2-functors and 2-natural transformations form also a 2-category.

Proof. It is lemmas 4.8 and 4.10.

□

As announced earlier, we can also define arrows between 2-natural transformations.

Definition 4.12. Let $F, G : \mathcal{C} \to \mathcal{D}$ be 2-functors between the 2-categories $\mathcal{C}$ and $\mathcal{D}$. Let also $\theta, \varphi : F \Rightarrow G$ be two 2-natural transformations. A modification $\Xi : \theta \rightsquigarrow \varphi$ is the data of, for all $A \in \mathcal{C}$, a 2-cell $\Xi_A : \theta_A \longrightarrow \varphi_A$ such that, for all $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ in $\mathcal{C}$, we have $\Xi_B \star F(\alpha) = G(\alpha) \star \Xi_A$ in $\mathcal{D}$.

$$
\begin{tikzcd}[column sep=large, row sep=huge]
F(A) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\Downarrow F(\alpha)"] \arrow[d, bend right, "\varphi_A"'] \arrow[d, bend left, "\theta_A"] \arrow[d, phantom, "\Leftarrow \Xi_A"] & F(B) \arrow[d, bend right, "\varphi_B"'] \arrow[d, bend left, "\theta_B"] \arrow[d, phantom, "\Leftarrow \Xi_B"] \\
G(A) \arrow[r, bend left, "G(f)"] \arrow[r, bend right, "G(g)"'] \arrow[r, phantom, "\Downarrow G(\alpha)"] & G(B)
\end{tikzcd}
$$

We also have identities, a composition law and a $\star$-law for modifications.

Lemma 4.13. Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be 2-functors between the 2-categories $\mathcal{C}$ and $\mathcal{D}$. Let also $\theta, \varphi, \psi : F \Rightarrow G$ and $\chi, \omega : G \Rightarrow H$ be 2-natural transformations and let $\Xi : \theta \rightsquigarrow \varphi$, $\Psi : \varphi \rightsquigarrow \psi$ and $\Omega : \chi \rightsquigarrow \omega$ be modifications. Then, $1_\theta : \theta \rightsquigarrow \theta$, $\Psi\Xi : \theta \rightsquigarrow \psi$ and $\Omega \star \Xi : \chi\theta \rightsquigarrow \omega\varphi$, defined by $(1_\theta)_A = 1_{\theta_A}$, $(\Psi\Xi)_A = \Psi_A \circ \Xi_A$ and $(\Omega \star \Xi)_A = \Omega_A \star \Xi_A$ for all $A \in \operatorname{ob}\mathcal{C}$, are also modifications.

$$
\begin{tikzcd}[column sep=large]
F \arrow[r, Rightarrow, shift left=2, "\theta"] \arrow[r, Rightarrow, shift right=2, "\varphi"'] \arrow[r, phantom, "\wr\Xi"] \arrow[r, Rightarrow, bend right=80, "\psi"'] \arrow[r, phantom, bend right=40, "\wr\Psi"] & G \arrow[r, Rightarrow, shift left=2, "\chi"] \arrow[r, Rightarrow, shift right=2, "\omega"'] \arrow[r, phantom, "\wr\Omega"] & H
\end{tikzcd}
$$

Proof. 1_θ is a modification since θ is a 2-natural modification and $\Omega \star \Xi$ is obviously a modification. For $\Psi\Xi$, let $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ be a 2-cell in $\mathcal{C}$:

$$
\begin{aligned}
(\Psi_B \circ \Xi_B) \star F(\alpha) &= (\Psi_B \circ \Xi_B) \star (1_{F(g)} \circ F(\alpha)) \\
&= (\Psi_B \star 1_{F(g)}) \circ (\Xi_B \star F(\alpha)) \quad \text{Interchange law} \\
&= (\Psi_B \star 1_{F(g)}) \circ (G(\alpha) \star \Xi_A) \quad \text{Since } \Xi \text{ is a modification} \\
&= (\Psi_B \star 1_{F(g)}) \circ \left[(G(\alpha) \circ 1_{G(f)}) \star (1_{\varphi_A} \circ \Xi_A)\right] \\
&= (\Psi_B \star 1_{F(g)}) \circ (G(\alpha) \star 1_{\varphi_A}) \circ (1_{G(f)} \star \Xi_A) \quad \text{Interchange law} \\
&= (\Psi_B \star 1_{F(g)}) \circ (1_{\varphi_B} \star F(\alpha)) \circ (1_{G(f)} \star \Xi_A) \quad \text{Since } \varphi \text{ is 2-natural} \\
&= \left[(\Psi_B \circ 1_{\varphi_B}) \star (1_{F(g)} \circ F(\alpha))\right] \circ (1_{G(f)} \star \Xi_A) \quad \text{Interchange law} \\
&= (\Psi_B \star F(\alpha)) \circ (1_{G(f)} \star \Xi_A) \\
&= (G(\alpha) \star \Psi_A) \circ (1_{G(f)} \star \Xi_A) \quad \text{Since } \Psi \text{ is a modification} \\
&= (G(\alpha) \circ 1_{G(f)}) \star (\Psi_A \circ \Xi_A) \quad \text{Interchange law} \\
&= G(\alpha) \star (\Psi_A \circ \Xi_A),
\end{aligned}
$$

which is what we had to prove.

$\square$

Notation 4.14. If $\mathcal{C}$ and $\mathcal{D}$ are small 2-categories, we write $[\mathcal{C},\mathcal{D}]$ for the small 2-category of 2-functors $\mathcal{C}\to\mathcal{D}$, 2-natural transformations and modifications. If $F,G:\mathcal{C}\to\mathcal{D}$ are 2-functors, $[F,G]$ will denote the small category of 2-natural transformations and modifications.

We conclude this section with a characterisation of invertible modifications.

Lemma 4.15. Let $F,G:\mathcal{C}\to\mathcal{D}$ be 2-functors between the 2-categories $\mathcal{C}$ and $\mathcal{D}$. Let also $\theta,\varphi:F\Rightarrow G$ be 2-natural transformations and $\Xi:\theta\rightsquigarrow\varphi$ a modification. Then, there exists a modification $\Psi:\varphi\rightsquigarrow\theta$ such that $\Xi\Psi=1_\varphi$ and $\Psi\Xi=1_\theta$ if and only if Ξ_A is a 2-isomorphism for all $A\in\operatorname{ob}\mathcal{C}$. In this case, we say that Ξ is an isomodification.

Proof. There is nothing to prove for the 'only if part'. Let us prove the 'if part'. Let $\Psi_A=(\Xi_A)^{-1}$ for all $A\in\operatorname{ob}\mathcal{C}$. We have to show that Ψ is a modification. Let
$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$
be a 2-cell in $\mathcal{C}$. We have to prove $\Psi_B\star F(\alpha)=G(\alpha)\star\Psi_A$. By the computations done in the proof of lemma 4.13, we know that $1_{\theta_B}\star F(\alpha)=(\Psi_B\circ\Xi_B)\star F(\alpha)=(\Psi_B\star F(\alpha))\circ(1_{G(f)}\star\Xi_A)$. But we also know that $1_{\theta_B}\star F(\alpha)=G(\alpha)\star 1_{\theta_A}=(G(\alpha)\star\Psi_A)\circ(1_{G(f)}\star\Xi_A)$ by the interchange law. Hence, it it suffices to show that $1_{G(f)}\star\Xi_A$ is a 2-isomorphism. But it is easy to see that its inverse is $1_{G(f)}\star\Psi_A$. Indeed:

$$\begin{aligned}(1_{G(f)}\star\Xi_A)\circ(1_{G(f)}\star\Psi_A)&=1_{G(f)}\star 1_{\varphi_A}=1_{G(f)\varphi_A}\\(1_{G(f)}\star\Psi_A)\circ(1_{G(f)}\star\Xi_A)&=1_{G(f)}\star 1_{\theta_A}=1_{G(f)\theta_A}.\end{aligned}$$

□

4.2 Pseudo-2-Functors and Biequivalences

In Chapter 5, we will classify cat-groups. A corollary of this classification will be that the 2-category CG will 'look the same' as an other 2-category, easier to understand. Of course, we have to be precise and so we would like to define what a 'biequivalence' between two 2-categories is. We want such a notion of biequivalence to be an actual equivalence relation on 2-categories. Moreover, in order to prove this corollary, we need a characterisation of a biequivalence in the same way we have in categories (i.e. an equivalence is a full, faithful and essentially surjective functor). Hence, we present in this section a definition of biequivalence satisfying both conditions. So as to define it, we need to introduce pseudo-2-functors, pseudo-2-natural transformations and pseudo-modifications.

Definition 4.16. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories. A pseudo-2-functor $F:\mathcal{C}\to\mathcal{D}$ is the data of:

- for all $A\in\operatorname{ob}\mathcal{C}$, an object $F(A)\in\operatorname{ob}\mathcal{D}$,
- for all $A,B\in\operatorname{ob}\mathcal{C}$, a functor $F:\mathcal{C}(A,B)\to\mathcal{D}(F(A),F(B))$,
- for all $A\in\operatorname{ob}\mathcal{C}$, a 2-isomorphism $\delta_A:\ 1_{F(A)}\xrightarrow{\sim}F(1_A)$,
- for all $A\xrightarrow{f}B\xrightarrow{g}C$ in $\mathcal{C}$, a 2-isomorphism $\gamma_{f,g}:\ F(g)F(f)\xrightarrow{\sim}F(gf)$

such that

- γ is natural: for all $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd}$ in $\mathcal{C}$, $\gamma_{f',g'}\ (F(\beta) \star F(\alpha)) = F(\beta \star \alpha)\ \gamma_{f,g}$,

$$\begin{tikzcd} F(g)F(f) \arrow[r, "F(\beta)\star F(\alpha)"] \arrow[d, "\gamma_{f,g}"'] \arrow[dr, phantom, "\circlearrowleft"] & F(g')F(f') \arrow[d, "\gamma_{f',g'}"] \\ F(gf) \arrow[r, "F(\beta\star\alpha)"'] & F(g'f') \end{tikzcd}$$

- for all $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in $\mathcal{C}$, $\gamma_{gf,h}\ (1_{F(h)} \star \gamma_{f,g}) = \gamma_{f,hg}\ (\gamma_{g,h} \star 1_{F(f)})$,

$$\begin{tikzcd} F(h)F(g)F(f) \arrow[r, "\gamma_{g,h}\star 1_{F(f)}"] \arrow[d, "1_{F(h)}\star\gamma_{f,g}"'] \arrow[dr, phantom, "\circlearrowleft"] & F(hg)F(f) \arrow[d, "\gamma_{f,hg}"] \\ F(h)F(gf) \arrow[r, "\gamma_{gf,h}"'] & F(hgf) \end{tikzcd}$$

- for all $A \xrightarrow{f} B$ in $\mathcal{C}$, $\gamma_{1_A,f}\ (1_{F(f)} \star \delta_A) = 1_{F(f)}$ and $\gamma_{f,1_B}\ (\delta_B \star 1_{F(f)}) = 1_{F(f)}$.

$$\begin{tikzcd} F(f) \arrow[r, "1_{F(f)}\star\delta_A"] \arrow[dr, "1_{F(f)}"'] & F(f)F(1_A) \arrow[d, "\gamma_{1_A,f}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & F(f) \end{tikzcd} \qquad \begin{tikzcd} F(f) \arrow[r, "\delta_B\star 1_{F(f)}"] \arrow[dr, "1_{F(f)}"'] & F(1_B)F(f) \arrow[d, "\gamma_{f,1_B}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & F(f) \end{tikzcd}$$

So, we notice that a pseudo-2-functor is a 2-functor except that, instead of requiring $F(g)F(f) = F(gf)$ and $F(1_A) = 1_{F(A)}$, we ask $F(g)F(f) \simeq F(gf)$ and $F(1_A) \simeq 1_{F(A)}$, with some coherent axioms between the 2-isomorphisms.

Lemma 4.17. A 2-functor $F : \mathcal{C} \to \mathcal{D}$ is a pseudo-2-functor with $\delta_A = 1_{1_{F(A)}}$ and $\gamma_{f,g} = 1_{F(gf)}$ for all suitable A, f and g in $\mathcal{C}$. Moreover, if $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ are pseudo-2-functors, then $\mathcal{C} \xrightarrow{GF} \mathcal{E}$ is also a pseudo-2-functor with $\delta_A^{GF} = G(\delta_A^F)\ \delta_{F(A)}^G$ and $\gamma_{f,g}^{GF} = G(\gamma_{f,g}^F)\ \gamma_{F(f),F(g)}^G$ for all suitable A, f and g in $\mathcal{C}$.

Proof. The first part is obvious. For the composition, it follows directly from the axioms. For example, we prove the second axiom:

$$\begin{aligned} \gamma_{gf,h}^{GF}\ \left(1_{GF(h)} \star \gamma_{f,g}^{GF}\right) &= G(\gamma_{gf,h}^F)\ \gamma_{F(gf),F(h)}^G\ \left(1_{GF(h)} \star G(\gamma_{f,g}^F)\right)\ \left(1_{GF(h)} \star \gamma_{F(f),F(g)}^G\right) \\ &= G(\gamma_{gf,h}^F)\ G(1_{F(h)} \star \gamma_{f,g}^F)\ \gamma_{F(g)F(f),F(h)}^G\ \left(1_{GF(h)} \star \gamma_{F(f),F(g)}^G\right) \\ &= G(\gamma_{f,hg}^F)\ G(\gamma_{g,h}^F \star 1_{F(f)})\ \gamma_{F(g)F(f),F(h)}^G\ \left(1_{GF(h)} \star \gamma_{F(f),F(g)}^G\right) \\ &= G(\gamma_{f,hg}^F)\ G(\gamma_{g,h}^F \star 1_{F(f)})\ \gamma_{F(f),F(h)F(g)}^G\ \left(\gamma_{F(g),F(h)}^G \star 1_{GF(f)}\right) \\ &= G(\gamma_{f,hg}^F)\ \gamma_{F(f),F(hg)}^G\ \left(G(\gamma_{g,h}^F) \star 1_{GF(f)}\right)\ \left(\gamma_{F(g),F(h)}^G \star 1_{GF(f)}\right) \\ &= \gamma_{f,hg}^{GF}\ \left(\gamma_{g,h}^{GF} \star 1_{GF(f)}\right). \end{aligned}$$

□

The arrows between pseudo-2-functors are called pseudo-2-natural transformations.

Definition 4.18. Let $F, G : \mathcal{C} \to \mathcal{D}$ be pseudo-2-functors. A pseudo-2-natural transformation $\theta : F \Rightarrow G$ is the data of:

- for all $A \in \mathrm{ob}\,\mathcal{C}$, a 1-cell $\theta_A : \; F(A) \longrightarrow G(A)$ in $\mathcal{D}$,
- for all $A \xrightarrow{f} B$ in $\mathcal{C}$, a 2-isomorphism $\tau_f : \; G(f)\theta_A \xrightarrow{\sim} \theta_B F(f)$

such that

- τ is natural: for all $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ in $\mathcal{C}$, $\tau_{f'}\,(G(\alpha) \star 1_{\theta_A}) = (1_{\theta_B} \star F(\alpha))\,\tau_f$,

$$\begin{tikzcd} G(f)\theta_A \arrow[r, "G(\alpha)\star 1_{\theta_A}"] \arrow[d, "\tau_f"'] \arrow[dr, phantom, "\circlearrowleft"] & G(f')\theta_A \arrow[d, "\tau_{f'}"] \\ \theta_B F(f) \arrow[r, "1_{\theta_B}\star F(\alpha)"'] & \theta_B F(f') \end{tikzcd}$$

- for all $A \in \mathrm{ob}\,\mathcal{C}$, $\tau_{1_A}\left(\delta^G_A \star 1_{\theta_A}\right) = \left(1_{\theta_A} \star \delta^F_A\right)$,

$$\begin{tikzcd} \theta_A \arrow[r, "\delta^G_A\star 1_{\theta_A}"] \arrow[dr, "1_{\theta_A}\star\delta^F_A"'] & G(1_A)\theta_A \arrow[d, "\tau_{1_A}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \theta_A F(1_A) \end{tikzcd}$$

- for all $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$,

$$\left(1_{\theta_C} \star \gamma^F_{f,g}\right)\left(\tau_g \star 1_{F(f)}\right)\left(1_{G(g)} \star \tau_f\right) = \tau_{gf}\left(\gamma^G_{f,g} \star 1_{\theta_A}\right).$$

$$\begin{tikzcd} G(g)G(f)\theta_A \arrow[r, "1_{G(g)}\star\tau_f"] \arrow[d, "\gamma^G_{f,g}\star 1_{\theta_A}"'] & G(g)\theta_B F(f) \arrow[r, "\tau_g\star 1_{F(f)}"] \arrow[d, phantom, "\circlearrowleft"] & \theta_C F(g)F(f) \arrow[d, "1_{\theta_C}\star\gamma^F_{f,g}"] \\ G(gf)\theta_A \arrow[rr, "\tau_{gf}"'] & & \theta_C F(gf) \end{tikzcd}$$

As for pseudo-2-functors, a pseudo-2-natural transformation is a 2-natural transformation except that the naturality is up to coherent 2-isomorphisms. Let us now state their first basic properties.

Lemma 4.19. Let $F, G : \mathcal{C} \to \mathcal{D}$ be 2-functors and $\theta : F \Rightarrow G$ be a 2-natural transformation. Then, θ is a pseudo-2-natural transformation with $\tau_f = 1_{\theta_B F(f)}$ for all $A \xrightarrow{f} B$ in $\mathcal{C}$.

Proof. Obvious since δ^F, δ^G, γ^F and γ^G's are identities.

$\square$

Lemma 4.20. Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be pseudo-2-functors and let $F \overset{\theta}{\Longrightarrow} G \overset{\varphi}{\Longrightarrow} H$ be pseudo-2-natural transformations. Then, $1_F : F \Longrightarrow F$ and $\varphi\theta : F \Longrightarrow H$ defined by $(1_F)_A = 1_{F(A)}$, $\tau_f^{1_F} = 1_{F(f)}$, $(\varphi\theta)_A = \varphi_A\theta_A$ and $\tau_f^{\varphi\theta} = (1_{\varphi_B} \star \tau_f^{\theta})(\tau_f^{\varphi} \star 1_{\theta_A})$ for all $A \in \operatorname{ob}\mathcal{C}$ and $A \xrightarrow{f} B$ in $\mathcal{C}$, are pseudo-2-natural transformations.

$$\tau_f^{\varphi\theta} :\ H(f)\varphi_A\theta_A \xrightarrow{\tau_f^{\varphi} \star 1_{\theta_A}} \varphi_B G(f)\theta_A \xrightarrow{1_{\varphi_B} \star \tau_f^{\theta}} \varphi_B\theta_B F(f)$$

Proof. Follows directly from definition 4.18. For example, here is the computation for the last axiom for $\varphi\theta$:

$$\begin{aligned}
&(1_{(\varphi\theta)_C} \star \gamma^F_{f,g})(\tau_g^{\varphi\theta} \star 1_{F(f)})(1_{H(g)} \star \tau_f^{\varphi\theta}) \\
&= (1_{\varphi_C} \star 1_{\theta_C} \star \gamma^F_{f,g})(1_{\varphi_C} \star \tau_g^{\theta} \star 1_{F(f)})(\tau_g^{\varphi} \star 1_{\theta_B} \star 1_{F(f)})(1_{H(g)} \star 1_{\varphi_B} \star \tau_f^{\theta})(1_{H(g)} \star \tau_f^{\varphi} \star 1_{\theta_A}) \\
&= (1_{\varphi_C} \star 1_{\theta_C} \star \gamma^F_{f,g})(1_{\varphi_C} \star \tau_g^{\theta} \star 1_{F(f)})(1_{\varphi_C} \star 1_{G(g)} \star \tau_f^{\theta})(\tau_g^{\varphi} \star 1_{G(f)} \star 1_{\theta_A})(1_{H(g)} \star \tau_f^{\varphi} \star 1_{\theta_A}) \\
&= (1_{\varphi_C} \star \tau_{gf}^{\theta})(1_{\varphi_C} \star \gamma^G_{f,g} \star 1_{\theta_A})(\tau_g^{\varphi} \star 1_{G(f)} \star 1_{\theta_A})(1_{H(g)} \star \tau_f^{\varphi} \star 1_{\theta_A}) \\
&= (1_{\varphi_C} \star \tau_{gf}^{\theta})(\tau_{gf}^{\varphi} \star 1_{\theta_A})(\gamma^H_{f,g} \star 1_{\varphi_A} \star 1_{\theta_A}) \\
&= \tau_{gf}^{\varphi\theta}(\gamma^H_{f,g} \star 1_{(\varphi\theta)_A}).
\end{aligned}$$

□

Lemma 4.21. Let $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\varphi"] & \mathcal{E} \end{tikzcd}$ be 2-categories, pseudo-2-functors and pseudo-2-natural transformations. Then, $\varphi \star \theta : GF \Longrightarrow G'F'$, defined by

$$(\varphi \star \theta)_A :\ G(F(A)) \xrightarrow{G(\theta_A)} G(F'(A)) \xrightarrow{\varphi_{F'(A)}} G'(F'(A))$$

and $\tau_f^{\varphi\star\theta} =$

$$\begin{tikzcd}
G'(F'(f))\varphi_{F'(A)}G(\theta_A) \arrow[r, "\tau^{\varphi}_{F'(f)} \star 1_{G(\theta_A)}"] & \varphi_{F'(B)}G(F'(f))G(\theta_A) \arrow[r, "1_{\varphi_{F'(B)}} \star \gamma^G_{\theta_A, F'(f)}"] & \varphi_{F'(B)}G(F'(f)\theta_A) \arrow[dl, "1_{\varphi_{F'(B)}} \star G(\tau_f^{\theta})"] \\
\varphi_{F'(B)}G(\theta_B)G(F(f)) & \varphi_{F'(B)}G(\theta_B F(f)) \arrow[l, "1_{\varphi_{F'(B)}} \star \left(\gamma^G_{F(f),\theta_B}\right)^{-1}"] &
\end{tikzcd}$$

for all $A \in \operatorname{ob}\mathcal{C}$ and $A \xrightarrow{f} B$ in $\mathcal{C}$, is also a pseudo-2-natural transformation.

Proof. This is also easy computations using the axioms of definitions 4.16 and 4.18.

□

Lemma 4.22. Let $F, G : \mathcal{C} \to \mathcal{D}$ be pseudo-2-functors and $\theta : F \Rightarrow G$ a pseudo-2-natural transformation. There exists a pseudo-2-natural transformation $\varphi : G \Rightarrow F$ such that $\theta\varphi = 1_G$ and $\varphi\theta = 1_F$ if and only if θ_A is an isomorphism for all $A \in \operatorname{ob}\mathcal{C}$. In this case, $\varphi_A = \theta_A^{-1}$ and $\tau_f^{\varphi} = 1_{\theta_B^{-1}} \star \left(\tau_f^{\theta}\right)^{-1} \star 1_{\theta_A^{-1}}$ for all $A \in \operatorname{ob}\mathcal{C}$ and $A \xrightarrow{f} B$ in $\mathcal{C}$. We call such a θ a pseudo-2-natural isomorphism.

Proof. The 'only if part' is trivial. Notice also that, if such a φ exists, we must have $\varphi_A = \theta_A^{-1}$ for all $A \in \mathrm{ob}\,\mathcal{C}$. Moreover, if $A \xrightarrow{f} B$ is in $\mathcal{C}$,

$$\begin{aligned}
1_{F(f)} &= \tau_f^{1_F} = \tau_f^{\varphi\theta} \\
&= (1_{\varphi_B} \star \tau_f^{\theta})(\tau_f^{\varphi} \star 1_{\theta_A}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\tau_f^{\varphi} &= \tau_f^{\varphi} \star 1_{\theta_A} \star 1_{\theta_A^{-1}} \\
&= 1_{\varphi_B} \star \left(\tau_f^{\theta}\right)^{-1} \star 1_{\theta_A^{-1}}.
\end{aligned}$$

Therefore, the definition of φ is forced. It remains to prove that this so defined φ is a pseudo-2-natural transformation, but it comes directly from the fact that θ is a pseudo-2-natural transformation.

□

Eventually, we define pseudo-modifications.

Definition 4.23. Let $F, G : \mathcal{C} \to \mathcal{D}$ be pseudo-2-functors and $\theta, \varphi : F \Rightarrow G$ two pseudo-2-natural transformations. A pseudo-modification $\Xi : \theta \rightsquigarrow \varphi$ is the data of, for all $A \in \mathrm{ob}\,\mathcal{C}$, a 2-cell $\Xi_A : \theta_A \longrightarrow \varphi_A$ such that, for all $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ in $\mathcal{C}$, $(\Xi_B \star F(\alpha))\ \tau_f^{\theta} = \tau_g^{\varphi}\ (G(\alpha) \star \Xi_A)$.

$$\begin{tikzcd}
G(f)\theta_A \arrow[r, "G(\alpha)\star\Xi_A"] \arrow[d, "\tau_f^{\theta}"'] \arrow[dr, phantom, "\circlearrowleft"] & G(g)\varphi_A \arrow[d, "\tau_g^{\varphi}"] \\
\theta_B F(f) \arrow[r, "\Xi_B \star F(\alpha)"'] & \varphi_B F(g)
\end{tikzcd}$$

Remark 4.24. We notice here that, if F and G are 2-functors and if θ and φ are 2-natural transformations, then, a pseudo-modification $\Xi : \theta \rightsquigarrow \varphi$ is exactly the same as a modification $\theta \rightsquigarrow \varphi$.

As for modifications, we have the following lemmas.

Lemma 4.25. Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be pseudo-2-functors and $\theta, \varphi, \psi : F \Rightarrow G$ and $\chi, \omega : G \Rightarrow H$ pseudo-2-natural transformations. Let also $\Xi : \theta \rightsquigarrow \varphi$, $\Psi : \varphi \rightsquigarrow \psi$ and $\Omega : \chi \rightsquigarrow \omega$ be pseudo-modifications. Then, $1_\theta : \theta \rightsquigarrow \theta$, $\Psi\Xi : \theta \rightsquigarrow \psi$ and $\Omega \star \Xi : \chi\theta \rightsquigarrow \omega\varphi$, defined by $(1_\theta)_A = 1_{\theta_A}$, $(\Psi\Xi)_A = \Psi_A \circ \Xi_A$ and $(\Omega \star \Xi)_A = \Omega_A \star \Xi_A$ for all $A \in \mathrm{ob}\,\mathcal{C}$, are also pseudo-modifications.

$$\begin{tikzcd}
F \arrow[r, Rightarrow, bend left, "\theta"] \arrow[r, Rightarrow, "\varphi"'] \arrow[r, phantom, shift left=2, "\wr\Xi"] \arrow[r, Rightarrow, bend right=80, "\psi"'] \arrow[r, phantom, shift right=4, "\wr\Psi"] & G \arrow[r, Rightarrow, bend left, "\chi"] \arrow[r, Rightarrow, bend right, "\omega"'] \arrow[r, phantom, "\wr\Omega"] & H
\end{tikzcd}$$

Proof. Essentially the same as lemma 4.13.

□

Lemma 4.26. Let $F, G : \mathcal{C} \to \mathcal{D}$ be pseudo-2-functors. Let also $\theta, \varphi : F \Rightarrow G$ be pseudo-2-natural transformations and $\Xi : \theta \rightsquigarrow \varphi$ a pseudo-modification. Then, there exists a pseudo-modification $\Psi : \varphi \rightsquigarrow \theta$ such that $\Xi\Psi = 1_\varphi$ and $\Psi\Xi = 1_\theta$ if and only if Ξ_A is a 2-isomorphism for all $A \in \mathrm{ob}\,\mathcal{C}$. In this case, Ξ is called a pseudo-isomodification.

Proof. Essentially the same as lemma 4.15.

□

We are now able to define when two 2-categories are 'biequivalent'. Recall that we would like such a notion of 'biequivalence' to be an equivalence relation on 2-categories and to be characterised in the same way it is for equivalences of categories.

Definition 4.27. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories. A biequivalence between $\mathcal{C}$ and $\mathcal{D}$ is the data of:

- two pseudo-2-functors $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$,
- four pseudo-2-natural transformations $\theta_1 : GF \Rightarrow 1_\mathcal{C}$, $\theta_2 : 1_\mathcal{C} \Rightarrow GF$,$\theta_3 : FG \Rightarrow 1_\mathcal{D}$ and $\theta_4 : 1_\mathcal{D} \Rightarrow FG$,
- four pseudo-isomodifications $\Xi_1 : \theta_1\theta_2 \rightsquigarrow 1_{1_\mathcal{C}}$, $\Xi_2 : \theta_2\theta_1 \rightsquigarrow 1_{GF}$, $\Xi_3 : \theta_3\theta_4 \rightsquigarrow 1_{1_\mathcal{D}}$ and $\Xi_4 : \theta_4\theta_3 \rightsquigarrow 1_{FG}$.

Definition 4.28. Let $F : \mathcal{C} \to \mathcal{D}$ be a pseudo-2-functor.

- F is essentially surjective on objects if, for all $B \in \mathrm{ob}\,\mathcal{D}$, there exists an object $A \in \mathrm{ob}\,\mathcal{C}$ and an isomorphism $h_B : F(A) \xrightarrow{\sim} B$.
- F is weakly essentially surjective on objects if, for all objects $B \in \mathrm{ob}\,\mathcal{D}$, there exists an object $A \in \mathrm{ob}\,\mathcal{C}$, two 1-cells $F(A) \xrightarrow{h_B} B \xrightarrow{k_B} F(A)$ and two 2-isomorphisms $\alpha_B : h_B k_B \xrightarrow{\sim} 1_B$ and $\beta_B : k_B h_B \xrightarrow{\sim} 1_{F(A)}$.
- F is essentially surjective on 1-cells if, for all $F(A) \xrightarrow{g} F(A')$ in $\mathcal{D}$, there exists a 1-cell $A \xrightarrow{f} A'$ in $\mathcal{C}$ and a 2-isomorphism $\varepsilon_g : F(f) \xrightarrow{\sim} g$.
- F is full on 2-cells if, for all $\begin{tikzcd} F(A) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(f')"'] \arrow[r, phantom, "\Downarrow\beta"] & F(A') \end{tikzcd}$ in $\mathcal{D}$, there exists a 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & A' \end{tikzcd}$ in $\mathcal{C}$ such that $F(\alpha) = \beta$.
- F is faithful on 2-cells if for all $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\alpha'"] & A' \end{tikzcd}$ in $\mathcal{C}$ such that $F(\alpha) = F(\alpha')$, we have $\alpha = \alpha'$.

With the same idea that any equivalence of categories can be turned into an adjoint equivalence, we have the following lemma.

Lemma 4.29. Let $F : \mathcal{C} \to \mathcal{D}$ be a pseudo-2-functor. If F is weakly essentially surjective on objects, then, for all objects $B \in \mathrm{ob}\,\mathcal{D}$, there exists an object $A \in \mathrm{ob}\,\mathcal{C}$, two 1-cells $F(A) \xrightarrow{h_B} B \xrightarrow{k_B} F(A)$ and two 2-isomorphisms $\alpha_B : h_B k_B \xrightarrow{\sim} 1_B$ and $\beta_B : k_B h_B \xrightarrow{\sim} 1_{F(A)}$ such that $\alpha_B \star 1_{h_B} = 1_{h_B} \star \beta_B$ and $1_{k_B} \star \alpha_B = \beta_B \star 1_{k_B}$.

Proof. Since F is weakly essentially surjective on objects, we have an object $A \in \mathrm{ob}\,\mathcal{C}$, two 1-cells $F(A) \xrightarrow{h_B} B \xrightarrow{k_B} F(A)$ and two 2-isomorphisms $\alpha_B : h_B k_B \xrightarrow{\sim} 1_B$ and $\beta'_B : k_B h_B \xrightarrow{\sim} 1_{F(A)}$. It suffices to construct a 2-isomorphism $\beta_B : k_B h_B \xrightarrow{\sim} 1_{F(A)}$ such that $\alpha_B \star 1_{h_B} = 1_{h_B} \star \beta_B$ and $1_{k_B} \star \alpha_B = \beta_B \star 1_{k_B}$. Firstly, we can compute

$$\begin{aligned}\beta'_B(\beta'_B \star 1_{k_B h_B}) &= (1_{1_{F(A)}} \star \beta'_B)(\beta'_B \star 1_{k_B h_B}) = \beta'_B \star \beta'_B \\ &= (\beta'_B \star 1_{1_{F(A)}})(1_{k_B h_B} \star \beta'_B) = \beta'_B(1_{k_B h_B} \star \beta'_B).\end{aligned}$$

Therefore, $\beta'_B \star 1_{k_B h_B} = 1_{k_B h_B} \star \beta'_B$. Similarly, we can prove that $\alpha_B \star 1_{h_B k_B} = 1_{h_B k_B} \star \alpha_B$. Now, we define the 2-isomorphism β_B as the composition of

$$k_B h_B \xrightarrow{1_{k_B h_B} \star \beta'^{-1}_B} k_B h_B k_B h_B \xrightarrow{1_{k_B} \star \alpha_B \star 1_{h_B}} k_B h_B \xrightarrow{\beta'_B} 1_{F(A).}$$

To show that $\alpha_B \star 1_{h_B} = 1_{h_B} \star \beta_B$, it suffices to compute

$$\begin{aligned}(\alpha_B \star 1_{h_B})(1_{h_B k_B h_B} \star \beta'_B) = \alpha_B \star 1_{h_B} \star \beta'_B &= (1_{h_B} \star \beta'_B)(\alpha_B \star 1_{h_B k_B h_B}) \\ &= (1_{h_B} \star \beta'_B)(1_{h_B k_B} \star \alpha_B \star 1_{h_B}).\end{aligned}$$

The other equality is similar.

□

Now, we prove the expected characterisation of biequivalences.

Proposition 4.30. Let $F : \mathcal{C} \to \mathcal{D}$ be a pseudo-2-functor. F is part of a biequivalence if and only if F is weakly essentially surjective on objects, essentially surjective on 1-cells and full and faithful on 2-cells.

Proof. We suppose first that we have a biequivalence $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$:

- F is weakly essentially surjective on objects since, if $B \in \mathrm{ob}\,\mathcal{D}$, we can set $A = G(B)$, $h_B = (\theta_3)_B$, $k_B = (\theta_4)_B$, $\alpha_B = (\Xi_3)_B$ and $\beta_B = (\Xi_4)_B$.

- To see that F is faithful on 2-cells, let $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\alpha\Downarrow\,\Downarrow\alpha'"] & A' \end{tikzcd}$ in $\mathcal{C}$ be such that $F(\alpha) = F(\alpha')$.

  So $GF(\alpha) = GF(\alpha')$ and $GF(\alpha) \star 1_{(\theta_2)_A} = GF(\alpha') \star 1_{(\theta_2)_A}$. Thus,

$$\begin{aligned}(1_{(\theta_2)_{A'}} \star \alpha)\ \tau^{\theta_2}_f &= \tau^{\theta_2}_{f'}\ (GF(\alpha) \star 1_{(\theta_2)_A}) \\ &= \tau^{\theta_2}_{f'}\ (GF(\alpha') \star 1_{(\theta_2)_A}) \\ &= (1_{(\theta_2)_{A'}} \star \alpha')\ \tau^{\theta_2}_f.\end{aligned}$$

  Hence, $1_{(\theta_1)_{A'}} \star 1_{(\theta_2)_{A'}} \star \alpha = 1_{(\theta_1)_{A'}} \star 1_{(\theta_2)_{A'}} \star \alpha'$ and

$$\begin{aligned}\alpha &= 1_{1_{A'}} \star \alpha \\ &= ((\Xi_1)_{A'} \star 1_{f'})\ (1_{(\theta_1\theta_2)_{A'}} \star \alpha)\ ({(\Xi_1)_{A'}}^{-1} \star 1_f) \\ &= ((\Xi_1)_{A'} \star 1_{f'})\ (1_{(\theta_1\theta_2)_{A'}} \star \alpha')\ ({(\Xi_1)_{A'}}^{-1} \star 1_f) \\ &= 1_{1_{A'}} \star \alpha' \\ &= \alpha'.\end{aligned}$$

  Therefore, F is faithful on 2-cells.

- We now prove that F is full on 2-cells. Let $F(A) \underset{F(f')}{\overset{F(f)}{\Downarrow \beta}} F(A')$ be a 2-cell in $\mathcal{D}$. We set $\alpha =$

$$
\begin{tikzcd}
f \arrow[r, "(\Xi_1)_{A'}{}^{-1} \star 1_f"] & (\theta_1\theta_2)_{A'} f \arrow[r, "1_{(\theta_1)_{A'}} \star \left(\tau_f^{\theta_2}\right)^{-1}"] & (\theta_1)_{A'} GF(f) (\theta_2)_A \arrow[d, "1_{(\theta_1)_{A'}} \star G(\beta) \star 1_{(\theta_2)_A}"] \\
f' & (\theta_1\theta_2)_{A'} f' \arrow[l, "(\Xi_1)_{A'} \star 1_{f'}"] & (\theta_1)_{A'} GF(f') (\theta_2)_A \arrow[l, "1_{(\theta_1)_{A'}} \star \tau_{f'}^{\theta_2}"]
\end{tikzcd}
$$

But we know, by naturality of τ^{θ_2}, that α is actually

$$
\begin{tikzcd}
f \arrow[r, "(\Xi_1)_{A'}{}^{-1} \star 1_f"] & (\theta_1\theta_2)_{A'} f \arrow[r, "1_{(\theta_1)_{A'}} \star \left(\tau_f^{\theta_2}\right)^{-1}"] & (\theta_1)_{A'} GF(f) (\theta_2)_A \arrow[d, "1_{(\theta_1)_{A'}} \star GF(\alpha) \star 1_{(\theta_2)_A}"] \\
f' & (\theta_1\theta_2)_{A'} f' \arrow[l, "(\Xi_1)_{A'} \star 1_{f'}"] & (\theta_1)_{A'} GF(f') (\theta_2)_A \arrow[l, "1_{(\theta_1)_{A'}} \star \tau_{f'}^{\theta_2}"]
\end{tikzcd}
$$

  Thus, by the same kind of proof we have just done, $G(\beta) = GF(\alpha)$. But G is also part of an equivalence. So G is faithful on 2-cells and $\beta = F(\alpha)$. Therefore, F is full on 2-cells.

- It remains to prove that F is essentially surjective on 1-cells: let $F(A) \xrightarrow{g} F(A')$ be a 1-cell in $\mathcal{D}$. We set $f = A \xrightarrow{(\theta_2)_A} GF(A) \xrightarrow{G(g)} GF(A') \xrightarrow{(\theta_1)_{A'}} A'$. Since we have $(\theta_1)_{A'} GF(f)(\theta_2)_A \simeq (\theta_1)_{A'}(\theta_2)_{A'} f \simeq f$, we know that

$$
\begin{aligned}
G(g) &\simeq (\theta_2)_{A'}\ (\theta_1)_{A'}\ G(g)\ (\theta_2)_A\ (\theta_1)_A \\
&\simeq (\theta_2)_{A'}\ f\ (\theta_1)_A \\
&\simeq (\theta_2)_{A'}\ (\theta_1)_{A'}\ GF(f)\ (\theta_2)_A\ (\theta_1)_A \\
&\simeq GF(f).
\end{aligned}
$$

  Hence, since G is full and faithful on 2-cells, $g \simeq G(f)$ and F is essentially surjective on 1-cells.

We suppose now that F is weakly essentially surjective on objects, essentially surjective on 1-cells and full and faithful on 2-cells. Let us construct a biequivalence $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$. By the axiom of choice, we can choose, for all $B \in \operatorname{ob}\mathcal{D}$, an object $G(B) \in \operatorname{ob}\mathcal{C}$, 1-cells $FG(B) \xrightarrow{h_B} B \xrightarrow{k_B} FG(B)$ and 2-isomorphisms α_B and β_B given by lemma 4.29. Since F is essentially surjective on 1-cells, we can also choose, for all $B \xrightarrow{g} B'$ in $\mathcal{D}$, a 1-cell $G(B) \xrightarrow{G(g)} G(B')$ in $\mathcal{C}$ and a 2-isomorphism $\varepsilon_{k_{B'} g h_B} : FG(g) \xrightarrow{\sim} k_{B'} g h_B$. Since F is full and faithful on 2-cells, we have, for all $B \underset{g'}{\overset{g}{\Downarrow \beta}} B'$ in $\mathcal{D}$, a unique 2-cell

$\begin{tikzcd} G(B) \arrow[r, bend left, "G(g)"{name=U}] \arrow[r, bend right, "G(g')"'{name=D}] \arrow[from=U, to=D, Rightarrow, "G(\beta)"] & G(B') \end{tikzcd}$ in $\mathcal{C}$ such that $FG(\beta) =$

$$FG(g) \xrightarrow{\varepsilon_{k_{B'}gh_B}} k_{B'}gh_B \xrightarrow{1_{k_{B'}} \star \beta \star 1_{h_B}} k_{B'}g'h_B \xrightarrow{\varepsilon^{-1}_{k_{B'}g'h_B}} FG(g')\ .$$

In addition, for all composable 1-cells $B \xrightarrow{g} B' \xrightarrow{g'} B''$ in $\mathcal{D}$, there exists a unique 2-isomorphism $G(g')G(g) \xrightarrow{\gamma^G_{g,g'}} G(g'g)$ such that $F(\gamma^G_{g,g'}) =$

$$\begin{tikzcd}
F(G(g')G(g)) \arrow[r, "\left(\gamma^F_{G(g),G(g')}\right)^{-1}"] & FG(g')FG(g) \arrow[r, "\varepsilon_{k_{B''}g'h_{B'}} \star \varepsilon_{k_{B'}gh_B}"] & k_{B''}g'h_{B'}k_{B'}gh_B \arrow[dl, "1_{k_{B''}} \star 1_{g'} \star \alpha_{B'} \star 1_g \star 1_{h_B}"] \\
FG(g'g) & k_{B''}g'gh_B \arrow[l, "\varepsilon^{-1}_{k_{B''}g'gh_B}"] &
\end{tikzcd}$$

Moreover, for all objects $B \in \operatorname{ob}\mathcal{D}$, there exists a unique 2-isomorphism $1_{G(B)} \xrightarrow{\delta^G_B} G(1_B)$ such that $F(\delta^G_B) =$

$$F(1_{G(B)}) \xrightarrow{\left(\delta^F_{G(B)}\right)^{-1}} 1_{FG(B)} \xrightarrow{\beta_B{}^{-1}} k_B h_B \xrightarrow{\varepsilon^{-1}_{k_B h_B}} FG(1_B)\ .$$

With some computations using the facts that F is faithful on 2-cells, $\alpha_B \star 1_{h_B} = 1_{h_B} \star \beta_B$ and $1_{k_B} \star \alpha_B = \beta_B \star 1_{k_B}$ for all $B \in \operatorname{ob}\mathcal{D}$, we can prove that this defines a pseudo-2-functor $G : \mathcal{D} \to \mathcal{C}$.

Now, we define θ_1, θ_2, θ_3 and θ_4 as follows:

- For all $A \in \operatorname{ob}\mathcal{C}$, $(\theta_1)_A : GF(A) \longrightarrow A$ is given by the essentially surjectivity of F on 1-cells: $\varepsilon_{h_{F(A)}} : F((\theta_1)_A) \xrightarrow{\sim} h_{F(A)}$. Then, for all $A \xrightarrow{f} A'$ in $\mathcal{C}$, we define the 2-isomorphism $\tau^{\theta_1}_f : f\ (\theta_1)_A \xrightarrow{\sim} (\theta_1)_{A'}\ GF(f)$ to be the unique 2-cell such that $F\left(\tau^{\theta_1}_f\right)$ is

$$\begin{tikzcd}
F(f\ (\theta_1)_A) \arrow[r, "\left(\gamma^F_{(\theta_1)_A,f}\right)^{-1}"] & F(f)F((\theta_1)_A) \arrow[r, "1_{F(f)} \star \varepsilon_{h_{F(A)}}"] & F(f)h_{F(A)} \arrow[d, "\alpha^{-1}_{F(A')} \star 1_{F(f)} \star 1_{h_{F(A)}}"'] \\
h_{F(A')}FGF(f) \arrow[d, "\varepsilon^{-1}_{h_{F(A')}} \star 1_{FGF(f)}"'] & & h_{F(A')}k_{F(A')}F(f)h_{F(A)} \arrow[ll, "1_{h_{F(A')}} \star \varepsilon^{-1}_{k_{F(A')}F(f)h_{F(A)}}"] \\
F((\theta_1)_{A'})FGF(f) \arrow[r, "\gamma^F_{GF(f),(\theta_1)_{A'}}"'] & F((\theta_1)_{A'}\ GF(f)). &
\end{tikzcd}$$

- For all $A \in \operatorname{ob}\mathcal{C}$, $(\theta_2)_A :\ A \longrightarrow GF(A)$ is also given by the essentially surjectivity of F on 1-cells: $\varepsilon_{k_{F(A)}} :\ F((\theta_2)_A) \xrightarrow{\sim} k_{F(A)}$. Then, for all $A \xrightarrow{f} A'$ in $\mathcal{C}$, we define the 2-isomorphism $\tau_f^{\theta_2} :\ GF(f)\ (\theta_2)_A \xrightarrow{\sim} (\theta_2)_{A'}\ f$ to be the unique 2-cell such that $F\left(\tau_f^{\theta_2}\right)$ is

$$\begin{tikzcd}
F(GF(f)\ (\theta_2)_A) \arrow[r, "\left(\gamma^F_{(\theta_2)_A, GF(f)}\right)^{-1}"] & FGF(f)F((\theta_2)_A) \arrow[r, "1_{FGF(f)}\star\varepsilon_{k_{F(A)}}"] & FGF(f)k_{F(A)} \arrow[d, "\varepsilon_{k_{F(A')}F(f)h_{F(A)}}\star 1_{k_{F(A)}}"'] \\
k_{F(A')}F(f) \arrow[d, "\varepsilon^{-1}_{k_{F(A')}}\star 1_{F(f)}"'] & & k_{F(A')}F(f)h_{F(A)}k_{F(A)} \arrow[ll, "1_{k_{F(A')}}\star 1_{F(f)}\star\alpha_{F(A)}"] \\
F((\theta_2)_{A'})F(f) \arrow[r, "\gamma^F_{f,(\theta_2)_{A'}}"'] & F((\theta_2)_{A'}\ f).
\end{tikzcd}$$

- For all $B \in \operatorname{ob}\mathcal{D}$, we set $(\theta_3)_B = h_B :\ FG(B) \longrightarrow B$ and for all $B \xrightarrow{g} B'$ in $\mathcal{D}$, we set $\tau_g^{\theta_3} =$

$$gh_B \xrightarrow{\alpha^{-1}_{B'}\star 1_g\star 1_{h_B}} h_{B'}k_{B'}gh_B \xrightarrow{1_{h_{B'}}\star\varepsilon^{-1}_{k_{B'}gh_B}} h_{B'}FG(g)\ .$$

- For all $B \in \operatorname{ob}\mathcal{D}$, we set $(\theta_4)_B = k_B :\ B \longrightarrow FG(B)$ and for all $B \xrightarrow{g} B'$ in $\mathcal{D}$, we set $\tau_g^{\theta_4} =$

$$FG(g)k_B \xrightarrow{\varepsilon_{k_{B'}gh_B}\star 1_{k_B}} k_{B'}gh_Bk_B \xrightarrow{1_{k_{B'}}\star 1_g\star\alpha_B} k_{B'}g\ .$$

We can check that the four of them are pseudo-2-natural transformations since F is faithful on 2-cells. It remains to construct Ξ_1, Ξ_2, Ξ_3 and Ξ_4:

- For all $A \in \operatorname{ob}\mathcal{C}$, let $(\Xi_1)_A :\ (\theta_1\theta_2)_A \xrightarrow{\sim} 1_A$ be the unique 2-cell such that $F((\Xi_1)_A) =$

$$\begin{tikzcd}
F((\theta_1)_A(\theta_2)_A) \arrow[r, "\left(\gamma^F_{(\theta_2)_A,(\theta_1)_A}\right)^{-1}"] & F((\theta_1)_A)F((\theta_2)_A) \arrow[r, "\varepsilon_{h_{F(A)}}\star\varepsilon_{k_{F(A)}}"] & h_{F(A)}k_{F(A)} \arrow[d, "\alpha_{F(A)}"] \\
& F(1_A) & 1_{F(A)}. \arrow[l, "\delta^F_A"]
\end{tikzcd}$$

- For all $A \in \operatorname{ob}\mathcal{C}$, let $(\Xi_2)_A :\ (\theta_2\theta_1)_A \xrightarrow{\sim} 1_{GF(A)}$ be the unique 2-cell such that $F((\Xi_2)_A) =$

$$\begin{tikzcd}
F((\theta_2)_A(\theta_1)_A) \arrow[r, "\left(\gamma^F_{(\theta_1)_A,(\theta_2)_A}\right)^{-1}"] & F((\theta_2)_A)F((\theta_1)_A) \arrow[r, "\varepsilon_{k_{F(A)}}\star\varepsilon_{h_{F(A)}}"] & k_{F(A)}h_{F(A)} \arrow[d, "\beta_{F(A)}"] \\
& F(1_{GF(A)}) & 1_{FGF(A)}. \arrow[l, "\delta^F_{GF(A)}"]
\end{tikzcd}$$

- For all $B \in \operatorname{ob}\mathcal{D}$, we set $(\Xi_3)_B = \alpha_B : \ (\theta_3)_B(\theta_4)_B = h_B k_B \longrightarrow 1_B$.

- For all $B \in \operatorname{ob}\mathcal{D}$, we set $(\Xi_4)_B = \beta_B : \ (\theta_4)_B(\theta_3)_B = k_B h_B \longrightarrow 1_{FG(B)}$.

It is also easy to check that they are pseudo-isomodifications, which concludes the proof. □

Remark 4.31. If, in the definition of biequivalence, we ask F and G to be 2-functors (instead of pseudo-2-functors) this proof is no longer valid. Indeed, we can prove that $G(g)G(g') \simeq G(gg')$ but we can not prove that $G(g)G(g') = G(gg')$ even if F is a 2-functor. This is why we have to use pseudo-2-functors if we want such a characterisation.

Corollary 4.32. Let $F : \mathcal{C} \to \mathcal{D}$ be a pseudo-2-functor. F is part of a biequivalence if and only if F is weakly essentially surjective on objects and for all $A, B \in \operatorname{ob}\mathcal{C}$, the functor $F : \mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$ is an equivalence of categories.

Proof. This is proposition 4.30 since an equivalence of categories is a full, faithful and essentially surjective functor. □

Corollary 4.33. Let $F : \mathcal{C} \to \mathcal{D}$ be a pseudo-2-functor essentially surjective on objects and 1-cells and full and faithful on 2-cells. Then, F is part of a biequivalence.

Proof. It is obvious that if F is essentially surjective on objects, then it is weakly essentially surjective on objects. Hence, it is proposition 4.30. □

We conclude this section with this result, announced earlier.

Proposition 4.34. Biequivalence is an equivalence relation on 2-categories, i.e. it is reflexive, symmetric and transitive.

Proof.

- **Reflexivity:** It suffices to consider the 2-functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.

- **Symmetry:** Follows directly from the definition 4.27.

- **Transitivity:** Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{F'} \mathcal{E}$ be pseudo-2-functors satisfying properties of proposition 4.30. We have to prove that $F'F$ also satisfies them. The facts that $F'F$ is essentially surjective on 1-cells and full and faithful on 2-cells follow directly from the corresponding properties for F and F'. Hence, it remains to show that $F'F$ is weakly essentially surjective on objects: Let $C \in \operatorname{ob}\mathcal{E}$. Since F' is weakly essentially surjective on objects, there exists an object $B \in \operatorname{ob}\mathcal{D}$, 1-cells $F'(B) \xrightarrow{h'_C} C \xrightarrow{k'_C} F'(B)$ in $\mathcal{E}$ and 2-isomorphisms $\alpha'_C : \ h'_C k'_C \xrightarrow{\sim} 1_C$ and $\beta'_C : \ k'_C h'_C \xrightarrow{\sim} 1_{F'(B)}$. Moreover, since F is weakly essentially surjective on objects, there exists an object A in $\mathcal{C}$, 1-cells $F(A) \xrightarrow{h_B} B \xrightarrow{k_B} F(A)$ and 2-isomorphisms $\alpha_B : \ h_B k_B \xrightarrow{\sim} 1_B$ and $\beta_B : \ k_B h_B \xrightarrow{\sim} 1_{F(A)}$. Let us set $H_C = h'_C F'(h_B)$ and $K_C = F'(k_B)k'_C$.

$$\begin{tikzcd} F'F(A) \arrow[r, "F'(h_B)"] \arrow[rr, bend right, "H_C"'] & F'(B) \arrow[r, "h'_C"] \arrow[d, phantom, "\circlearrowleft"] & C \arrow[r, "k'_C"] \arrow[rr, bend right, "K_C"'] & F'(B) \arrow[r, "F'(k_B)"] \arrow[d, phantom, "\circlearrowleft"] & F'F(A) \\ & {} & & {} & \end{tikzcd}$$

Let also $H_C K_C \xrightarrow{A_C} 1_C$ be

$$\begin{tikzcd}
h'_C F'(h_B)F'(k_B)k'_C \arrow[r, "1_{h'_C}\star\gamma^{F'}_{k_B,h_B}\star 1_{k'_C}"] & h'_C F'(h_B k_B)k'_C \arrow[r, "1_{h'_C}\star F'(\alpha_B)\star 1_{k'_C}"] & h'_C F'(1_B)k'_C \arrow[d, "1_{h'_C}\star\left(\delta^{F'}_B\right)^{-1}\star 1_{k'_C}"] \\
& 1_C & h'_C k'_C \arrow[l, "\alpha'_C"]
\end{tikzcd}$$

and $K_C H_C \xrightarrow{B_C} 1_{F'F(A)}$ be

$$\begin{tikzcd}
F'(k_B)k'_C h'_C F'(h_B) \arrow[r, "1_{F'(k_B)}\star\beta'_C\star 1_{F'(h_B)}"] & F'(k_B)F'(h_B) \arrow[r, "\gamma^{F'}_{h_B,k_B}"] & F'(k_B h_B) \arrow[d, "F'(\beta_B)"] \\
& 1_{F'F(A)} & F'(1_{F(A)}) \arrow[l, "\left(\delta^{F'}_{F(A)}\right)^{-1}"]
\end{tikzcd}$$

Since they are 2-isomorphisms, this proves that $F'F$ is weakly essentially surjective on objects.

$\square$

Notation 4.35. If $\mathcal{C}$ and $\mathcal{D}$ are two biequivalent 2-categories, we write $\mathcal{C} \simeq \mathcal{D}$.

5 Sinh's Theorem

After all those definitions, we are now able the start the classification of cat-groups. We know that a small cat-group $\mathcal{G}$ is a groupoid with a weak group structure on its objects. So, isomorphism classes of objects form a group $\Pi_0(\mathcal{G})$. Moreover, we know that $\Pi_1(\mathcal{G}) = \mathcal{G}(I, I)$ is an abelian group and we are going to describe an action of $\Pi_0(\mathcal{G})$ on it. Unfortunately, this information is not enough to reconstruct $\mathcal{G}$. Actually, there exists two different small cat-groups $\mathcal{G}$ and $\mathcal{G}'$ with $\Pi_0(\mathcal{G}) \simeq \Pi_0(\mathcal{G}')$ acting in the same way on $\Pi_1(\mathcal{G}) \simeq \Pi_1(\mathcal{G}')$. So, we will need to define one more invariant, called the Postnikov invariant of $\mathcal{G}$. This is an element a in the third cohomology group $H^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}))$ determined by the isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$. We will prove in this chapter that the triple $(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}), a)$ determines the small cat-group $\mathcal{G}$, up to monoidal equivalence.

We notice here that we want $\Pi_0(\mathcal{G})$ to be a group, hence, in particular, a set. This is why we can only classify small cat-groups. Fortunately, this is not a serious problem. Indeed, if we consider two particular (not necessarily small) cat-groups and if we want to compare them with the classification theorem we are going to prove, we can change our Grothendieck universe $\mathcal{U}$ in order to consider them as small cat-groups. We are not going to explain more details about that here, but this is why we will only work with small cat-groups in this chapter.

5.1 The 2-functors Π_0 and Π_1

Recall that we have defined CG as the 2-category of small cat-groups, monoidal functors and monoidal natural transformations. We now define $\mathscr{H}(\mathrm{CG})$ as the category of small cat-groups and classes of monoidally naturaly isomorphic monoidal functors. Hence, if $\mathcal{G}$ and $\mathcal{H}$ are small cat-groups, $\mathscr{H}(\mathrm{CG})(\mathcal{G}, \mathcal{H}) = \{\text{monoidal functors } F : \mathcal{G} \to \mathcal{H}\}/\!\sim$ where $F \sim F'$ if and only if there is a monoidal natural isomorphism $\alpha : F \Rightarrow F'$. Thus, two small cat-groups are isomorphic in $\mathscr{H}(\mathrm{CG})$ if and only if they are monoidally equivalent.

Definition 5.1. Let Gp be the 2-category of groups, where the only 2-cells are the identities. We have a 2-functor $\Pi_0 : \mathrm{CG} \to \mathrm{Gp}$ defined as follows. If $\mathcal{G}$ is a small cat-group, $\Pi_0(\mathcal{G})$ is the set $\mathcal{G}/\!\sim$ where $X \sim Y$ if and only if X and Y are isomorphic in $\mathcal{G}$, for all $X, Y \in \operatorname{ob}\mathcal{G}$. The group structure on $\Pi_0(\mathcal{G})$ is given by $1 = [I]$, $[X] \cdot [Y] = [X \otimes Y]$ and $[X]^{-1} = [X^*]$ for all $X, Y \in \operatorname{ob}\mathcal{G}$. It is easy to check that these operations are well-defined and that they turn $\Pi_0(\mathcal{G})$ into a group.

If $F : \mathcal{G} \to \mathcal{H}$ is a monoidal functor between cat-groups, then, $\Pi_0(F)$ is

$$\begin{aligned} \Pi_0(F) : \quad & \Pi_0(\mathcal{G}) \to \Pi_0(\mathcal{H}) \\ & [X] \mapsto [F(X)]. \end{aligned}$$

It is well-defined since F is a functor and it is a group morphism since F is monoidal.

Now, if $\alpha : F \Rightarrow G$ is a monoidal natural transformation in CG, then, $\Pi_0(F) = \Pi_0(G)$ since $\alpha_X : F(X) \xrightarrow{\sim} G(X)$ is an isomorphism for all $X \in \operatorname{ob}\mathcal{G}$. So, we can define $\Pi_0(\alpha)$ as $1_{\Pi_0(F)}$.

This defines a 2-functor $\Pi_0 : \mathrm{CG} \to \mathrm{Gp}$ and a functor $\Pi_0 : \mathscr{H}(\mathrm{CG}) \to \mathrm{Gp}$.

As we have said before, $\Pi_0(\mathcal{G})$ encodes the weak group structure on the objects of $\mathcal{G}$ given by $\otimes$. We are now going to introduce the 2-functor Π_1.

Definition 5.2. Let Ab be the 2-category of abelian groups, where the only 2-cells are the identities. We have a 2-functor $\Pi_1 : \mathrm{CG} \to \mathrm{Ab}$ defined as follows. If $\mathcal{G}$ is a small cat-group, we set $\Pi_1(\mathcal{G}) = \mathcal{G}(I, I)$ where the group law is the composition. It is an abelian group by proposition 2.11.

If $F : \mathcal{G} \to \mathcal{H}$ is a monoidal functors between cat-groups, then, $\Pi_1(F)$ is

$$\begin{aligned} \Pi_1(F): \quad & \mathcal{G}(I,I) \to \mathcal{H}(I,I) \\ & f \quad \mapsto F_I^{-1}\ F(f)\ F_I. \end{aligned}$$

Now, if $\alpha : F \Rightarrow G$ is a monoidal natural transformation in CG, then, $\Pi_1(F) = \Pi_1(G)$ since

$$\Pi_1(F)(f) = F_I^{-1}\ F(f)\ F_I = G_I^{-1}\ \alpha_I\ F(f)\ \alpha_I^{-1}\ G_I = G_I^{-1}\ G(f)\ G_I = \Pi_1(G)(f)$$

for all $f \in \mathcal{G}(I, I)$. Therefore, we can define $\Pi_0(\alpha)$ as $1_{\Pi_0(F)}$.

This defines a 2-functor $\Pi_1 : \mathrm{CG} \to \mathrm{Ab}$ and a functor $\Pi_1 : \mathscr{H}(\mathrm{CG}) \to \mathrm{Ab}$ in view of proposition 2.14.

Let us recall the isomorphisms γ and δ from proposition 3.14.

Reminder 5.3. If $\mathcal{G}$ is a cat-group and $X \in \mathrm{ob}\,\mathcal{G}$, we have two group isomorphisms:

1. $$\begin{aligned} \gamma_X :\ & \mathcal{G}(I,I) \to \mathcal{G}(X,X) :\ f \mapsto l_X\ (f \otimes 1_X)\ l_X^{-1} \\ \gamma_X^{-1} :\ & \mathcal{G}(X,X) \to \mathcal{G}(I,I) :\ g \mapsto i_X^{-1}\ (g \otimes 1_{X^*})\ i_X \end{aligned}$$

2. $$\begin{aligned} \delta_X :\ & \mathcal{G}(I,I) \to \mathcal{G}(X,X) :\ f \mapsto r_X\ (1_X \otimes f)\ r_X^{-1} \\ \delta_X^{-1} :\ & \mathcal{G}(X,X) \to \mathcal{G}(I,I) :\ g \mapsto e_X\ (1_{X^*} \otimes g)\ e_X^{-1} \end{aligned}$$

Moreover, these isomorphisms are natural, i.e. if $f \in \mathcal{G}(I, I)$ and $g \in \mathcal{G}(X, Y)$, then, $g\ \gamma_X(f) = \gamma_Y(f)\ g$ and $g\ \delta_X(f) = \delta_Y(f)\ g$ (this follows from the facts that l and r are natural). We also have that $\gamma_I = \delta_I = 1_{\mathcal{G}(I,I)}$ since $r_I = l_I$.

We can now define an action $\Pi_0(\mathcal{G}) \times \Pi_1(\mathcal{G}) \to \Pi_1(\mathcal{G})$.

Definition 5.4. Let $\mathcal{G}$ be a small cat-group. For all $[X] \in \Pi_0(\mathcal{G})$ and $f \in \Pi_1(\mathcal{G})$, we set

$$[X] \cdot f = \gamma_X^{-1}(\delta_X(f)) = i_X^{-1}(r_X \otimes 1_{X^*})((1_X \otimes f) \otimes 1_{X^*})(r_X^{-1} \otimes 1_{X^*})i_X \in \Pi_1(\mathcal{G}).$$

This definition does not depend on the representative $X \in [X]$ since if $X \simeq Y$ in $\mathcal{G}$, then, $\gamma_X^{-1}(\delta_X(f)) = \gamma_Y^{-1}(\delta_Y(f))$ by proposition 3.22. Indeed, if $X \xrightarrow{g} Y$ is an isomorphism in $\mathcal{G}$, then

$$\begin{aligned} & i_Y^{-1}\ (r_Y \otimes 1_{Y^*})\ ((1_Y \otimes f) \otimes 1_{Y^*})\ (r_Y^{-1} \otimes 1_{Y^*})\ i_Y \\ = {} & i_X^{-1}\ (g^{-1} \otimes ((g^{-1})^*)^{-1})\ (r_Y \otimes 1_{Y^*})\ ((1_Y \otimes f) \otimes 1_{Y^*})\ (r_Y^{-1} \otimes 1_{Y^*})\ (g \otimes (g^{-1})^*)\ i_X \\ = {} & i_X^{-1}\ (r_X \otimes 1_{X^*})\ ((g^{-1} \otimes 1_I) \otimes ((g^{-1})^*)^{-1})\ ((1_Y \otimes f) \otimes 1_{Y^*}) \\ & \quad ((g \otimes 1_I) \otimes (g^{-1})^*)\ (r_X^{-1} \otimes 1_{X^*})\ i_X \\ = {} & i_X^{-1}\ (r_X \otimes 1_{X^*})\ ((1_X \otimes f) \otimes 1_{X^*})\ (r_X^{-1} \otimes 1_{X^*})\ i_X. \end{aligned}$$

This defines an action of $\Pi_0(\mathcal{G})$ on $\Pi_1(\mathcal{G})$ since $[I] \cdot f = f$ and $[X] \cdot ([Y] \cdot f) = [X \otimes Y] \cdot f$ for all $X, Y \in \operatorname{ob} \mathcal{G}$ and $f \in \Pi_1(\mathcal{G})$. The last identity is due to the fact that we can choose the adjunction of example 2.23 to define $[X \otimes Y] \cdot f$ since, as we have just proved, it is equivalent to use the actual adjunction $(X \otimes Y, (X \otimes Y)^*, i_{X \otimes Y}, e_{X \otimes Y})$.

Definition 5.5. Let G be a group. A (left) G-module is an abelian group A with a left action of G on A such that $g \cdot (a_1 + a_2) = (g \cdot a_1) + (g \cdot a_2)$ for all $g \in G$ and $a_1, a_2 \in A$.

If $\mathcal{G}$ is a small cat-group, $\Pi_1(\mathcal{G})$ is a $\Pi_0(\mathcal{G})$-module since $[X] \cdot (gf) = ([X] \cdot g) \circ ([X] \cdot f)$ for all $X \in \operatorname{ob} \mathcal{G}$ and $f, g \in \mathcal{G}(I, I)$.

Proposition 5.6. Let $F : \mathcal{G} \to \mathcal{H}$ be a monoidal functor between cat-groups. Then, we have the following statements.

1. $\Pi_1(F)(f) = \gamma^{-1}_{F(I)}(F(f))$ for all $f \in \Pi_1(\mathcal{G})$.
2. $\Pi_1(F)(s \cdot f) = \Pi_0(F)(s) \cdot \Pi_1(F)(f)$ for all $s \in \Pi_0(\mathcal{G})$ and $f \in \Pi_1(\mathcal{G})$.
3. F is essentially surjective on objects if and only if $\Pi_0(F)$ is surjective.
4. F is faithful if and only if $\Pi_1(F)$ is injective.
5. F is full if and only if $\Pi_0(F)$ is injective and $\Pi_1(F)$ is surjective.
6. F is part of a monoidal equivalence if and only if $\Pi_0(F)$ and $\Pi_1(F)$ are isomorphisms.

Proof. 1. First, notice that, if $h \in \mathcal{H}(F(I), F(I))$, then

$$F_I^{-1}\ \gamma_{F(I)}(\gamma^{-1}_{F(I)}(h)) = \gamma_I(\gamma^{-1}_{F(I)}(h))\ F_I^{-1}$$

since γ is natural. Therefore, $h\ F_I = F_I\ \gamma^{-1}_{F(I)}(h)$ since $\gamma_I = 1_{\mathcal{G}(I,I)}$. If we set $h = F(f)$, we have $\Pi_1(F)(f) = \gamma^{-1}_{F(I)}(F(f))$.

2. Let $X \in s$. So $s = [X]$. By what we proved in point 1, we can compute

$$\begin{aligned}
&\gamma^{-1}_{F(I)}(F(\gamma_X^{-1}(\delta_X(f)))) \otimes 1_{F(X)} \\
&= l^{-1}_{F(X)}\ F(l_X)\ \widetilde{F}_{I,X}\ (F_I \otimes 1_{F(X)})\ \left(\gamma^{-1}_{F(I)}(F(\gamma_X^{-1}(\delta_X(f)))) \otimes 1_{F(X)}\right) \\
&= l^{-1}_{F(X)}\ F(l_X)\ \widetilde{F}_{I,X}\ \left(F(\gamma_X^{-1}(\delta_X(f))) \otimes 1_{F(X)}\right)\ (F_I \otimes 1_{F(X)}) \\
&= l^{-1}_{F(X)}\ F(l_X)\ F\left(\gamma_X^{-1}(\delta_X(f)) \otimes 1_X\right)\ \widetilde{F}_{I,X}\ (F_I \otimes 1_{F(X)}) \\
&= l^{-1}_{F(X)}\ F(\delta_X(f))\ F(l_X)\ \widetilde{F}_{I,X}\ (F_I \otimes 1_{F(X)}) \\
&= l^{-1}_{F(X)}\ F(\delta_X(f))\ l_{F(X)} \\
&= \gamma^{-1}_{F(X)}(F(\delta_X(f))) \otimes 1_{F(X)}.
\end{aligned}$$

Hence, by lemma 3.4.1, $\gamma^{-1}_{F(I)}(F(\gamma_X^{-1}(\delta_X(f)))) = \gamma^{-1}_{F(X)}(F(\delta_X(f)))$.

Moreover, we can also compute

$$\begin{aligned}
\delta_{F(X)}(\gamma^{-1}_{F(I)}(F(f))) &= \delta_{F(X)}(\gamma^{-1}_{F(I)}(F(f)))\ r_{F(X)}\ (1_{F(X)} \otimes F_I^{-1})\ \widetilde{F}^{-1}_{X,I}\ F(r_X^{-1}) \\
&= r_{F(X)}\ (1_{F(X)} \otimes \gamma^{-1}_{F(I)}(F(f)))\ (1_{F(X)} \otimes F_I^{-1})\ \widetilde{F}^{-1}_{X,I}\ F(r_X^{-1}) \\
&= r_{F(X)}\ (1_{F(X)} \otimes F_I^{-1})\ (1_{F(X)} \otimes F(f))\ \widetilde{F}^{-1}_{X,I}\ F(r_X^{-1}) \\
&= r_{F(X)}\ (1_{F(X)} \otimes F_I^{-1})\ \widetilde{F}^{-1}_{X,I}\ F(1_X \otimes f)\ F(r_X^{-1}) \\
&= F(r_X)\ F(1_X \otimes f)\ F(r_X^{-1}) \\
&= F(\delta_X(f)).
\end{aligned}$$

Therefore,

$$\gamma_{F(I)}^{-1}(F(\gamma_X^{-1}(\delta_X(f)))) = \gamma_{F(X)}^{-1}(\delta_{F(X)}(\gamma_{F(I)}^{-1}(F(f)))),$$

and so $\Pi_1(F)([X]\cdot f) = \Pi_0(F)([X])\cdot\Pi_1(F)(f)$.

3. By definition.

4. The 'only if part' is trivial. Let us prove the 'if part'. So, we suppose that $\Pi_1(F)$ is injective. In other words, the function $F : \mathcal{G}(I,I) \to \mathcal{H}(F(I),F(I))$ is also injective. Let $X \in \operatorname{ob}\mathcal{G}$ and $f, g \in \mathcal{G}(X,I)$ such that $F(f) = F(g)$. Thus, $F(gf^{-1}) = F(g)F(f)^{-1} = 1_{F(I)} = F(1_I)$. Hence, $gf^{-1} = 1_I$ and $g = f$. Therefore, the function $F : \mathcal{G}(X,I) \to \mathcal{H}(F(X),F(I))$ is injective. Now, we can prove that F is faithful. Let $f, g \in \mathcal{G}(X,Y)$ such that $F(f) = F(g)$. Then,

$$\begin{aligned}F(e_Y(1_{Y^*}\otimes f)) &= F(e_Y)\ F(1_{Y^*}\otimes f)\\ &= F(e_Y)\ \widetilde{F}_{Y^*,Y}\ (F(1_{Y^*})\otimes F(f))\ \widetilde{F}_{Y^*,X}^{-1}\\ &= F(e_Y)\ \widetilde{F}_{Y^*,Y}\ (F(1_{Y^*})\otimes F(g))\ \widetilde{F}_{Y^*,X}^{-1}\\ &= F(e_Y)\ F(1_{Y^*}\otimes g)\\ &= F(e_Y(1_{Y^*}\otimes g)).\end{aligned}$$

Since $F : \mathcal{G}(Y^*\otimes X, I) \to \mathcal{H}(F(Y^*\otimes X), F(I))$ is injective, $e_Y(1_{Y^*}\otimes f) = e_Y(1_{Y^*}\otimes g)$ and $1_{Y^*}\otimes f = 1_{Y^*}\otimes g$. We conclude by lemma 3.4.1.

5. Firstly, we suppose F is full. The fact that $\Pi_1(F)$ is onto is obvious. Let us prove that $\Pi_0(F)$ is injective. Suppose that there exits $X, Y \in \operatorname{ob}\mathcal{G}$ such that $[F(X)] = [F(Y)]$. So $F(X) \simeq F(Y)$ and $X \simeq Y$ since F is full. Thus $[X] = [Y]$ and $\Pi_0(F)$ is injective.

   Suppose now that $\Pi_0(F)$ is injective and $\Pi_1(F)$ is surjective. The last condition implies that the function $F : \mathcal{G}(I,I) \to \mathcal{H}(F(I),F(I))$ is surjective. Let $X \in \operatorname{ob}\mathcal{G}$. We first prove that $F : \mathcal{G}(X,I) \to \mathcal{H}(F(X),F(I))$ is onto: let $g \in \mathcal{H}(F(X),F(I))$. Hence $F(X) \simeq F(I)$ and $X \simeq I$ since $\Pi_0(F)$ is injective. So, there exists $h \in \mathcal{G}(X,I)$. Thus $gF(h)^{-1} \in \mathcal{H}(F(I),F(I))$. By assumptions, there exists $k \in \mathcal{G}(I,I)$ such that $F(k) = gF(h)^{-1}$. Therefore, $F(kh) = g$ and $F : \mathcal{G}(X,I) \to \mathcal{H}(F(X),F(I))$ is surjective.

   Now, we can prove that F is full. Let $g \in \mathcal{H}(F(X),F(Y))$. Since

$$F(e_Y)\ \widetilde{F}_{Y^*,Y}\ (1_{F(y^*)}\otimes g)\ \widetilde{F}_{Y^*,X}^{-1} :\ F(Y^*\otimes X) \longrightarrow F(I)\ ,$$

   there exists a $h \in \mathcal{G}(Y^*\otimes X, I)$ such that $F(h) = F(e_Y)\ \widetilde{F}_{Y^*,Y}\ (1_{F(y^*)}\otimes g)\ \widetilde{F}_{Y^*,X}^{-1}$. We consider the arrow

$$X \xrightarrow{l_X^{-1}} I\otimes X \xrightarrow{i_Y\otimes 1_X} (Y\otimes Y^*)\otimes X \xrightarrow{\quad a_{Y,Y^*,X}\quad} Y\otimes(Y^*\otimes X) \xrightarrow{1_Y\otimes h} Y\otimes I \xrightarrow{r_Y} Y.$$

   So, it remains to prove $F\left(r_Y\ (1_Y\otimes h)\ a_{Y^*,Y,X}\ (i_Y\otimes 1_X)\ l_X^{-1}\right) = g$:

$$\begin{aligned}&F(r_Y)\ F(1_Y\otimes h)\ F(a_{Y^*,Y,X})\ F(i_Y\otimes 1_X)\ F(l_X)^{-1}\\ &= F(r_Y)\ \widetilde{F}_{Y,I}\ (1_{F(Y)}\otimes F(e_Y))\ (1_{F(Y)}\otimes \widetilde{F}_{Y^*,Y})\ (1_{F(Y)}\otimes(1_{F(Y^*)}\otimes g))\\ &\quad (1_{F(Y)}\otimes \widetilde{F}_{Y^*,X}^{-1})\ \widetilde{F}_{Y,Y^*\otimes X}\ F(a_{Y^*,Y,X})\ F(i_Y\otimes 1_X)\ F(l_X)^{-1}\\ &= r_{F(Y)}\ (1_{F(Y)}\otimes F_I^{-1})\ (1_{F(Y)}\otimes F(e_Y))\ (1_{F(Y)}\otimes \widetilde{F}_{Y^*,Y})\ (1_{F(Y)}\otimes(1_{F(Y^*)}\otimes g))\\ &\quad a_{F(Y),F(Y^*),F(X)}\ (\widetilde{F}_{Y,Y^*}^{-1}\otimes 1_{F(X)})\ \widetilde{F}_{Y\otimes Y^*,X}^{-1}\ F(i_Y\otimes 1_X)\ F(l_X)^{-1}\end{aligned}$$

$$
\begin{aligned}
&= r_{F(Y)}\,(1_{F(Y)} \otimes F_I^{-1})\,(1_{F(Y)} \otimes F(e_Y))\,(1_{F(Y)} \otimes \widetilde{F}_{Y^*,Y})\,a_{F(Y),F(Y^*),F(Y)} \\
&\quad (1_{F(Y)\otimes F(Y^*)} \otimes g)\,(\widetilde{F}^{-1}_{Y,Y^*} \otimes 1_{F(X)})\,(F(i_Y) \otimes 1_{F(X)})\,\widetilde{F}^{-1}_{I,X}\,F(l_X)^{-1} \\
&= r_{F(Y)}\,(1_{F(Y)} \otimes F_I^{-1})\,(1_{F(Y)} \otimes F(e_Y))\,(1_{F(Y)} \otimes \widetilde{F}_{Y^*,Y})\,a_{F(Y),F(Y^*),F(Y)} \\
&\quad (\widetilde{F}^{-1}_{Y,Y^*} \otimes 1_{F(Y)})\,(1_{F(Y\otimes Y^*)} \otimes g)\,(F(i_Y) \otimes 1_{F(X)})\,\widetilde{F}^{-1}_{I,X}\,F(l_X)^{-1} \\
&= r_{F(Y)}\,(1_{F(Y)} \otimes F_I^{-1})\,(1_{F(Y)} \otimes F(e_Y))\,\widetilde{F}^{-1}_{Y,Y^*\otimes Y}\,F(a_{Y,Y^*,Y})\,\widetilde{F}_{Y\otimes Y^*,Y} \\
&\quad (F(i_Y) \otimes 1_{F(Y)})\,(1_{F(I)} \otimes g)\,\widetilde{F}^{-1}_{I,X}\,F(l_X)^{-1} \\
&= r_{F(Y)}\,(1_{F(Y)} \otimes F_I^{-1})\,\widetilde{F}^{-1}_{Y,I}\,F(1_Y \otimes e_Y)\,F(a_{Y,Y^*,Y})\,F(i_Y \otimes 1_Y)\,\widetilde{F}_{I,Y} \\
&\quad (1_{F(I)} \otimes g)\,(F_I \otimes 1_{F(X)})\,l^{-1}_{F(X)} \\
&= F(r_Y)\,F(1_Y \otimes e_Y)\,F(a_{Y,Y^*,Y})\,F(i_Y \otimes 1_Y)\,\widetilde{F}_{I,Y}\,(F_I \otimes 1_{F(Y)})\,(1_I \otimes g)\,l^{-1}_{F(X)} \\
&= F(l_Y)\,\widetilde{F}_{I,Y}\,(F_I \otimes 1_{F(Y)})\,l^{-1}_{F(Y)}\,g \\
&= g.
\end{aligned}
$$

6. Follows from points 3, 4 and 5.

$\square$

This proposition says in particular that, $\Pi_0(\mathcal{G})$, $\Pi_1(\mathcal{G})$ and the action of $\Pi_0(\mathcal{G})$ on $\Pi_1(\mathcal{G})$ are invariants under monoidal equivalences, i.e. they do not change (up to isomorphisms) if we change $\mathcal{G}$ by a monoidally equivalent cat-group $\mathcal{H}$. Let us now give some examples.

Example 5.7. Let G be a group. We have defined in example 3.9 the cat-group $\underline{\mathrm{D}}(G)$. By definition, we know that $\Pi_0(\underline{\mathrm{D}}(G)) \simeq G$ and $\Pi_1(\underline{\mathrm{D}}(G))$ is the trivial group.

Example 5.8. Let A be an abelian group. In example 3.10, we have defined the cat-group $A!$. We now see that $\Pi_0(A!)$ is the trivial group whereas $\Pi_1(A!) = A$.

Example 5.9. Let $A \xrightarrow{f} B$ be a morphism of abelian group. Examples 2.7 and 3.12 describe the cat-group $\underline{\mathrm{Coker}} f$. It is easy to see that $\Pi_0(\underline{\mathrm{Coker}} f) = B/\operatorname{Im} f = \operatorname{Coker} f$ and $\Pi_1(\underline{\mathrm{Coker}} f) = \operatorname{Ker} f$. Now, we understand why this cat-group is denoted $\underline{\mathrm{Coker}} f$. Moreover, we can see that the action of $\Pi_0(\underline{\mathrm{Coker}} f)$ on $\Pi_1(\underline{\mathrm{Coker}} f)$ is trivial. Indeed, if $[b] \in \Pi_0(\underline{\mathrm{Coker}} f)$ and $k \in \Pi_1(\underline{\mathrm{Coker}} f)$, we have

$$[b] \cdot k = \gamma_b^{-1}(\delta_b^{-1}(k)) = \gamma_b^{-1}(k) = -i_b + k + i_b = k.$$

5.2 Postnikov Invariant

As announced earlier, we have to define one more invariant: the Postnikov invariant. This is an element of the third cohomology group $H^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}))$. Of course, we are going to introduce the cohomology of groups in the beginning of this section. Then, we will propose two equivalent definitions of this invariant. One is from J. Baez and A. Lauda in [1], whereas the second one is due to H. X. Sinh in her thesis [14].

Definition 5.10. Let G be a group and A a G-module. We define, for each $n \in \mathbb{Z}$, the abelian group of n-cochains $C^n(G, A)$ as the trivial group 0 if $n \leqslant 0$ and, if $n > 0$, as the group of functions $f : G^n \to A$ such that $f(s_1, \ldots, s_{i-1}, 1, s_{i+1}, \ldots, s_n) = 0$ for all $i \in \{1, \ldots n\}$ and $s_1, \ldots, s_n \in G$. The group law is the obvious one, i.e. $(f+g)(s_1 \ldots, s_n) = f(s_1, \ldots, s_n) + g(s_1, \ldots, s_n)$ for all $s_1, \ldots, s_n \in G$.

Now, for each $n \in \mathbb{Z}$, we define the coboundary homomorphism

$$\delta_n : C^n(G, A) \to C^{n+1}(G, A).$$

If $n \leqslant 0$, δ_n is the trivial homomorphism. If $n > 0$, then,

$$\begin{aligned}
&\delta_n(f)(s_1, \ldots, s_{n+1}) \\
&= (-1)^{n+1}\Bigg(s_1 \cdot f(s_2, \ldots, s_{n+1}) + \sum_{i=1}^{n} (-1)^i f(s_1, \ldots, s_{i-1}, (s_i s_{i+1}), s_{i+2}, \ldots, s_{n+1}) \\
&\qquad + (-1)^{n+1} f(s_1, \ldots, s_n)\Bigg)
\end{aligned}$$

for all $f \in C^n(G, A)$ and $s_1, \ldots, s_{n+1} \in G$. Obviously, they are group homomorphisms.

Lemma 5.11. Let G be a group and A a G-module. We consider the coboundary homomorphisms of definition 5.10.

$$\cdots \longrightarrow C^n(G, A) \xrightarrow{\delta_n} C^{n+1}(G, A) \xrightarrow{\delta_{n+1}} C^{n+2}(G, A) \longrightarrow \cdots$$

Then, $\delta_{n+1} \circ \delta_n = 0$ for all $n \in \mathbb{Z}$.

Proof. If $n \leqslant 0$, it is trivial. So, we can suppose $n > 0$. Let $f \in C^n(G, A)$. For all $s_1, \ldots, s_{n+2} \in G$, we have

$$\begin{aligned}
&(-1)^{n+2}\, \delta_{n+1}(\delta_n(f))(s_1, \ldots, s_{n+2}) \\
&= s_1 \cdot (\delta_n f)(s_2, \ldots, s_{n+2}) - (\delta_n f)(s_1 s_2, \ldots, s_{n+2}) \\
&\quad + \sum_{j=2}^{n} (-1)^j (\delta_n f)(s_1, \ldots, s_j s_{j+1}, \ldots, s_{n+2}) \\
&\quad + (-1)^{n+1} (\delta_n f)(s_1, \ldots, s_{n+1} s_{n+2}) + (-1)^{n+2} (\delta_n f)(s_1, \ldots, s_{n+1}) \\
&= (-1)^{n+1} (s_1 s_2) \cdot f(s_3, \ldots, s_{n+2}) + (-1)^{n+1} \sum_{i=1}^{n} (-1)^i s_1 \cdot f(s_2, \ldots, s_{i+1} s_{i+2}, \ldots, s_{n+2}) \\
&\quad + s_1 \cdot f(s_2, \ldots, s_{n+1}) - (-1)^{n+1} (s_1 s_2) \cdot f(s_3, \ldots, s_{n+2}) + (-1)^{n+1} f(s_1 s_2 s_3, \ldots, s_{n+2}) \\
&\quad - (-1)^{n+1} \sum_{i=2}^{n} (-1)^i f(s_1 s_2, \ldots, s_{i+1} s_{i+2}, \ldots, s_{n+2}) - f(s_1 s_2, \ldots, s_{n+1}) \\
&\quad + \sum_{j=2}^{n} (-1)^{j+n+1} s_1 \cdot f(s_2, \ldots, s_j s_{j+1}, \ldots, s_{n+2}) \\
&\quad + \sum_{j=2}^{n} \sum_{i=1}^{j-2} (-1)^{j+n+1+i} f(s_1, \ldots, s_i s_{i+1}, \ldots, s_j s_{j+1}, \ldots, s_{n+2}) \\
&\quad + \sum_{j=2}^{n} (-1)^{n+2j} f(s_1, \ldots, s_{j-1} s_j s_{j+1}, \ldots, s_{n+2}) \\
&\quad + \sum_{j=2}^{n} (-1)^{n+2j+1} f(s_1, \ldots, s_j s_{j+1} s_{j+2}, \ldots, s_{n+2}) \\
&\quad + \sum_{j=2}^{n} \sum_{i=j+1}^{n} (-1)^{j+n+1+i} f(s_1, \ldots, s_j s_{j+1}, \ldots, s_{i+1} s_{i+2}, \ldots, s_{n+2}) +
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=2}^{n}(-1)^j f(s_1,\ldots,s_j s_{j+1},\ldots,s_{n+1})\\
&+s_1\cdot f(s_2,\ldots,s_{n+1}s_{n+2})+\sum_{i=1}^{n-1}(-1)^i f(s_1,\ldots,s_i s_{i+1},\ldots,s_{n+1}s_{n+2})\\
&+(-1)^n f(s_1,\ldots,s_n s_{n+1}s_{n+2})+(-1)^{n+1}f(s_1,\ldots,s_n)\\
&-s_1\cdot f(s_2,\ldots,s_{n+1})-\sum_{i=1}^{n}(-1)^i f(s_1,\ldots,s_i s_{i+1}\ldots,s_{n+1})-(-1)^{n+1}f(s_1,\ldots,s_n)\\
=\;&0
\end{aligned}
$$

since every term appears twice with opposite signs.

□

Definition 5.12. Let G be a group and A a G-module. By lemma 5.11, we have a cochain complex:

$$\cdots \longrightarrow C^{n-1}(G,A) \xrightarrow{\delta_{n-1}} C^n(G,A) \xrightarrow{\delta_n} C^{n+1}(G,A) \longrightarrow \cdots$$

We define $B^n(G,A) = \operatorname{Im}\delta_{n-1}$, $Z^n(G,A) = \operatorname{Ker}\delta_n$ and $H^n(G,A) = Z^n(G,A)/B^n(G,A)$ for all $n \in \mathbb{Z}$. In other words, $H^n(G,A)$ is the cokernel of the inclusion $\operatorname{Im}\delta_{n-1} \hookrightarrow \operatorname{Ker}\delta_n$. Elements of $B^n(G,A)$ are called n-coboundaries while elements of $Z^n(G,A)$ are called n-cocycles. The group $H^n(G,A)$ is the n^{th} cohomology group.

Remark 5.13. Let G be a group and A a G-module. We consider the group ring $\mathbb{Z}[G]$. We also view $\mathbb{Z}$ as a G-module where G acts trivially on $\mathbb{Z}$. Thus, since $\mathbb{Z}$ and A are G-modules, we can consider them as $\mathbb{Z}[G]$-modules. With this frame work, we can prove that $\operatorname{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},A) = H^n(G,A)$ for all $n \geqslant 2$. We do not give more details here, since we will not use this later.

Remark 5.14. Let G and G' be two groups, A a G-module and A' a G'-module. Let also $\varepsilon_0 : G \longrightarrow G'$ and $\varepsilon_1 : A \longrightarrow A'$ be two group homomorphisms such that $\varepsilon_1(g\cdot m) = \varepsilon_0(g)\cdot\varepsilon_1(m)$ for all $g \in G$ and $m \in A$. Then, A' is a G-module by the action $g\cdot m' = \varepsilon_0(g)\cdot m'$ for all $g \in G$ and $m' \in A'$. Moreover, for each $n \in \mathbb{Z}$, ε_0 and ε_1 induce two group homomorphisms $\overline{\varepsilon_0}^n : Z^n(G',A') \to Z^n(G,A')$ and $\overline{\varepsilon_1}^n : Z^n(G,A) \to Z^n(G,A')$, respecting the quotient. So they give rise to the two group morphisms

$$
\begin{aligned}
\overline{\varepsilon_0}^n : H^n(G',A') &\longrightarrow H^n(G,A')\\
[f'] &\longmapsto [(s_1,\ldots,s_n)\mapsto f'(\varepsilon_0(s_1),\ldots,\varepsilon_0(s_n))]
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\varepsilon_1}^n : H^n(G,A) &\longrightarrow H^n(G,A')\\
[f] &\longmapsto [(s_1,\ldots,s_n)\mapsto \varepsilon_1(f(s_1,\ldots,s_n))]
\end{aligned}
$$

for all $s_1,\ldots,s_n \in G$.

We will be in particular interested in the third cohomology group. This is the reason why we have the following definition.

Definition 5.15. Objects in the category $\widetilde{H^3}$ are the triples (G, A, a) where G is a group, A a G-module and $a \in H^3(G, A)$. Morphisms $\varepsilon : (G, A, a) \longrightarrow (G', A', a')$ in $\widetilde{H^3}$ are the pairs $\varepsilon = (\varepsilon_0, \varepsilon_1)$ of group homomorphisms $\varepsilon_0 : G \to G'$ and $\varepsilon_1 : A \to A'$ such that

$$\varepsilon_1(g \cdot m) = \varepsilon_0(g) \cdot \varepsilon_1(m)$$

for all $g \in G$ and $m \in A$ and satisfying

$$\overline{\varepsilon_1}^3(a) = \overline{\varepsilon_0}^3(a') \in H^3(G, A').$$

Compositions and identities are the obvious ones.

Remark 5.16.
- Morphisms in $\widetilde{H^3}$ are pairs of group homomorphisms $\varepsilon = (\varepsilon_0, \varepsilon_1)$ preserving the action of G on A and preserving the element $a \in H^3(G, A)$.
- ε is an isomorphism if and only if ε_0 and ε_1 are group isomorphisms.

We want to associate to each small group $\mathcal{G}$, a unique (up to isomorphism) object of $\widetilde{H^3}$. We already have $G = \Pi_0(\mathcal{G})$ and $A = \Pi_1(\mathcal{G})$. It remains to define $a \in H^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}))$. It will come from the associativity isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$. We present now two different ways to define it.

5.2.1 Baez-Lauda's Definition

Firstly, we present a definition of this Postnikov invariant due to Baez and Lauda in [1]. It is based on the fact that, up to monoidal equivalence, we can assume $X \simeq Y \Rightarrow X = Y$, $l_X = r_X = 1_X$ and $i_X = 1_I$ for all $X, Y \in \operatorname{ob}\mathcal{G}$.

Definition 5.17. A category $\mathcal{C}$ is skeletal if, for each pair of objects $X, Y \in \operatorname{ob}\mathcal{C}$ such that $X \simeq Y$, then, $X = Y$.

Definition 5.18. Let $\mathcal{G}$ be a cat-group. We say that $\mathcal{G}$ is a special cat-group if

- $\mathcal{G}$ is skeletal,
- $l_X = r_X = 1_X$ and $i_X = 1_I$ for all $X \in \operatorname{ob}\mathcal{G}$.

In [1], Baez and Lauda asked also that $e_X = 1_I$ for all $X \in \operatorname{ob}\mathcal{G}$. But, with this additional assumption, we will not be able to prove that every cat-group is monoidally equivalent to a special one. This is why we do not require it here.

Definition 5.19. Let $\mathcal{G}$ and $\mathcal{H}$ be two special cat-groups. A special monoidal functor $F : \mathcal{G} \to \mathcal{H}$ is a monoidal functor such that $F_I = 1_I$. Since the composition of special monoidal functors is still a special monoidal functor, we have the 2-category SpCG of small special cat-groups, special monoidal functors and monoidal natural transformations. Hence, we have an inclusion 2-functor SpCG $\hookrightarrow$ CG.

Example 5.20. Let G be a group. Clearly, $\underline{\mathrm{D}}(G)$ is a special cat-group since the only arrows are identities.

Example 5.21. Let A an abelian group. $A!$ is a special cat-group if we choose $i_I = 1_I$.

Example 5.22. Let $A \xrightarrow{f} B$ be a morphism of abelian groups. $\underline{\mathrm{Coker}}f$ is a special cat-group if and only if $f(a) = 0$ for all $a \in A$ and $i_b = 1_I$ for all $b \in B$.

Now, we want to prove that every cat-group is monoidally equivalent to a special one. To do so, we need the following lemma.

Lemma 5.23. Let $\mathcal{G}$ be a cat-group. Suppose that, for all objects $X, Y \in \operatorname{ob}\mathcal{G}$, we have an object $X \tilde{\otimes} Y \in \operatorname{ob}\mathcal{G}$ and an isomorphism $\gamma_{X,Y} : X\tilde{\otimes}Y \longrightarrow X \otimes Y$. Then, there exists a unique cat-group structure $\widetilde{\mathcal{G}} = (\mathcal{G}, \tilde{\otimes}, \tilde{I}, \tilde{l}, \tilde{r}, \tilde{a}, \tilde{*}, \tilde{i}, \tilde{e})$ on the category $\mathcal{G}$ such that

- the functor $\tilde{\otimes}$ is actually defined by the given object $X\tilde{\otimes}Y$ for all $X, Y \in \operatorname{ob}\mathcal{G}$,
- the functor $F : \mathcal{G} \to \widetilde{\mathcal{G}}$ defined as the identity $F = 1_{\mathcal{G}}$ on the underlying categories and equipped with $F_I = 1_I$, $\widetilde{F}_{X,Y} = \gamma_{X,Y}$ and $F^*_X = 1_{X^*}$ for all $X, Y \in \operatorname{ob}\mathcal{G}$ is a cat-group functor.

Moreover, this F is part of a monoidal equivalence.

Proof. Let us prove first the uniqueness. Since $F_I = 1_I$, we know that $\tilde{I} = I$. Moreover, by definition 2.12, we must have $\tilde{l}_X = l_X\ \gamma_{I,X}$, $\tilde{r}_X = r_X\ \gamma_{X,I}$, $f\tilde{\otimes}g = \gamma^{-1}_{X',Y'}\ (f \otimes g)\ \gamma_{X,Y}$ and

$$\tilde{a}_{X,Y,Z} = (1_X\tilde{\otimes}\gamma^{-1}_{Y,Z})\ \gamma^{-1}_{X,Y\otimes Z}\ a_{X,Y,Z}\ \gamma_{X\otimes Y,Z}\ (\gamma_{X,Y}\tilde{\otimes}1_Z)$$

for all $X, Y, Z, X', Y' \in \operatorname{ob}\mathcal{G}$, $f \in \mathcal{G}(X, X')$ and $g \in \mathcal{G}(Y, Y')$. In addition, by definition 3.16, we have $X^{\tilde{*}} = X^*$, $\tilde{i}_X = \gamma^{-1}_{X,X^*}\ i_X$ and $\tilde{e}_X = e_X\ \gamma_{X^*,X}$ for all $X \in \operatorname{ob}\mathcal{G}$ since $F^*_X = 1_{X^*}$. Therefore, such a cat-group structure is unique.

For the existence, it suffices to check that the definitions above imply that $\widetilde{\mathcal{G}}$ is a cat-group and that F is a cat-group functor. For example, for the Triangle Axiom, we can compute, for all $X, Y \in \operatorname{ob}\mathcal{G}$:

$$\begin{aligned}
(1_X\tilde{\otimes}\tilde{l}_Y)\ \tilde{a}_{X,\tilde{I},Y} &= \gamma^{-1}_{X,Y}\ (1_X \otimes \tilde{l}_Y)\ \gamma_{X,I\tilde{\otimes}Y}\ (1_X\tilde{\otimes}\gamma^{-1}_{I,Y})\ \gamma^{-1}_{X,I\otimes Y}\ a_{X,I,Y}\ \gamma_{X\otimes I,Y}\ (\gamma_{X,I}\tilde{\otimes}1_Y) \\
&= \gamma^{-1}_{X,Y}\ (1_X \otimes \tilde{l}_Y)\ (1_X \otimes \gamma^{-1}_{I,Y})\ a_{X,I,Y}\ (\gamma_{X,I} \otimes 1_Y)\ \gamma_{X\tilde{\otimes}I,Y} \\
&= \gamma^{-1}_{X,Y}\ (1_X \otimes l_Y)\ a_{X,I,Y}\ (\gamma_{X,I} \otimes 1_Y)\ \gamma_{X\tilde{\otimes}I,Y} \\
&= \gamma^{-1}_{X,Y}\ (r_X \otimes 1_Y)\ (\gamma_{X,I} \otimes 1_Y)\ \gamma_{X\tilde{\otimes}I,Y} \\
&= \gamma^{-1}_{X,Y}\ (\tilde{r}_X \otimes 1_Y)\ \gamma_{X\tilde{\otimes}I,Y} \\
&= \tilde{r}_X\tilde{\otimes}1_Y.
\end{aligned}$$

The fact that F is part of a monoidal equivalence follows from proposition 2.19.

□

Proposition 5.24. Every small cat-group is monoidally equivalent to a small special cat-group.

Proof. Let $\mathcal{G}$ be a small cat-group. Firstly, we prove that $\mathcal{G}$ is monoidally equivalent to a small skeletal cat-group. By the axiom of choice, we can choose, for all $s \in \Pi_0(\mathcal{G})$, a representative $X_s \in s$ such that $X_{[I]} = I$. For each $Y, Z \in \operatorname{ob}\mathcal{G}$, let $Y \otimes' Z = X_{[Y\otimes Z]}$. By lemma 5.23, we can suppose, without loss of generality, that $X_s \otimes X_t = X_{st}$ for all $s, t \in \Pi_0(\mathcal{G})$. Let $\mathcal{G}' \subseteq \mathcal{G}$ be the full subcategory with $\{X_s | s \in \Pi_0(\mathcal{G})\}$ as objects. Restrictions of $\otimes$, l, r and a to $\mathcal{G}'$ make $(\mathcal{G}', \otimes, I, l, r, a)$ be a monoidal category. Moreover, by proposition 3.5, we can extend $\mathcal{G}'$ to a cat-group. Since the inclusion $\mathcal{G}' \hookrightarrow \mathcal{G}$ is monoidal, essentially surjective on objects, full and faithful, $\mathcal{G}'$ is monoidally equivalent to $\mathcal{G}$. Since $\mathcal{G}'$ is skeletal, we can suppose, without loss of generality, that $\mathcal{G}$ is a small skeletal cat-group.

Now, let $X\tilde{\otimes}Y = X \otimes Y$ and

$$\gamma_{X,Y} = \begin{cases} l_Y^{-1} & \text{if X=I} \\ r_X^{-1} & \text{if Y=I} \\ 1_{X\otimes Y} & \text{otherwise} \end{cases}$$

for all $X, Y \in \operatorname{ob}\mathcal{G}$. Notice that this is well-defined since $l_I^{-1} = r_I^{-1}$. Now, we can replace $\mathcal{G}$ by $\widetilde{\mathcal{G}}$ of lemma 5.23 and we obtain $\tilde{l}_X = l_X\ \gamma_{I,X} = 1_X$ and $\tilde{r}_X = r_X\ \gamma_{X,I} = 1_X$ for all $X \in \operatorname{ob}\mathcal{G}$. Finally, we change $\widetilde{\mathcal{G}}$ by lemma 3.4.2 and corollary 3.28 in order to have $\tilde{i}_X = 1_I$ for all $X \in \operatorname{ob}\widetilde{\mathcal{G}}$.

□

Special cat-groups have some really nice properties, which will be helpful to construct the Postnikov invariant. We will often use the following lemma without referring to it.

Lemma 5.25. Let $\mathcal{G}$ be a special cat-group. Then, we have the following properties.

1. $fg = gf$ for all $f, g \in \mathcal{G}(X, X)$.
2. $(f_1 \otimes f_2) \otimes f_3 = f_1 \otimes (f_2 \otimes f_3)$ for all morphisms f_1, f_2, f_3 in $\mathcal{G}$.
3. $a_{X,Y,I} = a_{X,I,Y} = a_{I,X,Y} = 1_{X\otimes Y}$ for all $X, Y \in \operatorname{ob}\mathcal{G}$.
4. $f \otimes 1_I = f = 1_I \otimes f$ for all morphisms f in $\mathcal{G}$.
5. $f \otimes g = fg$ for all $f, g \in \mathcal{G}(I, I)$.

Proof. 1. This is corollary 3.15.

2. By point 1 and naturality of a. Notice that, since $\mathcal{G}$ is skeletal, for all $i \in \{1, 2, 3\}$, $f_i \in \mathcal{G}(X_i, X_i)$ for some $X_i \in \operatorname{ob}\mathcal{G}$.
3. $a_{X,I,Y} = 1_{X\otimes Y}$ follows from the Triangle Axiom. The two others identities come from lemma 2.10.
4. This is naturality of l and r.
5. We can compute $f \otimes g = (f \otimes 1_I)(1_I \otimes g) = fg$.

□

We have proved that any small cat-group is monoidally equivalent to a small special cat-group. Now, we prove that a monoidal functor between special cat-groups is naturaly isomorphic to a special one.

Proposition 5.26. Let $F : \mathcal{G} \to \mathcal{H}$ be a monoidal functor between special cat-groups. There exists a special monoidal functor $H : \mathcal{G} \to \mathcal{H}$ and a monoidal natural isomorphism $\alpha :\ F \overset{\sim}{\Longrightarrow} H$. Moreover, this H is part of a monoidal equivalence if and only if F is.

Proof. As a functor, let $H = F$. Since $\mathcal{H}$ is a special cat-group, $F(I) = I$. So, we can define $H_I = 1_I \in \mathcal{H}(I, I)$. Then, for all $X, Y \in \operatorname{ob}\mathcal{G}$, we set

$$\begin{tikzcd}[column sep=large] \widetilde{H}_{X,Y} :\ H(X) \otimes H(Y) = (F(X) \otimes I) \otimes F(Y) \arrow[r, "(1_{F(X)}\otimes F_I)\otimes 1_{F(Y)}"] & (F(X) \otimes F(I)) \otimes F(Y) \arrow[d, equal] \\ H(X \otimes Y) = F(X \otimes Y) & F(X) \otimes F(Y) \arrow[l, "\widetilde{F}_{X,Y}"] \end{tikzcd}$$

Note that for all $f \in \mathcal{G}(X,X)$ and $g \in \mathcal{G}(Y,Y)$, since $\widetilde{F}_{X,Y}(F(f) \otimes F(g)) = F(f \otimes g)\widetilde{F}_{X,Y}$, we have $F(f) \otimes F(g) = F(f \otimes g)$ by lemma 5.25.1. Naturality of $\widetilde{H}$ follows from this observation and lemma 5.25.1. To see that $\widetilde{H}$ commutes with the associativity isomorphisms a, it suffices to use lemmas 5.25.1 and 5.25.2. Then, we know that $\widetilde{H}$ commutes with the isomorphisms r and l since $\widetilde{F}_{X,I} = 1_{F(X)} \otimes F_I^{-1}$ and $\widetilde{F}_{I,X} = F_I^{-1} \otimes 1_{F(X)}$ imply $\widetilde{H}_{X,I} = 1_{H(X)} = \widetilde{H}_{I,X}$ for all $X \in \operatorname{ob}\mathcal{G}$ by lemma 5.25. Therefore, H is a special monoidal functor. Now, we define $\alpha : F \Longrightarrow H$ by $\alpha_X = 1_{F(X)} \otimes F_I^{-1}$ for all $X \in \operatorname{ob}\mathcal{G}$. Again, lemma 5.25 implies that α is natural and monoidal. Finally, since $H = F$ as functors, H is part of a monoidal equivalence if and only if F is.

□

Corollary 5.27. The two 2-categories CG and SpCG are biequivalent: CG $\simeq$ SpCG.

Proof. We consider the inclusion 2-functor SpCG $\hookrightarrow$ CG. Proposition 5.24 says that it is weakly essentially surjective on objects. It is essentially surjective on 1-cells by proposition 5.26. Moreover, this inclusion is trivially full and faithful on 2-cells. So, by proposition 4.30, CG $\simeq$ SpCG.

□

We are now able to define the Postnikov invariant of a small cat-group. We begin to define it for special ones.

Proposition 5.28. Let $\mathcal{G}$ be a small special cat-group. Since $\mathcal{G}$ is skeletal, $\Pi_0(\mathcal{G}) = \operatorname{ob}\mathcal{G}$ as sets. Then, the function

$$\begin{aligned} a' : \quad \Pi_0(\mathcal{G})^3 &\longrightarrow \Pi_1(\mathcal{G}) \\ (r,s,t) &\longmapsto a_{r,s,t} \otimes 1_{(rst)^{-1}} \end{aligned}$$

belongs to $Z^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}))$. We call $[a'] \in H^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}))$ the Postnikov invariant of $\mathcal{G}$.

Proof. By lemma 5.25.3, $a' \in C^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}))$. It remains to prove that $\delta_3(a') = 0$. This is nothing but the Pentagon Axiom: for all $r,s,t,u \in \Pi_0(\mathcal{G})$,

$$\begin{aligned}
&\delta_3(a')(r,s,t,u) = (r \cdot a'(s,t,u)) \circ a'(rs,t,u)^{-1} \circ a'(r,st,u) \circ a'(r,s,tu)^{-1} \circ a'(r,s,t) \\
&= (1_r \otimes a_{s,t,u} \otimes 1_{(stu)^{-1}} \otimes 1_{r^{-1}}) \circ (a^{-1}_{rs,t,u} \otimes 1_{(rstu)^{-1}}) \circ (a_{r,st,u} \otimes 1_{(rstu)^{-1}}) \\
&\quad \circ (a^{-1}_{r,s,tu} \otimes 1_{(rstu)^{-1}}) \circ (a_{r,s,t} \otimes 1_{(rst)^{-1}}) \\
&= (1_r \otimes a_{s,t,u} \otimes 1_{(rstu)^{-1}}) \circ (a_{r,st,u} \otimes 1_{(rstu)^{-1}}) \circ (a_{r,s,t} \otimes 1_u \otimes 1_{(rstu)^{-1}}) \\
&\quad \circ (a^{-1}_{rs,t,u} \otimes 1_{(rstu)^{-1}}) \circ (a^{-1}_{r,s,tu} \otimes 1_{(rstu)^{-1}}) \\
&= 1_{rstu} \otimes 1_{(rstu)^{-1}} \\
&= 1_I.
\end{aligned}$$

□

We want to associate to each small cat-group $\mathcal{G}$ an element of $H^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}))$. We do it in the following way.

Definition 5.29. Let $\mathcal{G}$ be a small cat-group. By proposition 5.24, we have a monoidal equivalence $F : \mathcal{G} \to \mathcal{H}$ where $\mathcal{H}$ is a small special cat-group with Postnikov invariant a'. The Postnikov invariant of $\mathcal{G}$ is the unique element $a \in H^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}))$ such that

$$(\Pi_0(F), \Pi_1(F)) : (\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}), a) \to (\Pi_0(\mathcal{H}), \Pi_1(\mathcal{H}), a')$$

is an isomorphism in $\widetilde{H^3}$ (see proposition 5.6).

Remark 5.30. • The element $a \in H^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}))$ making $(\Pi_0(F), \Pi_1(F))$ an arrow in $\widetilde{H^3}$ is unique since isomorphisms $\Pi_0(F)$ and $\Pi_1(F)$ induce isomorphisms $\overline{\Pi_0(F)}^3$ and $\overline{\Pi_1(F)}^3$.

- Definition 5.29 is compatible with definition 5.28 since, if $\mathcal{G}$ is special, it suffices to consider the monoidal equivalence $1_\mathcal{G} : \mathcal{G} \to \mathcal{G}$.

- Postnikov invariant of $\mathcal{G}$ might be defined only up to isomorphism. Indeed, if we consider an other monoidal equivalence $F' : \mathcal{G} \to \mathcal{H}'$ with $\mathcal{H}'$ a small special cat-group, this could define an $a' \neq a \in H^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}))$. Next proposition will tell us that, up to isomorphism in $\widetilde{H^3}$, these are actually the same element.

- This is an invariant since, if we have two monoidally equivalent small cat-groups $\mathcal{G}$ and $\mathcal{G}'$, the next proposition says that $(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}), a)$ and $(\Pi_0(\mathcal{G}'), \Pi_1(\mathcal{G}'), a')$ are isomorphic in $\widetilde{H^3}$.

- We should warn the reader here on a surprising fact. If the associativity isomorphisms $a_{X,Y,Z}$ of $\mathcal{G}$ are all identities, the Postnikov invariant will not necessarily vanish, since lemma 5.23 will not always let these associativity isomorphisms stay identities.

Proposition 5.31. 1. Let $\mathcal{G}$ be a small cat-group. Suppose $F_1 : \mathcal{G} \to \mathcal{H}_1$ and $F_2 : \mathcal{G} \to \mathcal{H}_2$ are two monoidal equivalences where $\mathcal{H}_1$ and $\mathcal{H}_2$ are small special cat-groups. They induce two Postnikov invariants $a_1, a_2 \in H^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}))$. Then, $(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}), a_1) \simeq (\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}), a_2)$ in $\widetilde{H^3}$.

2. If $\mathcal{G}$ and $\mathcal{G}'$ are two monoidally equivalent small cat-groups with Postnikov invariants a and a' respectively, then, $(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}), a) \simeq (\Pi_0(\mathcal{G}'), \Pi_1(\mathcal{G}'), a')$ in $\widetilde{H^3}$.

Proof. 1. Let a'_1 and a'_2 be respectively the Postnikov invariants of $\mathcal{H}_1$ and $\mathcal{H}_2$ defined in 5.28. It suffices to prove $(\Pi_0(\mathcal{H}_1), \Pi_1(\mathcal{H}_1), a'_1) \simeq (\Pi_0(\mathcal{H}_2), \Pi_1(\mathcal{H}_2), a'_2)$ in $\widetilde{H^3}$. We know there is a monoidal equivalence $F : \mathcal{H}_1 \to \mathcal{H}_2$. Due to proposition 5.26, we can assume without loss of generality that $F_I = 1_I$. Now, we consider the two morphisms $\varepsilon_0 = \Pi_0(F) : \Pi_0(\mathcal{H}_1) \to \Pi_0(\mathcal{H}_2)$ and $\varepsilon_1 = \Pi_1(F) : \Pi_1(\mathcal{H}_1) \to \Pi_1(\mathcal{H}_2)$. In view of proposition 5.6, it remains to prove $\overline{\varepsilon_1}^3(a'_1) = \overline{\varepsilon_0}^3(a'_2) \in H^3(\Pi_0(\mathcal{H}_1), \Pi_1(\mathcal{H}_2))$. Let

$$\begin{aligned} b : \quad \Pi_0(\mathcal{H}_1) \times \Pi_0(\mathcal{H}_1) &\longrightarrow \Pi_1(\mathcal{H}_2) \\ (s,t) &\longmapsto \widetilde{F}_{s,t} \otimes 1_{\varepsilon_0(st)^{-1}}. \end{aligned}$$

This b lies in $C^2(\Pi_0(\mathcal{H}_1), \Pi_1(\mathcal{H}_2))$. Indeed,

$$b(s,1) = \widetilde{F}_{s,1} \otimes 1_{\varepsilon_0(s)^{-1}} = 1_{F(s)} \otimes F_I^{-1} \otimes 1_{(F(s))^{-1}} = 1_I$$

for all $s \in \Pi_0(\mathcal{H}_1)$ by definition 2.12 and since $F_I = 1_I$. Similarly, we have $b(1,t) = 1_I$ for all $t \in \Pi_0(\mathcal{H}_1)$. Now, let $r, s, t \in \Pi_0(\mathcal{H}_1)$ and $r' = F(r)$, $s' = F(s)$ and $t' = F(t)$ in $\Pi_0(\mathcal{H}_2)$. Let us also denote $\hat{a}_1$ and $\hat{a}_2$ respectively for the families of associativity isomorphisms of $\mathcal{H}_1$ and $\mathcal{H}_2$. We can compute:

$$
\begin{aligned}
&\varepsilon_1(a_1'(r,s,t)) \ +_{\Pi_1(\mathcal{H}_2)} \ \delta_2(b)(r,s,t) \\
&= \Pi_1(F)((\hat{a}_1)_{r,s,t} \otimes 1_{(rst)^{-1}}) \circ (r \cdot b(s,t))^{-1} \circ b(rs,t) \circ b(r,st)^{-1} \circ b(r,s) \\
&= F((\hat{a}_1)_{r,s,t} \otimes 1_{(rst)^{-1}}) \circ (1_{r'} \otimes \widetilde{F}_{s,t}^{-1} \otimes 1_{(s't')^{-1}} \otimes 1_{r'^{-1}}) \\
&\quad \circ (\widetilde{F}_{rs,t} \otimes 1_{(r's't')^{-1}}) \circ (\widetilde{F}_{r,st}^{-1} \otimes 1_{(r's't')^{-1}}) \circ (\widetilde{F}_{r,s} \otimes 1_{(r's')^{-1}}) \\
&= (F(\hat{a}_1)_{r,s,t}) \otimes 1_{(r's't')^{-1}}) \circ (1_{r'} \otimes \widetilde{F}_{s,t}^{-1} \otimes 1_{(r's't')^{-1}}) \\
&\quad \circ (\widetilde{F}_{rs,t} \otimes 1_{(r's't')^{-1}}) \circ (\widetilde{F}_{r,st}^{-1} \otimes 1_{(r's't')^{-1}}) \circ (\widetilde{F}_{r,s} \otimes 1_{t'} \otimes 1_{(r's't')^{-1}}) \\
&= (\hat{a}_2)_{r',s',t'} \otimes 1_{(r's't')^{-1}} \\
&= a_2'(\varepsilon_0(r), \varepsilon_0(s), \varepsilon_0(t)).
\end{aligned}
$$

Therefore, $\overline{\varepsilon_1}^3(a_1') = \overline{\varepsilon_1}^3(a_1') + [\delta_2(b)] = \overline{\varepsilon_0}^3(a_2') \in H^3(\Pi_0(\mathcal{H}_1), \Pi_1(\mathcal{H}_2))$ which is what we wanted to prove.

2. By point 1, it suffices to consider the same small special cat-group $\mathcal{H}$ monoidally equivalent to $\mathcal{G}$ and $\mathcal{G}'$ in definition 5.29.

□

Definition 5.32. Due to this proposition, we have a function

$$
\begin{aligned}
\Pi : \quad \Big(\mathrm{ob}\,\mathscr{H}(\mathrm{CG})\Big)/\simeq \ &\longrightarrow \ \Big(\mathrm{ob}\,\widetilde{H^3}\Big)/\simeq \\
[\mathcal{G}] \ &\longmapsto [(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}), a)]
\end{aligned}
$$

where the equivalence relations $\simeq$ identify isomorphic objects and a is the Postnikov invariant of $\mathcal{G}$.

We will prove in section 5.3 that Π is actually a bijection. This will be the classification of cat-groups.

5.2.2 Sinh's Definition

We present now the definition of the Postnikov invariant due to Sinh in [14]. It is based on a coherent choice of representatives $X_s \in s$, called a 'stick'. We are going to prove it is actually equivalent to Baez-Lauda's definition.

Definition 5.33. Let $\mathcal{G}$ be a small cat-group. A stick $(X_s, j_X)_{s\in\Pi_0(\mathcal{G}),\, X\in\mathrm{ob}\,\mathcal{G}}$ in $\mathcal{G}$ is the data of:

- for each $s \in \Pi_0(\mathcal{G})$, a representative $X_s \in s$,
- for each $s \in \Pi_0(\mathcal{G})$ and $X \in s$, an isomorphism $X_s \xrightarrow{j_X} X$

such that,

- $X_{[I]} = I$,
- $j_{X_s} = 1_{X_s}$ for all $s \in \Pi_0(\mathcal{G})$,
- $j_{I\otimes X_s} = l_{X_s}^{-1}$ and $j_{X_s\otimes I} = r_{X_s}^{-1}$ for all $s \in \Pi_0(\mathcal{G})$.

Example 5.34. Every small special cat-group $\mathcal{G}$ has a unique stick defined by $j_X = 1_X$ for all objects X.

Example 5.35. Let A be a nontrivial abelian group and let us fix $a \neq 0 \in A$. We define $\mathcal{G}_a$ as the category $A!$ (see example 3.10). The functor $\otimes$ is defined as in $A!$, i.e. $I \otimes I = I$ and $a_1 \otimes a_2 = a_1 + a_2$ for all $a_1, a_2 \in A$. We set $l_I = r_I = a$ and $a_{I,I,I} = 1_I$. This makes $(\mathcal{G}_a, \otimes, I, l, r, a)$ a monoidal category which can be extended to a small cat-group by proposition 3.5. However, this cat-group does not have any stick since we have to choose $j_I = 0$ and $j_{I\otimes I} = -a$, which is a contradiction.

Example 5.35 shows that there are some small cat-groups for which there does not exist any stick. So, as we did for Baez-Lauda's definition, we will only define the invariant for small cat-groups having sticks. The definition for other small cat-groups can be made in the same way of definition 5.29. Indeed, we know we have a monoidal equivalence $F : \mathcal{G} \to \mathcal{H}$ where $\mathcal{H}$ is a small cat-group having a stick since small special cat-groups have the unique trivial stick of example 5.34. Moreover, we will see it does not depend of the monoidal equivalence $F : \mathcal{G} \to \mathcal{H}$ that we choose to define it on $\mathcal{G}$. Thus, we only need to define Sinh's Postnikov invariant for small cat-groups having sticks.

Definition 5.36. Let $\mathcal{G}$ be a small cat-group with a stick $(X_s, j_X)_{s\in\Pi_0(\mathcal{G}),\, X\in \mathrm{ob}\,\mathcal{G}}$. For all $r, s, t \in \Pi_0(\mathcal{G})$, we consider the morphism

$$\begin{tikzcd}
X_{rst} \arrow[r, "j_{X_{rs}\otimes X_t}"] & X_{rs} \otimes X_t \arrow[r, "j_{X_r\otimes X_s}\otimes 1_{X_t}"] & (X_r \otimes X_s) \otimes X_t \arrow[r, "a_{X_r,X_s,X_t}"] & X_r \otimes (X_s \otimes X_t) \arrow[d, "1_{X_r}\otimes j^{-1}_{X_s\otimes X_t}"] \\
 & & X_{rst} & X_r \otimes X_{st} \arrow[l, "j^{-1}_{X_r\otimes X_{st}}"]
\end{tikzcd}$$

So, we have a function, depending on the stick $(X_s, j_X)_{s\in\Pi_0(\mathcal{G}),\, X\in \mathrm{ob}\,\mathcal{G}}$,

$$\begin{aligned} \widetilde{a}' :\ & \Pi_0(\mathcal{G})^3 \longrightarrow \Pi_1(\mathcal{G}) \\ & (r,s,t) \longmapsto \gamma^{-1}_{X_{rst}} \left(j^{-1}_{X_r\otimes X_{st}} \left(1_{X_r} \otimes j^{-1}_{X_s\otimes X_t}\right) a_{X_r,X_s,X_t} \left(j_{X_r\otimes X_s} \otimes 1_{X_t}\right) j_{X_{rs}\otimes X_t} \right). \end{aligned}$$

We now have several things to check:

- This $\widetilde{a}'$ lies in $Z^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G})$. So we can define Sinh's Postnikov invariant as $\widetilde{a} = [\widetilde{a}'] \in H^3(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G})$.
- This $\widetilde{a}$ does not depend on the chosen stick (up to isomorphism in $\widetilde{H^3}$).
- If monoidally equivalent small cat-groups $\mathcal{G}$ and $\mathcal{G}'$ both have sticks, then, this construction induces the same element in $\widetilde{H^3}$ (also up to isomorphism).
- Baez-Lauda's definition and Sinh's definition give rise to the same object in $\widetilde{H^3}$ (again up to isomorphism).

All these facts are immediate corollaries of the following proposition.

Proposition 5.37. Let $\mathcal{G}$ be a small cat-group with a stick $(X_s, j_X)_{s\in\Pi_0(\mathcal{G}),\, X\in \mathrm{ob}\,\mathcal{G}}$. Then, there exists a small special cat-group $\mathcal{H}$ and a monoidal equivalence $F : \mathcal{G} \to \mathcal{H}$ such that, for all $r, s, t \in \Pi_0(\mathcal{G})$, we have

$$\Pi_1(F)(\widetilde{a}'(r,s,t)) = a'(\Pi_0(F)(r), \Pi_0(F)(s), \Pi_0(F)(t))$$

where $\widetilde{a}'$ is defined in 5.36 for $\mathcal{G}$ and a' is defined in 5.28 for $\mathcal{H}$.

Proof. Let $\mathcal{H}$ be the category with elements of $\Pi_0(\mathcal{G})$ as objects and arrows defined as:

$$\mathcal{H}(s,t) = \begin{cases} \{s\} \times \Pi_1(\mathcal{G}) & \text{if } s = t \\ \emptyset & \text{if } s \neq t. \end{cases}$$

Compositions are as in $\mathcal{G}$ and identities are $1_s = (s, 1_I)$ for all $s \in \Pi_0(\mathcal{G})$. Let $I^{\mathcal{H}} = 1$ in $\Pi_0(\mathcal{G})$, $s \otimes^{\mathcal{H}} t = st$ and

$$(s,f) \otimes^{\mathcal{H}} (t,g) = \left(st, \gamma_{X_{st}}^{-1}\left(j_{X_s \otimes X_t}^{-1}(\gamma_{X_s}(f) \otimes \gamma_{X_t}(g)) j_{X_s \otimes X_t}\right)\right)$$

for all $(s,f) \in \mathcal{H}(s,s)$ and $(t,g) \in \mathcal{H}(t,t)$. Since the γ's are group morphisms, $\otimes^{\mathcal{H}}$ is a functor $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$. We know from definition that $\gamma_{X_s}(f) \otimes 1_{X_t} = \gamma_{X_s \otimes X_t}(f)$ and

$$\begin{aligned} 1_{X_s} \otimes \gamma_{X_t}(g) = \delta_{X_s}(g) \otimes 1_{X_t} &= \gamma_{X_s}(\gamma_{X_s}^{-1}(\delta_{X_s}(g))) \otimes 1_{X_t} \\ &= \gamma_{X_s \otimes X_t}(\gamma_{X_s}^{-1}(\delta_{X_s}(g))) = \gamma_{X_s \otimes X_t}(s \cdot g). \end{aligned}$$

Thus, $\gamma_{X_s}(f) \otimes \gamma_{X_t}(g) = (\gamma_{X_s}(f) \otimes 1_{X_t})\,(1_{X_s} \otimes \gamma_{X_t}(g)) = \gamma_{X_s \otimes X_t}(f \circ (s \cdot g))$. Therefore, $(s,f) \otimes^{\mathcal{H}} (t,g) = (st, f \circ (s \cdot g))$ for all $(s,f) \in \mathcal{H}(s,s)$ and $(t,g) \in \mathcal{H}(t,t)$.

Now, we define $l^{\mathcal{H}}$, $r^{\mathcal{H}}$ and $a^{\mathcal{H}}$ as follows: for all $r, s, t \in \operatorname{ob} \mathcal{H}$, let $l_s^{\mathcal{H}} = (s, 1_I) = r_s^{\mathcal{H}}$ and

$$a_{r,s,t}^{\mathcal{H}} = \left(rst, \gamma_{X_{rst}}^{-1}\left(j_{X_r \otimes X_{st}}^{-1}\,(1_{X_r} \otimes j_{X_s \otimes X_t}^{-1})\, a_{X_r,X_s,X_t}\,(j_{X_r \otimes X_s} \otimes 1_{X_t})\, j_{X_{rs} \otimes X_t}\right)\right).$$

It is simple computations to check that $(\mathcal{H}, \otimes^{\mathcal{H}}, I^{\mathcal{H}}, l^{\mathcal{H}}, r^{\mathcal{H}}, a^{\mathcal{H}})$ is a monoidal category. If we set $s^{*^{\mathcal{H}}} = s^{-1}$ and $i_s^{\mathcal{H}} = (1, 1_I)$ for all $s \in \operatorname{ob} \mathcal{H}$, we know that we can extend $\mathcal{H}$ to have a small special cat-group $(\mathcal{H}, \otimes^{\mathcal{H}}, I^{\mathcal{H}}, l^{\mathcal{H}}, r^{\mathcal{H}}, a^{\mathcal{H}}, *^{\mathcal{H}}, i^{\mathcal{H}}, e^{\mathcal{H}})$.

We have an equivalence given by

$$\begin{aligned} F:\ \mathcal{G} &\longrightarrow \mathcal{H} \\ X &\longmapsto [X] \\ X \xrightarrow{f} Y &\longmapsto \left(s, \gamma_{X_s}^{-1}(j_Y^{-1} f j_X)\right) \\ (s = [X] = [Y]) & \end{aligned} \qquad \text{and} \qquad \begin{aligned} G : \mathcal{H} &\longrightarrow \mathcal{G} \\ s &\longmapsto X_s \\ (s,f) &\longmapsto \gamma_{X_s}(f). \end{aligned}$$

Indeed, $FG = 1_{\mathcal{H}}$ and $\alpha : 1_{\mathcal{G}} \Rightarrow GF$ defined by $\alpha_X = j_X^{-1}$ for all $X \in \operatorname{ob} \mathcal{G}$ is a natural isomorphism. Let $F_I = 1_{I^{\mathcal{H}}}$ and $\widetilde{F}_{X,Y} = F(j_X \otimes j_Y)$ for all $X, Y \in \operatorname{ob} \mathcal{G}$. It follows from easy calculations that F is then a monoidal equivalence. Now, it remains to compute

$$\begin{aligned} \Pi_1(F)(\widetilde{a}'(r,s,t)) &= F(\widetilde{a}'(r,s,t)) \\ &= \left([I], \widetilde{a}'(r,s,t)\right) \\ &= \left([I], a_{r,s,t}^{\mathcal{H}}\right) \\ &= \left([I], a_{r,s,t}^{\mathcal{H}} \circ (rst) \cdot (1_{(rst)^{-1}})\right) \\ &= a_{r,s,t}^{\mathcal{H}} \otimes^{\mathcal{H}} 1_{(rst)^{-1}} \\ &= a'(\Pi_0(F)(r), \Pi_0(F)(s), \Pi_0(F)(t)). \end{aligned}$$

□

With Baez-Lauda's definition, we can not easily know what the Postnikov invariant of a small cat-group $\mathcal{G}$ is. Indeed, we must find a small special cat-group monoidally equivalent to $\mathcal{G}$ to be able to compute it. Now, with Sinh's definition, if $\mathcal{G}$ has a stick (which is a much weaker condition then being special), we have a complete description of the Postnikov invariant of $\mathcal{G}$.

5.3 Classification of Cat-groups

We prove in this section Sinh's Classification Theorem of Cat-groups and its corollaries. Recall that we have defined a function

$$\begin{aligned}\Pi :\quad \Big(\operatorname{ob}\mathscr{H}(\mathrm{CG})\Big)/\simeq &\longrightarrow \quad \Big(\operatorname{ob}\widetilde{H^3}\Big)/\simeq\\ [\mathcal{G}] \quad &\longmapsto [(\Pi_0(\mathcal{G}),\Pi_1(\mathcal{G}),a)]\end{aligned}$$

(see 5.32). Sinh's Theorem says that this is a bijection. Recall also that two small cat-groups are isomorphic in $\mathscr{H}(\mathrm{CG})$ if and only if they are monoidally equivalent. So, a small cat-group $\mathcal{G}$ is uniquely determined (up to monoidal equivalence) by $\Pi_0(\mathcal{G})$, $\Pi_1(\mathcal{G})$, the action of $\Pi_0(\mathcal{G})$ on $\Pi_1(\mathcal{G})$ and its Postnikov invariant.

Theorem 5.38 (Sinh's Theorem, 1975)**.** Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two small cat-groups with Postnikov invariants a_1 and a_2 respectively. If $(\Pi_0(\mathcal{G}_1),\Pi_1(\mathcal{G}_1),a_1)\simeq(\Pi_0(\mathcal{G}_2),\Pi_1(\mathcal{G}_2),a_2)$ in $\widetilde{H^3}$, then, $\mathcal{G}_1$ and $\mathcal{G}_2$ are monoidally equivalent.

Proof. By definition 5.29, we can assume, without loss of generality, that $\mathcal{G}_1$ and $\mathcal{G}_2$ are small special cat-groups. Thus, a_1 and a_2 are defined as in proposition 5.28. Let $(\varepsilon_0,\varepsilon_1):\ (\Pi_0(\mathcal{G}_1),\Pi_1(\mathcal{G}_1),a_1)\xrightarrow{\sim}(\Pi_0(\mathcal{G}_2),\Pi_1(\mathcal{G}_2),a_2)$ be an isomorphism in $\widetilde{H^3}$. Since $\Pi_0(\mathcal{G}_1)=\operatorname{ob}\mathcal{G}_1$ and $\Pi_0(\mathcal{G}_2)=\operatorname{ob}\mathcal{G}_2$ as sets, we can construct

$$\begin{aligned}F:\quad \mathcal{G}_1 &\longrightarrow \mathcal{G}_2\\ s &\longmapsto \varepsilon_0(s)\\ s\xrightarrow{f} s &\longmapsto \varepsilon_1(f\otimes 1_{s^{-1}})\otimes 1_{\varepsilon_0(s)}.\end{aligned}$$

F is a functor since ε_1 is a group morphism. F is essentially surjective on objects since ε_0 is onto. F is faithful since ε_1 is injective and by lemma 3.4.1. Moreover, F is full since, if $\varepsilon_0(s)\xrightarrow{g}\varepsilon_0(s)$ is an arrow in $\mathcal{G}_2$, then $F(f)=g$ where $f=\varepsilon_1^{-1}\left(g\otimes 1_{\varepsilon_0(s^{-1})}\right)\otimes 1_s$ by lemma 5.25.4. Therefore, it remains to prove that F is monoidal. We will denote by $\hat{a}_1$ and $\hat{a}_2$ the families of associativity isomorphisms of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. Let us also denote by

$$\begin{aligned}a_i':\quad \Pi_0(\mathcal{G}_i)^3 &\longrightarrow \Pi_1(\mathcal{G}_i)\\ (r,s,t) &\longmapsto (\hat{a}_i)_{r,s,t}\otimes 1_{(rst)^{-1}}\end{aligned}$$

for $i\in\{1,2\}$ the functions of proposition 5.28. Hence, $a_1=[a_1']\in H^3(\Pi_0(\mathcal{G}_1),\Pi_1(\mathcal{G}_1))$ and $a_2=[a_2']\in H^3(\Pi_0(\mathcal{G}_2),\Pi_1(\mathcal{G}_2))$. Since $\overline{\varepsilon_1}^3(a_1)=\overline{\varepsilon_0}^3(a_2)\in H^3(\Pi_0(\mathcal{G}_1),\Pi_1(\mathcal{G}_2))$, there exists a function $b:\Pi_0(\mathcal{G}_1)^2\to\Pi_1(\mathcal{G}_2)$ such that $b(1,s)=b(s,1)=1_I$ and

$$\varepsilon_1(a_1'(r,s,t))+(\delta_2(b))(r,s,t)=a_2'(\varepsilon_0(r),\varepsilon_0(s),\varepsilon_0(t)) \tag{5.1}$$

for all $r,s,t\in\Pi_0(\mathcal{G}_1)$. Now, we can define $F_I=1_I$ and $\widetilde{F}_{s,t}=b(s,t)\otimes 1_{\varepsilon_0(st)}$ for all $s,t\in\operatorname{ob}\mathcal{G}_1$. The fact that $\widetilde{F}$ commutes with l and r follows from $b\in C^2(\Pi_0(\mathcal{G}_1),\Pi_1(\mathcal{G}_2))$ since $\mathcal{G}_1$ and $\mathcal{G}_2$ are special. Due to (5.1), $\widetilde{F}$ commutes also with $\hat{a}_1$ and $\hat{a}_2$: for all $r,s,t\in\operatorname{ob}\mathcal{G}_1$,

$$\begin{aligned}&F((\hat{a}_1)_{r,s,t})\ \widetilde{F}_{rs,t}\ (\widetilde{F}_{r,s}\otimes 1_{F(t)})\\ &=(\varepsilon_1(a_1'(r,s,t))\otimes 1_{\varepsilon_0(rst)})\ (b(rs,t)\otimes 1_{\varepsilon_0(rst)})\ (b(r,s)\otimes 1_{\varepsilon_0(rst)})\\ &=(b(r,st)\otimes 1_{\varepsilon_0(rst)})\ ((r\cdot b(s,t))\otimes 1_{\varepsilon_0(rst)})\ (a_2'(\varepsilon_0(r),\varepsilon_0(s),\varepsilon_0(t))\otimes 1_{\varepsilon_0(rst)})\quad\text{by (5.1)}\\ &=\widetilde{F}_{r,st}\ (1_{\varepsilon_0(r)}\otimes b(s,t)\otimes 1_{\varepsilon_0(r)^{-1}}\otimes 1_{\varepsilon_0(rst)})\ (\hat{a}_2)_{F(r),F(s),F(t)}\\ &=\widetilde{F}_{r,st}\ (1_{F(r)}\otimes\widetilde{F}_{s,t})\ (\hat{a}_2)_{F(r),F(s),F(t)}.\end{aligned}$$

Eventually, to see that $\widetilde{F}$ is natural, it suffices to check that $F(f \otimes g) = F(f) \otimes F(g)$ for all $f \in \mathcal{G}_1(s,s)$ and $g \in \mathcal{G}_1(t,t)$. But we know that

$$\begin{aligned}
\varepsilon_1(1_s \otimes g \otimes 1_{t^{-1}} \otimes 1_{s^{-1}}) &= \varepsilon_1(\gamma_s^{-1}(1_s \otimes g \otimes 1_{t^{-1}})) \\
&= \varepsilon_1(\gamma_s^{-1}(\delta_s(g \otimes 1_{t^{-1}}))) \\
&= \varepsilon_1(s \cdot (g \otimes 1_{t^{-1}})) \\
&= \varepsilon_0(s) \cdot \varepsilon_1(g \otimes 1_{t^{-1}}) \\
&= \gamma_{\varepsilon_0(s)}^{-1}(1_{\varepsilon_0(s)} \otimes \varepsilon_1(g \otimes 1_{t^{-1}})) \\
&= 1_{\varepsilon_0(s)} \otimes \varepsilon_1(g \otimes 1_{t^{-1}}) \otimes 1_{\varepsilon_0(s)^{-1}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
F(f \otimes g) &= \varepsilon_1(f \otimes g \otimes 1_{(st)^{-1}}) \otimes 1_{\varepsilon_0(st)} \\
&= \left[\varepsilon_1(f \otimes 1_t \otimes 1_{t^{-1}} \otimes 1_{s^{-1}}) \circ \varepsilon_1(1_s \otimes g \otimes 1_{t^{-1}} \otimes 1_{s^{-1}})\right] \otimes 1_{\varepsilon_0(st)} \\
&= \varepsilon_1(f \otimes 1_t \otimes 1_{t^{-1}} \otimes 1_{s^{-1}}) \otimes \varepsilon_1(1_s \otimes g \otimes 1_{t^{-1}} \otimes 1_{s^{-1}}) \otimes 1_{\varepsilon_0(s)} \otimes 1_{\varepsilon_0(t)} \\
&= \varepsilon_1(f \otimes 1_{s^{-1}}) \otimes 1_{\varepsilon_0(s)} \otimes \varepsilon_1(g \otimes 1_{t^{-1}}) \otimes 1_{\varepsilon_0(t)} \\
&= F(f) \otimes F(g).
\end{aligned}$$

Thus, F is a monoidal equivalence.

□

Proposition 5.39. Let (G, A, a) be an object of $\widetilde{H^3}$. There exists a small special cat-group $\mathcal{G}$ such that $(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}), a') \simeq (G, A, a)$ in $\widetilde{H^3}$ where a' is the Postnikov invariant of $\mathcal{G}$

Proof. Let $\mathcal{G}$ be the category with elements of G as objects and where morphisms are

$$\mathcal{G}(g_1, g_2) = \begin{cases} A & \text{if } g_1 = g_2 \\ \emptyset & \text{if } g_1 \neq g_2. \end{cases}$$

Composition is the addition in A and identities are $1_g = 0 \in A$ for all $g \in G$. $\mathcal{G}$ is obviously a small skeletal groupoid since A is a group. We set $g \otimes g' = gg'$ and $a \otimes a' = a + g \cdot a'$ for all $g, g' \in G$, $a \in \mathcal{G}(g,g)$ and $a' \in \mathcal{G}(g', g')$. This is a functor since the action of G is distributive with respect to the group law of A. Now, we set $I = 1 \in G$ and $l_g = r_g = 1_g = 0$ for all $g \in G$. l and r are natural by the axioms of a G-module.

Let us consider $a \in H^3(G, A)$. We know it is represented by a $\tilde{a} \in Z^3(G, A)$. So, $a = [\tilde{a}]$. Thus, we can define, for all $g_1, g_2, g_3 \in G$, $a^{\mathcal{G}}_{g_1,g_2,g_3} = \tilde{a}(g_1, g_2, g_3) : g_1g_2g_3 \longrightarrow g_1g_2g_3$. $a^{\mathcal{G}}$ is natural, since, for all $g_1 \xrightarrow{f_1} g_1$, $g_2 \xrightarrow{f_2} g_2$ and $g_3 \xrightarrow{f_3} g_3$ in $\mathcal{G}$, we have

$$f_1 \otimes (f_2 \otimes f_3) = f_1 \otimes (f_2 + g_2 \cdot f_3) = f_1 + g_1 \cdot f_2 + (g_1g_2) \cdot f_3 = (f_1 + g_1 \cdot f_2) \otimes f_3 = (f_1 \otimes f_2) \otimes f_3.$$

Moreover, $a^{\mathcal{G}}$ satisfies the Triangle Axiom since $\tilde{a}(g_1, 1, g_3) = 0$ for all $g_1, g_3 \in G$. It also satisfies the Pentagon Axiom since it is exactly the equation $\delta_3(\tilde{a}) = 0$. Hence, we have a monoidal category $(\mathcal{G}, \otimes, I, l, r, a^{\mathcal{G}})$. If we set $g^* = g^{-1}$ and $i_g = 1_I$ for all $g \in G$, we know we can extend it to a small special cat-group $(\mathcal{G}, \otimes, I, l, r, a^{\mathcal{G}}, *, i, e)$.

Obviously, we have $\Pi_0(\mathcal{G}) = G$ and $\Pi_1(\mathcal{G}) = A$. Let a' be the Postnikov invariant of $\mathcal{G}$ defined in 5.28, $\varepsilon_0 = 1_G$ and $\varepsilon_1 = 1_A$. To prove that $(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}), a') \xrightarrow{(\varepsilon_0, \varepsilon_1)} (G, A, a)$ is an isomorphism, it remains to prove $\overline{\varepsilon_1}^3(a') = \overline{\varepsilon_0}^3(a)$. So, let $g_1, g_2, g_3 \in G$:

$$\varepsilon_1\left(a^{\mathcal{G}}_{g_1,g_2,g_3} \otimes 1_{(g_1g_2g_3)^{-1}}\right) = \tilde{a}(g_1, g_2, g_3) \otimes 1_{(g_1g_2g_3)^{-1}} = \tilde{a}(\varepsilon_0(g_1), \varepsilon_0(g_2), \varepsilon_0(g_3)).$$

□

Notation 5.40. The small special cat-group we have constructed in this proof depends on the choice of the representative $\tilde{a} \in Z^3(G, A)$. So, for such a choice, we denote by $\mathcal{G}(G, A, \tilde{a})$ this cat-group.

Corollary 5.41. We have a bijection

$$\begin{aligned} \Pi : \quad \Big(\mathrm{ob}\, \mathscr{H}(\mathrm{CG})\Big)/\simeq &\longrightarrow \quad \Big(\mathrm{ob}\, \widetilde{H^3}\Big)/\simeq \\ [\mathcal{G}] \quad &\longmapsto [(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}), a)] \end{aligned}$$

where the equivalence relations $\simeq$ identify isomorphic objects and a is the Postnikov invariant of $\mathcal{G}$.

Proof. We have already proved it is well-defined. It is injective by Sinh's Theorem 5.38 and surjective by proposition 5.39.

□

We can actually do better. Indeed, we can compare the 2-category CG with another one.

Definition 5.42. We construct here the 2-category H^3:

- Objects are triples (G, A, a) where G is a group, A a G-module and $a \in Z^3(G, A)$.
- 1-cells $(G, A, a) \to (G', A', a')$ are triples $(\varepsilon_0, \varepsilon_1, b)$ where $\varepsilon_0 : G \to G'$ and $\varepsilon_1 : A \to A'$ are group morphisms such that $\varepsilon_1(g \cdot m) = \varepsilon_0(g) \cdot \varepsilon_1(m)$ for all $g \in G$ and $m \in A$ and $b \in C^2(G, A')$ is such that $\delta_2(b) = \overline{\varepsilon_0}^3(a') - \overline{\varepsilon_1}^3(a) \in Z^3(G, A')$.
- There is no 2-cells $(\varepsilon_0, \varepsilon_1, b) \to (\varepsilon'_0, \varepsilon'_1, b')$ if $\varepsilon_0 \neq \varepsilon'_0$ or $\varepsilon_1 \neq \varepsilon'_1$. Otherwise, 2-cells $(\varepsilon_0, \varepsilon_1, b) \to (\varepsilon_0, \varepsilon_1, b')$ are elements $c \in C^1(G, A')$ such that $\delta_1(c) = b - b'$.
- Composition of $(G, A, a) \xrightarrow{(\varepsilon_0, \varepsilon_1, b)} (G', A', a') \xrightarrow{(\eta_0, \eta_1, \beta)} (G'', A'', a'')$ is
$$(\eta_0\varepsilon_0, \eta_1\varepsilon_1, (g, g') \mapsto \beta(\varepsilon_0(g), \varepsilon_0(g')) + \eta_1 b(g, g')).$$
- $1_{(G,A,a)} = (1_G, 1_A, 0)$.
- Composition of $(\varepsilon_0, \varepsilon_1, b) \xrightarrow{c} (\varepsilon_0, \varepsilon_1, b') \xrightarrow{c'} (\varepsilon_0, \varepsilon_1, b'')$ is $c + c'$.
- $1_{(\varepsilon_0, \varepsilon_1, b)} = 0$.
- Finally, if we have
$$\begin{tikzcd} (G, A, a) \arrow[r, bend left, "{(\varepsilon_0,\varepsilon_1,b)}"] \arrow[r, bend right, "{(\varepsilon_0,\varepsilon_1,b')}"'] \arrow[r, phantom, "\Downarrow c"] & (G', A', a') \arrow[r, bend left, "{(\eta_0,\eta_1,\beta)}"] \arrow[r, bend right, "{(\eta_0,\eta_1,\beta')}"'] \arrow[r, phantom, "\Downarrow d"] & (G'', A'', a''), \end{tikzcd}$$
we set $d \star c = d\varepsilon_0 + \eta_1 c$.

It is only easy computations to check that this defines a 2-category.

Proposition 5.43. Let $(\varepsilon_0, \varepsilon_1, b) : (G, A, a) \to (G', A', a')$ be a morphism in H^3. Let also F be the functor

$$\begin{aligned} F : \mathcal{G}(G, A, a) &\longrightarrow \mathcal{G}(G', A', a') \\ g &\longmapsto \varepsilon_0(g) \\ g \xrightarrow{m} g &\longmapsto \varepsilon_0(g) \xrightarrow{\varepsilon_1(m)} \varepsilon_0(g) \ , \end{aligned}$$

$F_I = 1_I$ and $\widetilde{F}_{g_1,g_2} = b(g_1, g_2)$ for all $g_1, g_2 \in G$. Then, $(F, F_I, \widetilde{F})$ is a special monoidal functor. We denote it by $\mathcal{G}(\varepsilon_0, \varepsilon_1, b)$.

Proof. $\widetilde{F}$ is natural, since, for all $g_1 \xrightarrow{m_1} g_1$ and $g_2 \xrightarrow{m_2} g_2$ in $\mathcal{G}(G, A, a)$, we have

$$F(m_1 \otimes m_2) = \varepsilon_1(m_1 + g_1 \cdot m_2) = \varepsilon_1(m_1) + \varepsilon_0(g_1) \cdot \varepsilon_1(m_2) = F(m_1) \otimes F(m_2).$$

$\widetilde{F}$ commutes with l and r since $b(1, g) = b(g, 1) = 0$ for all $g \in G$. Eventually, the fact that $\widetilde{F}$ commutes with the associativity isomorphisms is exactly the equation

$$\delta_2(b) = \overline{\varepsilon_0}^3(a') - \overline{\varepsilon_1}^3(a).$$

$\square$

Let $(\varepsilon_0, \varepsilon_1, b) \xrightarrow{c} (\varepsilon_0, \varepsilon_1, b')$ be a 2-cell in H^3. We can define

$$\mathcal{G}(c) : \mathcal{G}(\varepsilon_0, \varepsilon_1, b) \to \mathcal{G}(\varepsilon_0, \varepsilon_1, b')$$

by $\mathcal{G}(c)_g : \ \varepsilon_0(g) \xrightarrow{c(g)} \varepsilon_0(g)$ for all $g \in G$. It is a monoidal natural transformation since $\delta_1(c) = b - b'$. We can easily see that this gives rise to a 2-functor

$$\begin{aligned} \mathcal{G} : \qquad H^3 &\longrightarrow \mathrm{SpCG} \hookrightarrow \mathrm{CG} \\ (G, A, a) &\longmapsto \mathcal{G}(G, A, a) \\ (\varepsilon_0, \varepsilon_1, b) &\longmapsto \mathcal{G}(\varepsilon_0, \varepsilon_1, b) \\ c &\longmapsto \mathcal{G}(c). \end{aligned}$$

We now prove our last theorem.

Theorem 5.44. We have a biequivalence $\mathrm{CG} \simeq H^3$.

Proof. By corollary 5.27, it is enough to prove $H^3 \simeq \mathrm{SpCG}$. We will use corollary 4.33 with the 2-functor $\mathcal{G} : H^3 \to \mathrm{SpCG}$. By definition, it is faithful on 2-cells. To see that $\mathcal{G}$ is full on 2-cells, let

$$\begin{tikzcd} (G, A, a) \arrow[r, shift left, "{(\varepsilon_0,\varepsilon_1,b)}"] \arrow[r, shift right, "{(\varepsilon'_0,\varepsilon'_1,b')}"'] & (G', A', a') \end{tikzcd}$$

be 1-cells in H^3 and $\alpha : \mathcal{G}(\varepsilon_0, \varepsilon_1, b) \Rightarrow \mathcal{G}(\varepsilon'_0, \varepsilon'_1, b')$ be a monoidal natural transformation. Thus, $\alpha_g : \ \varepsilon_0(g) \xrightarrow{\sim} \varepsilon'_0(g)$ for all $g \in G$ and so $\varepsilon_0 = \varepsilon'_0$ since $\mathcal{G}(G', A', a')$ is a special cat-group. Moreover, since α is natural,

$$\begin{tikzcd} \varepsilon_0(1) \arrow[r, "\varepsilon_1(m)"] \arrow[d, "\alpha_1"'] \arrow[dr, phantom, "\circlearrowleft"] & \varepsilon_0(1) \arrow[d, "\alpha_1"] \\ \varepsilon_0(1) \arrow[r, "\varepsilon'_1(m)"'] & \varepsilon_0(1) \end{tikzcd}$$

commutes for all $m \in A$, which implies $\varepsilon_1(m) = \varepsilon'_1(m)$. Therefore, $\varepsilon_1 = \varepsilon'_1$. Now, it remains to set $c(g) = \alpha_g$ for all $g \in G$ and we have $c : (\varepsilon_0, \varepsilon_1, b) \longrightarrow (\varepsilon'_0, \varepsilon'_1, b')$ in H^3 such that $\mathcal{G}(c) = \alpha$. Indeed, $c \in C^1(G, A')$ and $\delta_1(c) = b - b'$ since α is monoidal.

Now, we prove that $\mathcal{G}$ is essentially surjective on 1-cells. Let (G, A, a) and (G', A', a') be two objects in H^3 and $F : \mathcal{G}(G, A, a) \to \mathcal{G}(G', A', a')$ be a special monoidal functor. By naturality of $\widetilde{F}$,

$$\begin{tikzcd}
F(g_1) \otimes F((g_1)^{-1}g_2) \arrow[r, "\widetilde{F}_{g_1,(g_1)^{-1}g_2}"] \arrow[d, "F(m)\otimes 1_{F((g_1)^{-1}g_2)}"'] \arrow[dr, phantom, "\circlearrowleft"] & F(g_1 \otimes ((g_1)^{-1}g_2)) \arrow[d, "F(m\otimes 1_{(g_1)^{-1}g_2})"] \\
F(g_1) \otimes F((g_1)^{-1}g_2) \arrow[r, "\widetilde{F}_{g_1,(g_1)^{-1}g_2}"'] & F(g_1 \otimes ((g_1)^{-1}g_2))
\end{tikzcd}$$

commutes and so $F(g_1 \xrightarrow{m} g_1) = F(g_2 \xrightarrow{m} g_2) \in A'$ for all $g_1, g_2 \in G$ and $m \in A$. Hence, by proposition 5.43, $(\Pi_0(F), \Pi_1(F), (g_1, g_2) \mapsto \widetilde{F}_{g_1,g_2}) : (G, A, a) \to (G', A', a')$ is such that $\mathcal{G}(\Pi_0(F), \Pi_1(F), (g_1, g_2) \mapsto \widetilde{F}_{g_1,g_2}) = F$ since F is special. So, $\mathcal{G}$ is essentially surjective on 1-cells.

It remains to prove that $\mathcal{G}$ is essentially surjective on objects. Let $\mathcal{G}_1$ be a small special cat-group. We are going to prove that $\mathcal{G}(\Pi_0(\mathcal{G}_1), \Pi_1(\mathcal{G}_1), (X, Y, Z) \mapsto a_{X,Y,Z} \otimes 1_{(XYZ)^{-1}})$ is isomorphic to $\mathcal{G}_1$ in SpCG. Notice that $(X, Y, Z) \mapsto a_{X,Y,Z} \otimes 1_{(XYZ)^{-1}}$ is a representative of the Postnikov invariant of $\mathcal{G}_1$. Let

$$(G, A, a') = (\Pi_0(\mathcal{G}_1), \Pi_1(\mathcal{G}_1), (X, Y, Z) \mapsto a_{X,Y,Z} \otimes 1_{(XYZ)^{-1}}).$$

To prove this isomorphism, it suffices to notice that the monoidal functor

$$\begin{aligned} F : \qquad \mathcal{G}_1 &\longrightarrow \mathcal{G}(G, A, a') \\ X &\longmapsto X \\ X \xrightarrow{g} X &\longmapsto g \otimes 1_{X^{-1}} \end{aligned}$$

with $F_I = 1_I$ and $\widetilde{F}_{X,Y} = 1_{X\otimes Y}$ for all $X, Y \in \operatorname{ob}\mathcal{G}_1$ is the inverse in SpCG of

$$\begin{aligned} F' : \qquad \mathcal{G}(G, A, a') &\longrightarrow \mathcal{G}_1 \\ X &\longmapsto X \\ X \xrightarrow{f\in\Pi_1(\mathcal{G}_1)} X &\longmapsto f \otimes 1_X \end{aligned}$$

with $F'_I = 1_I$ and $\widetilde{F'}_{X,Y} = 1_{X\otimes Y}$ for all $X, Y \in \Pi_0(\mathcal{G}_1)$. To see that $\widetilde{F}$ and $\widetilde{F'}$ are natural, we can notice that

$$\begin{aligned} g \otimes g' &= (g \otimes 1_Y)(1_X \otimes g') \\ &= (g \otimes 1_{X^{-1}} \otimes 1_{XY})(1_X \otimes g' \otimes 1_{Y^{-1}} \otimes 1_{X^{-1}} \otimes 1_{XY}) \\ &= \big((g \otimes 1_{X^{-1}})(X \cdot (g' \otimes 1_{Y^{-1}})\big) \otimes 1_{XY} \end{aligned}$$

for all $X \xrightarrow{g} X$ and $Y \xrightarrow{g'} Y$ in $\mathcal{G}_1$ and

$$\begin{aligned} (f \otimes_{\mathcal{G}(G,A,a')} f') \otimes 1_{XY} &= (f + X \cdot f') \otimes 1_{XY} \\ &= (f \otimes 1_X \otimes 1_Y)(1_X \otimes f' \otimes 1_{X^{-1}} \otimes 1_{XY}) \\ &= (f \otimes 1_X \otimes 1_I \otimes 1_Y)(1_I \otimes 1_X \otimes f' \otimes 1_Y) \\ &= f \otimes 1_X \otimes f' \otimes 1_Y \end{aligned}$$

for all $X \xrightarrow{f} X$ and $Y \xrightarrow{f'} Y$ in $\mathcal{G}(G, A, a')$. Other properties to check are obvious. Therefore, $\mathcal{G}_1 \simeq \mathcal{G}(G, A, a')$ in SpCG and $\mathcal{G}$ is essentially surjective on objects.

□

This proof also describes the weak inverse $F : \mathrm{CG} \to H^3$ of $\mathcal{G} : H^3 \to \mathrm{SpCG} \hookrightarrow \mathrm{CG}$ on objects. Indeed, if $\mathcal{G}_1$ is a small cat-group, let $\mathcal{G}_2$ be the small special cat-group given by proposition 5.24. Let also $a_1 \in Z^3(\Pi_0(\mathcal{G}_1), \Pi_1(\mathcal{G}_1))$ be any representative of the Postnikov invariant of $\mathcal{G}_1$ and a_2 be the representative of the Postnikov invariant of $\mathcal{G}_2$ described in definition 5.28. Since, by definition 5.29 and proposition 5.31

$$(\Pi_0(\mathcal{G}_1), \Pi_1(\mathcal{G}_1), [a_1]) \simeq (\Pi_0(\mathcal{G}_2), \Pi_1(\mathcal{G}_2), [a_2])$$

in $\widetilde{H^3}$, then $(\Pi_0(\mathcal{G}_1), \Pi_1(\mathcal{G}_1), a_1) \simeq (\Pi_0(\mathcal{G}_2), \Pi_1(\mathcal{G}_2), a_2)$ in H^3. The proofs of theorem 5.44 and proposition 5.27 tell us that $(\Pi_0(\mathcal{G}_2), \Pi_1(\mathcal{G}_2), a_2)$ satisfies the property of weak essential surjectivity on objects of $\mathcal{G}$ for $\mathcal{G}_1$. But so does $(\Pi_0(\mathcal{G}_1), \Pi_1(\mathcal{G}_1), a_1)$ and we can define $F(\mathcal{G}_1)$ by

$$F(\mathcal{G}_1) = (\Pi_0(\mathcal{G}_1), \Pi_1(\mathcal{G}_1), a_1).$$

A. Joyal and R. Street in [7] and J. Baez and A. Lauda in [1] have given an other definition of H^3. Actually, it was the same as definition 5.42, but they have constructed 2-cells between $(\varepsilon_0, \varepsilon_1, b)$ and $(\varepsilon_0', \varepsilon_1', b')$, sometimes even if $\varepsilon_0 \neq \varepsilon_0'$ or $\varepsilon_1 \neq \varepsilon_1'$. However, they have still claimed that $H^3 \simeq \mathrm{CG}$ using the same 2-functor $\mathcal{G}$. It would have mean that there exists a monoidal natural transformation $\alpha : \mathcal{G}(\varepsilon_0, \varepsilon_1, b) \Rightarrow \mathcal{G}(\varepsilon_0', \varepsilon_1', b')$ even in some cases where $\varepsilon_0 \neq \varepsilon_0'$ or $\varepsilon_1 \neq \varepsilon_1'$. But we have seen in the proof of theorem 5.44 that this implies $\varepsilon_0 = \varepsilon_0'$ (by existence of morphisms $\varepsilon_0(g) \xrightarrow{\alpha_g} \varepsilon_0'(g)$) and $\varepsilon_1 = \varepsilon_1'$ (by naturality of α). Therefore, this essay gives a slight correction to their theorem by modifying the 2-category H^3.

6 Conclusion

After an introduction to monoidal categories (Chapter 2) and cat-groups (Chapter 3), the aim of this essay was to prove Sinh's Theorem, classifying cat-groups, which states that

$$\begin{aligned} \Pi : \quad \left(\operatorname{ob} \mathscr{H}(\mathrm{CG})\right)/\simeq &\longrightarrow \quad \left(\operatorname{ob} \widetilde{H^3}\right)/\simeq \\ [\mathcal{G}] \quad &\longmapsto [(\Pi_0(\mathcal{G}), \Pi_1(\mathcal{G}), a)] \end{aligned}$$

is a bijection (corollary 5.41). Moreover, we also wanted to deduce from it the biequivalence $\mathrm{CG} \simeq H^3$ (theorem 5.44). For this purpose, we first had to present, in Chapter 4, a definition of biequivalence. We wanted it to be an actual equivalence relation on 2-categories and to be characterised in terms of properties of a single pseudo-2-functor $F : \mathcal{C} \to \mathcal{D}$. Then, we noticed we had to slightly change the definition of H^3 proposed by Joyal and Street in [7] and Baez and Lauda in [1] in order to make this corollary correct.

Sinh's Theorem and its corollary 5.44 have an important application in Group Theory: Schreier's Classical Theorem. This is a result about group extensions and more precisely about obstruction. We refer the interested reader to [13] for more details.

Since they were introduced by P. Deligne, A. Fröhlich and C. T. C. Wall, cat-groups have been used in many subjects of Mathematics: Ring Theory, Algebraic Geometry, Group Cohomology, Algebraic Topology, and so forth. Nowadays, cat-groups are still studied on their own and inspire a lot of Mathematical researches.

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