# On localization and cellularization

## Motivation and history

# Abelian groups

Let $P$ be a set of primes and $\mathbb{Z}_{(P)}$ be the subring of $\mathbb{Q}$ consisting of $\dfrac{a}{b}$ with $b$ not divisible by $p \in P$. This ring is the *localization* of the ring of integers at $P$.

**Definition**

The functor $A \mapsto A \otimes \mathbb{Z}_{(P)}$ is called *localization* at $P$ in the category of abelian groups.

The category $\mathrm{Ab}$ of abelian groups has a full subcategory $\mathrm{Ab}_{(P)}$ of uniquely $q$-divisible groups, $q \notin P$. The forgetful functor $U : \mathrm{Ab}_{(P)} \to \mathrm{Ab}$ has a left adjoint $F : \mathrm{Ab} \to \mathrm{Ab}_{(P)}$, the "free uniquely $q$-divisible group on $A$". The above localization functor is the composite $UF : \mathrm{Ab} \to \mathrm{Ab}$.

# Localization at $P$

From another point of view, consider all homomorphisms

$q : \mathbb{Z} \to \mathbb{Z}$, given by multiplication by $q$, for $q \notin P$.

**Definition**

An abelian group $A$ is called $P$-*local* if

$q^* : \mathrm{Hom}(\mathbb{Z}, A) \to \mathrm{Hom}(\mathbb{Z}, A)$ is an isomorphism for all $q \notin P$.

So $A$ is $P$-local if and only if $A$ is uniquely $q$-divisible. There is a functor $L$ turning any abelian group into a $P$-local one, in a universal way:

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "\alpha"'] & LA \arrow[d, dashed] \\ & D \end{tikzcd}$$

## Magic square

Let $A$ be an abelian group, $P$ and $Q$ a partition of the set of all primes.

- $A_{(0)} = A \otimes \mathbb{Q}$ is the rationalization of $A$;
- $A_{(Q)} = A \otimes \mathbb{Z}_{(Q)}$ the localization at $Q$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_{(P)} \arrow[d] \\
A_{(Q)} \arrow[r] & A_{(0)}
\end{tikzcd}$$

**Theorem**

The above square is a pull-back square.

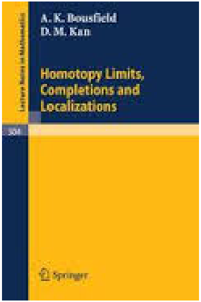
# Nilpotent groups

The theory of localization at sets of primes for nilpotent groups has been developed in the 1970's by Hilton, Mislin, and Roitberg.

Photo taken by Marge Dodrill, courtesy of Meg Hilton.

# Sullivan and the Yellow Monster

The same authors, and maybe first Sullivan, were actually interested in localizing spaces. At the same time, Bousfield and Kan were *completing* spaces at different primes.



# Why localize spaces?

There are many many reasons!

1. One can isolate $p$-local properties. (do you prefer $\mathbb{Z}/521059$ or $\mathbb{Z}/7 \oplus \mathbb{Z}/11 \oplus \mathbb{Z}/67 \oplus \mathbb{Z}/101$?)


2. One has magic *homotopy* pull-back squares.
3. One can classify finite loop spaces at a given prime (not globally).
4. One can try to isolate other "local" properties: homology type, connectivity, etc.

# Homological localization

Let $E_*$ be a homology theory (maybe ordinary homology $H\mathbb{F}_p$ or $H\mathbb{Z}$). Adams suggested to construct a space (or a spectrum) by inverting all $E_*$-equivalences using Verdier's machinery of category of fractions.

Bousfield went around the set-theoretical difficulties (1974). He found a single $E_*$-homology equivalence $f$ which is sufficient to invert in order to construct $X \to L_E X$.

# Homotopical localization I

Let $f : A \to B$ be any map.

**Definition**

A space $Z$ is *$f$-local* if $f^* : \mathrm{map}(B, Z) \to \mathrm{map}(A, Z)$ is a weak homotopy equivalence.

1. When $f$ is a universal homology equivalence, any simply connected space is $f$-local.
2. When $f : S^{n+1} \to *$, a space $X$ is $f$-local iff $\pi_{n+k} X = 0$ for $k > 0$.
3. When $f : B\mathbb{Z}/p \to *$ any finite complex is $f$-local (Sullivan conjecture).

# Homotopical localization II

**Definition**

A map $g : X \to Y$ is an *$f$-local equivalence* if $g^* : \mathrm{map}(Y, Z) \to \mathrm{map}(X, Z)$ is a weak homotopy equivalence for any $f$-local space $Z$.

1. When $f$ is a universal homology equivalence, the $f$-local equivalences are homology equivalences.
2. When $f : S^{n+1} \to *$, the $f$-local equivalences are those maps inducing isomorphisms on $\pi_{n+k}$ for $k > 0$.
3. When $f : B\mathbb{Z}/p \to *$, the $n$-connected cover map $X\langle n \rangle \to X$ is an $f$-local equivalence *up to p-completion* for $X$ a 1-connected finite complex with $\pi_2 X$ finite (Neisendorfer).

# Homotopical localization III

There is a functor in the category of spaces $L_f$, coaugmented and homotopy idempotent. The coaugmentation map $X \to L_f X$ is an $f$-local equivalence to an $f$-local space.

**Definition**

When $f$ is of the form $A \to *$, the functor $L_f$ is written $P_A$. It is called $A$-*periodization* or $A$-*nullification*.

1. When $A$ is a universal acyclic space, $P_A X \simeq X^+$, Quillen's plus construction.
2. When $A = S^{n+1}$, then $P_{S^{n+1}} X \simeq X[n]$, the $n$-th Postnikov section.
3. When $A = B\mathbb{Z}/p$, then $P_{B\mathbb{Z}/p} X$ is ... Miller's functor.

## Construction: idea

Let $X$ be a connected space, choose a base point $x$ and a map $\Sigma^k A \to X$ for every element in $[\Sigma^k A, X]$. Consider then the homotopy cofiber sequence

$$\bigvee_k \bigvee_{[\Sigma^k A, X]} \Sigma^k A \xrightarrow{ev} X \to X_1$$

Start again and kill one after the other all maps from $A$ and suspensions of $A$, maybe in a transfinite process. When it stops you have constructed, up to homotopy,

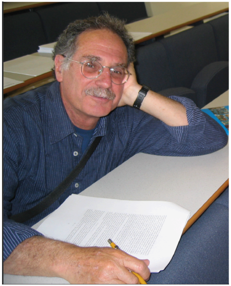
$$X \to P_A X$$

It has the property that $\mathrm{map}_*(A, P_A X) \simeq *$, so $P_A X$ is $A$-local:

$$\mathrm{map}(*, P_A X) = P_A X \simeq \mathrm{map}(A, P_A X)$$

# Functorial construction

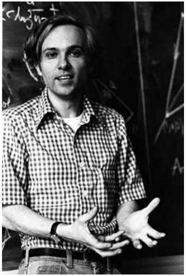
There is a functorial construction. It uses the full mapping space $\mathrm{map}(A, X)$ instead of considering only homotopy classes of maps. Farjoun gives a description in his book.



$$\begin{tikzcd} A \times X^A \coprod_{A \times X} X \arrow[r] \arrow[d] & X^A \arrow[d] \\ X \arrow[r] & L_1 X \end{tikzcd}$$

# Model categories

A (Quillen) model category is equipped with *weak equivalences*, fibrations, and cofibrations. To do homotopy theory Quillen introduced only five simple axioms! For example, fibrations have the right lifting property with respect to acyclic cofibrations.



$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "\wr"'] & E \arrow[d, two heads] \\ B \arrow[r] \arrow[ur] & X \end{tikzcd}$$

# Localization and model categories

There is a *left Bousfield localized* model structure on the category of spaces, where

1. weak equivalences are $f$-local equivalences;
2. cofibrations are unchanged;
3. fibrations are mysterious.

**Theorem**

The identity functors $sSet \leftrightarrows sSet_f$ form a Quillen adjunction. In the $f$-local model structure the fibrant replacement is given by $X \to L_f X$.

# Now move to blackboard