Notes on regular, exact and additive categories

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Chapter 1

Regular and exact categories

The notions of regular category and of exact categories (in the sense of Barr [1]) play an important role in modern categorical algebra. They are useful to axiomatise some exactness properties the categories Set of sets, Grp of groups, Ab of abelian groups, R-Mod of modules on a commutative ring R, Rng of rings, all have in common.
Some other categories, as for instance the category Top of topological spaces, do not share these same properties. In order to understand the notions of regular and exact categories it is useful to first examine various types of epimorphisms: this will be the subject of the first section.

1.1 Epimorphisms

Let $f\colon A \to B$ be an arrow in a category $\mathbb{C}$. One says that f is an **epimorphism** if, for any pair of parallel arrows $u, v\colon B \to C$ such that $u \circ f = v \circ f$, one has $u = v$.
In the category Set the epimorphisms are precisely the surjective maps. In the category Ab of abelian groups the epimorphisms are the surjective homomorphisms. This is also the case for in the category Grp (the proof is less simple than in Ab, see Exercise 1.1.8). In the category Top of topological spaces, the epimorphisms are the surjective continuous functions. There are some algebraic categories where epimorphisms fail to be surjective: for instance, the canonical inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the category Rng of rings, and it is not surjective.

The notion of **monomorphism** is defined dually: such is an $f\colon A \to B$ with the property that $u = v$ whenever $f \circ u = f \circ v$ for some parallel arrows $u, v\colon C \to A$. It is easy to check that the monomorphisms in Set are the injective maps, in Grp and in Ab the injective homomorphisms, in Top the continuous injective maps.

Any isomorphism is both a monomorphism and an epimorphism. The converse does not hold, in general, as the canonical inclusion $\mathbb{Z} \to \mathbb{Q}$ in Rng clearly shows. A category $\mathbb{C}$ where the equivalence

“mono + epi = iso”

holds is usually called a *balanced category*. For instance, Set, Grp and Ab are balanced, Rng and Top are not balanced.

1.1.1. Definition. *$f: A \to B$ is a* **strong epimorphism** *if, given any commutative square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[m]{} & D \end{array}$$

where $m: C \to D$ is a mono, then there exists a unique arrow $t: B \to C$ such that $m \circ t = h$ and $t \circ f = g$.

1.1.2. Lemma. *If $\mathbb{C}$ has binary products, every strong epimorphism is an epimorphism.*

Proof. Let $f: A \to B$ be a strong epi, $u, v: B \to C$ be such that $u \circ f = v \circ f$. Consider the diagonal $(1_C, 1_C) = \Delta: C \to C \times C$, and the commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u\circ f = v \circ f}\downarrow & & \downarrow{\scriptstyle (u,v)} \\ C & \xrightarrow[\Delta]{} & C \times C. \end{array}$$

Since Δ is a monomorphism, there exists a unique $t: B \to C$ such that $\Delta \circ t = (u, v)$ and $t \circ f = u \circ f = v \circ f$. It follows that

$$u = p_1 \circ (u, v) = p_1 \circ \Delta \circ t = t = p_2 \circ \Delta \circ t = p_2 \circ (u, v) = v.$$

□

1.1.3. Lemma. *An arrow $f: A \to B$ is an isomorphism if and only if $f: A \to B$ is a monomorphism and a strong epimorphism.*

Proof. If f is both a strong epimorphism and a monomorphism, one considers the commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle 1_B} \\ A & \xrightarrow[f]{} & B. \end{array}$$

The existence of the morphism $t: B \to A$ such that $f \circ t = 1_B$ and $t \circ f = 1_A$ shows that f is an isomorphism. Conversely, assume that f is an iso, and consider a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[m]{} & D. \end{array}$$

where m is a monomorphism. The arrow $t = g \circ f^{-1}: B \to C$ is the desired factorisation. □

1.1.4. Exercise. Show the following properties:

1. if $f: X \to Y$ and $g: Y \to Z$ are two strong epimorphisms, then $g \circ f: X \to Z$ is a strong epi.

2. If $f: X \to Y$ and $g: Y \to Z$ are such that $g \circ f: X \to Z$ is a strong epimorphism, then $g: Y \to Z$ is a strong epimorphism.

3. If $f = i \circ g$ is a strong epimorphism with i a monomorphism, then i is an isomorphism.

1.1.5. Definition. *One says that $f: A \to B$ is a* **regular epimorphism** *if $f: A \to B$ is the coequaliser of two arrows. In other words, f is a regular epimorphism if one can find $u, v: C \to A$ such that $f = \mathsf{Coeg}(u, v)$.*

1.1.6. Definition. *A* **split epimorphism** *is an arrow $f: A \to B$ such that there is an $i: B \to A$ with $f \circ i = 1_B$.*

Observe that in the category Set any epimorphism is split (= axiom of choice). This is not the case in the categories Grp and Ab, for instance. We are now going to prove the following chain of implications:

1.1.7. Proposition. *Let $\mathbb{C}$ be a category with binary products. One then has the implications*

$$\text{split epi} \Rightarrow \text{regular epi} \Rightarrow \text{strong epi} \Rightarrow \text{epi}$$

Proof. If $f: A \to B$ is split by an arrow $i: B \to A$, then $f = \mathsf{Coeg}(1_A, i \circ f)$. Indeed,

$$f \circ (i \circ f) = f = f \circ 1_A,$$

and, furthermore, if $g: A \to X$ is such that $g \circ (i \circ f) = g \circ 1_A$, then $\phi = g \circ i$ is the only arrow with the property that $\phi \circ f = g$.

Assume that $f: A \to B$ is a regular epimorphism. It is then the coequaliser of two arrows $u: T \to A$ and $v: T \to A$: consider then the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[m]{} & D \end{array}$$

with m a mono. The equalities

$$m \circ g \circ u = h \circ f \circ u = h \circ f \circ v = m \circ g \circ v$$

imply that $g \circ u = g \circ v$. The universal property of the coequaliser f implies that there is a unique $t: B \to C$ such that $t \circ f = g$. This arrow t is also such that $m \circ t = h$, so that f is a strong epimorphism.

The fact that any strong epimorphism is an epimorphism has been shown in Lemma 1.1.2. □

1.1.8. Exercise. The aim of this exercise (from [7]) is to prove that the epimorphisms in Grp are precisely the surjective homomorphisms.
Given an epimorphism $f: G \to H$ of groups, denote its image $f(G)$ by M. Assume that $M \neq H$.
We distinguish between the following two cases:

(a) M is a normal subgroup of H;

(b) M is not normal in H.

In case (a), the proof is the same as for abelian groups: show that the quotient H/M must be trivial.
In case (b) we can assume the index of M in H to be strictly greater than 2: otherwise, the subgroup would be normal. We consider the group $S(H)$ of permutations of the set H, and choose three distinct cosets M, Mu and Mv, with u and v elements of H. Define a permutation $\sigma \in S(H)$ as follows: for $z = xu \in Mu$, we put $\sigma(z) = \sigma(xu) = xv$ (for all $x \in M$); for $z = xv \in Mv$, we put $\sigma(z) = \sigma(xv) = xu$ (for all $x \in M$), and we let σ act as the identity on all other elements of H (i.e. $\sigma(z) = z$ if $z \notin Mu \cup Mv$). Let $\psi: H \to S(H)$ be the homomorphism which associates with an element h of H the permutation ψ_h, defined by

$$\psi_h(x) = hx, \quad \forall x \in H.$$

Let $\overline{\psi}: H \to S(H)$ be the homomorphism which associates, with an element h of H, the permutation

$$\overline{\psi}_h = \sigma^{-1} \circ \psi_h \circ \sigma.$$

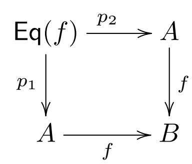
Show that $\psi f = \overline{\psi} f$, while $\overline{\psi} \neq \psi$, a contradiction.

1.2 Regular categories

Let us begin by considering some examples of *quotients* in the categories of sets, of groups and of topological spaces.
Let $f\colon A \to B$ be a map in Set, and

$$\mathsf{Eq}(f) = \{(x, y) \in A \times A \mid f(x) = f(y)\}$$

its *kernel pair*, i.e. the equivalence relation on A that identifies the elements of A having the same image by f. This equivalence relation can be obtained by building the pullback of f along f:



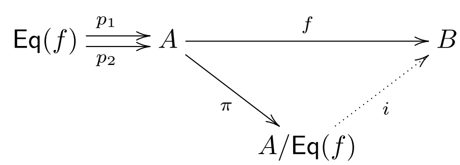
In general, one calls the relation

$$\mathsf{Eq}(f) \underset{p_2}{\overset{p_1}{\rightrightarrows}} A$$

the **kernel pair** of f.

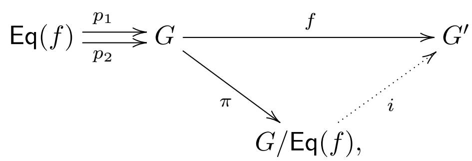
1.2.1. Exercise. Show that any regular epimorphism f in a category with kernel pairs is the coequaliser of its kernel pair $(\mathsf{Eq}(f), p_1, p_2)$.

In the category Set one can see that the canonical quotient $\pi\colon A \to A/\mathsf{Eq}(f)$ defined by $\pi(a) = \overline{a}$ actually is the coequaliser of p_1 and p_2. This yields a unique arrow $i\colon A/\mathsf{Eq}(f) \to B$ such that $i \circ \pi = f$:

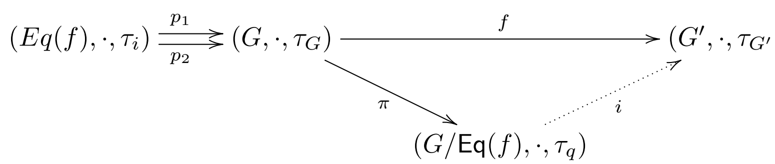


The map i is actually defined by $i(\overline{a}) = f(a)$ for any $\overline{a} \in A/\mathsf{Eq}(f)$. Remark that this gives a factorisation $i \circ \pi$ of the arrow f, where π is a regular epimorphism and i is a monomorphism in the category Set.
The same construction is possible in the category Grp of groups. Indeed, given a homomorphism $f\colon G \to G'$, one can consider the kernel pair $\mathsf{Eq}(f)$ which is again obtained by the pullback above, but this time computed in the category Grp. The equivalence relation $\mathsf{Eq}(f)$ is also a group, as a subgroup of the product $G \times G$ of the group G with itself. The canonical quotient $\pi\colon G \to G/\mathsf{Eq}(f)$ is a group homomorphism, and this allows one to build the following commutative

diagram in Grp:



where π is a regular epimorphism and i a monomorphism.
Let us then consider the category Grp(Top) of topological groups. It is possible to obtain the same kind of factorisation regular epimorphism-monomorphism also in this category, for any continuous homomorphism of topological groups. Given a continuous homomorphism $f\colon (G, \cdot, \tau_G) \to (G', \cdot, \tau_{G'})$ in Grp(Top), the kernel pair $(\mathsf{Eq}(f), \cdot, \tau_i)$ is a topological group for the topology τ_i induced by the product topology $\tau_{G\times G}$ of the topological group $(G\times G, \cdot, \tau_{G\times G})$. The quotients in Grp(Top) are actually computed as in Grp, by using the quotient topology. In this way one gets the following commutative diagram



where π is the canonical quotient, and τ_q the quotient topology. It turns out that π is the coequaliser of p_1 and p_2 in Grp(Top), and the induced morphism

$$i\colon (G/\mathsf{Eq}(f), \cdot, \tau_q) \to (G', \cdot, \tau_{G'})$$

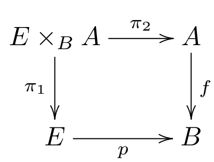
is a monomorphism (since it is injective).

There are many other categories where this same construction is possible, for instance in the category Rng of rings, Ab of abelian groups, R-Mod of modules on a commutative ring R.
All these are examples of regular categories in the following sense:

1.2.2. Definition. *A finitely complete category $\mathbb{C}$ is* **regular** *if*

- *coequalisers of kernel pairs exist in $\mathbb{C}$;*
- *regular epimorphisms are pullback stable.*

1.2.3. Examples. The categories Set of sets is regular. We have observed that the coequalisers of kernel pairs exist in Set, and it remains to check the pullback stability of regular epimorphisms. Consider a pullback



in Set where p is a surjective map (=regular epimorphism), and let us show that π_2 is also a surjective map. Let a be an element in A; there exists then an $e \in E$ such that $p(e) = f(a)$. This shows that there is an $(e, a) \in E \times_B A$ such that $\pi_2(e, a) = a$. The same argument still works in the category Grp, by taking into account the fact that regular epimorphisms therein are precisely the surjective homomorphisms, and that pullbacks are computed in Grp as in Set. For essentially the same reason the categories Rng, Ab and R-Mod are also regular categories.

The category Grp(Top) of topological groups is regular. The point here is that the canonical quotient $\pi\colon H \to H/\mathsf{Eq}(f)$ of a topological group (H, θ) by the equivalence relation $(\mathsf{Eq}(f), \tau_i)$ of an arrow f in the category Grp(Top) is an *open surjective homomorphism.*

Let us check this latter statement: if we write $K = \mathsf{ker}\,(\pi)$ for the kernel of π, then for any open $V \in \theta$ let us show that

$$\pi^{-1}(\pi(V)) = VK.$$

On the one hand if $z = vk$, where $v \in V$ and $k \in K$, one has

$$\pi(z) = \pi(vk) = \pi(v)\pi(k) = \pi(v) \in \pi(V),$$

so that $z \in \pi^{-1}(\pi(V))$.

If $z \in \pi^{-1}(\pi(V))$, then $\pi(z) = \pi(v_1)$, for a $v_1 \in V$, so that $v_1^{-1}z \in K$, and $z = v_1 k$, for a $k \in K$.

This implies that

$$\pi^{-1}(\pi(V)) = \bigcup_{k \in K} Vk \in \theta.$$

Indeed, the function $m_k\colon G \to G$ defined by $m_k(x) = xk$ for any $x \in G$ (with fixed $k \in K$) is a homeomorphism, hence $Vk = m_k(V) \in \theta$, since $V \in \theta$.

Conclusion: $\forall V \in \theta$, $\pi(V)$ is open, and this implies that π is an open map. It follows that in Grp(Top) the regular epimorphisms are the open surjective homomorphisms. To conclude that Grp(Top) is a regular category it suffices to verify that the open surjective homomorphisms are pullback stable. This is not difficult, and we leave the verification to the reader.

Counter-example

The category Top of topological spaces *is not regular.* To see that regular epimorphisms are not pullback-stable, consider the following example, given in [2], of the pullback

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{p_2} & Z \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where $X = (\{a, b, c, d\}, \tau_X\}$, $Y = (\{x, y, z\}, \tau_Y)$ and $Z = (\{l, m, n\}, \tau_Z)$, and the topologies are: $\tau_X = \{\emptyset, \{a, b, c, d\}, \{a, b\}\}$, $\tau_Y = \{\emptyset, \{x, y, z\}\}$ and $\tau_Z =$

$\{\emptyset, \{l, m, n\}, \{l, m\}\}$. The continuous maps f and g are defined by:

$$f(a) = x,\ f(b) = y = f(c),\ f(d) = z,$$

$$g(l) = x,\ g(m) = z = g(n).$$

Remark that $f: X \to Y$ is a quotient map. The pullback

$$X \times_Y Z = (\{(a, l), (d, m), (d, n)\}, \tau\}$$

is equipped with the topology

$$\tau = \{\emptyset, X \times_Y Z, \{(a, l)\}, \{(a, l), (d, m)\}\}.$$

Now, the projection p_2 is not a quotient map, since $p_2^{-1}(l) = \{(a, l)\}$ is open, while $\{l\}$ is not.

We are now going to show that any arrow in a regular category has a canonical factorisation as a regular epimorphism followed by a monomorphism, exactly as in the examples of the categories Set, Grp and $\mathsf{Grp(Top)}$ recalled here above. For this, the following simple lemma will be needed:

1.2.4. Lemma. *For an arrow $p: A \to B$ having a kernel pair $(\mathsf{Eq}(p), p_1, p_2)$ the following conditions are equivalent:*

1. *p is a monomorphism;*
2. *the projections $p_1: \mathsf{Eq}(p) \to A$ and $p_2: \mathsf{Eq}(p) \to A$ are equal.*

The following elementary properties of pullbacks will also be useful:

1.2.5. Lemma. *Consider a commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{k} & B & \xrightarrow{f} & C \\ {\scriptstyle a}\downarrow & (1) & \downarrow{\scriptstyle b} & (2) & \downarrow{\scriptstyle c} \\ A' & \xrightarrow[k']{} & B' & \xrightarrow[f']{} & C'. \end{array}$$

Then:

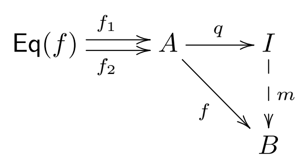
1. *if the squares (1) and (2) are pullbacks, then also the exterior rectangle (1) + (2) is a pullback;*
2. *if the exterior rectangle (1) + (2) and the square (2) are pullbacks, then the square (1) is a pullback.*

1.2.6. Theorem. *Let $\mathbb{C}$ be a regular category. Then*

1. *any arrow in $\mathbb{C}$ has a factorisation as a regular epimorphism followed by a monomorphism;*
2. *this factorisation is unique (up to isomorphism).*

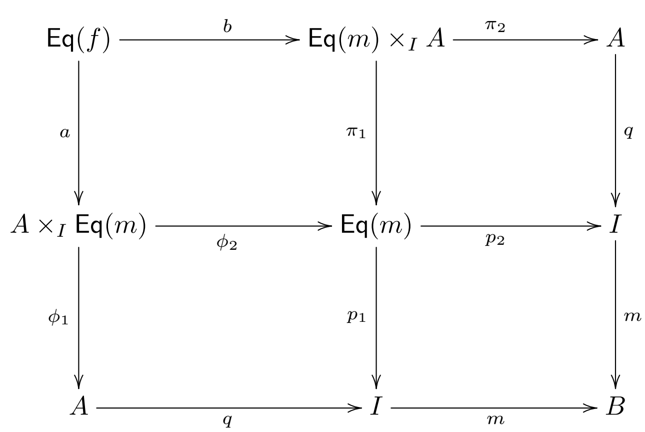
Proof.

1. Let $f : A \to B$ be an arrow in $\mathbb{C}$. Consider the diagram here below where $(\mathsf{Eq}(f), f_1, f_2)$ is the kernel pair of f, and q is the coequaliser of (f_1, f_2). There exists then a unique m such that $m \circ q = f$.



We are now going to show that m is a monomorphism: for this, it suffices to show that the projections $p_1 \colon \mathsf{Eq}(m) \to I$ and $p_2 \colon \mathsf{Eq}(m) \to I$ of the kernel pair of m are equal (Lemma 1.2.4).

Consider the diagram



where all the squares are pullbacks. By Lemma 1.2.5 we know that the whole square is a pullback, so that $f_1 = \phi_1 \circ a$ and $f_2 = \pi_2 \circ b$. The arrow $\phi_2 \circ a = \pi_1 \circ b$ is then an epimorphism, as a composite of epimorphisms. The fact that $\phi_1 \circ a = f_1$ and $\pi_2 \circ b = f_2$ implies that

$$p_1 \circ (\phi_2 \circ a) = q \circ \phi_1 \circ a = q \circ f_1 = q \circ f_2 = q \circ \pi_2 \circ b = p_2 \circ \pi_1 \circ b = p_2 \circ (\phi_2 \circ a);$$

since $\phi_2 \circ a$ is an epimorphism, it follows that $p_1 = p_2$.

2. Since any regular epi is a strong epi, the uniqueness of the factorisation can be easily checked by using the definition of strong epi.

□

1.2.7. Proposition. *Let $\mathbb{C}$ be a regular category, then the following properties are satisfied:*

1. *regular epis coincide with strong epis;*

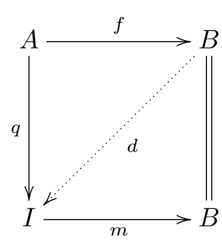
2. *if $g \circ f$ is a regular epi, then g is a regular epi;*

3. *if g and f are regular epis, then $g \circ f$ is a regular epi;*

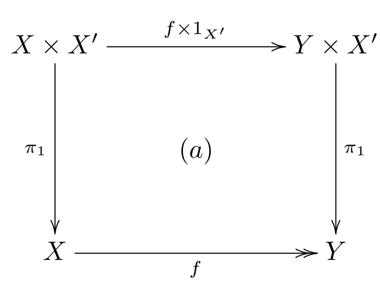
4. *f is an isomorphism if and only if f is a mono and a regular epi;*

5. *if $f: X \to Y$ and $g: X' \to Y'$ are regular epis, then $f \times g: X \times X' \to Y \times Y'$ is also a regular epi.*

Proof.
For 1., we only need to prove that any strong epimorphism $f: A \to B$ is regular. If $f = m \circ q$ with m a monomorphism and q a regular epimorphism



there is a unique arrow $d: B \to I$ such that $d \circ f = q$ and $m \circ d = 1_B$. The arrow d is the inverse of m, and f is then a regular epimorphism.
2., 3. and 4. follow from 1. and from the properties of strong epimorphisms.
5. If $f: X \to Y$ and $g: X' \to Y'$ are two regular epimorphisms, consider the commutative diagrams



$$\begin{array}{ccc} Y \times X' & \xrightarrow{1_Y \times g} & Y \times Y' \\ {\scriptstyle \pi_2}\downarrow & (b) & \downarrow{\scriptstyle \pi_2} \\ X' & \underset{g}{\twoheadrightarrow} & Y'. \end{array}$$

These squares are easily seen to be pullbacks, so that the arrows $f \times 1_{X'}$ and $1_Y \times g$ are regular epimorphisms (by regularity of $\mathbb{C}$), and then

$$f \times g = (1_Y \times g) \circ (f \times 1_{X'})$$

is a regular epimorphism by 3. □

We are now going to give an equivalent formulation of the notion of regular category:

1.2.8. Theorem. *Let $\mathbb{C}$ be a finitely complete category. Then $\mathbb{C}$ is a regular category if and only if*

1. *any arrow in $\mathbb{C}$ factorises as a regular epi followed by a mono;*

2. *these factorisations are pullback-stable.*

Proof. If $\mathbb{C}$ is regular, the validity of 1. and 2. follows from Theorem 1.2.6.
For the converse, assume that the conditions 1. and 2. hold in $\mathbb{C}$: then, in particular, regular epis are pullback stable. It remains to show that the coequalisers of kernel pairs exist in $\mathbb{C}$. Let $\mathsf{Eq}(f) \underset{f_2}{\overset{f_1}{\rightrightarrows}} X$ be the kernel pair of an arrow $f : X \to Y$ of $\mathbb{C}$. Consider $m \circ q$ the regular epi-mono factorisation of f. The fact that m is a mono implies that $\mathsf{Eq}(f) \underset{f_2}{\overset{f_1}{\rightrightarrows}} X$ is also the kernel pair of the regular epi q. The arrow q is then the coequaliser of its kernel pair $\mathsf{Eq}(f) \underset{f_2}{\overset{f_1}{\rightrightarrows}} X$ (see Exercise 1.2.1). □

By using Lemma 1.2.5 one can prove the following

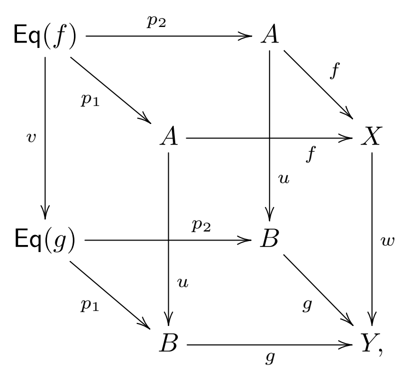
1.2.9. Lemma. *Let $\mathbb{C}$ be a category with pullbacks. One considers the com-*

mutative diagram here below where the horizontal rows are kernel pairs:

$$\begin{array}{ccccc}
\mathsf{Eq}(f) & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & A & \overset{f}{\longrightarrow} & X \\
{\scriptstyle v}\downarrow & (1) & \downarrow{\scriptstyle u} & (2) & \downarrow{\scriptstyle w} \\
\mathsf{Eq}(g) & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & B & \underset{g}{\longrightarrow} & Y
\end{array}$$

If (2) *is a pullback, then each commutative square in* (1) *is a pullback.*

Proof. Consider the commutative diagram



where the upper face of the cube and the right-hand face are pullbacks by assumption, so that also the rectangle

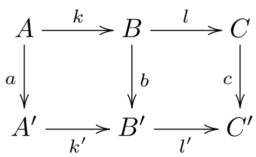
$$\begin{array}{ccccc}
\mathsf{Eq}(f) & \overset{v}{\longrightarrow} & \mathsf{Eq}(g) & \overset{p_2}{\longrightarrow} & B \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle g} \\
A & \underset{u}{\longrightarrow} & B & \underset{g}{\longrightarrow} & Y
\end{array}$$

is a pullback (the diagram is commutative). Since the right-hand square in this latter diagram is a pullback, Lemma 1.2.5.2 tells us that the left square is a pullback as well. A similar proof shows that also the square

$$\begin{array}{ccc}
\mathsf{Eq}(f) & \overset{v}{\longrightarrow} & \mathsf{Eq}(g) \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\
A & \underset{u}{\longrightarrow} & B
\end{array}$$

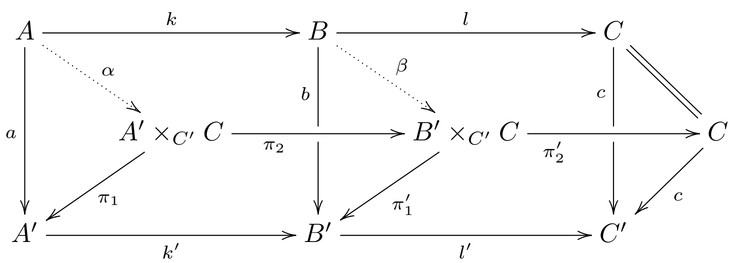
is a pullback. □

1.2.10. Lemma. *Consider a commutative diagram*

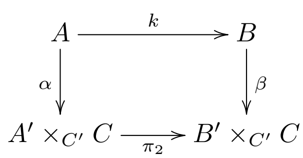


in a regular category $\mathbb{C}$*, where the left-hand square and the external rectangle are pullbacks. If the arrow* k' *is a regular epimorphism, then the right-hand square is a pullback.*

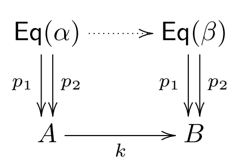
Proof. Consider the commutative diagram



where $(B' \times_{C'} C, \pi_1', \pi_2')$ is the pullback of l' and c, and $(A' \times_{C'} C, \pi_1, \pi_2)$ is the pullback of k' and π_1'. The fact that the external rectangle is a pullback implies that the induced arrow α is an isomorphism. The arrow π_2 is a regular epimorphism (because k' is one), so that $\pi_2 \circ \alpha = \beta \circ k$ is a regular epimorphism, and then β is a regular epimorphism (see Proposition 1.2.7). The arrow β is a monomorphism: this follows from the fact that the square



is a pullback, so that both the induced commutative squares

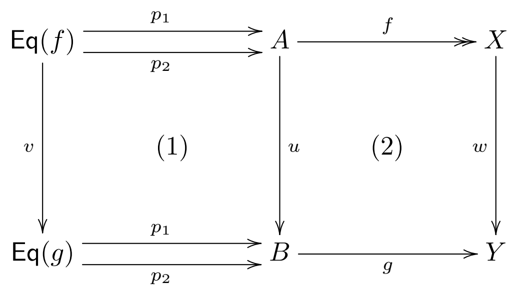


are pullbacks, with the dotted arrow a (regular) epimorphism. This implies that the projections $p_1\colon \mathsf{Eq}(\beta) \to B$ and $p_2\colon \mathsf{Eq}(\beta) \to B$ are equal, so that β is indeed a monomorphism.

□

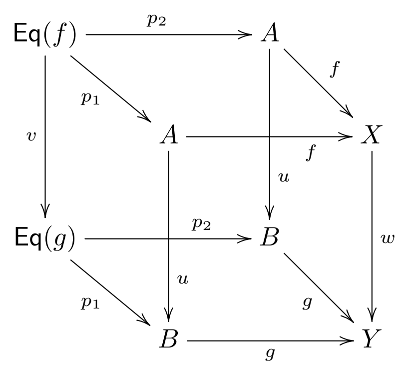
We are now ready to prove the following interesting result, often referred to as the *Barr-Kock Theorem* (although it was first observed by A. Grothendieck):

1.2.11. Theorem. *Let* $\mathbb{C}$ *be a regular category, and*

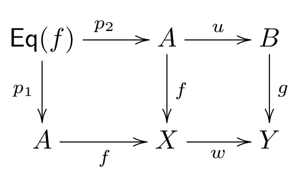


a commutative diagram with f *a regular epimorphism. If either of the left-hand commutative squares are pullbacks, then the right-hand square* (2) *is a pullback.*

Proof. Consider the following commutative diagram



The assumptions guarantee that, for instance, the left-hand face and the bottom face of the cube are pullbacks. By commutativity it follows that the rectangle



is also a pullback, as well as its left-hand square. Since f is a regular epi, by Lemma 1.2.10 it follows that the right-hand square is a pullback. □

1.2.12 Relations in regular categories

1.2.13. Definition. *Let $\mathbb{C}$ be a finitely complete category. A* **relation from** X **to** Y *in $\mathbb{C}$ is a graph*

$$X \xleftarrow{\ d_1\ } R \xrightarrow{\ d_2\ } Y$$

such that the factorisation $(d_1, d_2) : R \to X \times Y$ is a monomorphism; equivalently, the pair (d_1, d_2) is jointly monomorphic.

One usually identifies two relations $R \to X \times Y$ and $S \to X \times Y$ when they determine the same subobject of $X \times Y$ (= equivalence classes of monomorphisms with codomain $X \times Y$). If $X = Y$, one says that R is a relation "on X".

1. A relation R is **reflexive** when there is an arrow $\delta : X \to R$ such that $d_1 \circ \delta = 1_X = d_2 \circ \delta$.
2. R is **symmetric** if there is an arrow $\sigma : R \to R$ such that $d_1 \circ \sigma = d_2$ and $d_2 \circ \sigma = d_1$.
3. Consider the pullback

$$\begin{array}{ccc} R \times_X R & \xrightarrow{\ p_2\ } & R \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ R & \xrightarrow[\ d_2\ ]{} & X \end{array}$$

The relation R is **transitive** if there is an arrow $\tau : R \times_X R \to R$ such that $d_1 \circ \tau = d_1 \circ p_1$ et $d_2 \circ \tau = d_2 \circ p_2$.

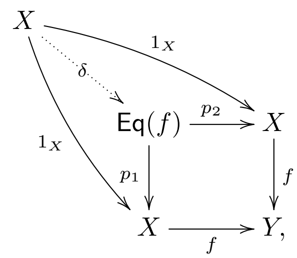
A relation R on X is an **equivalence relation** if R is reflexive, symmetric and transitive. Of course, this abstract notion of equivalence relation gives in particular the usual one in the category $\mathbb{C} = \mathsf{Set}$.
When $\mathbb{C} = \mathsf{Grp}$ is the category of groups, an equivalence relation $R \subset X \times X$ in the category Grp is an equivalence relation on the set X which is also a subgroup of the group $X \times X$.
In general, one has the following:

1.2.14. Lemma. *In a category with pullbacks the kernel pair* $\mathsf{Eq}(f) \underset{p_2}{\overset{p_1}{\rightrightarrows}} X$ *of an arrow $f : X \to Y$ is an equivalence relation on X in $\mathbb{C}$.*

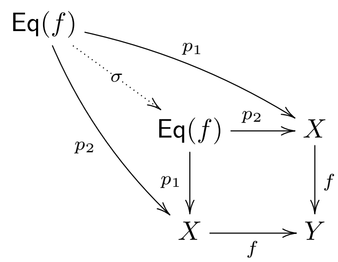
Proof. The arrows $p_1 \colon \mathsf{Eq}(f) \to X$ and $p_2 \colon \mathsf{Eq}(f) \to X$ are jointly monomorphic since they are projections of a pullback. The commutativity of the square

$$\begin{array}{ccc} X & \xrightarrow{\ 1_X\ } & X \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[\ f\ ]{} & Y \end{array}$$

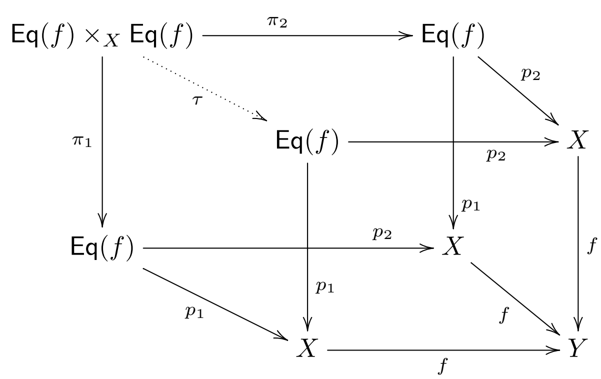
and the universal property of the kernel pair $(\mathsf{Eq}(f), p_1, p_2)$ implies that there is a unique $\delta\colon X \to \mathsf{Eq}(f)$ such that $p_1 \circ \delta = 1_X = p_2 \circ \delta$



and $\mathsf{Eq}(f)$ is then reflexive. Similarly, the commutativity of the external part of the diagram



implies that there is a unique arrow $\sigma\colon \mathsf{Eq}(f) \to \mathsf{Eq}(f)$ such that $p_1 \circ \sigma = p_2$ and $p_2 \circ \sigma = p_1$, hence $\mathsf{Eq}(f)$ is symmetric. For the transitivity of $\mathsf{Eq}(f)$ one considers the following commutative diagram



where the back face is a pullback. The commutativity of the diagram and the universal property of the kernel pair $(\mathsf{Eq}(f), p_1, p_2)$ shows that there is a unique τ such that $p_1 \circ \tau = p_1 \circ \pi_1$ and $p_2 \circ \tau = p_2 \circ \pi_2$, and this completes the proof.

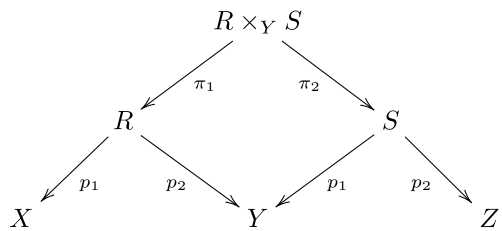
□

An important aspect of regular categories is that in these categories one can define a composition of relations, which has some nice properties.

In the category Set, if $R \to X \times Y$ is a relation from X to Y, and $S \to Y \times Z$ from Y to Z, one usually defines the relation $S \circ R \to X \times Z$ by setting

$$S \circ R = \{(x, z) \in X \times Z \,\text{such that}\, \exists\, y \in Y \,\text{with}\, xRySz\}.$$

This construction is actually possible in any regular category $\mathbb{C}$, thanks to the existence of regular images (see Theorem 1.2.6). One first builds the pullback



and one then factorises the arrow $(p_1 \circ \pi_1, p_2 \circ \pi_2)\colon R \times_Y S \to X \times Z$ as a regular epimorphism $q\colon R \times_Y S \to I$ followed by a monomorphism $i\colon I \to X \times Z$:

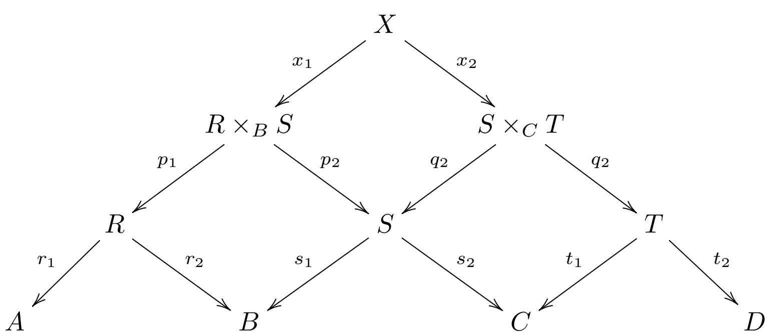
$$R \times_Y S \xrightarrow{\;q\;} I \xrightarrow{\;i\;} X \times Z$$

In Set, the set I is given by $(x, z) \in X \times Z$ such that there is a $(u, y, v) \in R \times_Y S$ with $u = x$ and $v = z$: this is precisely $S \circ R$!
This composition is actually associative:

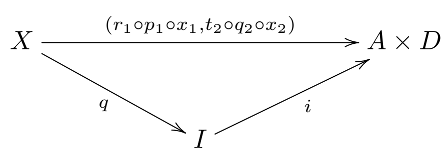
1.2.15. Theorem. *Let $\mathbb{C}$ be a regular category. If $R \to A \times B$, $S \to B \times C$ and $T \to C \times D$ are relations in $\mathbb{C}$, one has the equality*

$$T \circ (S \circ R) = (T \circ S) \circ R.$$

Proof. Consider the diagram obtained by building the following pullbacks:



The proof consists in showing that the relations $T \circ (S \circ R)$ and $(T \circ S) \circ R$ are both given by the regular image $i\colon I \to A \times D$ in the factorisation



as a regular epimorphism folllowed by a monomorphism of the arrow

$$(r_1 \circ p_1 \circ x_1, t_2 \circ q_2 \circ x_2).$$

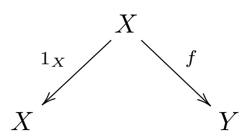
We leave it to the reader the verification of this fact, which uses the pullback stability of regular epimorphisms in a crucial way.

□

This result allows one to define a new category starting from any regular category $\mathbb{C}$, the category $\mathsf{Rel}(\mathbb{C})$ of relations de $\mathbb{C}$. The objects are the same as the ones in $\mathbb{C}$, an arrow from X to Y is simply a relation from X to Y, and composition is the relational one defined above. For any relation R from X to Y the discrete relation

$$\Delta_X : \; X \overset{1_X}{\underset{1_X}{\rightrightarrows}} X$$

is such that $R \circ \Delta_X = R$, and for any relation S from Z to X one has $\Delta_X \circ S = S$. It follows that the arrow Δ_X in $\mathsf{Rel}(\mathbb{C})$ is the identity on the object X.
There is actually a faithful functor $\Gamma \colon \mathbb{C} \to \mathsf{Rel}(\mathbb{C})$, associating the graph $\Gamma(f)$ with any arrow $f \colon X \to Y$ of $\mathbb{C}$, where $\Gamma(f) \colon X \to Y$ in $\mathsf{Rel}(\mathbb{C})$ is the *graph of f*, seen as a relation:



We are not going to develop the theory of relations in a regular category in these notes. The book [6] will be a useful reference for the interested reader.

1.2.16. Exercise. Show that any reflexive relation R in the category Grp of groups is an equivalence relations. Is this result true in the categories Set, or Ab? A finitely complete category $\mathbb{C}$ is called a *Mal'tsev category* if any internal reflexive relation in $\mathbb{C}$ is an equivalence relation [4]. As explained in this latter paper, for a regular category being a Mal'tsev category is equivalent to the following property: given any equivalence relations R and S on the same object X, then $R \circ S = S \circ R$ (2-permutability of the composition of equivalence relations).

1.3 Exact categories

1.3.1. Definition. *A category $\mathbb{C}$ is* **exact** *if:*

1. *$\mathbb{C}$ is regular;*

2. *any equivalence relation (R, d_1, d_2) in $\mathbb{C}$ is* **effective***: this means that (R, d_1, d_2) is the kernel pair of an arrow q in $\mathbb{C}$.*

1.3.2. Examples. The category Set is exact: each equivalence relation R on a set A is the kernel pair of the canonical quotient $\pi_R \colon A \to A/R$.

Also Grp is exact. Indeed, an equivalence relation R (in the categorical sense) on a group G in Grp is an equivalence relation in Set which is *compatible* with the group structure: one always has that $1R1$, the fact that xRy implies that $x^{-1}Ry^{-1}$ and, moreover, if xRy and uRv, then necessarily $xuRyv$. It follows that the canonical quotient $\pi\colon G \to G/R$ is a group homomorphism, and $\mathsf{Eq}(\pi) = R$.
One can check that, similarly, the categories Ab, R-Mod and Rng are all exact categories (any algebraic variety in the sense of universal algebra is an exact category, see [2], for instance). One can also show that the category HComp of compact Hausdorff spaces is an exact category.

The verification of the following simple fact is left to the reader:

1.3.3. Proposition. *Let $\mathbb{C}$ be an exact category, and $d_1, d_2\colon R \rightrightarrows X$ an equivalence relation on X in $\mathbb{C}$. Then the coequaliser of (d_1, d_2) exists, and $d_1, d_2\colon R \rightrightarrows X$ is the kernel pair of its coequaliser.*

Counter-examples

- Top is not exact (it is not even regular!). The effective equivalence relations $R \subset X \times X$ are the equivalence relations on X having the topology induced by the product space $X \times X$.

- Grp(Top) is a regular category which is not exact.

- Let $\mathsf{Ab}_{\mathsf{t.f.}}$ be the category of torsion-free abelian groups. This is the full subcategory of Ab whose objects are the abelian groups satisfying the implication $nx = 0 \Rightarrow x = 0$, for any $n \in \mathbb{N}^*$. Limits in $\mathsf{Ab}_{\mathsf{t.f.}}$ are computed as in the category Ab of abelian groups. On the other hand, any homomorphism $f\colon A \to B$ in $\mathsf{Ab}_{\mathsf{t.f.}}$ has a factorisation $m \circ q\colon A \to I \to B$ in Ab, where q is a surjective homomorphism and m a monomorphism. Since B is torsion-free, I is torsion-free as well. The morphism $q\colon A \to I$ is the coequaliser of the projections $\pi_1\colon \mathsf{Eq}(f) \to A$ and $\pi_2\colon \mathsf{Eq}(f) \to A$ in $\mathsf{Ab}_{\mathsf{t.f.}}$, and it is then a regular epimorphism in $\mathsf{Ab}_{\mathsf{t.f.}}$. Consequently, the category $\mathsf{Ab}_{\mathsf{t.f.}}$ is regular. However, $\mathsf{Ab}_{\mathsf{t.f.}}$ is not exact. To see this, let us consider the relation R_2 on $\mathbb{Z}$ defined by
$$mR_2n \quad \Leftrightarrow \quad \exists k \in \mathbb{Z} \text{ such that } m - n = 2k.$$
This is an equivalence relation in $\mathsf{Ab}_{\mathsf{t.f.}}$ which is not effective. It is easy to see that the coequaliser of the projections $\pi_1\colon R_2 \to \mathbb{Z}$ and $\pi_2\colon R_2 \to \mathbb{Z}$ is zero. This implies that R_2 can not be effective, since if it were so, it would be $\mathbb{Z} \times \mathbb{Z}$.

Chapter 2

Additive categories

In the category Ab of abelian groups it is possible to define an "addition" on the set of parallel arrows from one object to another. Indeed, if $f\colon A \to B$ and $g\colon A \to B$ are two arrows in Ab, one defines

$$(f+g)(x) = f(x) + g(x) \qquad \forall x \in A,$$

and this function $f+g\colon A \to B$ is again an arrow in Ab. This operation provides $\mathsf{Ab}(A,B)$ with an abelian group structure, whose neutral element is the zero morphism $0\colon A \to 0 \to B$ (this latter being the unique morphism that factors through the trivial group 0). Observe that in the category Grp it is not possible to "add" two parallel morphisms, simply because the function $f+g$ is not a group homomorphism, in general.
The example of the category Ab of abelian groups (or any category $\mathsf{R\text{-}Mod}$ of R-modules) will be the guiding one in this chapter, where the notion of additive category is briefly introduced.

2.1 Preadditive categories

2.1.1. Definition. *A category* $\mathbb{C}$ *is* **preadditive** *if:*

1. $\mathbb{C}(X,Y)$ *is an abelian group* $\forall X,Y$ *of* $\mathbb{C}$ *;*
2. $\forall X,Y,Z$ *in* $\mathbb{C}$ *the composition mappings* $\circ : \mathbb{C}(X,Y) \times \mathbb{C}(Y,Z) \to \mathbb{C}(X,Z)$ *are group homomorphisms in each variable : for all* $f, f' \in \mathbb{C}(X,Y)$, $g, g' \in \mathbb{C}(Y,Z)$ *one has*

$$g \circ (f + f') = g \circ f + g \circ f',$$
$$(g + g') \circ f = g \circ f + g' \circ f.$$

The notion of preadditive category is self-dual: $\mathbb{C}$ is preadditive if and only if $\mathbb{C}^{op}$ is preadditive.

2.1.2. Examples.

- Ab is preadditive;
- R-Mod, the category of (left) modules on a commutative unitary ring R is preadditive;
- $\mathsf{Ab}_{\mathsf{s.t.}}$ the category of torsion-free abelian groups is preadditive;
- the category Ab(Top) of topological abelian groups is preadditive;
- any unitary ring R can be seen as a preadditive category $\mathbb{C}$ having a unique object, written $*$. The elements r of R are the arrows $r\colon * \to *$, so that $\mathbb{C}(*, *)$ is a an abelian group $(\mathsf{R}, +, 0)$. On the other hand the composite $r \circ s\colon * \to *$ of two arrows is the product $r \cdot s$ of r and s as elements of the ring R.

The categories Set, Grp and Top are not preadditive.
The proof of the following result is left to the reader:

2.1.3. Lemma. *Let $\mathbb{C}$ be a preadditive category, and X an object in $\mathbb{C}$. The following conditions are equivalent:*

1. *X is initial;*
2. *X is terminal;*
3. *X is a zero object.*

When this is the case, the morphism $f\colon A \to B$ which factors through the zero object $X = 0$ is the neutral element for the group structure of $\mathbb{C}(A, B)$.

A category $\mathbb{C}$ having a zero object is called a **pointed** category. For instance, the categories Grp, Ab, R-Mod, Mon (of monoids), Grp(Top), Ab(Top) and Set_* (of pointed sets) are pointed, whereas the categories Set and Top are not pointed. The category CRng of commutative unitary rings is not pointed.

If X and Y are two objects in a pointed category, one writes $0_{X,Y}$, or simply 0, for the unique arrow $X \to 0 \to Y$, called the **zero morphism**.

2.1.4. Definition. *A diagram*

$$X \underset{i_1}{\overset{p_1}{\leftrightarrows}} Z \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y$$

is a **biproduct** *of X and Y if $p_1 \circ i_1 = 1_X$, $p_2 \circ i_2 = 1_Y$ and $i_1 \circ p_1 + i_2 \circ p_2 = 1_Z$.*

2.1.5. Remark. The notion of biproduct is self-dual.

2.1.6. Example. In the category Ab, one has the biproduct

$$A \underset{i_1}{\overset{p_1}{\rightleftarrows}} A \bigoplus B \underset{i_2}{\overset{p_2}{\rightleftarrows}} B$$

where $A \bigoplus B = \{(a, b), \mid a \in A, b \in B\}$, $p_1(a, b) = a$, $p_2(a, b) = b$, $i_1(a) = (a, 0)$, $i_2(b) = (0_A, b)$. It is clear that p_k, i_k (for $k = 1, 2$) are group homomorphisms, and $p_1 \circ i_1 = 1_A$, $p_2 \circ i_2 = 1_B$, $i_1 \circ p_1 + i_2 \circ p_2 = 1_{A \bigoplus B}$.

2.1.7. Theorem. *Let $\mathbb{C}$ be a preadditive category. The following conditions are equivalent:*

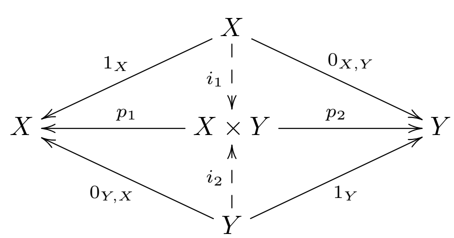
1. *the product of X and Y exists in $\mathbb{C}$;*
2. *the biproduct of X and Y exists in $\mathbb{C}$.*

When this is the case, given a biproduct

$$X \underset{i_1}{\overset{p_1}{\rightleftarrows}} Z \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y$$

the diagram $X \xrightarrow{i_1} Z \xleftarrow{i_2} Y$ *is a coproduct.*

Proof. 1. $\Rightarrow$ 2. Let $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$ be the product of X and Y. Consider the diagram



The factorisations i_1 and i_2 are such that $p_1 \circ i_1 = 1_X$, $p_2 \circ i_1 = 0_{X,Y}$ and $p_2 \circ i_2 = 1_Y$, $p_1 \circ i_2 = 0_{Y,X}$. One has

$$p_1 \circ (i_1 \circ p_1 + i_2 \circ p_2) = p_1 \circ i_1 \circ p_1 + p_1 \circ i_2 \circ p_2 = p_1 + 0 = p_1 \circ 1_{X \times Y}$$

and

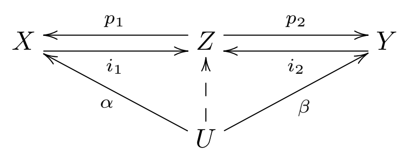
$$p_2 \circ (i_1 \circ p_1 + i_2 \circ p_2) = p_2 \circ i_1 \circ p_1 + p_2 \circ i_2 \circ p_2 = 0 + p_2 = p_2 \circ 1_{X \times Y}.$$

Since (p_1, p_2) is jointly monomorphic, it follows that $i_1 \circ p_1 + i_2 \circ p_2 = 1_{X \times Y}$, hence X and Y have a biproduct.

Conversely, assume that Z is the biproduct of X and Y. First note that $p_2 \circ i_1 = 0$: indeed,

$$\begin{aligned} p_2 \circ i_1 &= p_2 \circ (i_1 \circ p_1 + i_2 \circ p_2) \circ i_1 \\ &= p_2 \circ i_1 \circ p_1 \circ i_1 + p_2 \circ i_2 \circ p_2 \circ i_1 \\ &= p_2 \circ i_1 + p_2 \circ i_1, \end{aligned}$$

thus $p_2 \circ i_1 = 0_{X,Y}$. One similarly shows that $p_1 \circ i_2 = 0_{Y,X}$.
Let us then prove that Z is the product of X and Y. Let U be an object of $\mathbb{C}$, and let $\alpha : U \longrightarrow X$, $\beta : U \longrightarrow Y$ be any two morphisms of $\mathbb{C}$ as depicted in the following diagram:



Let $\lambda = i_1 \circ \alpha + i_2 \circ \beta : U \longrightarrow Z$. One has:

$$p_2 \circ \lambda = p_2 \circ (i_1 \circ \alpha + i_2 \circ \beta) = p_2 \circ i_1 \circ \alpha + p_2 \circ i_2 \circ \beta = p_2 \circ i_1 \circ \alpha + \beta$$

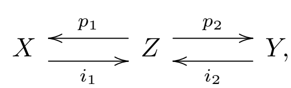
so that

$$p_2 \circ \lambda = 0 + \beta = \beta.$$

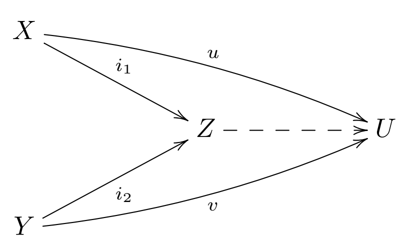
The proof of the equality $p_1 \circ \lambda = \alpha$ is similar. Let us now show that λ is unique. If $\lambda' : U \to Z$ is such that $p_1 \circ \lambda' = \alpha$ and $p_2 \circ \lambda' = \beta$, then

$$\begin{aligned} \lambda &= i_1 \circ \alpha + i_2 \circ \beta \\ &= i_1 \circ (p_1 \circ \lambda') + i_2 \circ (p_2 \circ \lambda') \\ &= (i_1 \circ p_1 + i_2 \circ p_2) \circ \lambda' \\ &= \lambda'. \end{aligned}$$

Given a biproduct



let us check that the diagram $X \xrightarrow{i_1} Z \xleftarrow{i_2} Y$ is a coproduct.
Consider the diagram



and let us show that there is a unique $\theta : Z \to U$ such that $\theta \circ i_1 = u$ and $\theta \circ i_2 = v$. One sets $\theta = u \circ p_1 + v \circ p_2$, and then observes that

$$\theta \circ i_1 = (u \circ p_1 + v \circ p_2) \circ i_1 = u \circ p_1 \circ i_1 + v \circ p_2 \circ i_1 = u + 0 = u.$$

One proceeds similarly to check that $\theta \circ i_2 = v$. The uniqueness of the factorisation is easy to check, and we leave the verification to the reader. $\square$

2.2 Additive categories

2.2.1. Definition. *A category* $\mathbb{C}$ *is* **additive** *if*

1. $\mathbb{C}$ *is preadditive;*
2. $\mathbb{C}$ *is pointed;*
3. $\mathbb{C}$ *has biproducts for any pair of objects* X *and* Y *of* $\mathbb{C}$*.*

Notation : From now on we shall denote by $X \bigoplus Y$ the (object part of the) biproduct of X and Y.

2.2.2. Examples. The categories Ab, $\mathsf{Ab}_{\mathsf{s.t.}}$, $\mathsf{Ab}(\mathsf{Top})$ and $\mathsf{R\text{-}Mod}$ are additive.

We now recall the notions of kernel and cokernel of an arrow.

2.2.3. Definition. *Let* $\mathbb{C}$ *be a finitely complete pointed category. If* $f\colon A \to B$ *is an arrow in* $\mathbb{C}$*, the* **kernel** *of* f*, denoted by* $\mathsf{ker}\,(f)$*, is the equaliser of* f *and* $0_{A,B}$ *:*

$$K[f] \xrightarrow{\mathsf{ker}\,(f)} A \underset{0_{A,B}}{\overset{f}{\rightrightarrows}} B.$$

Equivalently, one can define the kernel of f *as the arrow* $\mathsf{ker}\,(f)\colon K[f] \to A$ *in the pullback*

$$\begin{array}{ccc} K[f] & \xrightarrow{\mathsf{ker}\,(f)} & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & B \end{array}$$

2.2.4. Exercise. Check that the notion of kernel just defined extends the usual one in the categories Grp, Ab and $\mathsf{Grp}(\mathsf{Top})$.

The dual notion is the following:

2.2.5. Definition. *Let* $\mathbb{C}$ *be a finitely cocomplete pointed category. If* $f\colon A \to B$*, one defines the* **cokernel** *of* f*, denoted by* $\mathsf{coker}(f)$*, as the coequaliser of* f *and* $0_{A,B}$*:*

$$A \underset{0_{A,B}}{\overset{f}{\rightrightarrows}} B \xrightarrow{\mathsf{coker}(f)} \mathsf{Cok(f)}.$$

Equivalently, one can define the cokernel of f *as the arrow* $\mathsf{coker}(f)\colon B \to \mathsf{Cok(f)}$ *in the pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow{\scriptstyle \mathsf{coker}(f)} \\ 0 & \longrightarrow & \mathsf{Cok(f)}. \end{array}$$

2.2.6. Definition. *A* **normal epimorphism** *is an arrow in a pointed category $\mathbb{C}$ which is the cokernel of an arrow in $\mathbb{C}$.*

For instance, in the category Grp any epimorphism is normal: if $f\colon A \to B$ is a surjective homomorphism, then f is the cokernel of the canonical inclusion $K[f] \to A$.
In general, one has the implication

$$\textit{normal epimorphism} \;\Rightarrow \textit{regular epimorphism.}$$

The notion of **normal monomorphism** is defined dually. Normal monomorphisms in the category Grp correspond to normal subgroups, so that monomorphisms are not normal in Grp, in general. On the contrary monomorphisms are always normal in the categories Ab and R-Mod, of course.

2.2.7. Proposition. *Let $\mathbb{C}$ be an additive category. Consider the diagram*

$$K[f] \overset{k}{\rightarrowtail} X \underset{s}{\overset{f}{\rightleftarrows}} Y$$

where f is a split epimorphism, and k its kernel. Then $X \cong K[f] \bigoplus Y$.

Proof. By assumption there is the biproduct

$$K[f] \underset{i_1}{\overset{p_1}{\leftrightarrows}} K[f] \bigoplus Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y$$

and, by Theorem 2.1.7, $(K[f] \bigoplus Y, p_1, p_2)$ is a product and $(K[f] \bigoplus Y, i_1, i_2)$ a coproduct.
Let $\begin{pmatrix} k \\ s \end{pmatrix} : K[f] \bigoplus Y \to X$ be the unique morphism such that $\begin{pmatrix} k \\ s \end{pmatrix} \circ i_1 = k$ and $\begin{pmatrix} k \\ s \end{pmatrix} \circ i_2 = s$. We would like to show that $\begin{pmatrix} k \\ s \end{pmatrix}$ is an isomorphism.

On the one hand one has the equalities

$$f \circ (1_X - s \circ f) = f - f \circ s \circ f = f - f = 0$$

and, since $k = \mathsf{ker}\,(f)$, there is then a factorisation $g : X \to K[f]$ such that $k \circ g = 1_X - s \circ f$.
Let us then write $(g, f) : X \to K[f] \bigoplus Y$ for the unique morphism such that $p_1 \circ (g, f) = g$ and $p_2 \circ (g, f) = f$.
To show that $g \circ k = 1_{K[f]}$ we observe that

$$(k \circ g) \circ k = (1_X - s \circ f) \circ k = k - s \circ f \circ k = k - 0 = k$$

and that k is a monomorphism, hence $g \circ k = 1_{K[f]}$.
We then observe that

$$p_1 \circ (g, f) \circ \begin{pmatrix} k \\ s \end{pmatrix} \circ i_1 = g \circ k = 1_{K[f]} = p_1 \circ i_1$$

and

$$p_2 \circ (g, f) \circ \begin{pmatrix} k \\ s \end{pmatrix} \circ i_1 = f \circ k = 0_{K[f],Y} = p_2 \circ i_1$$

and the fact that (p_1, p_2) is jointly monomorphic yields

$$(g, f) \circ \begin{pmatrix} k \\ s \end{pmatrix} \circ i_1 = i_1 \qquad (*)$$

Let us prove that $g \circ s = 0$:

$$(k \circ g) \circ s = (1_X - s \circ f) \circ s = s - s \circ f \circ s = s - s = 0$$

and from the fact that k is a monomorphism one deduces that $g \circ s = 0$. It follows that

$$p_1 \circ (g, f) \circ \begin{pmatrix} k \\ s \end{pmatrix} \circ i_2 = g \circ s = 0 = p_1 \circ i_2$$

and

$$p_2 \circ (g, f) \circ \begin{pmatrix} k \\ s \end{pmatrix} \circ i_2 = f \circ s = 1_Y = p_2 \circ i_2,$$

therefore

$$(g, f) \circ \begin{pmatrix} k \\ s \end{pmatrix} \circ i_2 = i_2 \qquad (**)$$

since (p_1, p_2) is jointly monomorphic.
The pair (i_1, i_2) is jointly epimorphic, and $(*) + (**)$ give

$$(g, f) \circ \begin{pmatrix} k \\ s \end{pmatrix} = 1_{K[f] \bigoplus Y}.$$

Finally,

$$\begin{aligned}
\begin{pmatrix} k \\ s \end{pmatrix} \circ (g, f) &= \begin{pmatrix} k \\ s \end{pmatrix} \circ 1_{K[f] \bigoplus Y} \circ (g, f) \\
&= \begin{pmatrix} k \\ s \end{pmatrix} \circ (i_1 \circ p_1 + i_2 \circ p_2) \circ (g, f) \\
&= \begin{pmatrix} k \\ s \end{pmatrix} \circ i_1 \circ p_1 \circ (g, f) + \begin{pmatrix} k \\ s \end{pmatrix} \circ i_2 \circ p_2 \circ (g, f) \\
&= k \circ g + s \circ f \\
&= 1_X - s \circ f + s \circ f \\
&= 1_X
\end{aligned}$$

showing that there is an isomorphism $X \cong K[f] \bigoplus Y$.

□

The additive structure on a pointed category with biproducts is essentially unique:

2.2.8. Proposition. *Let $\mathbb{C}$ be a pointed category with biproducts. Then two additive structures on $\mathbb{C}$ are isomorphic.*

Proof. The proof consists in 3 steps.

- Step 1: one shows that $p_1 - p_2 : X \bigoplus X \to X$ is determined by the "limit-colimit" structure of the biproduct $X \bigoplus X$.

  Define $\sigma_X = p_1 - p_2 \colon X \bigoplus X \to X$, and let us show that σ_X is the cokernel of the diagonal $\Delta_X = (1_X, 1_X) : X \to X \bigoplus X$. One has:

  $$\sigma_X \circ \Delta_X = (p_1 - p_2) \circ \Delta_X = p_1 \circ \Delta_X - p_2 \circ \Delta_X = 1_X - 1_X = 0.$$

  Universal property: let $f : X \bigoplus X \to Y$ be such that $f \circ \Delta_X = 0$. Consider the biproduct

  $$X \underset{i_1}{\overset{p_1}{\rightleftarrows}} X \bigoplus X \underset{i_2}{\overset{p_2}{\rightleftarrows}} X,$$

  and observe that Δ_X can be expressed in terms of i_1 and i_2:

  $$\begin{aligned} \Delta_X &= 1_{X \bigoplus X} \circ \Delta_X \\ &= (i_1 \circ p_1 + i_2 \circ p_2) \circ \Delta_X \\ &= i_1 \circ p_1 \circ \Delta_X + i_2 \circ p_2 \circ \Delta_X \\ &= i_1 \circ 1_X + i_2 \circ 1_X \\ &= i_1 + i_2. \end{aligned}$$

  It follows that $f \circ \Delta_X = 0 = f \circ (i_1 + i_2) = f \circ i_1 + f \circ i_2$ so that $f \circ i_1 = -f \circ i_2$.
  Define $\varphi = f \circ i_1 : X \to Y$, and one has the equalities

  $$\begin{aligned} \varphi \circ \sigma_X &= f \circ i_1 \circ (p_1 - p_2) \\ &= f \circ i_1 \circ p_1 - f \circ i_1 \circ p_2 \\ &= f \circ (i_1 \circ p_1 + i_2 \circ p_2) \\ &= f. \end{aligned}$$

  The arrow φ is the unique arrow from X to Y such that $\varphi \circ \sigma_X = f$. Indeed, assume that $h : X \to Y$ is such that $h \circ \sigma_X = f$. Then

  $$h = h - 0 = h \circ 1_X - h \circ 0 = h \circ p_1 \circ i_1 - h \circ p_2 \circ i_1 = h \circ (p_1 - p_2) \circ i_1 = h \circ \sigma_X \circ i_1 = f \circ i_1,$$

  therefore $\sigma_X = \mathsf{coker}(\Delta_X)$.

- Step 2: let us show that, for any graph $Y \underset{g}{\overset{f}{\rightrightarrows}} X$, the difference f-g is determined by the "limit-colimit" structure of the biproduct. One has

  $$(p_1 - p_2) \circ (f, g) = p_1 \circ (f, g) - p_2 \circ (f, g) = f - g$$

  and since both $p_1 - p_2$ and $(f, g) \colon Y \to X \times X$ are determined by the "limit-colimit" structure of the biproduct, then this is also the case for f-g.

- Step 3: one observes that $f + g = f - (0 - g)$, and this completes the proof.

□

The following notion plays a fundamental role in homological algebra:

2.2.9. Definition. *A category* $\mathbb{C}$ *is* **abelian** *if*

1. $\mathbb{C}$ *is pointed;*

2. $\mathbb{C}$ *has binary products and binary coproducts;*

3. *any arrow in* $\mathbb{C}$ *has a kernel and a cokernel;*

4. *in* $\mathbb{C}$ *any monomorphism is a kernel and any epimorphism a cokernel.*

We conclude with notes stating a well known result due to M. Tierney, clarifying the relationship between exact categories and abelian categories (see the second chapter in [5], or [2], for an elementary introduction to the notion of abelian category):

2.2.10. Theorem. *A category is abelian if and only if it is exact and additive.*

The interested reader will find a detailed proof of this theorem, that uses several useful notions from modern categorical algebra, in the article [3].

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