A factorisation system \((E, M)\) in a category \(C\) is a pair of classes of arrows in \(C\) s.t.:

1) \(E, M\) contain all identities
2) \(E, M\) are stable by composition with isos
3) \(E\) and \(M\) are orthogonal: for any commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{e \in E} & B \\
\downarrow u & & \downarrow v \\
C & \xleftarrow{m \in M} & D
\end{array}
\]

such that \(M \cdot q = v\), \(q \cdot e = u\)

4) any arrow \(A \xrightarrow{f} B\) has a factorisation

\[
\begin{array}{ccc}
A & \xrightarrow{e \in E} & C \\
\downarrow & & \downarrow \\
C & \xleftarrow{m \in M} & D
\end{array}
\]

\(f = m \cdot e\), with \(e \in E\) and \(m \in M\)

Remark: The factorisation in 4) is unique.
The class $\mathcal{M}$ is always stable under pullbacks:

\[
\begin{array}{ccc}
& & B \\
& g \downarrow & \downarrow m \in \mathcal{M} \\
A & \xrightarrow{f} & C \\
\downarrow e & & \downarrow \quad g \\
B & \xrightarrow{m \in \mathcal{M}} & C \\
\end{array}
\]

**Lemma 15**

Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system in $\mathcal{E}$. Then:

1) $f \in \mathcal{E}$ (i.e., $f$ is orthogonal to all $m \in \mathcal{M}$)

2) $f \in \mathcal{M}$ (i.e., $f$ is orthogonal to all $e \in \mathcal{E}$)

**Proof**

1) Assume that $f$ is orthogonal to all $m \in \mathcal{M}$, and consider its factorisation $A \xrightarrow{f} B$, where

\[
A \xrightarrow{f} C \\
\downarrow e \quad \downarrow l_c \\
B \xrightarrow{m \in \mathcal{M}} C \\
\]

By considering the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{e \in \mathcal{E}} & B \\
\downarrow e \quad \downarrow l_c & & \downarrow m \in \mathcal{M} \\
B & \xrightarrow{m \in \mathcal{M}} & C \\
\end{array}
\]

we see that $g \cdot m = 1_B$.

Since $\mathcal{E}$ is stable under composition with $1_A$, we conclude that $f = m \cdot e \in \mathcal{E}$.
Examples

1) **Ab**

Any torsion theory \((\mathcal{T}, \mathcal{F})\) in \(\text{Ab}\) induces a 
**MONOTONE-LIGHT FACTORIZATION SYSTEM**.

Indeed, \((N)\) holds since only sums is normal.

One can show that \(\mathcal{F}\) contains all free groups (whenever \((\mathcal{T}, \mathcal{F}) \neq \text{(Ab, 0)})\).

Indeed, if there is an \(F_i \in \mathcal{F}\) one has a free group in \(\mathcal{T}\):

\[
0 \rightarrow T(F_i) \rightarrow F_i \rightarrow \text{Ker}(F_i) \rightarrow 0
\]

\(T(F_i) \in \mathcal{F}\), since \(F_i \in \mathcal{F}\!\!\)!

But \(\mathcal{T}\) is stable in \(\text{Ab}\) under coproducts

\[\Rightarrow \mathcal{T} = \text{Ab} \quad \text{(since it would contain all free groups)}\]

a contradiction! For any \(A \in \text{Ab}\), \(\exists F \rightarrow PA\), with \(F \in \mathcal{F}\).

**Conclusion:**

\[\mathcal{E} = \{ f: A \rightarrow B \text{ surj. hom.} \mid \text{Ker}(f) \in \mathcal{T}\} = \mathcal{E}^1\]

\[\mathcal{M}^* = \{ f: A \rightarrow B \mid \text{Ker}(f) \in \mathcal{F}\} \]

\((\mathcal{E}^1, \mathcal{M}^*)\) is a **MONOTONE-LIGHT FACTORIZATION SYSTEM**.
Any torsion theory \((\mathcal{T}, \mathcal{F})\) in \(\text{GRP}\) induces a **monotone-light factorisation system**.

\((\mathcal{N})\) holds because any radical

\[ T(G) \rightarrow G \]

in \(\text{GRP}\) is "fully invariant", so that for any normal mono \(K \rightarrow A\), the composite

\[ T(K) \rightarrow K \rightarrow A \]

is a normal subgroup in \(A\).

Since any subgroup of a free group is free (**Nielsen-Schreier Theorem**), the same argument as for \(\text{Ab}\) shows that any torsion-free subcategory \(\mathcal{F}\) contains all free groups.

\[ \forall A \in \text{GRP}, \exists \mathcal{F} \xrightarrow{P} \text{A with F} \in \mathcal{F} \]

\[ \mathcal{E}^\prime = \{ f: A \rightarrow B \text{ surj. hom.} \mid K[f] \text{ in Perf} \} \]

\[ \mathcal{M}^* = \{ f: A \rightarrow B \mid K[f] \text{ in HypoAb} \} \]

is a monotone-light factorisation system in \(\text{GRP}\).
3) The torsion theory \((\text{GRP(Cov)}, \text{GRP(TotDis)})\) in the category \(\text{GRP(Haus)}\) of Hausdorff groups satisfies monocyly \((N)\).

Moreover any Hausdorff group is a quotient of a totally disconnected group:

\[
\forall H \in \text{GRP(Haus)} \quad \exists D \in \text{GRP(TotDis)}
\]

with \(\Delta \twoheadrightarrow H\) a regular epimorphism (Arkhangelskii, 1981).

One then gets the factorization

\[
\begin{array}{c}
K \xrightarrow{\pi} K/\Gamma_0(K) \in \text{GRP(TotDis)} \\
\downarrow \quad \downarrow \\
\Gamma_0(K) \xrightarrow{\pi} A \xrightarrow{f} A/\Gamma_0(K) \\
\downarrow \quad \downarrow \\
f \downarrow \quad \downarrow \\
B \leftarrow m \in M^*
\end{array}
\]

\[
\mathcal{E}' = \left\{ f: A \to B \text{ open surj. hom} \mid K[f] \in \text{GRP(Cov)} \right\}
\]

\[
M^* = \left\{ f: A \to B \mid K[f] \in \text{GRP(TotDis)} \right\}
\]

This is a monotone-light localization system!
4) \textbf{CRNG}

The torsion theory \((\text{Nil}_{\text{CRNG}}, \text{Red}_{\text{CRNG}})\)
in the category \text{CRNG} of commutative rings
is hereditary: for any monomorphism \(R \xrightarrow{m} N\)
where \(N \in \text{Nil}_{\text{CRNG}}\), then \(R \in \text{Nil}_{\text{CRNG}}\).

This implies that condition \((N)\) holds.

Since free commutative rings are reduced,
\(\forall R \in \text{CRNG} \text{ there is } F \xrightarrow{p} D R\),
with \(F \in \text{Red}_{\text{CRNG}}\) and \(p\) a surjective homomorphism.

It follows that
\[
\mathcal{E}^1 = \{ f : A \to B \text{ surj. hom. } | K[f] \in \text{Nil}_{\text{CRNG}} \}
\]
\[
\mathcal{M}^* = \{ f : A \to B \mid K[f] \in \text{Red}_{\text{CRNG}} \}
\]
is a monotone-light factorization system
in \text{CRNG}.
The category $\text{M}^* = \{ \text{light maps} \}$ is itself reflective in $\text{Arr}(E)$:

\[ \text{M}^* \hookrightarrow \text{Arr}(E) \]

the reflection of an arrow $f: A \to B$ is given by

\[
\begin{array}{ccc}
A & \xrightarrow{e \in E'} & I \\
\downarrow f & & \downarrow \\
B & \xrightarrow{\text{M}^* \in \text{M}^*} & \text{M}^*
\end{array}
\]

This yields a new torsion theory in $\text{Arr}(E)$

\[ (\mathcal{T}_1, \mathcal{F}_1) \]

where $\mathcal{T}_1 = \{ T \to 0 \mid T \in \mathcal{T} \}$

$\mathcal{F}_1 = \{ A \xrightarrow{\phi} B \mid \text{Ker} \phi \in \mathcal{F} \}$

This torsion theory satisfies again the assumption of the theorem, yielding a chain of monotone-light factorisation systems in the categories $\text{Arr}^*(E)$, $\forall n \geq 1$. 
REFERENCES

1) A. Carboni, G. Janelidze, G.M. Kelly, R. Pare, 
ON LOCALIZATION AND STABILIZATION OF FACTORIZATION SYSTEMS, 

2) C. Cassidy, M. Hébert, G.M. Kelly, 

3) T. Everaert, M. Gran 
Monotone-light factorisation systems and torsion theories, 