Semi-abelian categories, semi-localisations and torsion theories

Introduction

1950  S. Mac Lane, Duality for Groups, Bull. Amer. Math. Society

1955  D. Buchsbaum, Exact Categories and Duality, Trans. Amer. Math. Society

1957  A. Grothendieck, Sur quelques points d'algèbre homologique, Tohoku Math. J.

The notion of abelian category has become very important in homological algebra.

What can be said about the structural properties of the non-abelian category \( \text{GRP} \)?

Would it be possible to find a 4th morphism in:

\[ \text{Ab} : \text{abelian category} \rightarrow \text{GRP} : ? \]

Aim: Find an "axiomatic context" for

- isomorphism theorems
- non-abelian homological algebra
- radical and torsion theories
- commutator theory
Several proposals of "Non-Abelian" contexts:

1954  Amitsur  \{  \}  Radical Theory

1959  Kurosh  \{  \}  Non-Abelian Homology

1956  Higgins

1961  Frölich  \{  \}  Isomorphism Theorems

1970  Gerstenhaber

1971  Wyler

1968  Huq  \{  \}  Commutator Theory

In 2001 the notion of semi-Abelian category was introduced by G. Janelidze, L. Márki and W. Tholen.

Terminology:

\( \mathcal{C} \) is Abelian \( \iff \) \( \mathcal{C} \) is semi-Abelian

\( \mathcal{C}^{\text{op}} \) is semi-Abelian
OUTLINE

I) REGULAR AND HOMOLOGICAL CATEGORIES
   DEFINITIONS, EXAMPLES AND PROPERTIES

II) SEMI-ABELIAN CATEGORIES
   - DEFINITION, EXAMPLES, RELATIONSHIP WITH ABELIAN CATEGORIES
   - TORSION THEORIES, EXAMPLES, PROPERTIES
   - REFLECTIVE SUBCATEGORIES
   - CLOSURE OPERATORS

III) SEMI-LOCALISATIONS
   - FACTORIZATION SYSTEMS, SEMI-LEFT-EXACT REFLECTORS
   - ABSTRACT CHARACTERISATION OF SEMI-LOCALISATIONS
   - CATEGORICAL GALOIS THEORY
Aim of this "mini-course":

1) Explain what a semi-abelian category is, and study some of its basic properties.

2) Introduce the notion of torsion theories in the semi-abelian context, and examine some new non-abelian examples.

3) Relate torsion theories to:
   - Semi-left-exact reflections
   - Closure operators
   - Factorisation systems
A finitely complete category $\mathcal{C}$ is **regular** if

1) any arrow $f: A \to B$ has a factorization

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow \Phi \downarrow \downarrow \\
I 
\end{array}
\]

where $\Phi$ is a **regular epi** and $i$ is a **monomorphism**.

2) these regular epi-mono factorizations are pullback-stable:
**Examples**

**Set**

The regular epi/mono factorization of a map $\xymatrix{ x \ar[r]^f & y }$ is given by

$\xymatrix{ x \ar[r]^f \ar[r]^g & y \ar[r]_{\pi} & \text{Im}(f) }$

where $\text{Im}(f) = \{ f(x) \mid x \in x \}$.

In **SET**: regular epis $\equiv$ surjective maps.

Surjective maps are easily seen to be pullback stable.

**GRP** category of groups

Ab " " abelian groups

Mon " " monoids

Lattics " " lattices

**GRP (Top)** " " topological groups

**GRP (Comp)** " " connected groups

**sSET, sGRP**

**Counter-Examples**

Top category of topological spaces \{ not regular \}

CAT category of categories
The composite $S \circ R$ of $S$ and $R$ is given by the regular image of the unique arrow:

$$R \times S \xrightarrow{\langle r_1 \cdot \pi_1, r_2 \cdot \pi_2 \rangle} X \times Z$$

Accordingly:

$$S \circ R$$

is a relation from $X$ to $Z$.

**SET**

$$R \times S = \left\{ (x, y) \in R \times (y, z) \in S \mid R_2(x, y) = 1, (y, z) \in S \right\}$$

$$= \left\{ (x, y, z) \in X \times Y \times Z \mid (x, y) \in R, (y, z) \in S \right\}$$

$$S \circ R = \left\{ (x, z) \in X \times Z \mid \exists (a, b, c) \in R \times S \text{ with } a = x, c = z \right\}$$

$$= \left\{ (x, z) \in X \times Z \mid \exists b \in Y \text{ with } (x, b) \in R, (b, z) \in S \right\}$$
What is a **relation** on a group \((X, _, 1)\) in the category **GRP**?

It is a diagram:

\[
\begin{array}{c}
R \xrightarrow{r_1} X \xleftarrow{r_2} X \\
\downarrow \quad \downarrow \quad \downarrow \\
\downarrow r_1 \quad \downarrow r_2 \quad \downarrow r_2 \\
X \quad \quad \quad X
\end{array}
\]

determining a relation on the underlying set of \(X\), with the property that \(R\) is a **subgroup** of \(X \times X\).

- \((1, 1) \in R\)

- If \((x, y) \in R\) then \((x^{-1}, y^{-1}) \in R\)

- If \((x, y) \in R\) then \((ux, vy) \in R\)

**Exercise**

Let \(R \subseteq X \times X\) be a **reflexive relation** in **GRP**, to that \(R_1 \cdot e = 1_X = R_2 \cdot e\).

Prove that \(R\) is then **symmetric** and **transitive**, thus an **equivalence relation**.
**DEFINITION** (Carboni, Lambek, Pedicchio, 1991)

A category $\mathcal{C}$ with finite limits is a **Mal’tsev category** if any reflexive relation in $\mathcal{C}$ is an equivalence relation.

**Examples**

- any **abelian category**
- $\text{GRP}, \text{RNG}, \text{LIE}_k, \text{GRP}(\text{TOP}), \text{GRP}(\text{COMP}), \text{BOOLE}$

**Remark**

The categories $\text{SET}, \text{SSET}, \text{TOP}, \text{MON}, \text{LATTICES}$ are not Mal’tsev categories.

**Theorem** (Carboni, Lambek, Pedicchio)

For a regular category $\mathcal{C}$ the following are equivalent:

1) $\mathcal{C}$ is a Mal’tsev category

2) $R \circ S = S \circ R$ for any $R \in \text{Eq}(\mathcal{C})$, $S \in \text{Eq}(\mathcal{C})$

3) any reflexive relation in $\mathcal{C}$ is **symmetric**

4) any reflexive relation in $\mathcal{C}$ is **transitive**
Given an arrow $f: A \to B$ in $\mathcal{C}$ its **KERNEL PAIR** is the relation $(\text{Eq}(f), p_1, p_2)$ in the pullback:

\[
\begin{array}{c}
\text{Eq}(f) \xrightarrow{p_2} A \\
p_1 \downarrow \quad \downarrow f \\
A \quad \xrightarrow{f} B
\end{array}
\]

**SET**

\[
\text{Eq}(f) = \{ (a_1, a_2) \in A \times A \mid f(a_1) = f(a_2) \}
\]

is the **EQUIVALENCE RELATION** on $A$ obtained by identifying two elements in $A$ when $f(a_1) = f(a_2)$.

**DEFINITION**

A regular category $\mathcal{C}$ is **EXACT** if any equivalence relation in $\mathcal{C}$ is **EFFECTIVE**, i.e. a **KERNEL PAIR**:

\[
R \xrightarrow{\pi_1} A \xleftarrow{\pi_2} R \xrightarrow{\pi_1} A \xrightarrow{f} B
\]

such that $R \cong \text{Eq}(f)$

**EXAMPLES**

GRP, Rng, Mon, LATTICES, BOOLE, ANY ABELIAN CATEGORY
**Homological Categories**

**Definition** (Borceux-Bourn, 2004)

A **regular category** is **homological** if

1) $\mathcal{C}$ is **pointed**: $0$

2) $\mathcal{C}$ is **protomodular**: the split short exact lemma holds in $\mathcal{C}$, i.e., given

\[
\begin{array}{ccccccccc}
0 & \rightarrow & K & \rightarrow & A & \rightarrow & B & \rightarrow & 0 \\
\downarrow u & & \downarrow v & & \downarrow \epsilon & & \downarrow w & & \\
0 & \rightarrow & K' & \rightarrow & A' & \rightarrow & B' & \rightarrow & 0 \\
\end{array}
\]

$u, w$ **isos** $\Rightarrow v$ **iso**

**Examples**

- any **abelian category**

- $\text{Grp}, \text{Rng}, \text{Lie}_k, \text{Grp}(\text{Top}), \text{Grp}(\text{Comp})$

- $\mathbb{X}$-Mod

**Objects** $\xrightarrow{\alpha} B \\
\alpha(a) = b\alpha(a)b^{-1}

**Arrows** $\xrightarrow{\alpha}$

\[
\begin{array}{ccccccccc}
A & \xrightarrow{\alpha} & B & \\
\downarrow \iota & & \downarrow \iota_0 & & \downarrow \iota_1 & & \downarrow \iota_1 & \\
A' & \xrightarrow{\iota} & B' \\
\end{array}
\]

$\iota, \iota_1$ **isos** $\Rightarrow \iota_0$ **iso**
\( X-\text{Mod} \cong \text{GRPD}(GRP) \)

\[
\begin{array}{c}
\text{objects} & \text{INTERNAL GROUPOIDS} \\
\text{in groups} & \text{IN GROUPS} \\
\text{arrows} & \text{INTERNAL FUNCTORS} \\
(\xi_0, \xi_1) : X \to Y \\
\end{array}
\]

\[
\begin{array}{c}
X_1 \times X_1 \xrightarrow{\text{max}} X_1 \xrightarrow{e} X_0 \\
X_0 \xrightarrow{i_1} X_1 \xrightarrow{e} X_0 \\
X_1 \xrightarrow{i_1} X_1 \xleftarrow{\xi_0} X_0 \\
X_0 \xleftarrow{\xi_0} X_0 \\
\end{array}
\]

The equivalence is monoidal by the **normalisation functor**:

given

\[
\begin{array}{c}
X : X_1 \times X_1 \xrightarrow{\Pi_1} X_1 \xrightarrow{e} X_0 \\
\end{array}
\]

one associates the crossed module

\[
K[\xi] \xrightarrow{K\xi} X_1 \xleftarrow{c} X_0
\]

where

\[
x_k = e(x) \cdot K \cdot e(x)^{-1} \quad \text{for } K \in K[\xi], \ x \in X_0
\]

**FACT:** if \( C \) is homological,

then \( \text{GRPD}(C) \) is homological!

**REMARK**

Any homological category is a Mal'tsev category.

(See Borceux - Bourn, 2004)
Proposition 1 (Bourn, 1991)

\( \mathcal{C} \) is finitely complete and pointed. TFCAE:

1) the Split Short Five Lemma holds in \( \mathcal{C} \)
2) given a commutative diagram

\[
\begin{align*}
A & \longrightarrow C & \longrightarrow & \mathcal{E} \\
\downarrow & & & \downarrow \\
B & \longrightarrow & D & \longrightarrow & F
\end{align*}
\]

where \( P \cdot \lambda = 1_D \), 1. is a pullback \( \Rightarrow \) 2. is a pullback \( \Rightarrow 1. + 2. \) is a pullback

Proof

2) \( \Rightarrow 1. \) Given a commutative diagram

\[
\begin{align*}
0 & \longrightarrow K & \longrightarrow A & \longrightarrow & B & \longrightarrow & 0 \\
\downarrow & & & \downarrow & & & \downarrow \\
0 & \longrightarrow K' & \longrightarrow A' & \longrightarrow & B' & \longrightarrow & 0
\end{align*}
\]

where \( \pi \) and \( \varpi \) are isos, one forms the cube

\[
\begin{array}{c}
K \\
\downarrow \pi \\
\downarrow \varpi \\
O \\
\downarrow \sim \\
K' \\
\downarrow \varpi' \\
\downarrow \varpi \\
O \\
\downarrow \sim \\
A' \\
\downarrow \varpi' \\
\downarrow \varpi \\
B' \\
\downarrow \sim \\
B
\end{array}
\]

\( \Rightarrow \)

\[
\begin{align*}
K & \longrightarrow A & \longrightarrow & A' \\
\downarrow & & & \downarrow \\
0 & \longrightarrow B & \longrightarrow & B'
\end{align*}
\]

1. is a pullback \( \Rightarrow \) 2. is a pullback \( \Rightarrow 1. + 2. \) is a pullback
1) Assume that the Split Short Five Lemma holds, and consider the diagram

\[
\begin{array}{ccc}
A & \rightarrow & C & \rightarrow & E \\
\downarrow & & \downarrow & & \downarrow \\
B & \rightarrow & D & \rightarrow & F
\end{array}
\]

where \(1. + 2.\) are pullback.

One then forms the diagram

\[
\begin{array}{ccc}
K[P] & \rightarrow & A & \rightarrow & C & \rightarrow & E \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
K[P] & \rightarrow & P' & \rightarrow & P & \rightarrow & F
\end{array}
\]

where \(\alpha'\) is an iso (by assumption) \(\Rightarrow\) \(\alpha''\) is an iso.

One then gets the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & K[P] & \rightarrow & C & \rightarrow & D & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & K[P'] & \rightarrow & P & \rightarrow & D & \rightarrow & 0
\end{array}
\]

where \(\alpha''\) is an iso \(\Rightarrow\) \(\alpha\) is an iso.

\(\Rightarrow\) 2. is a pullback.
**Corollary 2 (Bourn, 1991)**

In a pointed protomodular category $\mathcal{E}$

\[ \text{[} f: x \to y \text{ is a MONO} \text{]} \iff [ K(f) \cong 0 ] \]

**Proof**

\( \Rightarrow \) If \( f: x \to y \) is a MONO, then the square

\[
\begin{array}{ccc}
0 & \to & x \\
\downarrow & & \downarrow f \\
0 & \to & y
\end{array}
\]

is a pullback, and \( K(f) \cong 0 \).

\( \Leftarrow \) Conversely, when \( K(f) \cong 0 \) form the diagram

\[
\begin{array}{ccc}
K(f) \cong 0 & \to & x \\
\downarrow & \cong & \downarrow f \\
0 & \to & y
\end{array}
\]

where 1. and 1 + 2. are pullbacks.

From protomodularity it follows that 2.

is a pullback, and \( f: x \to x \) is a Mono.

\( \square \)
**Proposition 3**

Let \( \mathcal{C} \) be a **regular pointed category**. Then \( \mathcal{C} \) is **homological**.

For any commutative diagram

\[
\begin{array}{ccc}
A & \rightarrow & C \\
\downarrow & & \downarrow P \\
B & \rightarrow & D
\end{array}
\]

\[
\begin{array}{ccc}
& & \rightarrow \\
P & \rightarrow & \rightarrow \\
& \downarrow & \downarrow \\
& D & \rightarrow F
\end{array}
\]

where \( P \) is a **regular epi**

1. **is a pullback**
2. \( 1 + 2 \). **is a pullback**

**Proof**

**Exercise** (hint: use the Barr-Kock thm)
**DEFINITION**

A pair of arrows $A \xrightarrow{\alpha} C \xleftarrow{\beta} B$ is **Jointly Extremal Epimorphic** if, for every **Mono** $J \xrightarrow{j} C$ such that

![Diagram](https://via.placeholder.com/150)

then $j$ is an ISO.

**EXERCISE**

In a category with equalisers

$[\text{Jointly Extremal Epimorphic}] \Rightarrow [\text{Jointly Epimorphic}]$

where **jointly Epimorphic** means

$A \xrightarrow{\alpha} C \xleftarrow{\beta} B$

$u \downarrow \downarrow v$

$\downarrow D$

$u \cdot \alpha = v \cdot \alpha$

$u \cdot \beta = v \cdot \beta$

$\implies u = v$
Consider a pullback

\[
\begin{array}{ccc}
E \times_A B & \xrightarrow{P_2} & A \\
\downarrow P_1 & & \uparrow P \\
E & \xrightarrow{\delta} & B
\end{array}
\]

along a split epimorphism \( P : A \to B \) in a homological category. Then the pair \((P_2, \delta)\) is jointly extremal epimorphic.

**Proof**

1st step: The property holds for the special case \( E = 0 \):

\[
\begin{array}{ccc}
\text{Ker}(P) & \xrightarrow{\delta} & A \\
\downarrow & & \downarrow f \\
0 & \xrightarrow{\delta} & B
\end{array}
\]

Assume that \((\text{Ker}(P), \delta)\) factor through a mono \( J \xrightarrow{\delta} A\). One can then form the diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{a} & J & \xrightarrow{b} & B & \xrightarrow{0} \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \xrightarrow{\delta} & A & \xrightarrow{\delta} & B & \xrightarrow{0}
\end{array}
\]

The split short five lemma implies that \( \delta \) is an iso.
\textbf{2nd Step} Consider then any pullback along a split epimorphism:

\[
\begin{array}{ccc}
E \times A & \overset{p_2}{\rightarrow} & A \\
\downarrow & & \downarrow \gamma \downarrow \beta \downarrow \\
E & \overset{\iota}{\rightarrow} & B
\end{array}
\]

and assume that \( p_2 \) and \( \iota \) factor through \( \gamma \).

Complete the diagram by taking the kernel of \( p_1 \):

\[
\begin{array}{ccc}
K(p_1) & \overset{\text{Ker}(p_1)}{\rightarrow} & E \times A \\
\downarrow & & \downarrow p_1 \\
0 & \rightarrow & E \rightarrow B
\end{array}
\]

Clearly \( K(p_1) \cong K(p) \) and

\[
\begin{array}{ccc}
K(p) & \overset{p_2 \cdot \text{Ker}(p_1)}{\rightarrow} & A \\
\downarrow & & \downarrow \beta \downarrow \gamma \downarrow \\
0 & \rightarrow & B
\end{array}
\]

factor through \( \gamma \rightarrow A \). Since the pair \( (p_2 \cdot \text{Ker}(p_1), \beta) \) is \textit{jointly extremal epimorphic} one concludes that \( \gamma \rightarrow A \) is an iso, as desired. \( \Box \)

\textbf{Exercise} Show that the property used in Lemma is actually equivalent to protomodularity.
Given a split epi $x \xrightarrow{\theta} y$ in a homological category, when we consider its kernel $k : k \rightarrow y$,

$$
\begin{array}{ccc}
K & \xrightarrow{k} & X \\
\downarrow & & \downarrow \\
0 & \rightarrow & Y
\end{array}
$$

the pair $(k, \theta)$ is jointly extremal epimorphic.

This means that $x$ is then the supremum of $k \rightarrow x$ and $y \rightarrow x$, as subobjects of $x$.

This shows a difference with the additive context.

Indeed, in the additive context one has an isomorphism:

$$
\begin{array}{ccc}
K & \xrightarrow{k} & X \\
\downarrow & & \downarrow \\
0 & \rightarrow & Y
\end{array}
$$
In a pointed protoideal category any REGULAR EPI is a NORMAL EPI.

Proof

Let $A \xrightarrow{f} B$ be a regular epi and $K \xrightarrow{k} A$ its KERNEL. We are going to show that $f = \text{coker}(k)$.

Form the diagram

$$
\begin{array}{ccc}
K \times K & \xrightarrow{\Delta} & \text{Eq}(f) \\
\Delta \downarrow & & & \downarrow \\
K & \xrightarrow{f} & A \\
\downarrow & \downarrow & \downarrow \\
o & \xrightarrow{\alpha} & B
\end{array}
$$

and consider an arrow $A \xrightarrow{\alpha} C$ such that $\alpha \cdot K = 0$.

Then:

$$(\alpha \cdot P_1) \cdot \hat{K} = \alpha \cdot K \cdot P_1 = 0 = (\alpha \cdot P_2) \cdot \hat{K}$$

$$(\alpha \cdot P_1) \cdot \Delta = \alpha = (\alpha \cdot P_2) \cdot \Delta$$

$$(\hat{K}, \Delta) \text{ JUNCTLY EPIMORPHIC}$$

$$\Rightarrow \alpha \cdot P_1 = \alpha \cdot P_2.$$

Since the regular epi $f$ is the coequalizer of its kernel pair $\text{Eq}(f) \xrightarrow{P_1} \xrightarrow{P_2} A$, it follows that there is a unique $\gamma : B \to C$ such that $\gamma \cdot f = \alpha$.

$\square$
REMARK

In a **homological category** the notion of **short exact sequence** reduces to a regular epi:

\[
A \xrightarrow{f} B
\]

equipped with its kernel \(K(f)\):

\[
0 \longrightarrow K(f) \xrightarrow{k} A \xrightarrow{f} B \longrightarrow 0
\]

As we have just shown, any regular epi is the cokernel of its kernel!

**Proposition 5** is the categorical version of the first isomorphism theorem in **Grp**

if \(\varphi : G \rightarrow G'\) is a surjective homomorphism,

\[
0 \longrightarrow K(f) \xrightarrow{k} G \xrightarrow{\varphi} G' \xrightarrow{\varphi' \circ k} 0
\]

then
**THEOREM 6** (Bourn, 2001)

In a regular and pointed category $\mathcal{C}$, if $\mathcal{C}$ is homological, then the short five lemma holds in $\mathcal{C}$.

**Proof:**

$\Rightarrow$ Clear, any split epi is a normal epi in a homological category.

$\Leftarrow$ Apply Proposition 3 to the "cube" derived from the commutative diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & K \\ & \downarrow u & \downarrow v \\ 0 & \longrightarrow & A \\
\end{array}
\quad
\begin{array}{ccc}
& & f \\
\downarrow w & & \\
& & 0
\end{array}
\quad
\begin{array}{ccc}
0 & \longrightarrow & B \\
\downarrow w & & \\
0 & \longrightarrow & 0
\end{array}
$$

Where $u$ and $w$ are isomorphisms.

$\square$
The Short Five Lemma holds in any homological category, and is crucial to establish other "homological lemmas":

- Noether's Isomorphism Theorems
- $3 \times 3$ - Lemma
- Five Lemma
- Snake Lemma

The proofs are not too difficult, however, they are different from the ones in the Abelian case, simply because the notion of homological category is not self-adjoint!


**Exercise** In a homological category prove the Five Lemma: given

\[
\begin{array}{ccccccc}
0 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & D & \rightarrow & E & \rightarrow & 0 \\
& & u_1 & \downarrow & u_2 & \cong & u_3 & \downarrow & u_4 & \cong & u_5 & \downarrow & 0 \\
& & u_2, u_4 \text{ isos} & & & & & & & & & & \text{Epi, Ret, Mono} \\
\end{array}
\]

\[
\begin{array}{ccccccc}
0 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & D & \rightarrow & E & \rightarrow & 0 \\
& & & & & & & & & & & & \text{Epi, Ret, Mono} \\
\end{array}
\]

\[
\{ u_2, u_4 \text{ isos} \} \Rightarrow u_3 \text{ iso}
\]