CATEGORICAL GROUPS

A SPECIAL TOPIC

IN HIGHER DIMENSIONAL

CATEGORICAL ALGEBRA

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Contents

Introduction

The aim of this mini-course is to introduce categorical groups as a topic of research, interesting in and of itself, and to show that they are a higher dimensional gadget useful for studying familiar constructions and for understanding better certain classical results in (homological) algebra.
If you wish to enjoy this mini-course, please come armed with basic knowledge of monoidal categories, 2-categories, and internal categories. Sections 7.1, 7.2 and 8.1 of [1] and Sections 6.1 and 6.4 of [2] cover all the needed material. Monoidal categories and 2-categories are also covered by [5], which will be made available to the participants.

Categorical Groups

In the first lesson, I will introduce the notion of a categorical group with several examples from homological algebra, ring theory, algebraic topology and algebraic K-theory. I will also discuss the links among categorical groups, internal groupoids in groups, and group extensions.

Abelian 2-categories

The second lesson will be devoted to the theory of symmetric categorical groups, starting from strong homotopy kernels and cokernels and ending at long exact sequences of homology categorical groups, which leads to the axiomatic notion of an abelian 2-category.

Homological Algebra

In the third lesson, two applications of the higher dimensional point of view introduced in Lessons 1 and 2 will be discussed. First, I will show that strong homotopy kernels reveal the Snail Lemma, a generalization of the Snake Lemma that remains completely hidden if we look at the Snake Lemma from the classical "1-dimensional" point of view. Second, we will see that Sinh's homotopical classification of categorical groups subsumes the Mac Lane - Schreier theory of group extensions.

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Lesson I: Categorical groups

My point of view on research in mathematics:

"Va savoir pourquoi une descente vue d'en bas ressemble tellement à une montée."
Goofy, *Le super-héros est fatigué*, Disney Studio (foreign market stories), 1969.

The aim of Lesson 1 is:

1. To discover the definition of categorical group starting from a very basic example, the cokernel of a morphism between abelian groups.
2. To illustrate the notion of categorical group with some examples coming from homological algebra, ring theory, algebraic topology, and algebraic K-theory.
3. To understand the relation between categorical groups, internal groupoids in groups, crossed modules, and group extensions.

Chapter 1

The definition

If I write the equation

$$\forall\, A, B \in \mathcal{C},\ A \times B \simeq A + B$$

you immediately think to abelian groups or, more in general, to abelian categories or, more in general, to additive categories. If I write the equation

$$\forall\, f \colon A \to B \in \mathcal{C},\ \mathrm{Ker}(f) \simeq \mathrm{Coker}(f)$$

you immediately think that I'm a bit crazy: indeed, if for every arrow f one has $\mathrm{Ker}(f) \simeq \mathrm{Coker}(f)$, then every object in $\mathcal{C}$ is isomorphic to the zero object.

The problem with the "wrong equation" $\mathrm{Ker}(f) \simeq \mathrm{Coker}(f)$ is that, in some sense, the two terms live in different places. Let us look at what happens in the category Ab of abelian groups.

1.1 Example. Let $f \colon A \to B$ be an arrow in Ab. Its cokernel is the set of equivalence classes

$$\mathrm{Coker}(f) = \{[b] \mid b \in B\}$$

with $b_1 \equiv b_2$ if there exists $a \in A$ such that $b_1 + f(a) = b_2$, and operations defined on representatives, that is, $[b_1] + [b_2] = [b_1 + b_2]$ and $0 = [0]$. But what is the structure of $\mathrm{Coker}(f)$ before passing to the quotient?
We can describe a category $\underline{\mathrm{Coker}}(f)$ as follows: the objects are the elements of B, and an arrow $a \colon b_1 \to b_2$ is an element $a \in A$ such that $b_1 + f(a) = b_2$. The composition is the sum in A and the identity on an object $b \in B$ is the zero element of A. Moreover, $\underline{\mathrm{Coker}}(f)$ has a monoidal structure

$$(a \colon b_1 \to b_2) \otimes (a' \colon b_1' \to b_2') = (a + a' \colon b_1 + b_1' \to b_2 + b_2')$$

with unit object the zero element of B. Finally, each arrow in $\underline{\mathrm{Coker}}(f)$ is invertible with respect to the composition, the inverse of $a \colon b_1 \to b_2$ is $-a \colon b_2 \to b_1$, and each object of $\underline{\mathrm{Coker}}(f)$ is invertible with respect to the tensor product, the inverse of b is $-b$.

Clearly, the set of isomorphism classes of objects of $\underline{\mathrm{Coker}}(f)$ is nothing but the usual cokernel of f:

$$\pi_0(\underline{\mathrm{Coker}}(f)) = \mathrm{Coker}(f)$$

But $\underline{\mathrm{Coker}}(f)$ is not just an intermediate step to construct the usual cokernel: it contains more information than $\mathrm{Coker}(f)$. Indeed, the group of automorphisms of the unit object of $\underline{\mathrm{Coker}}(f)$ precisely is the kernel of f:

$$\pi_1(\underline{\mathrm{Coker}}(f)) = \mathrm{Ker}(f)$$

and this is the right form of the "wrong equation" I considered at the beginning.

With Example 1.1 in mind, we can give the definition of categorical group.

1.2 Definition. The 2-category $\underline{CG}$ of categorical groups.

1. The objects of $\underline{CG}$ are categorical groups (also called cat-groups, 2-groups, gr-categories, categories with a group structure). A categorical group is a monoidal category $\mathbb{G} = (\mathbb{G}, \otimes, I, a, l, r)$ such that
   (a) each arrow is an isomorphism, and
   (b) each object is weakly invertible with respect to the tensor product:
   $$\forall\, X \in \mathbb{G} \;\; \exists\, X^* \in \mathbb{G} : X \otimes X^* \simeq I \simeq X^* \otimes X$$
2. The arrows of $\underline{CG}$ are monoidal functors $F = (F, F_2, F_I) \colon \mathbb{G} \to \mathbb{H}$ with
   $$F_2^{X,Y} \colon FX \otimes FY \to F(X \otimes Y) \quad F_I \colon I \to FI$$
3. The 2-arrows of $\underline{CG}$ are monoidal natural transformation $\alpha \colon F \Rightarrow G$

$$\begin{tikzcd} FX \otimes FY \arrow[r, "F_2^{X,Y}"] \arrow[d, "\alpha_X \otimes \alpha_Y"'] & F(X \otimes Y) \arrow[d, "\alpha_{X \otimes Y}"] \\ GX \otimes GY \arrow[r, "G_2^{X,Y}"'] & G(X \otimes Y) \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "F_I"] \arrow[dr, "G_I"'] & FI \arrow[d, "\alpha_I"] \\ & GI \end{tikzcd}$$

1.3 Exercise.

1. Categorical groups look like groups.
   (a) Let $\mathbb{G}$ be a monoidal groupoid such that for any object X there exists an object X^* such that $X \otimes X^* \simeq I$. Prove that $X^* \otimes X \simeq I$ holds true for any X.
   (b) Let $F \colon \mathbb{G} \to \mathbb{H}$ be a functor between categorical groups and assume that $FX \otimes FY \simeq F(X \otimes Y)$ for every $X, Y \in \mathbb{G}$. Prove that $I \simeq FI$.
2. Categorical groups are more than groups.

(a) Let $\mathbb{G}$ be a monoidal groupoid such that for any object X there exists an object X^* and an arrow $\eta_X : I \to X \otimes X^*$. Prove that there exists a unique arrow $\varepsilon_X : X^* \otimes X \to I$ such that (omitting the associativity isomorphisms)

$$\begin{tikzcd} I \otimes X \arrow[r, "\eta_X \otimes \mathrm{id}"] & X \otimes X^* \otimes X \arrow[d, "\mathrm{id} \otimes \varepsilon_X"] \\ X \arrow[u, "l_X"] \arrow[r, "r_X"'] & X \otimes I \end{tikzcd} \qquad \begin{tikzcd} X^* \otimes I \arrow[r, "\mathrm{id} \otimes \eta_X"] & X^* \otimes X \otimes X^* \arrow[d, "\varepsilon_X \otimes \mathrm{id}"] \\ X^* \arrow[u, "r_{X^*}"] \arrow[r, "l_{X^*}"'] & I \otimes X^* \end{tikzcd}$$

The 4-tuple $(X, X^*, \eta_X, \varepsilon_X)$ is called a duality in $\mathbb{G}$.

(b) Let $F : \mathbb{G} \to \mathbb{H}$ be a functor between categorical groups equipped and let $F_2^{X,Y} : FX \otimes FY \to F(X \otimes Y)$ be a natural and coherent family of arrows. Prove that there exists a unique arrow $F_I : I \to FI$ such that

$$\begin{tikzcd} I \otimes FX \arrow[r, "F_I \otimes \mathrm{id}"] & FI \otimes FX \arrow[d, "F_2^{I,X}"] \\ FX \arrow[u, "l_{FX}"] \arrow[r, "F(l_X)"'] & F(I \otimes X) \end{tikzcd} \qquad \begin{tikzcd} FX \otimes I \arrow[r, "\mathrm{id} \otimes F_I"] & FX \otimes FI \arrow[d, "F_2^{X,I}"] \\ FX \arrow[u, "r_{FX}"] \arrow[r, "F(r_X)"'] & F(X \otimes I) \end{tikzcd}$$

3. What about monoidal natural transformations between monoidal functors between categorical groups?

4. Prove that in a categorical group the dual of an object is essentially unique: given dualities $(X, X^*, \eta_X, \varepsilon_X)$ and $(X, \widehat{X}, \alpha_X, \beta_X)$, there exists a unique $x : X^* \to \widehat{X}$ such that

$$\begin{tikzcd} X \otimes X^* \arrow[dd, "\mathrm{id} \otimes x"'] & & X^* \otimes X \arrow[dl, "\varepsilon_X"'] \arrow[dd, "x \otimes \mathrm{id}"] \\ & I \arrow[ul, "\eta_X"'] \arrow[dl, "\alpha_X"] & \\ X \otimes \widehat{X} & & \widehat{X} \otimes X \arrow[ul, "\beta_X"] \end{tikzcd}$$

5. Prove that in a categorical group $\mathbb{G}$ the choice, for every object X, of a duality $(X, X^*, \eta_X, \varepsilon_X)$ induces a “monoidal” equivalence $(-)^* : \mathbb{G} \to \mathbb{G}$. Solution: for a given arrow $f : X \to Y$, define $f^* : X^* \to Y^*$ as follows:

$$X^* \xrightarrow{\mathrm{id} \otimes \eta_Y} X^* \otimes Y \otimes Y^* \xrightarrow{\mathrm{id} \otimes f^{-1} \otimes \mathrm{id}} X^* \otimes X \otimes Y^* \xrightarrow{\varepsilon_X \otimes \mathrm{id}} Y^*$$

1.4 Definition. The 2-category $\underline{SCG}$ of symmetric categorical groups.

1. The objects of $\underline{SCG}$ are symmetric categorical groups (also called Picard categories), that is, categorical groups which are symmetric as monoidal categories. The symmetry will be denoted by $c_{X,Y} \colon X \otimes Y \to Y \otimes X$.

2. The arrows of $\underline{SCG}$ are symmetric monoidal functors, that is, monoidal functors $F \colon \mathbb{G} \to \mathbb{H}$ compatible with the symmetry:

$$\begin{tikzcd} FX \otimes FY \arrow[r, "F_2^{X,Y}"] \arrow[d, "c_{FX,FY}"'] & F(X \otimes Y) \arrow[d, "F(c_{X,Y})"] \\ FY \otimes FX \arrow[r, "F_2^{Y,X}"'] & F(Y \otimes X) \end{tikzcd}$$

3. The 2-arrows of $\underline{SCG}$ are monoidal natural transformations.

(There is also the 2-category $\underline{BCG}$ of braided categorical groups, with same arrows and 2-arrows as $\underline{SCG}$, but it is less relevant in these lessons.)

1.5 Exercise.

1. Define
$$\pi_0 \colon \underline{CG} \to Grp \qquad \pi_1 \colon \underline{CG} \to Ab$$
where Grp is the category of groups and Ab is the category of abelian groups. Prove that π_0 and π_1 are 2-functors: if $\alpha \colon F \Rightarrow G$ is a 2-arrow in $\underline{CG}$, then $\pi_0(F) = \pi_0(G)$ and $\pi_1(F) = \pi_1(G)$.
Find convenient candidates
$$[-]_0 \colon Grp \to \underline{CG} \qquad [-]_1 \colon Ab \to \underline{CG}$$
in order to have $\pi_0 \dashv [-]_0$ and $[-]_1 \dashv \pi_1$.

2. Let $F \colon \mathbb{G} \to \mathbb{H}$ be an arrow in $\underline{CG}$. Prove that

   (a) F is essentially surjective iff $\pi_0(F)$ is surjective,

   (b) F is faithful iff $\pi_1(F)$ is injective,

   (c) F is full iff $\pi_0(F)$ is injective and $\pi_1(F)$ is surjective.

   Deduce that F is an equivalence in $\underline{CG}$ iff $\pi_0(F)$ and $\pi_1(F)$ are group isomorphisms. (This last point is not completely obvious. Why?)

References for Chapter 1

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Chapter 2

Examples

I start with two strictly related examples of categorical groups arising in homological algebra.

2.1 Example. Fix two groups A and G with A abelian and consider the category $\underline{\mathrm{Ext}}(G, A)$: objects are extensions, that is, exact sequences of the form

$$0 \longrightarrow A \xrightarrow{\chi} B \xrightarrow{\sigma} G \longrightarrow 0$$

(with the group B not necessarily abelian) and arrows are group homomophisms $\beta\colon B \to B'$ such that

$$\begin{tikzcd} & B \arrow[dd, "\beta"'] \arrow[dr, "\sigma"] & \\ A \arrow[ur, "\chi"] \arrow[dr, "\chi'"'] & & G \\ & B' \arrow[ur, "\sigma'"'] & \end{tikzcd}$$

Observe that, by the Five Lemma, $\underline{\mathrm{Ext}}(G, A)$ is a groupoid. Fix now an action (or operator) $\varphi\colon G \to \mathrm{Aut}(A)$ and consider the full subcategory $\underline{\mathrm{OpExt}}(G, A, \varphi)$ of $\underline{\mathrm{Ext}}(G, A)$ of those extensions such that

$$\begin{tikzcd} A \arrow[r, "\chi"] & B \arrow[r, "\sigma"] \arrow[d, "\bar{\chi}"'] & G \arrow[dl, "\varphi"] \\ & \mathrm{Aut}(A) & \end{tikzcd}$$

where $\bar{\chi}\colon B \to \mathrm{Aut}(A)$ is the action induced by the fact that A is isomorphic to the kernel of σ. The groupoid $\underline{\mathrm{OpExt}}(G, A, \varphi)$ is in fact a symmetric categorical group: the unit object is the semi-direct product extension

$$0 \longrightarrow A \xrightarrow{i_A} A \rtimes_{\varphi} G \xrightarrow{p_G} G \longrightarrow 0$$

with operation

$$(a_1, x_1) \rtimes_\varphi (a_2, x_2) = (a_1 + x_1 \cdot a_2, x_1 x_2)$$

and the tensor product is the Baer sum

(To be inserted)

Clearly, $\pi_0(\underline{\mathrm{OpExt}}(G, A, \varphi))$ is the usual abelian group $\mathrm{OpExt}(G, A, \varphi)$ of isomorphism classes of extensions with fixed operator φ.

2.2 Example. As in Example 2.1, fix two groups A and G with A abelian and an action $\varphi \colon G \to \mathrm{Aut}(A)$. We can construct two abelian groups (with point-wise sum in A)

$$\mathrm{C}^1(G, A) = \{g \colon G \to A \mid g(1) = 0\}$$

$$\mathrm{Z}^2(G, A, \varphi) = \left\{ \begin{array}{ll} f \colon G \times G \to A \mid & f(x, 1) = 0 = f(0, y) \\ & x \cdot f(y, z) + f(x, yz) = f(x, y) + f(xy, z) \end{array} \right\}$$

and a group homomorphism

$$\delta \colon \mathrm{C}^1(G, A) \to \mathrm{Z}^2(G, A, \varphi) \ , \ \ (\delta g)(x, y) = x \cdot g(y) - g(xy) + g(x)$$

By considering the cokernel of δ as in Example 1.1, we get a symmetric categorical group $\underline{\mathrm{Coker}}(\delta)$ which should be called the second cohomology categorical group of G with coefficients in A. Moreover, there exists a symmetric monoidal functor

$$\mathcal{E} \colon \underline{\mathrm{Coker}}(\delta) \to \underline{\mathrm{OpExt}}(G, A, \varphi) \qquad (f \colon G \times G \to A) \mapsto (A \to A \rtimes_f G \to G)$$

where the operation in $A \rtimes_f G$ is the semi-direct product deformed by the factor-set f:

$$(a_1, x_1) \rtimes_f (a_2, x_2) = (a_1 + x_1 \cdot a_2 + f(x_1, x_2), x_1 x_2)$$

Theorem: the functor $\mathcal{E}$ is an equivalence of symmetric categorical groups.

2.3 Exercise. The theorem stated in Example 2.2 is a "pay-one-take-two" result. Apply π_0 and π_1 to the equivalence $\mathcal{E} \colon \underline{\mathrm{Coker}}(\delta) \to \underline{\mathrm{OpExt}}(G, A, \varphi)$ and get two classical isomorphisms.
Solution: - Using π_0 you get the cohomological description of extensions : $\mathrm{H}^2(G, A, \varphi) \simeq \mathrm{OpExt}(G, A, \varphi)$.
- Using π_1 you get the isomorphism $\mathrm{Der}(G, A, \varphi) \simeq \mathrm{Aut}_{\mathrm{id}}(A \rtimes_\varphi G)$ between the group of derivations and the group of those automorphisms of $A \rtimes_\varphi G$ inducing the identity on A and G.

Now two examples coming from ring theory. The first one is obvious.

2.4 Example. Fix a commutative ring R with unit. The category R-*Mod* of left R-modules is a symmetric monoidal category with tensor product $\otimes_R$ and unit object R. The Picard categorical group of R is the subcategory $\underline{\mathrm{Pic}}(R)$ of R-*Mod* of those modules which are weakly invertible with respect to the tensor product, and taking only isomorphisms as arrows. It is an exercise to check that $\pi_0(\underline{\mathrm{Pic}}(R))$ is the Picard group $\mathrm{Pic}(R)$ of R, usually defined as the group of projective modules of constant rank 1.
This example can be obviously generalized: from any (symmetric) monoidal category $\mathcal{C}$ we get a (symmetric) categorical group $\underline{\mathrm{Pic}}(\mathcal{C})$ by taking weakly invertible objects and isomorphisms between them.

2.5 Example. Fix a commutative ring R with unit. There are several (not obviously) equivalent ways to define the Brauer group of R. My favorite one is of course to construct first a symmetric categorical group and then taking its π_0. We start with the bicategory $\underline{\underline{\mathrm{Bim}}}(R)$: objects are R-algebras (that is, monoids in the monoidal category R-*Mod*), arrows $M\colon A \to B$ are A-B-bimodules, and 2-arrows are homomorphisms of bimodules. The identity arrow on an algebra A is A itself, and the composition of two bimodules $M\colon A \to B$ and $N\colon B \to C$ is the tensor product over B, that is, the coequalizer

$$M \otimes B \otimes N \overset{\mu_M \otimes \mathrm{id}}{\underset{\mathrm{id} \otimes \mu_N}{\rightrightarrows}} M \otimes N \xrightarrow{q} M \otimes_B N$$

(μ is the action) which inherits a bimodule structure $M \otimes_b N\colon A \to C$ from those of M and N because coequalizers in R-*Mod* are stable under tensor product. As for any bicategory, if in $\underline{\underline{\mathrm{Bim}}}(R)$ we identify arrows when they are connected by an invertible 2-arrow, we get a category $\underline{\mathrm{Bim}}(R)$ which in fact is a symmetric monoidal category (the tensor product and the unit object are as in R-*Mod*). We can now define the Brauer categorical group of R as

$$\underline{\mathrm{Br}}(R) = \underline{\mathrm{Pic}}(\underline{\mathrm{Bim}}(R))$$

More explicitly, objects in $\underline{\mathrm{Br}}(R)$ are Azumaya R-algebras, and arrows are Morita equivalences:
- An isomorphism $A \simeq B$ in $\underline{\mathrm{Bim}}(R)$ is a bimodule $M\colon A \to B$ such that there exists another bimodule $N\colon B \to A$ such that $M \otimes_b N \simeq A$ and $N \otimes_A M \simeq B$ as bimodules. By the Eilenberg-Watts theorem, this is the same as giving a Morita equivalence, that is, an equivalence of categories A-*Mod* $\simeq$ B-*Mod*.
- An algebra A in $\underline{\mathrm{Bim}}(R)$ is weakly invertible with respect to the tensor product if there exists another algebra B such that $A \otimes_R B$ is isomorphic in $\underline{\mathrm{Bim}}(R)$ to R, that is, $A \otimes_R B$ is Morita-equivalent to R. Such an algebra A is usually called an Azumaya algebra.
Finally, $\pi_0(\underline{\mathrm{Br}}(R)) = \mathrm{Br}(R)$ is the usual Brauer group of R described as the group of Morita-equivalence classes of Azumaya algebras.

2.6 Exercise. Check that $\pi_1(\underline{\mathrm{Br}}(R)) = \mathrm{Pic}(R)$ and $\pi_1(\mathrm{Pic}(R)) = \mathrm{U}(R)$, the group of units of R (the elements of R invertible with respect to the multiplicative structure).

Now the expected example from algebraic topology.

2.7 Example. Recall that, for a pointed topological space Y, the following homotopy invariants are defined:

1. $\pi_0(Y)$, the pointed set of connected components;
2. $\pi_1(Y) = \pi_0(\Omega Y)$, the fundamental group of Y, where Ω is the loop functor;
3. $\pi_n(Y) = \pi_0(\Omega^n Y)$, which is an abelian group if $n \geq 2$.

These are "1-dimensional" homotopy invariants (pointed sets, groups, abelian groups). We define now "2-dimensional" homotopy invariants:

1. $\Pi_1(Y)$, the fundamental pointed groupoid of Y, the objects are the points of Y, the arrows are the homotopy rel end-points classes of paths;
2. $\Pi_2(Y) = \Pi_1(\Omega Y)$, the fundamental categorical group of Y;
3. $\Pi_3(Y) = \Pi_1(\Omega^2 Y)$, which is a braided categorical group;
4. $\Pi_{n+1}(Y) = \Pi_1(\Omega^n Y)$, which is a symmetric categorical group if $n \geq 3$.

Here, homotopy invariant means that

$$\Pi_1 \colon \underline{Top}_* \to \underline{Grpd}_*,\ \Pi_2 \colon \underline{Top}_* \to \underline{CG},\ \Pi_3 \colon \underline{Top}_* \to \underline{BCG},\ \Pi_{n+1} \colon \underline{Top}_* \to \underline{SCG}$$

are 2-functors, where $\underline{Top}_*$ is the 2-category of pointed topological spaces, continuous maps preserving the base point, and homotopy classes of homotopies.

2.8 Exercise. Go back from 2-dimensional to 1-dimensional homotopy invariants. Check that

$$\pi_0(\Pi_1(Y)) = \pi_0(Y) \text{ and } \pi_1(\Pi_1(Y)) = \pi_1(Y)$$

More in general, check that

$$\pi_0(\Pi_{n+1}(Y)) = \pi_n(Y) \text{ and } \pi_1(\Pi_{n+1}(Y)) = \pi_{n+1}(Y)$$

(Sorry, notation does not help here.)

The last example is step zero in algebraic K-theory. The idea is simple, but calculations are far to be easy.

2.9 Example. Recall that the full inclusion of the category of abelian groups in the category of commutative monoids has a left adjoint:

$$\mathit{CMon} \underset{i}{\overset{r}{\rightleftarrows}} \mathit{Ab} \quad r \dashv i$$

The left adjoint r can be described as the cokernel of the diagonal

$$M \xrightarrow{\Delta} M \times M \xrightarrow{q} \mathrm{Coker}(\Delta) = r(M)$$

The cokernel is taken in the category *CMon*, but it is an abelian group because the diagonal is a final morphism, see Exercise 2.10. More explicitly, $r(M)$ can be described as

$$r(M) = \frac{M \times M}{\equiv}$$

where $(b, a) \equiv (d, c)$ if there exist $x, y \in M$ such that $x \cdot a = y \cdot c$ and $x \cdot b = y \cdot d$

$$\begin{tikzcd}
 & \bullet \arrow[dl, "b"'] \arrow[dr, "a"] & \\
\bullet & \bullet \arrow[u, "x"'] \arrow[d, "y"] & \bullet \\
 & \bullet \arrow[ul, "d"] \arrow[ur, "c"'] &
\end{tikzcd}$$

Now we can give the classical definition of the first abelian group in algebraic K-theory: if $\mathbb{M}$ is a symmetric monoidal groupoid, then $\mathrm{K}_0(\mathbb{M}) = r(\pi_0(\mathbb{M}))$.
What I want to do now is to present $\mathrm{K}_0(\mathbb{M})$ as π_0 of a convenient symmetric categorical groups.
Theorem: the inclusion $\underline{SCG} \to \underline{SMG}$ of the 2-category of symmetric categorical groups in the 2-category of symmetric monoidal groupoids has a left biadjoint, that I call $\mathbb{K}_0$.

$$\begin{tikzcd}
\underline{SMG} \arrow[r, shift left, "\mathbb{K}_0"] \arrow[d, shift right, "\pi_0"'] & \underline{SCG} \arrow[l, shift left, "i"] \arrow[d, shift right, "\pi_0"'] \\
CMon \arrow[r, shift left, "r"] \arrow[u, shift right, "{[-]_0}"'] & Ab \arrow[l, shift left, "i"] \arrow[u, shift right, "{[-]_0}"']
\end{tikzcd}$$

Since in the previous diagram the part built up with right adjoints is obviously commutative, the part built up with left adjoints is also commutative (eventually up to a natural isomorphism). Therefore, we can describe $\mathrm{K}_0(\mathbb{M})$ as π_0 of a symmetric categorical group, as desired:

$$\pi_0(\mathbb{K}_0(\mathbb{M})) \simeq r(\pi_0(\mathbb{M})) = \mathrm{K}_0(\mathbb{M})$$

2.10 Exercise.

1. A morphism $f \colon M \to N$ in *CMon* is final when for all $n \in N$ there exist $n' \in N$ and $m \in M$ such that $n \cdot n' = f(m)$. Prove that the cokernel of f in *CMon* is an abelian group if and only if f is final.

2. Compare the explicit description of $r(M)$ given in Example 2.9 with the general construction of the category of fractions $\mathbb{C}[\Sigma^{-1}]$ of a class Σ of morphisms having a right calculus of fractions.
   Solution: if you put $\mathbb{C} = [M]_1$, the category with one object and the elements of M as arrows, and $\Sigma = M$, then $\mathbb{C}[\Sigma^{-1}] = [r(M)]_1$.

3. If you know the classical definition of the abelian group $\mathrm{K}_1(\mathbb{M})$, check that $\pi_1(\mathbb{K}_0(\mathbb{M})) = \mathrm{K}_1(\mathbb{M})$.

References for Chapter 2

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Chapter 3

Facets of categorical groups

Among categorical groups, there are the strict ones. A categorical group is strict if it is strict as a monoidal category, that is, the coherent natural isomorphisms r, l, a are identities

$$X \otimes I = X = I \otimes X \ , \ \ X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$$

and moreover every object is strictly invertible with respect to the tensor product

$$\forall\, X \in \mathbb{G} \ \exists\, X^* \in \mathbb{G} : X \otimes X^* = I = X^* \otimes X$$

Analogously, a monoidal functor (F, F_2, F_I) is strict if F_2 and F_0 are identities

$$FX \otimes FY = F(X \otimes Y) \ , \ \ I = FI$$

In this way we get the 2-category $\underline{StrCG}$ of strict categorical groups, strict monoidal functors, and monoidal natural transformations. The aim of this section is to show that the 2-category $\underline{StrCG}$ has different descriptions (somehow more popular that strict categorical groups) and to understand correctly the inclusion

$$\underline{StrCG} \to \underline{CG}$$

3.1 Definition. An internal groupoid in *Grp* is a groupoid such that the set of objects is a group, the set of arrows is a group, and all the strictural maps (domain, codomain, ...) are group homomorphisms. In a similar way one defines internal functors and internal natural transformations in *Grp*. The notation is:

$$\mathbb{G}\colon \quad G_1 \times_{c,d} G_1 \xrightarrow{m} G_1 \underset{c}{\overset{d}{\rightrightarrows}} G_0 \quad G_1 \xrightarrow{i} G_1$$

where the following diagram is a pullback

$$\begin{tikzcd} G_1 \times_{c,d} G_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & G_1 \arrow[d, "d"] \\ G_1 \arrow[r, "c"'] & G_0 \end{tikzcd}$$

$$F\colon \mathbb{G} \to \mathbb{H}\colon \quad \begin{tikzcd} G_1 \arrow[r, "F_1"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & H_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ G_0 \arrow[r, "F_0"'] & H_0 \end{tikzcd}$$

$$\alpha\colon F \Rightarrow G\colon \quad \begin{tikzcd} G_1 \arrow[r, shift left, "F_1"] \arrow[r, shift right, "G_1"'] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & H_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ G_0 \arrow[ur, "\alpha"] \arrow[r, shift left, "F_0"] \arrow[r, shift right, "G_0"'] & H_0 \end{tikzcd}$$

3.2 Definition. A crossed module of groups is a diagram

$$\mathbb{G}\colon \quad G_0 \flat G \xrightarrow{*} G \xrightarrow{\partial} G_0$$

where G_0 and G are groups, $*$ and ∂ are group homomorhisms, and the following is a kernel diagram

$$G_0 \flat G \xrightarrow{k} G_0 + G \xrightarrow{[\mathrm{id},0]} G_0$$

The homomorphisms $*$ and ∂ are required to satisfy the following conditions

$$\begin{tikzcd} G \flat G \arrow[r, "\partial \flat\, \mathrm{id}"] \arrow[d, "\mathcal{I}_G"'] & G_0 \flat G \arrow[r, "\mathrm{id}\, \flat \partial"] \arrow[d, "*"] & G_0 \flat G_0 \arrow[d, "\mathcal{I}_{G_0}"] \\ G \arrow[r, "\mathrm{id}"'] & G \arrow[r, "\partial"'] & G_0 \end{tikzcd}$$

where $\mathcal{I}_G$ is given, for any group G, by the followong composition

$$\mathcal{I}_G\colon \ G \flat G \xrightarrow{k} G + G \xrightarrow{[\mathrm{id},\mathrm{id}]} G$$

A morphism $F\colon \mathbb{G} \to \mathbb{H}$ of crossed modules is given by a pair of group homomorphisms $f\colon G \to H$ and $f_0\colon G_0 \to H_0$ such that

$$\begin{tikzcd} G_0 \flat G \arrow[r, "*"] \arrow[d, "f_0 \flat f"'] & G \arrow[r, "\partial"] \arrow[d, "f"'] & G_0 \arrow[d, "f_0"] \\ H_0 \flat H \arrow[r, "*"'] & H \arrow[r, "\partial"'] & H_0 \end{tikzcd}$$

(If you don't like the operator bemolle $\flat$, you can replace it by cartesian product, the homomorphism $*\colon G_0 \flat G \to G$ by a group action $\cdot\colon G_0 \times G \to G$, and $\mathcal{I}_G\colon G \flat G \to G$ by conjugation $G \times G \to G$.)

The following proposition states the announced equivalent descriptions of <u>*StrCG*</u>. We denote by *XMod* the category of crossed modules of groups, and by <u>*Grpd(Grp)*</u> the 2-category of internal groupoids in *Grp*.

3.3 Proposition.

1. *The 2-categories $\underline{\mathit{Grpd}}(\mathit{Grp})$ and $\underline{\mathit{StrCG}}$ are equal.*
2. *The categories $\underline{\mathit{Grpd}}(\mathit{Grp})$ (forget 2-arrows) and XMod are equivalent.*

Proof. 1. Obvious: if $\mathbb{G}$ is an internal groupoid in *Grp*, then the monoidal structure on objects and on arrows is provided by the group operation in G_0 and G_1. Conversely, if $\mathbb{G}$ is a strict categorical group, then the set of objects is a group with respect to the tensor product, and the same holds for the set of arrows.
2. Given an internal groupoid $\mathbb{G}$ as in Definition 3.1, we get a crossed module

$$G_0 \flat G \xrightarrow{\ *\ } G \xrightarrow{\partial = g \cdot d} G_0$$

from the following diagram, where both rows are kernels

$$\begin{tikzcd} G_0 \flat G \arrow[r, "k"] \arrow[d, "*"'] & G_0 + G \arrow[r, "{[\mathrm{id},0]}"] \arrow[d, "{[e,g]}"'] & G_0 \arrow[d, "\mathrm{id}"] \\ G \arrow[r, "g"'] & G_1 \arrow[r, "c"'] & G_0 \end{tikzcd}$$

Conversely, if we start with a crossed module $\mathbb{G}$ as in Definition 3.2, we can consider the coequalizer

$$\begin{tikzcd} G_0 \flat G \arrow[rr, "k"] \arrow[dr, "*"'] & & G_0 + G \arrow[r, "q"] & G \rtimes G_0 \\ & G \arrow[ur, "i_G"'] & & \end{tikzcd}$$

and taking $G_1 = G \rtimes G_0$ we get an internal groupoid

$$\begin{tikzcd} G_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G_0 \arrow[l, "e" description] \end{tikzcd}$$

with $e = i_{G_0} \cdot q \colon G_0 \to G_0 + G \to G_1$ and d and c such that

$$\begin{tikzcd} G \arrow[r, "i_G \cdot q"] \arrow[dr, "\partial"'] & G_1 \arrow[d, "d"] & G_0 \arrow[l, "i_{G_0} \cdot q"'] \arrow[dl, "\mathrm{id}"] \\ & G_0 & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "i_G \cdot q"] \arrow[dr, "0"'] & G_1 \arrow[d, "c"] & G_0 \arrow[l, "i_{G_0} \cdot q"'] \arrow[dl, "\mathrm{id}"] \\ & G_0 & \end{tikzcd}$$

□

Now that we have a better understanding of the 2-category $\underline{\mathit{StrCG}}$, our next goal is to have a better understanding of the inclusion $\underline{\mathit{StrCG}} \to \underline{\mathit{CG}}$. For this, let us consider an intermediate 2-category

$$\underline{\mathit{StrCG}} \to \underline{\mathit{MON}} \to \underline{\mathit{CG}}$$

The objects of $\underline{MON}$ are strict categorical groups (that is, internal groupoids in *Grp*), whereas arrows and 2-arrows are not necessarily strict monoidal functors and monoidal natural transformations.

3.4 Proposition. *The full inclusion* $\underline{MON} \to \underline{CG}$ *is a biequivalence of 2-categories.*

Proof. This is a refinement of the classical (highly non trivial) theorem asserting that every monoidal category is monoidally equivalent to a strict monoidal category. $\square$

Thanks to Proposition 3.4, we can concentrate our attention on the not full inclusion $\underline{StrCG} \to \underline{MON}$. The next example provides a new link between categorical groups and group extensions, and gives a way to understand the distance between monoidal functors and strict monoidal (or internal) functors.

3.5 Example. Fix two groups G and H and consider the following diagram

$$\begin{tikzcd}
\underline{\mathrm{Ext}}(G,H) \arrow[r, shift left] & \underline{MON}([G]_0, \underline{\mathrm{Hol}}(H)) \arrow[l, shift left] \\
\underline{\mathrm{SectExt}}(G,H) \arrow[u, "\text{full}"] \arrow[r, shift left] & \underline{Grpd(Grp)}([G]_0, \underline{\mathrm{Hol}}(H)) \arrow[u, "\text{full}"'] \arrow[l, shift left] \\
 & \underline{Grpd(Grp)}([G]_0, [\mathrm{Aut}(H)]_0) \arrow[u] \\
\underline{\mathrm{SplitExt}}(G,H) \arrow[uu, "\text{not full}"] \arrow[r, shift right, "\mathrm{Ker}"'] & Grp(G, \mathrm{Aut}(H)) \arrow[u, "\simeq"'] \arrow[l, shift right, "\rtimes"']
\end{tikzcd}$$

On the left column from the top to the bottom, $\underline{\mathrm{Ext}}(G,H)$ is the groupoid of group extensions of the form

$$0 \longrightarrow H \xrightarrow{\chi} B \xrightarrow{\sigma} G \longrightarrow 0$$

with arrows defined as in Example 2.1, and $\underline{\mathrm{SectExt}}(G,H)$ is the full subgroupoid of those extensions such that $\sigma\colon B \to G$ admits a section in *Grp*. An object in the groupoid $\underline{\mathrm{SplitExt}}(G,H)$ is an extension together with a specified section $i\colon G \to B$ of $\sigma\colon B \to H$, and arrows are morphisms of extensions commuting also with the specified sections

$$\begin{tikzcd}
 & B \arrow[dr, "\sigma"] \arrow[dd, "\beta"] & \\
H \arrow[ur, "\chi"] \arrow[dr, "\chi'"'] & & G \\
 & B' \arrow[ur, "\sigma'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[dd, "\beta"'] & \\
 & G \arrow[ul, "i"'] \arrow[dl, "i'"] \\
B' &
\end{tikzcd}$$

The four objects on the right column are hom-groupoids (*Grp* is considered as a 2-category with only identity 2-arrows, so that the corresponding hom-groupoid is just a set). The strict categorical group $\underline{\mathrm{Hol}}(H)$ is the internal groupoid corresponding, in the equivalence $\underline{\mathit{Grpd}}(\mathit{Grp}) \simeq \mathit{XMod}$ of Proposition 3.3, to the crossed module of inner automorphisms

$$\mathcal{I} \colon H \to \mathrm{Aut}(H) : \;\; \mathcal{I}_x(y) = x \cdot y \cdot x^{-1} \;, \;\; \mathrm{Aut}(H) \times H \to H : \;\; (f, x) \mapsto f(x)$$

The rows are equivalences. The bottom one associates to a group homomorphism $\varphi \colon G \to \mathrm{Aut}(H)$ the semi-direct product extension with its canonical section

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "i_H"] & H \rtimes_\varphi G \arrow[r, shift right, "p_G"'] & G \arrow[l, shift right, "i_G"'] \arrow[r] & 0 \end{tikzcd}$$

and, in the opposite direction, associates to a split extension

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "\chi"] & B \arrow[r, shift right, "\sigma"'] & G \arrow[l, shift right, "i"'] \arrow[r] & 0 \end{tikzcd}$$

the action $i \cdot \bar{\chi} \colon G \to B \to \mathrm{Aut}(H)$, where $\bar{\chi}$ is the action induced by the fact that H is the kernel of σ. The equivalence on the top of the diagram, and its restriction to extensions with section, can be described as follows. Consider an extension

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "\chi"] & B \arrow[r, "\sigma"] & G \arrow[r] & 0 \end{tikzcd}$$

and fix a set-theoretical section $s \colon G \to B$ of σ. We get a monoidal functor $F \colon [G]_0 \to \underline{\mathrm{Hol}}(H)$ defined on objects by

$$F_0 \colon G \to \mathrm{Aut}(H) \;, \;\; F_0(x)(h) = s(x) \cdot h \cdot s(x)^{-1}$$

and on arrows by

$$F_1 \colon G \to H \rtimes \mathrm{Aut}(H) \;, \;\; F_1(x) = (1, F_0(x))$$

and with monoidal structure $F_2^{x,y} \colon F_0(x) \circ F_0(y) \to F_0(x \cdot y)$ given by

$$F_2 \colon G \times G \to H \rtimes \mathrm{Aut}(H) \;, \;\; F_2(x,y) = (s(x) \cdot s(y) \cdot s(x \cdot y)^{-1}, F_0(x \cdot y))$$

Since in the previous diagram we have equivalences

$$\underline{\mathrm{Ext}}(G,H) \simeq \underline{\mathit{MON}}([G]_0, \underline{\mathrm{Hol}}(H)) \qquad \underline{\mathrm{SectExt}}(G,H) \simeq \underline{\mathit{Grpd}}(\mathit{Grp})([G]_0, \underline{\mathrm{Hol}}(H))$$

the "distance" between monoidal functors $[G]_0 \to \underline{\mathrm{Hol}}(H)$ and internal functors $[G]_0 \to \underline{\mathrm{Hol}}(H)$ is the same as the "distance" between extensions of G by H and extensions of G by H admitting a group-theoretical section: not every extension admits a group-theoretical section just because the axiom of choice holds in *Set* but not in *Grp*.

The final comment in Example 3.5 suggests that, if we want to understand correctly the inclusion

$$J\colon \underline{\mathit{Grpd}}(\mathit{Grp}) = \underline{\mathit{StrCG}} \to \underline{\mathit{MON}}$$

we have to look at the role of the axiom of choice, and what is well-known is that we need the axiom of choice to pass from weak equivalences to equivalences. To make this analogy precise, we need two preliminary facts.

3.6 Remark. Let $E\colon \mathbb{G} \to \mathbb{H}$ be a weak equivalence in $\underline{\mathit{Grpd}}(\mathit{Grp})$, that is, an internal functor wich is full, faithful and essentially surjective. Therefore, using the axiom of choice in *Set*, the functor E has a quasi-inverse $E^*\colon \mathbb{H} \to \mathbb{G}$, but in general such a quasi-inverse is not an internal functor (because the axiom of choice does not hold in *Grp*). What is true is that for every weak equivalence E in $\underline{\mathit{Grpd}}(\mathit{Grp})$, the functor $J(E)$ is an equivalence in $\underline{\mathit{MON}}$. (Recall point 2 of Exercise 1.5: if a functor is an equivalence and is monoidal, then any quasi-inverse can be equipped with a monoidal structure.)

3.7 Remark. Consider now any monoidal functor $F\colon \mathbb{G} \to \mathbb{H}$ between strict categorical groups.We can construct the comma category (or strong homotopy pullback)

$$\begin{tikzcd} F \downarrow \mathrm{Id} \arrow[r, "F'"] \arrow[d, "E"'] \arrow[dr, phantom, "\overset{\varphi}{\Rightarrow}"] & \mathbb{H} \arrow[d, "\mathrm{Id}"] \\ \mathbb{G} \arrow[r, "F"'] & \mathbb{H} \end{tikzcd}$$

An object of $F \downarrow \mathrm{Id}$ is a triple $(X \in \mathbb{G}, x\colon FX \to H, H \in \mathbb{H})$. An arrow $(X, x, H) \to (X', x', H')$ is a pair of arrows $f\colon X \to X', h\colon H \to H'$ such that

$$\begin{tikzcd} FX \arrow[r, "x"] \arrow[d, "Ff"'] & H \arrow[d, "h"] \\ FX' \arrow[r, "x'"'] & H' \end{tikzcd}$$

The functors F' and E are the obvious projections, and the natural transformation φ is defined by

$$\varphi_{(X,x,H)} = x\colon FX = F(E(X, x, H)) \to F'(X, x, H) = H$$

It is easy to check that $F \downarrow \mathrm{Id}$ is a strict categorical group, E and F' are internal functors, E is a weak equivalence, and φ is monoidal. In other words, we have obtained a span decomposition (or tabulation) with the left leg a weak equivalence

$$\begin{tikzcd} & F \downarrow \mathrm{Id} \arrow[dl, "E \in \underline{\mathit{Grpd}}(\mathit{Grp})"'] \arrow[dr, "F' \in \underline{\mathit{Grpd}}(\mathit{Grp})"] \arrow[d, phantom, "\simeq"] & \\ \mathbb{G} \arrow[rr, "F \in \underline{\mathit{MON}}"'] & {} & \mathbb{H} \end{tikzcd}$$

Putting together Remark 3.6 and Remark 3.7, we have (almost) proved the following result, which expresses in a precise way how far is $\underline{MON}$ (or $\underline{CG}$) from $\underline{Grpd}(Grp)$.

3.8 Proposition. *The 2-functor* $J \colon \underline{Grpd}(Grp) \to \underline{MON}$ *is the bicategory of fractions of* $\underline{Grpd}(Grp)$ *with respect to the class of weak equivalences. This means that:*

1. $J(E)$ *is an equivalence in* $\underline{MON}$ *for every weak equivalence* E *in* $\underline{Grpd}(Grp)$, *and*
2. J *is universal with respect to such a condition.*

References for Chapter 3

Bla

Lesson II: Abelian 2-categories

My point of view on the teaching of mathematics:

"Il n'y a jamais mauvais élève, seulement mauvais enseignant."
Jackie Chan, *The Karate Kid*, Columbia Pictures, 2010.

The aim of Lesson 2 is:

1. To study some constructions, essentially strong homotopy kernels and strong homotopy cokernels, which can be performed in the 2-category $\underline{SCG}$ and which lead to a convenient notion of exactness in $\underline{SCG}$.
2. To give the definition of abelian 2-category, based on results obtained for symmetric categorical groups, and to show some meaningful results which hold true in any abelian 2-category.
3. To discuss the (not so obvious) relation between abelian categories and abelian 2-categories.

Chapter 4

Kernels, cokernels, and exactness

4.1 Definition. Let $F\colon \mathbb{G} \to \mathbb{H}$ be an arrow in $\underline{CG}$. The strong homotopy kernel of F is the following diagram in $\underline{CG}$:

$$
\begin{tikzcd}
 & \mathbb{G} \arrow[dr, "F"] & \\
\underline{\mathrm{Ker}}(F) \arrow[ur, "\mathrm{K}(F)"] \arrow[rr, "0"'] \arrow[rr, phantom, shift left=3, "k(F)\Uparrow"] & & \mathbb{H}
\end{tikzcd}
$$

Objects of $\underline{\mathrm{Ker}}(F)$ are pairs $(X \in \mathbb{G}, x\colon I \to FX)$, and arrows $f\colon (X,x) \to (Y,y)$ of $\underline{\mathrm{Ker}}(F)$ are arrows $f\colon X \to Y$ in $\mathbb{G}$ such that

$$
\begin{tikzcd}
FX \arrow[rr, "Ff"] & & FY \\
 & I \arrow[ul, "x"] \arrow[ur, "y"'] &
\end{tikzcd}
$$

The functor $\mathrm{K}(F)\colon \underline{\mathrm{Ker}}(F) \to \mathbb{G}$ is defined by

$$\mathrm{K}(F)\colon\ (X,x) \xrightarrow{f} (Y,y) \ \mapsto\ X \xrightarrow{f} Y$$

The finctor $0\colon \underline{\mathrm{Ker}}(F) \to \mathbb{H}$ is the constant functor sending every arrow on the identity of the unit object. The natural transformation $k(F)\colon 0 \Rightarrow \mathrm{K}(F)\cdot F$ is defined by

$$k(F)_{(X,x)} = x\colon 0(X,x) = I \to FX = F(\mathrm{K}(F))(X,x)$$

Finally, observe that if F is in $\underline{BCG}$ or in $\underline{SCG}$, then so is its strong homotopy kernel.

The basic questions about the strong homotopy kernel are: what $\underline{\mathrm{Ker}}(F)$ says about F ? What is the universal property of $\underline{\mathrm{Ker}}(F)$? The answer to the first question is an exercise.

4.2 Exercise. Let $F \colon \mathbb{G} \to \mathbb{H}$ be an arrow in $\underline{CG}$. Prove that

1. F is faithful if and only if $\pi_1(\underline{\mathrm{Ker}}(F)) = 0$.
2. F is full if and only if $\pi_0(\underline{\mathrm{Ker}}(F)) = 0$.
3. F is full and faithful if and only if $\underline{\mathrm{Ker}}(F) \simeq \mathbb{I}$ (the category with only one arrow).

Now we look at the universal property.

4.3 Proposition. *Let $F \colon \mathbb{G} \to \mathbb{H}$ be an arrow in $\underline{CG}$. The diagram*

$$\begin{tikzcd}
 & \mathbb{G} \arrow[dr, "F"] & \\
\underline{\mathrm{Ker}}(F) \arrow[ur, "\mathrm{K}(F)"] \arrow[rr, "0"'] \arrow[rr, phantom, bend left=20, "k(F)\Uparrow"] & & \mathbb{H}
\end{tikzcd}$$

constructed in Definition 4.1 satisfies the following universal properties.

1. *It is a strong homotopy kernel:*
   (a) *For every diagram in $\underline{CG}$ of the form*

   $$\begin{tikzcd}
    & \mathbb{G} \arrow[dr, "F"] & \\
   \mathbb{X} \arrow[ur, "M"] \arrow[rr, "0"'] \arrow[rr, phantom, bend left=20, "\varphi\Uparrow"] & & \mathbb{H}
   \end{tikzcd}$$

   there exists a unique $M' \colon \mathbb{X} \to \underline{\mathrm{Ker}}(F)$ such that $M' \cdot \mathrm{K}(F) = M$ and $M' \cdot k(F) = \varphi$.

   (b) *For every diagram in $\underline{CG}$ of the form*

   $$\begin{tikzcd}
    & \underline{\mathrm{Ker}}(F) \arrow[dr, "\mathrm{K}(F)"] & \\
   \mathbb{X} \arrow[ur, "M"] \arrow[dr, "N"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathbb{G} \\
    & \underline{\mathrm{Ker}}(F) \arrow[ur, "\mathrm{K}(F)"'] &
   \end{tikzcd}$$

   with α compatible with $k(F)$, that is, such that

   $$\begin{tikzcd}
   M \cdot \mathrm{K}(F) \cdot F \arrow[r, Rightarrow, "\alpha \cdot F"] & N \cdot \mathrm{K}(F) \cdot F \\
   M \cdot 0 \arrow[u, Rightarrow, "M \cdot k(F)"] \arrow[r, equal] & N \cdot 0 \arrow[u, Rightarrow, "N \cdot k(F)"']
   \end{tikzcd}$$

there exists a unique $\beta \colon M \Rightarrow N$ *such that* $\beta \cdot \mathrm{K}(F) = \alpha$.

2. *It is a bikernel: For every diagram in* $\underline{CG}$ *of the form*

$$\begin{tikzcd} & \mathbb{G} \arrow[dr, "F"] & \\ \mathbb{X} \arrow[ur, "M"] \arrow[rr, "0"'] \arrow[rr, phantom, shift left=4, "\varphi\Uparrow"] & & \mathbb{H} \end{tikzcd}$$

there exists a fill-in, that is,

$$\begin{tikzcd} & \underline{\mathrm{Ker}}(F) \arrow[dr, "\mathrm{K}(F)"] & \\ \mathbb{X} \arrow[ur, "M'"] \arrow[rr, "M"'] \arrow[rr, phantom, shift left=4, "\varphi'\Downarrow"] & & \mathbb{G} \end{tikzcd} \quad \text{such that} \quad \begin{tikzcd} M' \cdot \mathrm{K}(F) \cdot F \arrow[r, Rightarrow, "\varphi' \cdot F"] & M \cdot F \\ M' \cdot 0 \arrow[u, Rightarrow, "M' \cdot k(F)"] \arrow[r, equal] & 0 \arrow[u, Rightarrow, "\varphi"'] \end{tikzcd}$$

and for any other fill-in

$$\begin{tikzcd} & \underline{\mathrm{Ker}}(F) \arrow[dr, "\mathrm{K}(F)"] & \\ \mathbb{X} \arrow[ur, "M''"] \arrow[rr, "M"'] \arrow[rr, phantom, shift left=4, "\varphi''\Downarrow"] & & \mathbb{G} \end{tikzcd} \quad \begin{tikzcd} M'' \cdot \mathrm{K}(F) \cdot F \arrow[r, Rightarrow, "\varphi'' \cdot F"] & M \cdot F \\ M'' \cdot 0 \arrow[u, Rightarrow, "M'' \cdot k(F)"] \arrow[r, equal] & 0 \arrow[u, Rightarrow, "\varphi"'] \end{tikzcd}$$

there exists a unique $\psi \colon M' \Rightarrow M''$ *such that*

$$\begin{tikzcd} M' \cdot \mathrm{K}(F) \arrow[rr, Rightarrow, "\psi \cdot \mathrm{K}(F)"] \arrow[dr, Rightarrow, "\varphi'"'] & & M'' \cdot \mathrm{K}(F) \arrow[dl, Rightarrow, "\varphi''"] \\ & M & \end{tikzcd}$$

Proof. 1. (a) Define $M' \colon \mathbb{X} \to \underline{\mathrm{Ker}}(F)$ by

$$f \colon X \to Y \quad \mapsto \quad M(f) \colon (MX, \varphi_X \colon I \to FMX) \to (MY, \varphi_Y \colon I \to FMY)$$

The fact that $M(f)$ is an arrow of $\underline{\mathrm{Ker}}(F)$ is precisely the naturality of φ:

$$\begin{tikzcd} FMX \arrow[rr, "F(M(f))"] & & FMY \\ & I \arrow[ul, "\varphi_X"] \arrow[ur, "\varphi_Y"'] & \end{tikzcd}$$

(b) For $M \colon \mathbb{X} \to \underline{\mathrm{Ker}}(F)$, let me write

$$MX = (\underline{MX}, m(X) \colon I \to F(\underline{MX}))$$

and similarly for NX, so that the components of $\alpha\colon M \cdot \mathrm{K}(F) \Rightarrow N \cdot \mathrm{K}(F)$ are arrows in $\mathbb{G}$ of the form $\alpha_X \colon \underline{MX} \to \underline{NX}$. We are looking for $\beta\colon M \Rightarrow N$, and the condition $\beta \cdot \mathrm{K}(F) = \alpha$ means that $\beta_X = \alpha_X$, so that the proof reduces to check that $\alpha_X \colon (\underline{MX}, m(X)) \to (\underline{NX}, n(X))$ is an arrow in $\underline{\mathrm{Ker}}(F)$. This means that

$$\begin{tikzcd} F(\underline{MX}) \arrow[rr, "F(\alpha_X)"] & & F(\underline{NX}) \\ & I \arrow[ul, "m(X)"] \arrow[ur, "n(X)"'] & \end{tikzcd}$$

must commute, and this commutativity is precisely the compatibility condition between α and $k(F)$.
2. The universal property of type "strong homotopy kernel" implies the universal property of type "bikernel". Indeed, for any $\mathbb{X} \in \underline{CG}$, we can consider the following diagram in $\underline{Grpd}_*$, where $\underline{\mathrm{Ker}}(- \cdot F)$ is the strong homotopy kernel of $- \cdot F$ in $\underline{Grpd}_*$ and J is the canonical comparison

$$\begin{tikzcd} \underline{\mathrm{Ker}}(- \cdot F) \arrow[r, "\mathrm{K}(- \cdot F)"] & \underline{CG}(\mathbb{X}, \mathbb{G}) \arrow[r, "- \cdot F"] & \underline{CG}(\mathbb{X}, \mathbb{H}) \\ & \underline{CG}(\mathbb{X}, \underline{\mathrm{Ker}}(F)) \arrow[ul, "J"] \arrow[u, "- \cdot \mathrm{K}(F)"'] & \end{tikzcd}$$

Now, it is easy to check that the universal property of the strong homotopy kernel means that J is an isomorphism of categories (that is, fully faithful and bijective on objects), whereas the universal property of the bikernel means that J is an equivalence of categories (that is, fully faithful and essentially surjective on objects). $\square$

4.4 Remark. Here is a quite "ideological" comment on the difference between the strong homotopy kernel and the bikernel. Despite the fact that the universal property of the strong homotopy kernel is somehow easier to use, I prefer the universal property of the bikernel. Indeed, the universal property of the strong homotopy kernel determines it up to isomorphism of categories, whereas the universal property of the bikernel determines it only up to equivalence of categories. Now, the notion of equivalence is a genuine 2-categorical notion, whereas the notion of isomorphism of categories is not a 2-categorical notion (for example, it is not stable under natural isomorphism). A more technical point in favour of the bikernel comes from Lemma 4.11, which in particular implies that every faithful arrow in $\underline{SCG}$ is equivalent (but in general not isomorphic) to the kernel of its cokernel.

For the dual construction, we have to work in $\underline{SCG}$. To work in $\underline{CG}$ or even in $\underline{BCG}$ is not enough. The idea to describe the strong homotopy cokernel of an arrow in $\underline{SCG}$ is the same idea followed in Example 1.1: with abelian groups, you first construct a category, and then identify objects when they are isomorphic. With symmetric categorical groups, you first construct a bicategory, and then identify arrows when they are 2-isomorphic.

4.5 Definition. Let $F\colon \mathbb{G} \to \mathbb{H}$ be an arrow in $\underline{SCG}$. The strong homotopy cokernel of F is the following diagram in $\underline{SCG}$:

$$\begin{tikzcd} & \mathbb{H} \arrow[dr, "\mathrm{C}(F)"] & \\ \mathbb{G} \arrow[ur, "F"] \arrow[rr, "0"'] & \Downarrow c(F) & \underline{\mathrm{Coker}}(F) \end{tikzcd}$$

The objects of $\underline{\mathrm{Coker}}(F)$ are those of $\mathbb{H}$. A pre-arrow $(X, f)\colon A \to B$ in $\underline{\mathrm{Coker}}(F)$ is a pair with $X \in \mathbb{G}$ and $f\colon A \to FX \otimes B$ in $\mathbb{H}$. An arrow $[X, f]\colon A \to B$ is an equivalence classe of pre-arrows, with two pre-arrows

$$f\colon A \to FX \otimes B \quad \text{and} \quad f'\colon A \to FX' \otimes B$$

being equivalent if there exists $x\colon X \to X'$ in $\mathbb{G}$ such that

$$\begin{tikzcd} FX \otimes B \arrow[rr, "F(x)\otimes \mathrm{id}"] & & FX' \otimes B \\ & A \arrow[ul, "f"] \arrow[ur, "f'"'] & \end{tikzcd}$$

The functor $\mathrm{C}(F)\colon \mathbb{H} \to \underline{\mathrm{Coker}}(F)$ is the identity on objects and it sends an arrow $f\colon A \to B$ on the arrow represented by the pair

$$(I, f \cdot l_B \cdot (F_I \otimes \mathrm{id})\colon A \to B \to I \otimes B \to FI \otimes B)$$

The natural transformation $c(F)\colon F \cdot \mathrm{C}(F) \Rightarrow 0$ has component $c(F)_X$ represented by the pair

$$(X, r_{FX}\colon FX \to FX \otimes I)$$

The symmetry of $\mathbb{H}$ enters in the picture in order to define the tensor product in $\underline{\mathrm{Coker}}(F)$: the tensor product of two arrows $[X, f]\colon A \to B$ and $[Y, g]\colon C \to D$ is the arrow represented by the pair

$$\left(\begin{tikzcd}[ampersand replacement=\&] X \otimes B,\ A \otimes C \arrow[r, "f\otimes g"] \& FX \otimes B \otimes FY \otimes D \arrow[r, "\mathrm{id}\otimes c_{B,FY}\ \mathrm{id}"] \& FX \otimes FY \otimes B \otimes D \arrow[d, "F_2^{X,Y}\otimes \mathrm{id}\otimes \mathrm{id}"] \\ \& \& F(X \otimes Y) \otimes B \otimes D \end{tikzcd} \right)$$

The associativity, unit, and commutativity constraints of $\underline{\mathrm{Coker}}(F)$ come from those in $\mathbb{H}$ via the functor $\mathrm{C}(F)$. It is to prove that the commutativity of $\mathbb{H}$ is natural also with respect to the arrows in $\underline{\mathrm{Coker}}(F)$ that we need a symmetry, a braiding is not enough.

Once again, the basic questions about the strong homotopy cokernel are: what $\underline{\mathrm{Coker}}(F)$ says about F ? What is the universal property of $\underline{\mathrm{Coker}}(F)$?

4.6 Exercise. Let $F\colon \mathbb{G} \to \mathbb{H}$ be an arrow in $\underline{SCG}$. Prove that

1. F is essentially surjective if and only if $\pi_0(\underline{\mathrm{Coker}}(F)) = 0$.
2. F is full if and only if $\pi_1(\underline{\mathrm{Coker}}(F)) = 0$.
3. F is full and essentially surjective if and only if $\underline{\mathrm{Coker}}(F) \simeq \mathbb{I}$.

Here is another simple exercise which can help to grasp the difference between kernels and cokernels in (abelian) groups, and kernels and cokernels in (symmetric) categorical groups.

4.7 Exercise.

1. In $\underline{CG}$: $\underline{\mathrm{Ker}}(\mathbb{G} \to \mathbb{I}) = \mathbb{G}$, $\underline{\mathrm{Ker}}(\mathbb{I} \to \mathbb{G}) = [\pi_1(\mathbb{G})]_0$
2. In $\underline{SCG}$: $\underline{\mathrm{Coker}}(\mathbb{G} \to \mathbb{I}) = [\pi_0(\mathbb{G})]_1$, $\underline{\mathrm{Coker}}(\mathbb{I} \to \mathbb{G}) = \mathbb{G}$.

Now we look at the universal property. The explicit statements are dual of those of Proposition 4.3 and we leave them to the reader.

4.8 Proposition. *Let $F\colon \mathbb{G} \to \mathbb{H}$ be an arrow in $\underline{SCG}$. The diagram*

$$\begin{tikzcd} & \mathbb{H} \arrow[dr, "\mathrm{C}(F)"] \arrow[d, phantom, "\Downarrow c(F)"] & \\ \mathbb{G} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & \underline{\mathrm{Coker}}(F) \end{tikzcd}$$

constructed in Definition 4.5 satisfies the following universal properties.

1. *It is a strong homotopy cokernel.*
2. *It is a bicokernel.*

Proof. (Not really a proof) Consider a diagram in $\underline{SCG}$ of the form

$$\begin{tikzcd} & \mathbb{H} \arrow[dr, "M"] \arrow[d, phantom, "\varphi\Downarrow"] & \\ \mathbb{G} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & \mathbb{X} \end{tikzcd}$$

The factorization $M'\colon \underline{\mathrm{Coker}}(F) \to \mathbb{X}$ of M through $\mathrm{C}(F)$ sends an arrow $[X,f]\colon A \to B$ on the arrow

$$MA \xrightarrow{M(f)} M(FX \otimes B) \xrightarrow{(M_2^{FX,B})^{-1}} MFX \otimes MB \xrightarrow{\varphi_X \otimes \mathrm{id}} I \otimes MB \xrightarrow{l^{-1}_{MB}} MB$$

□

The reader will find the following lemma useful in order to give a complete proof of Proposition 4.8.

4.9 Lemma. *Let $F\colon \mathbb{G} \to \mathbb{H}$ be an arrow in $\underline{SCG}$ and $[X,f]\colon A \to B$ an arrow in $\underline{\mathrm{Coker}}(F)$. We get the following diagram in $\underline{\mathrm{Coker}}(F)$:*

$$\begin{tikzcd}
A \arrow[r, "{[X,f]}"] \arrow[d, "\mathrm{C}(F)(f)"'] & B \arrow[d, "\mathrm{C}(F)(l_B)"] \\
FX \otimes B \arrow[r, "c(F)_X \otimes \mathrm{id}"'] & I \otimes B
\end{tikzcd}$$

One more exercise on kernels and cokernels in $\underline{SCG}$. It provides the definitive solution to the "wrong equation" used at the beginning of Lesson 1 (compare with Example 1.1).

4.10 Exercise.

1. Let $F\colon \mathbb{G} \to \mathbb{H}$ be an arrow in $\underline{SCG}$. Prove that $\pi_0(\underline{\mathrm{Ker}}(F)) \simeq \pi_1(\underline{\mathrm{Coker}}(F))$.
2. Let $f\colon A \to B$ be in Ab. Recall that $\pi_0 \dashv [-]_0$ and $[-]_1 \dashv \pi_1$. (see Exercise 1.5) and apply point 1 to $[f]_0\colon [A]_0 \to [B]_0$ and to $[f]_1\colon [A]_1 \to [B]_1$.
   Solution: $\pi_1(\underline{\mathrm{Coker}}[f]_0) = \mathrm{Ker}(f)$, $\pi_0(\underline{\mathrm{Ker}}[f]_1) = \mathrm{Coker}(f)$.
3. More is true: if $f\colon A \to B$ is in Ab, then $\underline{\mathrm{Coker}}[f]_0 \simeq \underline{\mathrm{Ker}}[f]_1$.

Now that we dispose of kernels and cokernels in $\underline{SCG}$, we can study the notion of exactness. The prototype of exact sequences should be of course

$$\begin{tikzcd}
 & \mathbb{G} \arrow[dr, "F"] & \\
\underline{\mathrm{Ker}}(F) \arrow[ur, "\mathrm{K}(F)"] \arrow[rr, "0"'] \arrow[rr, phantom, shift left=4, "k(F)\Uparrow"] & & \mathbb{H}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{H} \arrow[dr, "\mathrm{C}(F)"] & \\
\mathbb{G} \arrow[ur, "F"] \arrow[rr, "0"'] \arrow[rr, phantom, shift left=4, "\Downarrow c(F)"] & & \underline{\mathrm{Coker}}(F)
\end{tikzcd}$$

and a sequence of the form

$$\begin{tikzcd}
 & \mathbb{B} \arrow[dr, "G"] & \\
\mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] \arrow[rr, phantom, shift left=4, "\simeq\varphi"] & & \mathbb{C}
\end{tikzcd}$$

will be declared "2-exact" if in some sense it looks like one of the prototypes. We make precise this idea with the following lemma.

4.11 Lemma. *Consider an arrow $F\colon \mathbb{A} \to \mathbb{B}$ in $\underline{SCG}$ and construct the diagram*

$$\begin{tikzcd}
 & & \underline{\mathrm{Ker}}(\mathrm{C}(F)) \arrow[dr, "\mathrm{K}(\mathrm{C}(F))"] \arrow[d, phantom, "\simeq"] & & \\
\underline{\mathrm{Ker}}(F) \arrow[r, "\mathrm{K}(F)"] & \mathbb{A} \arrow[ur, "F_1"] \arrow[rr, "F"] \arrow[dr, "\mathrm{C}(\mathrm{K}(F))"'] & \arrow[d, phantom, "\simeq"] & \mathbb{B} \arrow[r, "\mathrm{C}(F)"] & \underline{\mathrm{Coker}}(F) \\
 & & \underline{\mathrm{Coker}}(\mathrm{K}(F)) \arrow[ur, "F_2"'] & &
\end{tikzcd}$$

1. F_1 is full and essentially surjective, and $\mathrm{K}(\mathrm{C}(F))$ *is faithful.*

2. $\mathrm{C}(\mathrm{K}(F))$ *is essentially surjective, and F_2 is full and faithful.*

The above factorizations of F form two different factorization systems (in a 2-categorical sense) in $\underline{SCG}$.

4.12 Exercise. Check that in $\underline{SCG}$ the factorizations of Lemma 4.11 can be obtained also via the following diagrams.

1. The (full and essentially surjective, faithful) factorization:

$$\begin{tikzcd}
 & & & \underline{\mathrm{Coker}}(\varepsilon_{\underline{\mathrm{Ker}}(F)} \cdot \mathrm{K}(F)) \arrow[dr] \arrow[d, phantom, "\simeq"] & \\
{[\pi_1 \underline{\mathrm{Ker}}(F)]_1} \arrow[r, "\varepsilon_{\underline{\mathrm{Ker}}(F)}"] & \underline{\mathrm{Ker}}(F) \arrow[r, "\mathrm{K}(F)"] & \mathbb{A} \arrow[ur] \arrow[rr, "F"] & & \mathbb{B}
\end{tikzcd}$$

2. The (essentially surjective, full and faithful) factorization:

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "F"] \arrow[dr] & \arrow[d, phantom, "\simeq"] & \mathbb{B} \arrow[r, "\mathrm{C}(F)"] & \underline{\mathrm{Coker}}(F) \arrow[r, "\eta_{\underline{\mathrm{Coker}}(F)}"] & {[\pi_0 \underline{\mathrm{Coker}}(F)]_0} \\
 & \underline{\mathrm{Ker}}(\mathrm{C}(F) \cdot \eta_{\underline{\mathrm{Coker}}(F)}) \arrow[ur] & & &
\end{tikzcd}$$

4.13 Definition. Consider a sequence in $\underline{CG}$ or in $\underline{SCG}$

$$\begin{tikzcd}
 & \mathbb{B} \arrow[dr, "G"] \arrow[d, phantom, "\simeq\varphi"] & \\
\mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] & \, & \mathbb{C}
\end{tikzcd}$$

We say that the sequence (F, φ, G) is 2-exact if the canonical factorization $F' \colon \mathbb{A} \to \underline{\mathrm{Ker}}(G)$ of F through $\mathrm{K}(G)$ is full and essentially surjective.

Here is the expected result which makes the notion of 2-exactness self-dual in the symmetric case.

4.14 Proposition. *Consider the following diagram in* $\underline{SCG}$

$$\begin{tikzcd}
 & \mathbb{A} \arrow[rr, "0"] \arrow[dl, "F'"'] \arrow[dr, "F"] \arrow[d, phantom, "\simeq"] & \arrow[d, phantom, "\simeq\varphi"] & \mathbb{C} & \\
\underline{\mathrm{Ker}}(G) \arrow[rr, "\mathrm{K}(G)"'] & \, & \mathbb{B} \arrow[ur, "G"] \arrow[rr, "\mathrm{C}(F)"'] & \arrow[u, phantom, "\simeq"] & \underline{\mathrm{Coker}}(F) \arrow[ul, "G'"']
\end{tikzcd}$$

The following conditions are equivalent:

1. *F' is full and essentially surjective.*
2. *G' is full and faithful.*

4.15 Exercise. For an arrow $F\colon \mathbb{A} \to \mathbb{B}$ in $\underline{CG}$, check that

1. $\mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{0} \mathbb{I}$ is 2-exact iff F is full and essentially surjective,
2. $\mathbb{I} \xrightarrow{0} \mathbb{A} \xrightarrow{F} \mathbb{B}$ is 2-exact iff F is full and faithful.

The simple but fundamental fact about 2-exactness is the following result.

4.16 Lemma. *If the sequence in $\underline{CG}$*

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "G"] & \\ \mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] \arrow[rr, phantom, shift left=4, "\simeq\varphi"] & & \mathbb{C} \end{tikzcd}$$

is 2-exact, then

$$\pi_0\mathbb{A} \xrightarrow{\pi_0(F)} \pi_0\mathbb{B} \xrightarrow{\pi_0(G)} \pi_0\mathbb{C} \quad \text{and} \quad \pi_1\mathbb{A} \xrightarrow{\pi_1(F)} \pi_1\mathbb{B} \xrightarrow{\pi_1(G)} \pi_1\mathbb{C}$$

are exact sequences in the usual sense.

4.17 Remark. The converse of Lemma 4.16 is not tru, here is a counter-example. The sequence in $\underline{SCG}$

$$\mathbb{I} \to \mathbb{I} \to [\mathbb{Z}_2]_1$$

is not 2-exact. Nevertheless, its image under π_0 is $0 \to 0 \to 0$ and its image under π_1 is $0 \to 0 \to \mathbb{Z}_2$, and both are exact sequences.

4.18 Corollary.

1. *Consider an arrow in $\underline{CG}$ together with its kernel*

$$\underline{\mathrm{Ker}}(F) \xrightarrow{\mathrm{K}(F)} \mathbb{A} \xrightarrow{F} \mathbb{B}$$

We get an exact sequence of abelian groups and groups

$$\pi_1\underline{\mathrm{Ker}}(F) \to \pi_1\mathbb{A} \to \pi_1\mathbb{B} \to \pi_0\underline{\mathrm{Ker}}(F) \to \pi_0\mathbb{A} \to \pi_0\mathbb{B}$$

2. *Consider an arrow in $\underline{SCG}$ together with its kernel and its cokernel*

$$\underline{\mathrm{Ker}}(F) \xrightarrow{\mathrm{K}(F)} \mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{\mathrm{C}(F)} \underline{\mathrm{Coker}}(F)$$

We get an exact sequence of abelian groups

$$\begin{tikzcd} \pi_1\underline{\mathrm{Ker}}(F) \arrow[r] & \pi_1\mathbb{A} \arrow[r] & \pi_1\mathbb{B} \arrow[r] & \pi_1\underline{\mathrm{Coker}}(F) \arrow[dlll, equal] \\ \pi_0\underline{\mathrm{Ker}}(F) \arrow[r] & \pi_0\mathbb{A} \arrow[r] & \pi_0\mathbb{B} \arrow[r] & \pi_0\underline{\mathrm{Coker}}(F) \end{tikzcd}$$

Proof. 1. We construct the connecting homomorphism $\delta\colon \pi_1\mathbb{B} \to \pi_0\underline{\mathrm{Ker}}(F)$ as follows:

$$\delta\colon\ (b\colon I_{\mathbb{B}} \to I_{\mathbb{B}}) \ \mapsto\ [I_{\mathbb{A}}, b \cdot F_I \colon I_{\mathbb{B}} \to I_{\mathbb{B}} \to FI_{\mathbb{A}}]$$

We leave to the reader to check the exactness in $\pi_1\mathbb{B}$ and in $\pi_0\underline{\mathrm{Ker}}(F)$.
2. Obvious from Exercise 4.10 and Lemma 4.16. $\square$

The exact sequence of point 1 of Corollary 4.18 will appear again in the first part of Lesson 3.

4.19 Remark. As already observed in the proof of Proposition 4.3, the construction of $\underline{\mathrm{Ker}}(F)$ from $F\colon \mathbb{A} \to \mathbb{B}$ makes sense even if F is not an arrow in $\underline{\mathit{CG}}$, but just a pointed functor between pointed groupoids. Moreover, the universal properties stated in Proposition 4.3 are still valid. (What fails in this more general case is Exercise 4.2: it is no longer true that $\underline{\mathrm{Ker}}(F)$ measures the fulness and the faithfulness of F.) In the same way, the formal definition of 2-exactness still makes sense for a sequence $(F\varphi, G)$ in $\underline{\mathit{Grpd}}_*$, and point 1 of Corollary 4.18 remains true: from

$$\underline{\mathrm{Ker}}(F) \xrightarrow{\mathrm{K}(F)} \mathbb{A} \xrightarrow{F} \mathbb{B}$$

in $\underline{\mathit{Grpd}}_*$, we get an exact sequence of groups and pointed sets

$$\pi_1\underline{\mathrm{Ker}}(F) \to \pi_1\mathbb{A} \to \pi_1\mathbb{B} \to \pi_0\underline{\mathrm{Ker}}(F) \to \pi_0\mathbb{A} \to \pi_0\mathbb{B}$$

More in general, Lemma 4.16 holds true for 2-exact sequences in $\underline{\mathit{Grpd}}_*$.

To illustrate the notion of 2-exactness, we go back to some examples.

4.20 Example. Here is the expected example from algebraic topology. Consider a pointed continuous map between pointed topological spaces together with its homotopy kernel

$$\mathrm{K}(f) \xrightarrow{k(f)} X \xrightarrow{f} Y$$

Recall that the homotopy kernel is the subspace of $X \times Y^{[0,1]}$ of the pairs of the form $(x \in X, y\colon * \to f(x))$.

1. There is a 2-exact sequence of pointed groupoids

$$\Pi_1(\mathrm{K}(f)) \xrightarrow{\Pi_1(k(f))} \Pi_1(X) \xrightarrow{\Pi_1(f)} \Pi_1(Y)$$

2. In fact, there is a long 2-exact sequence of symmetric categorical groups, braided categorical groups, categorical groups, and pointed groupoids

$$\ldots \to \Pi_2(\mathrm{K}(f)) \to \Pi_2(X) \to \Pi_2(Y) \to \Pi_1(\mathrm{K}(f)) \to \Pi_1(X) \to \Pi_1(Y)$$

3. Therefore, by applying $\pi_0 \colon \underline{\mathit{Grpd}}_* \to \mathit{Set}_*$, we get the usual homotopy exact sequence

$$\ldots \to \pi_1(\mathrm{K}(f)) \to \pi_1(X) \to \pi_1(Y) \to \pi_0(\mathrm{K}(f)) \to \pi_0(X) \to \pi_0(Y)$$

Proof. 1. To be inserted.
2. We have to prove the 2-exactness of

$$\ldots \to \Pi_2(\mathrm{K}(f)) \to \Pi_2(X) \to \Pi_2(Y) \to \Pi_1(\mathrm{K}(f)) \to \Pi_1(X) \to \Pi_1(Y)$$

that is, of

$$\ldots \to \Pi_1(\Omega\,\mathrm{K}(f)) \to \Pi_1(\Omega X) \to \Pi_1(\Omega Y) \to \Pi_1(\mathrm{K}(f)) \to \Pi_1(X) \to \Pi_1(Y)$$

But the Puppe sequence

$$\ldots \to \Omega^2\,\mathrm{K}(f) \to \Omega^2 X \to \Omega^2 Y \to \Omega\,\mathrm{K}(f) \to \Omega X \to \Omega Y \to \mathrm{K}(f) \to X \to Y$$

is homotopy equivalent to the sequence of iterated homotopy kernels

$$\ldots \to \mathrm{K}(kkkf) \to \mathrm{K}(kkf) \to \mathrm{K}(kf) \to \mathrm{K}(f) \to X \to Y$$

We can therefore conclude by applying point 1 to each point of this last sequence.
3. Obvious from point 2, because π_0 sends 2-exact sequences in exact sequences (Lemma 4.16 and Remark 4.19). $\square$

4.21 Remark. Point 1 of Corollary 4.18 is the tip of an iceberg. In this course I don't need the whole iceberg, but in the next example I need one step more than in Corollary 4.18. If $F \colon \mathbb{A} \to \mathbb{B}$ is a pointed 2-functor between pointed 2-groupoids, we can construct the comma-2-groupoid

$$\underline{\underline{\underline{\mathrm{Ker}}}}(F) \xrightarrow{\mathrm{K}(F)} \mathbb{A} \xrightarrow{F} \mathbb{B}$$

If we apply π_0 (which identifies arrows when they are 2-isomorphic) and π_1 (the hom-groupoid at the base point), we get a 6-term 2-exact sequence of categorical groups and pointed groupoids

$$\pi_1\underline{\underline{\underline{\mathrm{Ker}}}}(F) \to \pi_1\mathbb{A} \to \pi_1\mathbb{B} \to \pi_0\underline{\underline{\underline{\mathrm{Ker}}}}(F) \to \pi_0\mathbb{A} \to \pi_0\mathbb{B}$$

If we apply again π_0 and π_1 and use the (easy to check) fact that

$$\pi_0(\pi_1\mathbb{A}) = \pi_1(\pi_0\mathbb{A})$$

we get a 9-term exact sequence of abelian groups, groups and pointed sets

$$\begin{tikzcd}
\pi_1^2\underline{\underline{\underline{\mathrm{Ker}}}}(F) \arrow[r] & \pi_1^2\mathbb{A} \arrow[r] & \pi_1^2\mathbb{B} \arrow[r] & \pi_1\pi_0\underline{\underline{\underline{\mathrm{Ker}}}}(F) \arrow[r] \arrow[dlll, equal] & \pi_1\pi_0\mathbb{A} \arrow[r] \arrow[dlll, equal] & \pi_1\pi_0\mathbb{B} \arrow[dlll, equal] \\
\pi_0\pi_1\underline{\underline{\underline{\mathrm{Ker}}}}(F) \arrow[r] & \pi_0\pi_1\mathbb{A} \arrow[r] & \pi_0\pi_1\mathbb{B} \arrow[r] & \pi_0^2\underline{\underline{\underline{\mathrm{Ker}}}}(F) \arrow[r] & \pi_0^2\mathbb{A} \arrow[r] & \pi_0^2\mathbb{B}
\end{tikzcd}$$

4.22 Example. Let $f\colon R \to S$ be a morphism of unital commutative rings. A classical result asserts that there exists an exact sequence connecting the groups of units, the Picard groups and the Brauer groups

$$\mathrm{U}(R) \to \mathrm{U}(S) \to ?? \to \mathrm{Pic}(R) \to \mathrm{Pic}(S) \to ?? \to \mathrm{Br}(R) \to \mathrm{Br}(R)$$

This sequence is a special case of the sequence constructed in Remark 4.21. Start with the "Azumaya complex" $\underline{\underline{\mathrm{Az}}}(R)$, whose objects are Azumaya R-algebras, arrows are Morita equivalences (or, equivalently, invertible bimodules), and 2-arrows are natural isomorphisms (or, equivalently, bimodule isomorphisms). $\underline{\underline{\mathrm{Az}}}(R)$ is a sub-bigroupoid of the bicategory $\underline{\underline{\mathrm{Bim}}}(R)$ introduced in Example 2.5. Comparing with Example 2.4, Example 2.5 and Exercise 2.6, we have:

- $\pi_0\underline{\underline{\mathrm{Az}}}(R) = \underline{\mathrm{Br}}(R)$, the Brauer categorical group of R,

- $\pi_1\underline{\underline{\mathrm{Az}}}(R) = \underline{\mathrm{Pic}}(R)$, the Picard categorical group of R,

- $\pi_0^2\underline{\underline{\mathrm{Az}}}(R) = \pi_0\underline{\mathrm{Br}}(R) = \mathrm{Br}(R)$, the Brauer group of R,

- $\pi_1^2\underline{\underline{\mathrm{Az}}}(R) = \pi_1\underline{\mathrm{Pic}}(R) = \mathrm{U}(R)$, the group of units of R,

- $\pi_0\pi_1\underline{\underline{\mathrm{Az}}}(R) = \pi_0\underline{\mathrm{Pic}}(R) = \mathrm{Pic}(R)$ (or $\pi_1\pi_0\underline{\underline{\mathrm{Az}}}(R) = \pi_1\underline{\mathrm{Br}}(R) = \mathrm{Pic}(R)$), the Picard group of R.

Moreover, a ring homomorphism $f\colon R \to S$ induces a 2-functor

$$F\colon \underline{\underline{\mathrm{Az}}}(R) \to \underline{\underline{\mathrm{Az}}}(S)\ ,\quad A \mapsto S \otimes_R A$$

Take now its kernel in the sense of Remark 4.21

$$\underline{\underline{\mathrm{Ker}}}(F) \to \underline{\underline{\mathrm{Az}}}(R) \to \underline{\underline{\mathrm{Az}}}(S)$$

Following the construction of Remark 4.21, we get first a 6-term 2-exact sequence of symmetric categorical groups

$$\pi_1\underline{\underline{\mathrm{Ker}}}(F) \to \underline{\mathrm{Pic}}(R) \to \underline{\mathrm{Pic}}(S) \to \pi_0\underline{\underline{\mathrm{Ker}}}(F) \to \underline{\mathrm{Br}}(R) \to \underline{\mathrm{Br}}(S)$$

and finally a 9-term exact sequence of abelian groups

$$\begin{tikzcd}
\pi_1^2\underline{\underline{\mathrm{Ker}}}(F) \arrow[r] & \mathrm{U}(R) \arrow[r] & \mathrm{U}(S) \arrow[r] & \pi_1\pi_0\underline{\underline{\mathrm{Ker}}}(F) \arrow[dllll] & \\
\mathrm{Pic}(R) \arrow[r] & \mathrm{Pic}(S) \arrow[r] & \pi_0^2\underline{\underline{\mathrm{Ker}}}(F) \arrow[r] & \mathrm{Br}(R) \arrow[r] & \mathrm{Br}(S)
\end{tikzcd}$$

References for Chapter 4

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Chapter 5

Abelian 2-categories

In order to unify and generalize results concerning diagram lemmas and exact sequences from modules to sheaves of modules, someone invented (or discovered?) abelian categories. Since several basic results on diagram lemmas and exact sequences have an analogue for symmetric categorical groups, let us try to invent abelian 2-categories. Of course, the basic example of an abelian 2-category should be the 2-category $\underline{SCG}$.

An abelian category can be defined in several equivalent ways. Here is probably the most elementary one.

5.1 Definition. A category $\mathcal{C}$ is abelian if the following conditions are satisfied.

1. $\mathcal{C}$ has a zero object,
2. $\mathcal{C}$ has binary products and binary coproducts,
3. $\mathcal{C}$ has kernels and cokernels,
4. In $\mathcal{C}$ each mono is a kernel and each epi is a cokernel.

Let us try to transpose this definition to symmetric categorical groups. In $\underline{SCG}$ the one-arrow category $\mathbb{I}$ plays the role of zero object. $\underline{SCG}$ has binary products and binary coproducts (in any possible sense), and they coincide. $\underline{SCG}$ has kernels and cokernels (in the convenient 2-categorical sense explained in Chapter 4). Before understanding the last condition in Definition 5.1, we have to understand what is a mono in a 2-category. Here is a possible answer: an arrow $F\colon \mathbb{A} \to \mathbb{B}$ in a 2-category is a mono if for every diagram of the form

$$\begin{tikzcd}
& \mathbb{A} \arrow[dr, "F"] & \\
\mathbb{X} \arrow[ur, "M"] \arrow[dr, "N"'] \arrow[rr, phantom, "\Downarrow \alpha"] & & \mathbb{B} \\
& \mathbb{A} \arrow[ur, "F"'] &
\end{tikzcd}$$

there exists a unique $\beta\colon M \Rightarrow N$ such that $\beta \cdot F = \alpha$.

5.2 Exercise. Show that an arrow $F\colon \mathbb{A} \to \mathbb{B}$ in $\underline{SCG}$ is a mono if and only if it is full and faithful. Dualize from monos to epis.

Consider now a full and faithful arrow $F\colon \mathbb{A} \to \mathbb{B}$ in $\underline{SCG}$ and apply Lemma 4.11

$$\begin{tikzcd}
& \underline{\mathrm{Ker}}(\mathrm{C}(F)) \arrow[dr, "\mathrm{K}(\mathrm{C}(F))"] \arrow[d, phantom, "\simeq"] & & \\
\mathbb{A} \arrow[ur, "F_1"] \arrow[rr, "F"'] & {} & \mathbb{B} \arrow[r, "\mathrm{C}(F)"] & \underline{\mathrm{Coker}}(F)
\end{tikzcd}$$

The comparison F_1 is an equivalence, and then F is a kernel (that is, it satisfies the universal property of the bikernel of $\mathrm{C}(F)$).

It seems that we have done: all the conditions of Definition 5.1 can be transposed to a 2-category and $\underline{SCG}$ satisfies them. No! The problem is that in condition 4 of Definition 5.1 there is an invisible part, and you can do nothing with the definition of abelian category if this invisible part does not hold. Here is the real condition 4.

4 In $\mathcal{C}$ each mono is a kernel and <u>each kernel is a mono</u>, each epi is a cokernel and <u>each cokernel is an epi</u>.

Unfortunately, in $\underline{SCG}$ kernels are not monos (and cokernels are not epis). Indeed, given a kernel in $\underline{SCG}$

$$\underline{\mathrm{Ker}}(F) \xrightarrow{\mathrm{K}(F)} \mathbb{A} \xrightarrow{F} \mathbb{B}$$

the arrow $\mathrm{K}(F)$ is always faithful, but it is almost never full (it is full if and only if $\pi_1\mathbb{B} = 0$).

Let us try with some other possible definitions of abelian category.

5.3 Definition. A category $\mathcal{C}$ is abelian if

1. it is additive, and

2. Barr-exact

or, equivalently, if

1. it is non-empty,

2. pre-additive (that is, enriched in Ab), and

3. Barr-exact

or, equivalently, if

1. it is semi-abelian, and

2. its dual is semi-abelian

or, equivalently, if

1. it has binary products and binary coproducts,

2. it has a zero object, and

3. it is Puppe-exact: for every arrow $f\colon A \to B$, the canonical comparison w_f in the following diagram is an isomorphism

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "k_f"] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "c_f"] & \mathrm{Coker}(f) \\
& \mathrm{Coker}(k_f) \arrow[r, "w_f"'] & \mathrm{Ker}(c_f) \arrow[u] &
\end{tikzcd}$$

With all but the last version of the definition of abelian category, the problem is that, despite some recent attempts, it is far to be clear what it means for a 2-category to be Barr-exact. What about Puppe-exactness? Here also there is a problem (keep in mind Lemma 4.11): if we start with an arrow $F\colon \mathbb{A} \to \mathbb{B}$ in $\underline{SCG}$, the comparison $w_F\colon \underline{\mathrm{Coker}}(\mathrm{K}(F)) \to \underline{\mathrm{Ker}}(\mathrm{C}(F))$ simply does not exist. It exists if and only if F is full, and in this case w_F is in fact an equivalence.

$$\begin{tikzcd}
& & \underline{\mathrm{Ker}}(\mathrm{C}(F)) \arrow[dr, "\mathrm{K}(\mathrm{C}(F))"] & & \\
\underline{\mathrm{Ker}}(F) \arrow[r, "\mathrm{K}(F)"] & \mathbb{A} \arrow[ur, "F_1"] \arrow[dr, "\mathrm{C}(\mathrm{K}(F))"'] \arrow[r, phantom, "\simeq"] & {} & \mathbb{B} \arrow[r, "\mathrm{C}(F)"] \arrow[l, phantom, "\simeq"'] & \underline{\mathrm{Coker}}(F) \\
& & \underline{\mathrm{Coker}}(\mathrm{K}(F)) \arrow[uu, dashed, "w_F"'] \arrow[ur, "F_2"'] & &
\end{tikzcd}$$

For this last problem we have a possible solution, but we need three new kinds of bilimits (and of bicolimits).
Convention: from now on, 2-category mean track 2-category, that is, we assume that all 2-arrows are invertible.

5.4 Definition. Let $\mathcal{B}$ be a 2-category with zero object.

1. The pip of an arrow $F\colon \mathbb{A} \to \mathbb{B}$ is a diagram of the form

$$\begin{tikzcd}
\underline{\mathrm{Pip}}(F) \arrow[r, bend left, "0"{name=U}] \arrow[r, bend right, "0"'{name=D}] & \mathbb{A} \arrow[r, "F"] & \mathbb{B}
\arrow[from=U, to=D, phantom, "\Downarrow \pi_F"]
\end{tikzcd}$$

such that $\pi_F \cdot F = \mathrm{id}_0$, and universal (in the sense of bilimits) with respect to such a condition. This means that, for any other diagram of the form

$$\begin{tikzcd} \mathbb{X} \arrow[r, bend left, "0"] \arrow[r, bend right, "0"'] \arrow[r, phantom, "\Downarrow\lambda"] & \mathbb{A} \arrow[r, "F"] & \mathbb{B} \end{tikzcd}$$

such that $\lambda \cdot F = \mathrm{id}_0$, there exists an arrow $L\colon \mathbb{X} \to \underline{\mathrm{Pip}}(F)$ such that $L \cdot \pi_F = \lambda$. Moreover, if $L'\colon \mathbb{X} \to \underline{\mathrm{Pip}}(F)$ is another arrow such that $L' \cdot \pi_F = \lambda$, then there exists a unique 2-arrow $\varphi\colon L \Rightarrow L'$ such that $\varphi \cdot \pi_F = \lambda$.

2. The root of a 2-arrow $\alpha\colon 0 \Rightarrow 0$ is a diagram of the form

$$\begin{tikzcd} \underline{\mathrm{Root}}(\alpha) \arrow[r, "\mathrm{R}_\alpha"] & \mathbb{A} \arrow[r, bend left, "0"] \arrow[r, bend right, "0"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{B} \end{tikzcd}$$

such that $\mathrm{R}_\alpha \cdot \alpha = \mathrm{id}_0$, and universal with respect to such a condition. This means that, for any other diagram of the form

$$\begin{tikzcd} \mathbb{X} \arrow[r, "T"] & \mathbb{A} \arrow[r, bend left, "0"] \arrow[r, bend right, "0"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{B} \end{tikzcd}$$

such that $T \cdot \alpha = \mathrm{id}_0$, there exists a pair $(T'\colon \mathbb{X} \to \underline{\mathrm{Root}}(\alpha), \tau'\colon T' \cdot \mathrm{R}_\alpha \Rightarrow T)$. Moreover, for any other pair $(T''\colon \mathbb{X} \to \underline{\mathrm{Root}}(\alpha), \tau''\colon T'' \cdot \mathrm{R}_\alpha \Rightarrow T)$, there exists a unique 2-arrow $\varphi\colon T' \Rightarrow T''$ such that

$$\begin{tikzcd} T' \cdot \mathrm{R}_\alpha \arrow[rr, Rightarrow, "\varphi \cdot \mathrm{R}_\alpha"] \arrow[dr, Rightarrow, "\tau'"'] & & T'' \cdot \mathrm{R}_\alpha \arrow[dl, Rightarrow, "\tau''"] \\ & T & \end{tikzcd}$$

3. The relative kernel of a sequence

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "G"] & \\ \mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] \arrow[rr, phantom, bend left, "\Uparrow\varphi"] & & \mathbb{C} \end{tikzcd}$$

is a diagram of the form

$$\begin{tikzcd} & \mathbb{A} \arrow[dr, "F"] & \\ \underline{\mathrm{Ker}}(F,\varphi) \arrow[ur, "\mathrm{K}(F,\varphi)"] \arrow[rr, "0"'] \arrow[rr, phantom, bend left, "\Uparrow k(F,\varphi)"] & & \mathbb{B} \end{tikzcd}$$

with $k(F,\varphi)$ and φ compatible

$$\begin{tikzcd} \mathrm{K}(F,\varphi)\cdot F\cdot G & & \mathrm{K}(F,\varphi)\cdot 0 \arrow[ll, Leftarrow, "\mathrm{K}(F,\varphi)\cdot\varphi"'] \arrow[d, equal] \\ 0\cdot G \arrow[u, Rightarrow, "k(F,\varphi)\cdot G"] \arrow[rr, equal] & & 0 \end{tikzcd}$$

and universal with respect to such a condition. This means that, for any other compatible diagram

$$\begin{tikzcd} & \mathbb{A} \arrow[dr, "F"] & \\ \mathbb{X} \arrow[ur, "E"] \arrow[rr, "0"'] \arrow[rr, phantom, bend left=20, "\Uparrow\psi"] & & \mathbb{B} \end{tikzcd} \qquad \begin{tikzcd} E\cdot F\cdot G & E\cdot 0 \arrow[l, Leftarrow, "E\cdot\varphi"'] \arrow[d, equal] \\ 0\cdot G \arrow[u, Rightarrow, "\psi\cdot G"] \arrow[r, equal] & 0 \end{tikzcd}$$

there exists a fill-in

$$\begin{tikzcd} & \underline{\mathrm{Ker}}(F,\varphi) \arrow[dr, "\mathrm{K}(F,\varphi)"] & \\ \mathbb{X} \arrow[ur, "E'"] \arrow[rr, "E"'] \arrow[rr, phantom, bend left=20, "\Downarrow\psi'"] & & \mathbb{A} \end{tikzcd} \qquad \begin{tikzcd} E'\cdot\mathrm{K}(F,\varphi)\cdot F \arrow[r, Rightarrow, "\psi'\cdot F"] & E\cdot F \\ E'\cdot 0 \arrow[u, Rightarrow, "E'\cdot k(F,\varphi)"] \arrow[r, equal] & 0 \arrow[u, Rightarrow, "\psi"'] \end{tikzcd}$$

Moreover, for any other fill-in

$$\begin{tikzcd} & \underline{\mathrm{Ker}}(F,\varphi) \arrow[dr, "\mathrm{K}(F,\varphi)"] & \\ \mathbb{X} \arrow[ur, "E''"] \arrow[rr, "E"'] \arrow[rr, phantom, bend left=20, "\Downarrow\psi''"] & & \mathbb{A} \end{tikzcd} \qquad \begin{tikzcd} E''\cdot\mathrm{K}(F,\varphi)\cdot F \arrow[r, Rightarrow, "\psi''\cdot F"] & E\cdot F \\ E''\cdot 0 \arrow[u, Rightarrow, "E''\cdot k(F,\varphi)"] \arrow[r, equal] & 0 \arrow[u, Rightarrow, "\psi"'] \end{tikzcd}$$

there exists a unique 2-arrow $\varepsilon\colon E' \Rightarrow E''$ such that

$$\begin{tikzcd} E'\cdot\mathrm{K}(F,\varphi) \arrow[rr, Rightarrow, "\varepsilon\cdot\mathrm{K}(F,\varphi)"] \arrow[dr, Rightarrow, "\psi'"'] & & E''\cdot\mathrm{K}(F,\varphi) \arrow[dl, Rightarrow, "\psi''"] \\ & E & \end{tikzcd}$$

5.5 Definition. Let $\mathcal{B}$ be a 2-category with zero object.

1. The copip of an arrow $F\colon \mathbb{A} \to \mathbb{B}$ is a diagram of the form

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] & \mathbb{B} \arrow[r, bend left, "0"] \arrow[r, bend right, "0"'] \arrow[r, phantom, "\Downarrow\sigma_F"] & \underline{\mathrm{Copip}}(F) \end{tikzcd}$$

such that $F\cdot\sigma_F = \mathrm{id}_0$, and universal with respect to such a condition.

2. The coroot of a 2-arrow $\alpha\colon 0 \Rightarrow 0$ is a diagram of the form

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "0"] \arrow[r, bend right, "0"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{B} \arrow[r, "\mathrm{C}_\alpha"] & \underline{\mathrm{Coroot}}(\alpha) \end{tikzcd}$$

such that $\alpha \cdot \mathrm{C}_\alpha = \mathrm{id}_0$, and universal with respect to such a condition.

3. The relative cokernel of a sequence

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "G"] \arrow[d, phantom, "\Uparrow\varphi"] & \\ \mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] & & \mathbb{C} \end{tikzcd}$$

is a diagram of the form

$$\begin{tikzcd} & \mathbb{C} \arrow[dr, "\mathrm{C}(\varphi,G)"] \arrow[d, phantom, "c(\varphi,G)\Downarrow"] & \\ \mathbb{B} \arrow[ur, "G"] \arrow[rr, "0"'] & & \underline{\mathrm{Coker}}(\varphi,G) \end{tikzcd}$$

with $c(\varphi, G)$ and φ compatible

$$\begin{tikzcd} F \cdot G \cdot \mathrm{C}(\varphi,G) \arrow[r, Rightarrow, "\varphi\cdot\mathrm{C}(\varphi,G)"] \arrow[d, Rightarrow, "F\cdot c(\varphi,G)"'] & 0 \cdot \mathrm{C}(\varphi,G) \arrow[d, equal] \\ F \cdot 0 \arrow[r, equal] & 0 \end{tikzcd}$$

and universal with respect to such a condition.

The existence of roots, pips and relative kernels in $\underline{\mathcal{CG}}$ and in $\underline{\mathcal{SCG}}$, as well as the existence of coroots, copips and relative cokernels in $\underline{\mathcal{SCG}}$ is guaranteed by the following proposition (where all (co)limits are intended in the sense of bi(co)limits).

5.6 Proposition. *Let $\mathcal{B}$ be a 2-category with a zero object (denoted by $\mathbb{I}$).*

1. *If $\mathcal{B}$ has kernels, then it has relative kernels.*

2. *If $\mathcal{B}$ has relative kernels, then it has kernels, roots and pips.*

3. *The same happens with the dual notions of relative cokernels, cokernels, coroots and copips.*

Proof. 1. The relative cokernel of a sequence

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "G"] \arrow[d, phantom, "\Uparrow\varphi"] & \\ \mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & \mathbb{C} \end{tikzcd}$$

is the kernel of the factorization $F' \colon \mathbb{A} \to \underline{\mathrm{Ker}}(G)$ of F through the kernel of G.
2. Consider a sequence and its relative kernel

$$\begin{tikzcd} \underline{\mathrm{Ker}}(F,\varphi) \arrow[rr, "0"] \arrow[dr, "\mathrm{K}(F,\varphi)"'] & {} \arrow[d, phantom, "\Downarrow k(F,\varphi)"] & \mathbb{B} \arrow[dr, "G"] \arrow[d, phantom, "\Uparrow\varphi"] & \\ & \mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & \mathbb{C} \end{tikzcd}$$

1. If $\mathbb{C} = \mathbb{I}$, then $\underline{\mathrm{Ker}}(F,\varphi) \simeq \underline{\mathrm{Ker}}(F)$.

2. If $\mathbb{B} = \mathbb{I}$, then $\underline{\mathrm{Ker}}(F,\varphi) \simeq \underline{\mathrm{Root}}(\varphi)$.

3. If $\mathbb{A} = \mathbb{I}$, then $\underline{\mathrm{Ker}}(F,\varphi) \simeq \underline{\mathrm{Pip}}(G)$.

□

5.7 Example. Using Proposition 5.6, we get the following explicit descriptions.

1. In $\underline{\mathcal{CG}}$, $\underline{\mathrm{Root}}(\alpha)$ is the full sub-category of $\mathbb{A}$ of the objects A such that $\alpha_A = \mathrm{id} \colon I \to I$.

2. In $\underline{\mathcal{CG}}$, $\underline{\mathrm{Pip}}(F)$ is the discrete categorical group $[\pi_1(\underline{\mathrm{Ker}}(F)]_0$. In particular, $\underline{\mathrm{Pip}}(\mathbb{A} \to \mathbb{I}) = [\pi_1\mathbb{A}]_0 = \underline{\mathrm{Ker}}(\mathbb{I} \to \mathbb{A})$.

3. In $\underline{\mathcal{SCG}}$, $\underline{\mathrm{Coroot}}(\alpha)$ has the same objects than $\mathbb{B}$ and arrows are equivalence classes of arrows of $\mathbb{B}$, with $f, f' \colon X \to Y$ equivalent if there exists $A \in \mathbb{A}$ such that

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Y \simeq I \otimes Y \arrow[rr, "\alpha_A \otimes \mathrm{id}"'] & & I \otimes Y \simeq Y \end{tikzcd}$$

4. In $\underline{\mathcal{SCG}}$, $\underline{\mathrm{Copip}}(F)$ is the connected categorical group $[\pi_0(\underline{\mathrm{Coker}}(F)]_1$. In particular, $\underline{\mathrm{Copip}}(\mathbb{I} \to \mathbb{A}) = [\pi_0\mathbb{A}]_1 = \underline{\mathrm{Coker}}(\mathbb{A} \to \mathbb{I})$.

5.8 Definition. A 2-category $\mathcal{B}$ is abelian if

1. it has a zero object,

2. it has binary products and binary coproducts, and

3. it is Puppe-exact: for every arrow $F\colon \mathbb{A} \to \mathbb{B}$, the canonical comparisons w_F and v_F appearing in the following diagrams are equivalences

$$\begin{tikzcd}
\underline{\mathrm{Pip}}(F) \arrow[rr, bend left, "0"] \arrow[rr, bend right, "0"'] \arrow[rr, phantom, "\Downarrow \pi_F"] & & \mathbb{A} \arrow[r, "F"] \arrow[d, "\mathrm{C}_{\pi_F}"'] \arrow[dr, phantom, "\simeq"] & \mathbb{B} \arrow[r, "\mathrm{C}(F)"] & \underline{\mathrm{Coker}}(F) \\
& & \underline{\mathrm{Coroot}}(\pi_F) \arrow[r, "w_F"'] & \underline{\mathrm{Ker}}(\mathrm{C}(F)) \arrow[u, "\mathrm{K}(\mathrm{C}(F))"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathrm{Ker}}(F) \arrow[r, "\mathrm{K}(F)"] & \mathbb{A} \arrow[r, "F"] \arrow[d, "\mathrm{C}(\mathrm{K}(F))"'] \arrow[dr, phantom, "\simeq"] & \mathbb{B} \arrow[rr, bend left, "0"] \arrow[rr, bend right, "0"'] \arrow[rr, phantom, "\sigma_F \Downarrow"] & & \underline{\mathrm{Copip}}(F) \\
& \underline{\mathrm{Coker}}(\mathrm{K}(F)) \arrow[r, "v_F"'] & \underline{\mathrm{Root}}(\sigma_F) \arrow[u, "\mathrm{R}_{\sigma_F}"'] & &
\end{tikzcd}$$

Before giving some formal consequences of the definition of abelian 2-category, and in particular in order to generalize Lemma 4.11 on factorizations and Proposition 4.14 on 2-exact sequences, we need a point of terminology.

5.9 Definition. Consider an arrow $F\colon \mathbb{A} \to \mathbb{B}$ in a 2-category $\mathcal{B}$ and, for any $\mathbb{X} \in \mathcal{B}$, the induced functors

$$\mathcal{B}(-,F)\colon \mathcal{B}(\mathbb{X},\mathbb{A}) \to \mathcal{B}(\mathbb{X},\mathbb{B}) \qquad \mathcal{B}(F,-)\colon \mathcal{B}(\mathbb{B},\mathbb{X}) \to \mathcal{B}(\mathbb{A},\mathbb{X})$$

1. F is faithful when $\mathcal{B}(-,F)$ is faithful for all $\mathbb{X} \in \mathcal{B}$,
2. F is fully faithful when $\mathcal{B}(-,F)$ is full and faithful for all $\mathbb{X} \in \mathcal{B}$,
3. F is cofaithful when $\mathcal{B}(F,-)$ is faithful for all $\mathbb{X} \in \mathcal{B}$,
4. F is fully cofaithful when $\mathcal{B}(F,-)$ is full and faithful for all $\mathbb{X} \in \mathcal{B}$.

5.10 Exercise. (Compare with Exercise 5.2.) Show that an arrow in $\underline{\mathcal{SCG}}$ is

1. faithful in the sense of Definition 5.9 iff it is faithful in the usual sense,
2. fully faithful in the sense of Definition 5.9 iff it is full and faithful in the usual sense,
3. cofaithful in the sense of Definition 5.9 iff it is essentially surjective in the usual sense,
4. fully cofaithful in the sense of Definition 5.9 iff it is full and essentially surjective in the usual sense.

In the next Theorem, I list some of the fundamental results which can be proved in any abelian 2-category.

5.11 Theorem. *Let $\mathcal{B}$ be an abelian 2-category.*

1. (a) *$\mathcal{B}$ is preadditive in a 2-categorical sense, that is, it is enriched in $\underline{SCG}$.*
   (b) *$\mathcal{B}$ is additive, that is, binary products and binary coproducts are equivalent.*

2. *$\mathcal{B}$ admits two factorization systems:*
   (a) *Every arrow in $\mathcal{B}$ has a (fully cofaithful, faithful) factorization (use the Coroot-Ker decomposition of Definition 5.8).*
   (b) *Every arrow in $\mathcal{B}$ has a (cofaithful, fully faithful) factorization (use the Coker-Root decomposition of Definition 5.8).*

3. *In $\mathcal{B}$ there are well-defined notions of relative homology and relative 2-exactness of a complex. A complex in $\mathcal{B}$ is a diagram of the form*

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "X"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=30, "\alpha\Downarrow"] & \mathbb{A} \arrow[r, "F"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=30, "\varphi\Uparrow"] & \mathbb{B} \arrow[r, "G"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=30, "\Downarrow\gamma"] & \mathbb{C} \arrow[r, "Y"] & \mathbb{Y}
\end{tikzcd}$$

with α and φ compatible, and φ and γ compatible

$$\begin{tikzcd}
X \cdot F \cdot G \arrow[r, Rightarrow, "\alpha \cdot G"] \arrow[d, Rightarrow, "X \cdot \varphi"'] & 0 \cdot G \arrow[d, equal] \\
X \cdot 0 \arrow[r, equal] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
F \cdot G \cdot Y \arrow[r, Rightarrow, "F \cdot \gamma"] \arrow[d, Rightarrow, "\varphi \cdot Y"'] & F \cdot 0 \arrow[d, equal] \\
0 \cdot Y \arrow[r, equal] & 0
\end{tikzcd}$$

Consider the factorizations

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "X"] \arrow[dr, "0"'] \arrow[dr, phantom, shift left=3, "\bar{\alpha}\Downarrow"] & \mathbb{A} \arrow[r, "F"] \arrow[d, "F'"] \arrow[dr, phantom, "\simeq"] & \mathbb{B} \arrow[r, "G"] \arrow[dr, "\mathrm{C}(\alpha,F)"'] \arrow[dr, phantom, shift left=3, "\simeq"] & \mathbb{C} \arrow[r, "Y"] \arrow[dr, phantom, "\Rightarrow\bar{\gamma}"] & \mathbb{Y} \\
& \underline{\mathrm{Ker}}(G,\gamma) \arrow[ur, "\mathrm{K}(G,\gamma)"'] \arrow[d] \arrow[drr, phantom, "\simeq"] & & \underline{\mathrm{Coker}}(\alpha,F) \arrow[u, "G'"'] \arrow[ur, "0"'] & \\
& \underline{\mathrm{Coker}}(\bar{\alpha},F') \arrow[rr, dashed] & & \underline{\mathrm{Ker}}(G',\bar{\gamma}) \arrow[u] &
\end{tikzcd}$$

   (a) *The canonical comparison $\underline{\mathrm{Coker}}(\bar{\alpha},F') \to \underline{\mathrm{Ker}}(G',\bar{\gamma})$ is an equivalence. The object $\underline{\mathrm{Coker}}(\bar{\alpha},F')$ can be called the relative homology of the complex.*
   (b) *The following conditions are equivalent. When they are satisfied, we say that the complex is relative 2-exact.*
      i. *$\mathrm{K}(G,\gamma)$ is the kernel of $\mathrm{C}(\alpha,F)$.*

ii. $\mathrm{C}(\alpha, F)$ *is the cokernel of* $\mathrm{K}(G, \gamma)$.

iii. *The relative homology of the complex is trivial:* $\underline{\mathrm{Coker}}(\bar{\alpha}, F') \simeq \mathbb{I}$.

4. *(This is a special case of point 3, obtained by taking* $\mathbb{X} \simeq \mathbb{I} \simeq Y$*.) In* $\mathcal{B}$ *there are well-defined notions of homology and 2-exactness of a sequence. Consider a sequence* (F, φ, G) *and the factorizations*

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "F"] \arrow[rrrr, bend left, "0"] \arrow[rrrr, phantom, bend left=15, "\simeq\varphi"] \arrow[dr, "F'"'] & \arrow[d, phantom, "\simeq"] & \mathbb{B} \arrow[rr, "G"] \arrow[dr, "\mathrm{C}(F)"] & \arrow[d, phantom, "\simeq"] & \mathbb{C} \\
& \underline{\mathrm{Ker}}(G) \arrow[ur, "\mathrm{K}(G)"] \arrow[d] \arrow[rr, phantom, "\simeq"] & & \underline{\mathrm{Coker}}(F) \arrow[ur, "G'"'] & \\
& \underline{\mathrm{Coker}}(F') \arrow[rr, dashed] & & \underline{\mathrm{Ker}}(G') \arrow[u] &
\end{tikzcd}$$

(a) *The canonical comparison* $\underline{\mathrm{Coker}}(F') \to \underline{\mathrm{Ker}}(G')$ *is an equivalence. The object* $\underline{\mathrm{Coker}}(F')$ *can be called the homology of the sequence.*

(b) *The following conditions are equivalent. When they are satisfied, we say that the sequence is 2-exact.*

i. $\mathrm{K}(G)$ *is the kernel of* $\mathrm{C}(F)$.

ii. $\mathrm{C}(F)$ *is the cokernel of* $\mathrm{K}(G)$.

iii. F' *is fully cofaithful.*

iv. G' *is fully faithful.*

v. *The homology of the sequence is trivial:* $\underline{\mathrm{Coker}}(F') \simeq \mathbb{I}$.

5. *In* $\mathcal{B}$ *there is a well-defined notion of extension. Consider a sequence*

$$\begin{tikzcd}
& \mathbb{B} \arrow[dr, "G"] \arrow[d, phantom, "\Uparrow\varphi"] & \\
\mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & \mathbb{C}
\end{tikzcd}$$

The following conditions are equivalent. When they are satisfied, we say that the sequence is an extension.

(a) (F, φ) *is the kernel of* G *and* (G, φ) *is the cokernel of* F.

(b) *The complex*

$$\begin{tikzcd}
\mathbb{I} \arrow[r, "0"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\simeq"] & \mathbb{A} \arrow[r, "F"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\varphi\Downarrow"] & \mathbb{B} \arrow[r, "G"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\simeq"] & \mathbb{C} \arrow[r, "0"] & \mathbb{I}
\end{tikzcd}$$

is relative 2-exact in each point.

6. *The long homology sequence. Consider an extension of* $\mathbb{N}$*-complexes in* $\mathcal{B}$

$$\begin{tikzcd} & \mathbb{B}_\bullet \arrow[dr, "G_\bullet"] & \\ \mathbb{A} \arrow[ur, "F_\bullet"] \arrow[rr, "0"'] \arrow[rr, phantom, bend left=20, "\Uparrow \varphi_\bullet"] & & \mathbb{C}_\bullet \end{tikzcd}$$

There is a long sequence constructed with the relative homology of each complex (as in point 3)

$$\begin{tikzcd} \ldots \mathbb{H}^n(\mathbb{A}_\bullet) \arrow[r] \arrow[rr, bend right, "0"', "\simeq"] & \mathbb{H}^n(\mathbb{B}_\bullet) \arrow[r] \arrow[rr, bend left, "0", "\simeq"'] & \mathbb{H}^n(\mathbb{C}_\bullet) \arrow[r] \arrow[rr, bend right, "0"', "\simeq"] & \mathbb{H}^{n+1}(\mathbb{A}_\bullet) \arrow[r] & \mathbb{H}^{n+1}(\mathbb{B}_\bullet) \ldots \end{tikzcd}$$

and such a sequence is 2-exact in each point.

5.12 Exercise. The definition of extension given in point 5 of Theorem 5.11 deserves some comments.

1. (Compare with Exercise 4.15.) For an arrow $F\colon \mathbb{A} \to \mathbb{B}$ in $\mathcal{B}$, show that

   (a) $\mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{0} \mathbb{I}$ is relative 2-exact iff F is cofaithful,

   (b) $\mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{0} \mathbb{I}$ is 2-exact iff F is fully cofaithful,

   (c) $\mathbb{I} \xrightarrow{0} \mathbb{A} \xrightarrow{F} \mathbb{B}$ is relative 2-exact iff F is faithful,

   (d) $\mathbb{I} \xrightarrow{0} \mathbb{A} \xrightarrow{F} \mathbb{B}$ is 2-exact iff F is fully faithful.

2. Show that 2-exactness implies relative 2-exactness.

3. Consider the "trivial extension"

$$\mathbb{I} \to \mathbb{A} \to \mathbb{A} \times \mathbb{B} \to \mathbb{B} \to \mathbb{I}$$

   Show that this sequence is relative 2-exact in each point (so, it is an extension), but it is not 2-exact in $\mathbb{A}$ and in $\mathbb{B}$.

5.13 Example.

1. <u>*SCG*</u> is an abelian 2-category.

2. If $\mathcal{D}$ is any (small) 2-category and $\mathcal{B}$ is an abelian 2-category, the 2-category of 2-functors 2-$\underline{\mathrm{Funct}}[\mathcal{D}, \mathcal{B}]$ is abelian.

3. If $\mathcal{D}$ is a preadditive (that is, enriched in $\underline{SCG}$) small 2-category and $\mathcal{B}$ is an abelian 2-category, the 2-category of additive 2-functors 2-$\underline{\mathrm{Add}}[\mathcal{D}, \mathcal{B}]$ is abelian. (Recall that a 2-functor $F \colon \mathcal{D} \to \mathcal{B}$ is additive if all the induced hom-functors
$$F_{X,Y} \colon \mathcal{D}(X, Y) \to \mathcal{B}(FX, FY)$$
are symmetric monoidal functors.)

4. In particular, in point 3 we can take as $\mathcal{D}$ a "categorical ring", that is, a preadditive 2-category with only one object.

5. The full sub-2-category of $\underline{SCG}$ of those symmetric categorical groups $\mathbb{A}$ such that, for every object $X \in \mathbb{A}$,
$$c_{X,X} \colon X \otimes X \to X \otimes X$$
is the identity, is abelian. This is a special case of point 4, taking as $\mathcal{D}$ the discrete categorical ring $[\mathbb{Z}]_0$.

6. A more sophisticated example coming from algebraic geometry. The 2-category of pre-stacks and the 2-category of stacks on a topological space (or on a Grothendieck site) are abelian.

References for Chapter 5

Bla

Chapter 6

Abelian categories and abelian 2-categories

Let me start with an exercise.

6.1 Exercise. Consider a complex of abelian groups

$$A_\bullet\colon \quad \ldots \to A_{n-1} \to A_n \to A_{n+1} \to \ldots$$

with homology groups $\mathrm{H}^n(A_\bullet)$. We can embed it in the abelian 2-category $\underline{\mathit{SCG}}$ in two different ways:

$$[A_\bullet]_0\colon \quad \ldots \to [A_{n-1}]_0 \to [A_n]_0 \to [A_{n-1}]_0 \to \ldots$$

$$[A_\bullet]_1\colon \quad \ldots \to [A_{n-1}]_1 \to [A_n]_1 \to [A_{n+1}]_1 \to \ldots$$

and then construct the relative homology categorical groups of the two complexes as in Theorem 5.11

$$\mathbb{H}^n([A_\bullet]_0) \quad \text{and} \quad \mathbb{H}^n([A_\bullet]_1)$$

Check that

1. $\pi_0(\mathbb{H}^n([A_\bullet]_0)) = \mathrm{H}^n(A_\bullet) = \pi_1(\mathbb{H}^{n+1}([A_\bullet]_0))$
2. $\pi_0(\mathbb{H}^n([A_\bullet]_1)) = \mathrm{H}^{n+1}(A_\bullet) = \pi_1(\mathbb{H}^{n+1}([A_\bullet]_1))$

The previous exercise suggests the missing item in Example 5.13. If I want to generalize Exercise 6.1 replacing the category Ab of abelian groups with an arbitrary abelian category $\mathcal{A}$, I have to construct an abelian 2-category $\mathcal{B}(\mathcal{A})$ out from $\mathcal{A}$ and two embeddings $[-]_0, [-]_1\colon \mathcal{A} \to \mathcal{B}(\mathcal{A})$.

The first candidate is $\mathcal{B}(\mathcal{A}) = \underline{\mathit{Grpd}}(\mathcal{A})$ or, equivalently, $\mathcal{B}(\mathcal{A}) = \mathcal{A}^{\to}$. The 2-category $\underline{\mathit{Grpd}}(\mathcal{A})$ is defined precisely as $\underline{\mathit{Grpd}}\mathit{(Grp)}$ in Definition 3.1, but all the diagrams are in $\mathcal{A}$ instead of being in *Grp*. The 2-category $\mathcal{A}^{\rightarrow}$ has objects, arrows and 2-arrows depicted in the following diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A_0 \arrow[ur, "\alpha"] \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] & B_0 \end{tikzcd}$$

with the conditions

$$a \cdot f_0 = f \cdot b\ ,\quad a \cdot g_0 = g \cdot b\ ,\quad a \cdot \alpha = f - g\ ,\quad \alpha \cdot b = f_0 - g_0$$

The biequivalence $\underline{\mathit{Grpd}}(\mathcal{A}) \simeq \mathcal{A}^{\rightarrow}$ is a special case of the Dold-Kan correspondence. If you prefer, the biequivalence $\underline{\mathit{Grpd}}(\mathcal{A}) \simeq \mathcal{A}^{\rightarrow}$ can be seen as the abelian version of the equivalence $\underline{\mathit{Grpd}}(\mathit{Grp}) \simeq \mathit{XMod}$ of Proposition 3.3.

Question: if $\mathcal{A}$ is an abelian category, is $\underline{\mathit{Grpd}}(\mathcal{A})$ an abelian 2-category? The answer is: no, almost never. To make clear the answer, I need one more exercise and a Lemma.

6.2 Exercise. (Compare with Exercise 5.10.) Let $\mathcal{A}$ be an abelian category and consider an internal functor $F \colon \mathbb{A} \to \mathbb{B}$

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & B_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

1. F is fully faithful in the sense of Definition 5.9 if and only if it is internally full and faithful. This means that the following is a limit diagram

$$\begin{tikzcd} & & A_1 \arrow[dll, "d"'] \arrow[d, "f_1"] \arrow[drr, "c"] & & \\ A_0 \arrow[dr, "f_0"'] & & B_1 \arrow[dl, "d"] \arrow[dr, "c"'] & & A_0 \arrow[dl, "f_0"] \\ & B_0 & & B_0 & \end{tikzcd}$$

2. F is cofaithful in the sense of Definition 5.9 if and only if it is internally essentially surjective. This means that in the following diagram (where the square is a pullback), the composite $t_2 \cdot c$ is an epi

$$\begin{tikzcd} A_0 \times_{F_0,d} B_1 \arrow[r, "t_2"] \arrow[d, "t_1"'] & B_1 \arrow[r, "c"] \arrow[d, "d"] & B_0 \\ A_0 \arrow[r, "F_0"'] & B_0 & \end{tikzcd}$$

3. F is an equivalence if and only if it is internally full and faithful and the composite $t_2 \cdot c$ is a split epi.

6.3 Lemma. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be an arrow in an abelian 2-category $\mathcal{B}$. The following coditions are equivalent:*

1. *F is faithful and fully cofaithful.*
2. *F is cofaithful and fully faithful.*
3. *F is an equivalence.*

Exercise 6.2 and Lemma 6.3 tell us that in an abelian 2-category the difference between weak equivalences and equivalences disappears (we already meet this fact in $\underline{\mathit{SCG}}$, see Exercise 1.5), whereas this is not the case in $\underline{\mathit{Grpd}}(\mathcal{A})$.

6.4 Corollary. *Let $\mathcal{A}$ be an abelian category. The following conditions are equivalent:*

1. *The 2-category $\underline{\mathit{Grpd}}(\mathcal{A})$ is abelian.*
2. *In $\underline{\mathit{Grpd}}(\mathcal{A})$ every weak equivalence is an equivalence.*
3. *In $\mathcal{A}$ each object is projective.*

6.5 Example. If the abelian category $\mathcal{A}$ is the category of vector spaces on a field, then the 2-category $\underline{\mathit{Grpd}}(\mathcal{A})$ is abelian.

The equivalence between condition 1 and condition 2 in Corollary 6.4 (and the non-abelian situation already studied in Proposition 3.8) suggests how to get an abelian 2-category from an abelian category.

6.6 Proposition. *Let $\mathcal{A}$ be an abelian category. The bicategory of fractions of $\underline{\mathit{Grpd}}(\mathcal{A})$ with respect to weak equivalences*

$$\underline{\mathit{Grpd}}(\mathcal{A}) \to \underline{\mathit{Grpd}}(\mathcal{A})[\Sigma]$$

is an abelian 2-category.

Proof. The proof consists in giving an explicit description of the bicategory of fractions $\underline{\mathit{Grpd}}(\mathcal{A})[\Sigma]$ using "butterflies"

$$\Gamma\colon \underline{\mathit{Grpd}}(\mathcal{A}) \simeq \mathcal{A}^{\to} \to \underline{\mathit{Bfly}}(\mathcal{A})$$

I will leave to the reader to check that the bicategory $\underline{\mathit{Bfly}}(\mathcal{A})$ is abelian.
The objects of $\underline{\mathit{Bfly}}(\mathcal{A})$ are those of $\mathcal{A}^{\to}$, that is, the arrows of $\mathcal{A}$. An arrow $F\colon \mathbb{A} \to \mathbb{B}$ in $\underline{\mathit{Bfly}}(\mathcal{A})$ is a butterfly, that is, a diagram of the form

$$\begin{tikzcd}
A \arrow[dr, "m"] \arrow[dd, "a"'] & & B \arrow[dl, "n"'] \arrow[dd, "b"] \\
& F \arrow[dl, "s"] \arrow[dr, "t"'] & \\
A_0 & & B_0
\end{tikzcd}$$

with $m \cdot t = 0$ and

$$0 \to B \to F \to A_0 \to 0$$

exact. The 2-arrows $f \colon F \Rightarrow F'$ are as in the following diagram

$$\begin{tikzcd}
 & F \arrow[ddl, "s"'] \arrow[ddr, "t"] \arrow[ddd, "f"] & \\
A \arrow[ur, "m"] \arrow[d, "a"'] \arrow[ddr, "m'"] & & B \arrow[ul, "n"'] \arrow[d, "b"] \arrow[ddl, "n'"'] \\
A_0 & & B_0 \\
 & F' \arrow[ul, "s'"] \arrow[ur, "t'"'] &
\end{tikzcd}$$

Since in particular f is a morphism of extensions, it is an isomorphism. Composition of butterflies is depicted in the following diagram, where $q \colon F \times_{t,s'} F' \to Q$ is the cokernel of $\langle n, m' \rangle \colon B \to F \times_{t,s'} F'$

$$\begin{tikzcd}
 & & Q \arrow[ddddll, bend right, "\overline{r_1 \cdot s}"'] \arrow[ddddrr, bend left, "\overline{r_2 \cdot t'}"] & & \\
 & & F \times_{t,s'} F' \arrow[u, "q"] \arrow[ddl, "r_1"'] \arrow[ddr, "r_2"] & & \\
A \arrow[urr, "\langle m,0 \rangle"] \arrow[dr, "m"] \arrow[dd, "a"'] & & B \arrow[u, "\langle n,m' \rangle"] \arrow[dl, "n"] \arrow[dr, "m'"'] \arrow[dd, "b"] & & C \arrow[ull, "\langle 0,n' \rangle"'] \arrow[dl, "n'"'] \arrow[dd, "c"] \\
 & F \arrow[dl, "s"] \arrow[dr, "t"'] & & F' \arrow[dl, "s'"] \arrow[dr, "t'"'] & \\
A_0 & & B_0 & & C_0
\end{tikzcd}$$

The identity butterfly on an object $\mathbb{A}$ is depicted in the following diagram

$$\begin{tikzcd}
A \arrow[dr, "\langle a, -\mathrm{id} \rangle"] \arrow[dd, "a"'] & & A \arrow[dl, "i_2"'] \arrow[dd, "a"] \\
 & A_0 \times A \arrow[dl, "p_1"] \arrow[dr, "{[\mathrm{id}, a]}"'] & \\
A_0 & & A_0
\end{tikzcd}$$

There is a 2-functor $\Gamma\colon \mathcal{A}^{\to} \to \underline{\mathit{Bfly}}(\mathcal{A})$ defined by

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
A \arrow[dr, "{\langle a,-f\rangle}"] \arrow[dd, "a"'] & & B \arrow[dl, "i_2"'] \arrow[dd, "b"] \\
& A_0 \times B \arrow[dl, "p_1"'] \arrow[dr, "{[f_0,b]}"] & \\
A_0 & & B_0
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A_0 \arrow[ur, "\alpha"] \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] & B_0
\end{tikzcd}
\quad \mapsto \quad
A_0 \times B \xrightarrow{\begin{pmatrix} \mathrm{id} & 0 \\ \alpha & \mathrm{id} \end{pmatrix}} A_0 \times B
$$

Now I list the main steps to prove the universal property of $\Gamma\colon \mathcal{A}^{\to} \to \underline{\mathit{Bfly}}(\mathcal{A})$.

1. An arrow in $\mathcal{A}^{\to}$
$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}
$$
is a weak equivalence if and only if the square is a pullback and a pushout, if and only if the sequence
$$0 \longrightarrow A \xrightarrow{\langle a,-f\rangle} A_0 \times B \xrightarrow{[f_0,b]} B_0 \longrightarrow 0$$
is exact. The arrow is an equivalence if and only if the same sequence is split exact.

2. A butterfly $F\colon \mathbb{A} \to \mathbb{B}$ is an equivalence if and only if the sequence
$$0 \to A \to F \to B_0 \to 0$$
is exact. When this is the case, a quasi-inverse $F^*\colon \mathbb{B} \to \mathbb{A}$ of F is given by the flipped butterfly.

3. There is an equivalence of categories
$$\mathcal{A}^{\to}(\mathbb{A},\mathbb{B}) \simeq \underline{\mathit{Bfly}}(\mathcal{A})(\mathbb{A},\mathbb{B})_{\mathrm{Split}}$$
where a butterfly $F\colon \mathbb{A} \to \mathbb{B}$ is split when the sequence
$$0 \to B \to F \to A_0 \to 0$$

is split exact. Indeed, if $s^* \colon A_0 \to F$ and $n^* \colon F \to B$ are such that $s^* \cdot s = \mathrm{id}_{A_0}$ and $n \cdot n^* = \mathrm{id}_B$, then we get the following arrow in $\mathcal{A}^{\to}$

$$\begin{tikzcd} A \arrow[r, "-m\cdot n^*"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A_0 \arrow[r, "s^*\cdot t"'] & B_0 \end{tikzcd}$$

4. For every butterfly $F \colon \mathbb{A} \to \mathbb{B}$, there is a span in $\mathcal{A}^{\to}$

$$\begin{tikzcd} A \arrow[d, "a"'] & A \times B \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "{[m,n]}"'] & B \arrow[d, "b"] \\ A_0 & F \arrow[l, "s"] \arrow[r, "t"'] & B_0 \end{tikzcd}$$

with the left leg a weak equivalence. Moreover, this provides a tabulation of F:

$$\begin{tikzcd} & \Gamma[m,n] \arrow[dl, "\Gamma(\text{left leg})"'] \arrow[dr, "\Gamma(\text{right leg})"] \arrow[d, phantom, "\simeq"] & \\ \mathbb{A} \arrow[rr, "F"'] & {} & \mathbb{B} \end{tikzcd}$$

□

Now that we are able to construct the abelian 2-category $\underline{Bfly}(\mathcal{A})$ from an abelian category $\mathcal{A}$, we can look at the converse problem: is it possible to construct an abelian category starting from a given abelian 2-category? The hint to answer this question comes from Exercise 4.7: in $\underline{SCG}$ we have that

$$\underline{\mathrm{Ker}}(\mathbb{I} \to \mathbb{G}) = [\pi_1\mathbb{G}]_0 \ , \ \ \underline{\mathrm{Coker}}(\mathbb{G} \to \mathbb{I}) = [\pi_0\mathbb{G}]_1$$

To generalize this situation, we need a simple construction inspired to the category $\mathcal{A}^{\to}$.

6.7 Definition. Let $\mathcal{B}$ be a 2-category. The 2-category $\mathcal{B}^{\to}$ has objects, arrows and 2-arrows depicted in the following diagram:

$$\begin{tikzcd}[row sep=huge] \mathbb{A} \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Uparrow\alpha"] \arrow[d, "a"'] \arrow[dr, phantom, "\varphi\Rightarrow \quad \psi\Rightarrow"] & \mathbb{B} \arrow[d, "b"] \\ \mathbb{A}_0 \arrow[r, bend left, "g_0"] \arrow[r, bend right, "f_0"'] \arrow[r, phantom, "\Uparrow\alpha_0"] & \mathbb{B}_0 \end{tikzcd}$$

The objects are a and b, the arrows are the triples $(f, f_0, \varphi\colon a \cdot f_0 \Rightarrow f \cdot b)$ and $(g, g_0, \psi\colon a \cdot g_0 \Rightarrow g \cdot b$, and the 2-arrow is the pair $(\alpha\colon f \Rightarrow g, \alpha_0\colon f_0 \Rightarrow g_0)$ such that

$$\begin{tikzcd} a \cdot f_0 \arrow[r, Rightarrow, "a\cdot\alpha_0"] \arrow[d, Rightarrow, "\varphi"'] & a \cdot g_0 \arrow[d, Rightarrow, "\psi"] \\ f \cdot b \arrow[r, Rightarrow, "\alpha\cdot b"'] & g \cdot b \end{tikzcd}$$

6.8 Remark. Assume now that the 2-category $\mathcal{B}$ has kernels and cokernels, in the sense of bilimits. The universal properties of the kernel and the cokernel give a biadjunction

$$\mathcal{B}^{\to} \underset{\underline{\mathrm{Ker}}}{\overset{\underline{\mathrm{Coker}}}{\rightleftarrows}} \mathcal{B}^{\to} \qquad \underline{\mathrm{Coker}} \dashv \underline{\mathrm{Ker}}$$

In particular, for each object $\mathbb{X} \in \mathcal{B}$, the above biadjunction restricts to a biadjunction

$$\mathbb{X} \downarrow \mathcal{B} \underset{\underline{\mathrm{Ker}}}{\overset{\underline{\mathrm{Coker}}}{\rightleftarrows}} \mathcal{B} \downarrow \mathbb{X} \qquad \underline{\mathrm{Coker}} \dashv \underline{\mathrm{Ker}}$$

Finally, for $\mathbb{X} = \mathbb{I}$, we get a biadjunction

$$\begin{tikzcd} \mathbb{I} \downarrow \mathcal{B} \arrow[r, shift right, "\underline{\mathrm{Ker}}"'] \arrow[d, "\simeq"'] & \mathcal{B} \downarrow \mathbb{I} \arrow[l, shift right, "\underline{\mathrm{Coker}}"'] \arrow[d, "\simeq"] \\ \mathcal{B} \arrow[r, shift right, "\Omega"'] & \mathcal{B} \arrow[l, shift right, "\Sigma"'] \end{tikzcd}$$

which we take as definition of $\Omega\colon \mathcal{B} \to \mathcal{B}$ and $\Sigma\colon \mathcal{B} \to \mathcal{B}$, with $\Sigma \dashv \Omega$.
Clearly, when $\mathcal{B} = \underline{\mathcal{SCG}}$ we get, as expected,

$$\Sigma(\mathbb{G}) = \underline{\mathrm{Coker}}(\mathbb{G} \to \mathbb{I}) = [\pi_0\mathbb{G}]_1\ , \ \ \Omega(\mathbb{G}) = \underline{\mathrm{Ker}}(\mathbb{I} \to \mathbb{G}) = [\pi_1\mathbb{G}]_0$$

6.9 Exercise. Let $F\colon \mathbb{A} \to \mathbb{B}$ an arrow in an abelian 2-category. The following sequence is 2-exact in each point

$$\begin{tikzcd} \mathbb{I} \arrow[r] & \underline{\mathrm{Pip}}(F) \arrow[r] & \Omega\mathbb{A} \arrow[r] & \Omega\mathbb{B} \arrow[r] & \underline{\mathrm{Ker}}(F) \arrow[r] & \mathbb{A} \arrow[dlllll] \\ \mathbb{B} \arrow[r] & \underline{\mathrm{Coker}}(F) \arrow[r] & \Sigma\mathbb{A} \arrow[r] & \Sigma\mathbb{B} \arrow[r] & \underline{\mathrm{Copip}}(F) \arrow[r] & \mathbb{I} \end{tikzcd}$$

6.10 Definition. Let $\mathcal{B}$ be a 2-category with a zero object. An object $\mathbb{B} \in \mathbb{B}$ is

1. discrete if for every $\mathbb{X} \in \mathcal{B}$ the groupoid $\mathcal{B}(\mathbb{X}, \mathbb{B})$ is discrete,
2. connected if for every $\mathbb{X} \in \mathcal{B}$ the groupoid $\mathcal{B}(\mathbb{B}, \mathbb{X})$ is discrete.

6.11 Proposition. *Let $\mathcal{B}$ be an abelian 2-category. For an object $\mathbb{B} \in \mathcal{B}$ we put*

$$\pi_0(\mathbb{B}) = \Omega(\Sigma(\mathbb{B})) \qquad \pi_1(\mathbb{B}) = \Sigma(\Omega(\mathbb{B}))$$

1. *$\mathbb{B}$ is discrete iff the unit $\mathbb{B} \to \pi_0(\mathbb{B})$ is an equivalence iff $\pi_1(\mathbb{B}) \simeq \mathbb{I}$.*
2. *$\mathbb{B}$ is connected iff the counit $\pi_1(\mathbb{B}) \to \mathbb{B}$ is an equivalence iff $\pi_0(\mathbb{B}) \simeq \mathbb{I}$.*
3. *The sub-2-category $\mathrm{Dis}(\mathcal{B})$ of the discrete objects is a category (only identity 2-arrows) and it is abelian. Moreover, $\pi_0 \colon \mathcal{B} \to \mathrm{Dis}(\mathcal{B})$ is left adjoint to the inclusion $\mathrm{Dis}(\mathcal{B}) \to \mathcal{B}$.*
4. *The sub-2-category $\mathrm{Conn}(\mathcal{B})$ of the connected objects is a category (only identity 2-arrows) and it is abelian. Moreover, $\pi_1 \colon \mathcal{B} \to \mathrm{Conn}(\mathcal{B})$ is right adjoint to the inclusion $\mathrm{Conn}(\mathcal{B}) \to \mathcal{B}$.*
5. *The biadjunction $\Sigma \dashv \Omega$ restricts to an equivalence $\mathrm{Conn}(\mathcal{B}) \simeq \mathrm{Dis}(\mathcal{B})$.*

$$\begin{tikzcd}
\mathcal{B} \arrow[r, shift right, "\Omega"'] \arrow[d, shift left, "\pi_1"] & \mathcal{B} \arrow[l, shift right, "\Sigma"'] \arrow[d, shift right, "\pi_0"'] \\
\mathrm{Conn}(\mathcal{B}) \arrow[u, shift left] \arrow[r, shift right, "\Omega"'] & \mathrm{Dis}(\mathcal{B}) \arrow[u, shift right] \arrow[l, shift right, "\Sigma"']
\end{tikzcd}$$

6.12 Example.

1. As expected, $\mathrm{Dis}(\underline{SCG}) \simeq Ab \simeq \mathrm{Conn}(\underline{SCG})$.
2. If $\mathbb{A}$ is a categorical ring, then $\pi_0(\mathbb{A})$ is a ring and $\mathrm{Dis}(2\text{-}\underline{\mathrm{Add}}[\mathbb{A}, \underline{SCG}])$ is equivalent to the category of $\pi_0(\mathbb{A})$-modules.
3. If in point 2 we take as categorical ring $\mathbb{A}$ the discrete categorical ring $[\mathbb{Z}]_0$, then $\mathrm{Dis}(2\text{-}\underline{\mathrm{Add}}[\mathbb{A}, \underline{SCG}]) \simeq Ab$. Therefore, we have two non-equivalent abelian 2-categories producing Ab as the sub-category of discrete objects (use point 5 of Example 5.13).
4. For any abelian category $\mathcal{A}$, we can close the circle: $\mathrm{Dis}(\underline{Bfly}(\mathcal{A})) \simeq \mathcal{A}$.
5. Problem: Let $\mathcal{B}$ be an abelian 2-category. What is the relation between $\mathcal{B}$ and $\underline{Bfly}(\mathrm{Dis}(\mathcal{B}))$?

References for Chapter 6

Bla

Lesson III: Homological algebra

My point of view on applications of mathematics:

"Mais à quoi sert de faire des maths si on ne peut pas compter les uns sur les autres."
Youssoupha, *L'amour*, Bomaye Musik, 2012.

The aim of Lesson 3 is to give two examples showing that categorical groups and, more in general, higher dimensional categorical structures can be used in algebra to discover new facts or to understand well-known facts from a different point of view. The two examples are:

1. The snail lemma, which is a generalization of the snake lemma.
2. The Sinh homotopy classification of categorical groups, which is a generalization of the Mac Lane - Schreier theory of group extensions.

Chapter 7

The snail lemma

Let me start recalling the classical snake lemma in Ab. In fact, it holds in any abelian category, and even more in general.

7.1 Snake Lemma. *Consider the following diagram in Ab*

$$\begin{tikzcd}
\mathrm{Ker}(c) \arrow[r] \arrow[d, "k(c)"'] & \mathrm{Ker}(a) \arrow[r] \arrow[d, "k(a)"'] & \mathrm{Ker}(b) \arrow[d, "k(b)"] \\
\mathrm{Ker}(f) \arrow[r, "k(f)"] \arrow[d, "c"'] & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
\mathrm{Ker}(f_0) \arrow[r, "k(f_0)"'] \arrow[d, "q(c)"'] & A_0 \arrow[r, "f_0"'] \arrow[d, "q(a)"'] & B_0 \arrow[d, "q(b)"] \\
\mathrm{Coker}(c) \arrow[r] & \mathrm{Coker}(a) \arrow[r] & \mathrm{Coker}(b)
\end{tikzcd}$$

If $f\colon A \to B$ is surjective, then there exists a morphism $\delta\colon \mathrm{Ker}(b) \to \mathrm{Coker}(c)$ making exact the sequence

$$\mathrm{Ker}(c) \to \mathrm{Ker}(a) \to \mathrm{Ker}(b) \to \mathrm{Coker}(c) \to \mathrm{Coker}(a) \to \mathrm{Coker}(b)$$

Proof. To be inserted. $\square$

The problem we want to study in this chapter is: do we really need the assumption that $f\colon A \to B$ is an epi?

The starting point for the snake lemma is the diagram

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "k(f)"] \arrow[d, "c"'] & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
\mathrm{Ker}(f_0) \arrow[r, "k(f_0)"'] & A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

Such a diagram can be seen as an arrow $F\colon \mathbb{A} \to \mathbb{B}$ in the category $Ab^{\to}$ together with its kernel (that is, the level-wise kernel). But $Ab^{\to}$ is not just a category, we know that it is a 2-category and that it can be embedded in $\underline{SCG}$ (see the beginning of Chapter 6)

$$Ab^{\to} \simeq \underline{Grpd}(Ab) \to \underline{SCG}$$

Moreover, when $F\colon \mathbb{A} \to \mathbb{B}$ is seen as an arrow in $\underline{SCG}$, we can take its strong homotopy kernel $\underline{\mathrm{Ker}}(F)$ and we get an exact sequence of abelian groups (Corollary 4.18)

$$\pi_1\underline{\mathrm{Ker}}(F) \to \pi_1\mathbb{A} \to \pi_1\mathbb{B} \to \pi_0\underline{\mathrm{Ker}}(F) \to \pi_0\mathbb{A} \to \pi_0\mathbb{B}$$

Let us now work out this construction in detail. The internal groupoid in Ab corresponding to an object $\mathbb{A} = (a\colon A \to A_0)$ of $Ab^{\to}$ is

$$\begin{tikzcd}(A_0 \times A) \times_{c,d} (A_0 \times A) = A_0 \times A \times A \arrow[r, "m"] & A_0 \times A \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A_0 \arrow[l, "e"' description]\end{tikzcd}$$

$$d(x_0, x) = x_0 \ , \ \ c(x_0, x) = x_0 + a(x) \ , \ \ e(x_0) = (x_0, 0)$$
$$m(x_0, x, y) = (x_0, x + y) \ , \ \ i(x_0, x) = (x_0 + a(x), -x)$$

and an easy computation shows that

$$\pi_1(\mathbb{A}) = \mathrm{Ker}(a) \quad \text{and} \quad \pi_0(\mathbb{A}) = \mathrm{Coker}(a)$$

On arrows, the passage from $Ab^{\to}$ to $\underline{SCG}$ gives

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A_0 \times A \arrow[r, "f_0 \times f"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & B_0 \times B \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

Following the general description of the strong homotopy kernel given in Definition 4.1, we get that $(\underline{\mathrm{Ker}}(F))_0$ is nothing but the pullback

$$\begin{tikzcd} A_0 \times_{f_0,b} B \arrow[r] \arrow[d] & B \arrow[d, "b"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

and $(\underline{\mathrm{Ker}}(F))_1$ with the domain and the codomain maps is

$$\begin{tikzcd} A_0 \times_{f_0,b} B \times A \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & A_0 \times_{f_0,b} B \end{tikzcd}$$

$$d(x_0, y, x) = (x_0, y) \ , \ \ c(x_0, y, x) = (x_0 + a(x), y + f(x))$$

Going back to $Ab^{\to}$, the strong homotopy kernel $\underline{\mathrm{Ker}}(F)$ corresponds to the unique factorization through the pullback

$$\langle a,f\rangle\colon A \to A_0 \times_{f_0,b} B$$

as in the following diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\langle a,f\rangle"] \arrow[dd, "a"'] & & B \arrow[dd, "b"] \\
& A_0 \times_{f_0,b} B \arrow[ur] \arrow[dl] & \\
A_0 \arrow[rr, "f_0"'] & & B_0
\end{tikzcd}$$

We have discovered the

7.2 Snail Lemma. *Consider the following diagram in Ab*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

There is an exact sequence of abelian groups

$$\mathrm{Ker}\langle a,f\rangle \to \mathrm{Ker}(a) \to \mathrm{Ker}(b) \to \mathrm{Coker}\langle a,f\rangle \to \mathrm{Coker}(a) \to \mathrm{Coker}(b)$$

Proof. Apply Corollary 4.18 to

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, "\langle a,f\rangle"'] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\
A_0 \times_{f_0,b} B \arrow[r] & A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

which corresponds to the strong homotopy kernel $\underline{\mathrm{Ker}}(F) \to \mathbb{A} \to \mathbb{B}$ via the embedding $Ab^{\to} \simeq \underline{Grpd}(Ab) \to \underline{SCG}$. $\square$

Let me insist on the fact that, contrarily to what happens in the snake lemma, the connecting morphism $\mathrm{Ker}(b) \to \mathrm{Coker}\langle a,f\rangle$ of the snail sequence exists for obvious general reasons, explaiend in the following diagram, with no assumption on $f\colon A \to B$.

$$\begin{tikzcd}
 & & \mathrm{Ker}(b) \arrow[d, "\langle 0,k(b)\rangle"] \arrow[ddd, bend right, "0"'] \arrow[drr, "k(b)"] & & \\
A \arrow[rr, "\langle a,f\rangle"'] \arrow[rrrr, bend left, "f"] \arrow[ddrr, "a"'] & & A_0 \times_{f_0,b} B \arrow[rr] \arrow[dr, "q"] \arrow[dd] & & B \arrow[dd, "b"] \\
 & & & \mathrm{Coker}\langle a,f\rangle & \\
 & & A_0 \arrow[rr, "f_0"'] & & B_0
\end{tikzcd}$$

Now we go back to the snake lemma and show how it can be deduced from the snail lemma. For this, consider the diagram

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "k(f)"] \arrow[ddd, "c"'] & A \arrow[rrr, "f"] \arrow[dr, "\langle a,f\rangle"] \arrow[ddd, "a"'] & & & B \arrow[ddd, "b"] \\
 & & A_0 \times_{f_0,b} B \arrow[urr] \arrow[dr, "q"] \arrow[ddl] & & \\
 & & & \mathrm{Coker}\langle a,f\rangle & \\
\mathrm{Ker}(f_0) \arrow[r, "k(f_0)"'] \arrow[uurr, "\langle k(f_0),0\rangle"] \arrow[d, "q(c)"'] & A_0 \arrow[rrr, "f_0"'] & & & B_0 \\
\mathrm{Coker}(c) \arrow[uuurrr, "\lambda"'] & & & &
\end{tikzcd}$$

7.3 Corollary. *In the previous diagram:*

1. *The comparison* $\lambda\colon \mathrm{Coker}(c) \to \mathrm{Coker}\langle a, f\rangle$ *is always a mono.*
2. *If* $f\colon A \to B$ *is an epi , then the comparison* λ *is an epi (and then it is an isomorphism).*

Therefore, if f is an epi, the snake sequence exists and the exactness of the snail sequence implies the exactness of the snake sequence.

$$\begin{tikzcd}
\mathrm{Ker}(c) \arrow[r] \arrow[d, equal] & \mathrm{Ker}(a) \arrow[r] \arrow[d, equal] & \mathrm{Ker}(b) \arrow[r] \arrow[d, equal] & \mathrm{Coker}(c) \arrow[r] \arrow[d, "\lambda"'] & \mathrm{Coker}(a) \arrow[r] \arrow[d, equal] & \mathrm{Coker}(b) \arrow[d, equal] \\
\mathrm{Ker}\langle a,f\rangle \arrow[r] & \mathrm{Ker}(a) \arrow[r] & \mathrm{Ker}(b) \arrow[r] & \mathrm{Coker}\langle a,f\rangle \arrow[r] & \mathrm{Coker}(a) \arrow[r] & \mathrm{Coker}(b)
\end{tikzcd}$$

To end this chapter, I would like to understand, from the point of view of internal groupoids in Ab, the assumption that $f\colon A \to B$ is an epi. Let me write once again $F\colon \mathbb{A} \to \mathbb{B}$ for the arrow in $Ab^{\to}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

Consider its kernel in the category $Ab^{\to}$, its strong homotopy kernel in the 2-category $Ab^{\to}$ and the canonical comparison Λ:

$$\begin{tikzcd} \mathrm{Ker}(F) \arrow[rr] \arrow[dr, "\Lambda"'] & & \mathbb{A} \arrow[r, "F"] & \mathbb{B} \\ & \underline{\mathrm{Ker}}(F) \arrow[ur] & & \end{tikzcd}$$

We can extract from the diagram preceding Corollary 7.3 the following diagram

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "k(f)"] \arrow[d, "c"'] & A \arrow[d, "\langle a,f\rangle"] \\ \mathrm{Ker}(f_0) \arrow[r, "\langle k(f_0),0\rangle"'] \arrow[d, "q"'] & A_0 \times_{f_0,b} B \arrow[d, "q"] \\ \mathrm{Coker}(c) \arrow[r, "\lambda"'] & \mathrm{Coker}\langle a,f\rangle \end{tikzcd}$$

which shows that the comparison $\lambda\colon \mathrm{Coker}(c) \to \mathrm{Coker}\langle a,f\rangle$ entering in the corollary is nothing but $\pi_0(\Lambda\colon \mathrm{Ker}(F) \to \underline{\mathrm{Ker}}(F))$. This simple remark is the key to understand the condition on $f\colon A \to B$ in the snake lemma, but here we need the notion of fibration of groupoids.

7.4 Definition. A functor $F\colon \mathbb{A} \to \mathbb{B}$ between groupoids is a fibration when, for every arrow $g\colon B \to FA$ in $\mathbb{B}$, there exists an arrow $f\colon A' \to A$ in $\mathbb{A}$ such that $F(f) = g$.

Here is why fibrations enter in the picture.

7.5 Proposition. *In the biequivalence $Ab^{\to} \simeq \underline{Grpd}(Ab)$:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] & \mapsto & A_0 \times A \arrow[r, "f_0 \times f"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & B_0 \times B \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ A_0 \arrow[r, "f_0"'] & B_0 & & A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

the internal functor is a fibration if and only if the arrow $f\colon A \to B$ is an epi.

And, finally, here is a general property of fibrations which explains why fibrations permit to pass from the snail lemma to the snake lemma.

7.6 Proposition. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a fibration between groupoids in Ab (more in general, between pointed groupoids). The comparison Λ between its kernel and its strong homotopy kernel*

$$\begin{tikzcd} \mathrm{Ker}(F) \arrow[rr] \arrow[dr, "\Lambda"'] & & \mathbb{A} \arrow[r, "F"] & \mathbb{B} \\ & \underline{\mathrm{Ker}}(F) \arrow[ur] & & \end{tikzcd}$$

is a weak equivalence. Therefore, $\pi_0(\Lambda)$ and $\pi_1(\Lambda)$ are isomorphisms.

References for Chapter 7

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Chapter 8

Homotopical classification of categorical groups

In this chapter we go back to the relation between group extensions and categorical groups, relation which already appears in Chapter 3. Let us fix two groups G and H and recall from Example 3.5 that there is an equivalence of categories

$$\underline{\mathrm{SplitExt}}(G,H) \simeq \mathit{Grp}(G,\mathrm{Aut}(H))\ ,\quad E \mapsto \varphi_E$$

$$\begin{tikzcd} E\colon\ H \arrow[r, "i"] & X \arrow[r, shift right, "s"'] \arrow[d, "\chi"'] & G \arrow[l, shift right, "j_s"'] \arrow[dl, "j_s\cdot\chi=\varphi_E"] \\ & \mathrm{Aut}(H) & \end{tikzcd}$$

Now the question is: what about $\underline{\mathrm{Ext}}(G,H)$? There exists a functor

$$\underline{\mathrm{Ext}}(G,H) \to \mathit{Grp}(G,\mathrm{Out}(H))\ ,\quad E \mapsto \psi_E$$

$$\begin{tikzcd} E\colon\ H \arrow[r, "i"] \arrow[dr, "\mathcal{I}"'] & X \arrow[r, "s"] \arrow[d, "\chi"] & G \arrow[d, "\psi_E"] \\ & \mathrm{Aut}(H) \arrow[r, "q"'] & \mathrm{Out}(H) \end{tikzcd}$$

but this functor is not an equivalence; Indeed, passing to isomorphism classes of extensions, the induced map $[E] \mapsto \psi_E$ is (well-defined but) neither surjective nor injective. Now we can make the previous question more precise:

Question 1: Can we replace $\mathit{Grp}(G,\mathrm{Out}(H))$ with something else in order to obtain an equivalence?

Question 2: For which morphisms $\psi\colon G \to \mathrm{Out}(H)$ there exists at least one extension E such that $\psi_E = \psi$?

Question 3: How many different extensions of G by H can produce the same morphism $\psi\colon G \to \mathrm{Out}(H)$?

From Example 3.5 we already know the answer to Question 1: there is an equivalence of categories

$$\underline{\mathrm{Ext}}(G,H) \simeq \underline{CG}([G]_0, \underline{\mathrm{Hol}}(H))$$

The answers to Question 2 and Question 3 are provided by the classical Schreier - Mac Lane theory of group extensions, that I summarize in the next proposition.

8.1 Proposition. *Fix two groups G and H. Consider*

$$z(H) = \mathrm{Ker}(\mathcal{I}) \longrightarrow H \xrightarrow{\mathcal{I}} \mathrm{Aut}(H) \xrightarrow{q} \mathrm{Coker}(\mathcal{I}) = \mathrm{Out}(H)$$

and the action

$$\mathrm{Out}(H) \times z(H) \to z(H)\ ,\ \ [f], a \mapsto f(a)$$

Fix also a morphism $\psi\colon G \to \mathrm{Out}(H)$. Consider the induced action

$$\overline{\psi}\colon\ G \times z(H) \xrightarrow{\psi\times\mathrm{id}} \mathrm{Out}(H) \times z(H) \longrightarrow z(H)$$

and the cohomology groups

$$\begin{tikzcd}
\mathrm{C}^1(G,z(H)) \arrow[r, "\partial"] \arrow[dr] & \mathrm{C}^2(G,z(H)) \arrow[r, "\partial"] \arrow[dr] & \mathrm{C}^3(G,z(H)) \arrow[r, "\partial"] & \mathrm{C}^4(G,z(H)) \\
& \mathrm{Z}^2(G,z(H),\overline{\psi}) \arrow[u] \arrow[d, "q"'] & \mathrm{Z}^3(G,z(H),\overline{\psi}) \arrow[u] \arrow[d, "q"] & \\
& \mathrm{H}^2(G,z(H),\overline{\psi}) & \mathrm{H}^3(G,z(H),\overline{\psi}) &
\end{tikzcd}$$

1. *Fix two set-theoretical maps $\varphi\colon G \to \mathrm{Aut}(H)$ and $f\colon G\times G \to H$ such that*

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & \mathrm{Aut}(H) \arrow[d, "q"] \\
& \mathrm{Out}(H)
\end{tikzcd}$$

$$\varphi(1) = \mathrm{id}_H\ ,\ f(x,1) = 1 = f(1,y)\ ,\ \varphi(x)\cdot\varphi(y)\cdot\varphi(xy)^{-1} = \mathcal{I}(f(x,y))$$

and put

$$k\colon G\times G\times G \to H\ ,\ \ k(x,y,z) = \varphi(x)(f(x,y)) + f(x,yz) - f(xy,z) - f(x,y)$$

Then

(a) *$k \in \mathrm{Z}^3(G, z(H), \overline{\psi})$ (the element k is called the obstruction to the realization of ψ)*

(b) *$q(k) \in \mathrm{H}^2(G, z(H), \overline{\psi})$ depends only on $\psi\colon G \to \mathrm{Out}(H)$ and not on φ and f*

(c) *there exists an extension $E\colon H \to X \to G$ such that $\psi_E = \psi$, if and only if $q(k) = 0$.*

2. *Write $\underline{\mathrm{OpExt}}(G, H, \psi)$ for the full sub-category of $\underline{\mathrm{Ext}}(G, H)$ of those extensions E such that $\psi_E = \psi$. If $\underline{\mathrm{OpExt}}(G, H, \psi)$ is not empty, then there is a simply transitive action*

$$\mathrm{H}^2(G, z(H), \overline{\psi}) \times \pi_0(\underline{\mathrm{OpExt}}(G, H, \psi)) \to \pi_0(\underline{\mathrm{OpExt}}(G, H, \psi))$$

and therefore $\mathrm{H}^2(G, z(H), \overline{\psi}) \simeq \pi_0(\underline{\mathrm{OpExt}}(G, H, \psi))$.

We have answered the three questions, but I'm unhappy because the three questions are clearly related, whereas the answers to Question 2 and Question 3 seem to be unrelated to the answer to Question 1. The simple but crucial idea to correct this problem is that, if we look to an extension $E\colon H \to X \to G$ as a monoidal functor $E\colon [G]_0 \to \underline{\mathrm{Hol}}(H)$, then $\psi_E\colon G \to \mathrm{Out}(H)$ is $\pi_0(E)$, and $\pi_1(E)$ is trivial:

$$\begin{tikzcd}
0 \arrow[r, "\pi_1(E)=0"] \arrow[d] & z(H) \arrow[d] \\
G \arrow[r, "E_1"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\mathrm{id}"] & H \rtimes \mathrm{Aut}(H) \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\
G \arrow[r, "E_0"'] \arrow[d, "\mathrm{id}"'] & \mathrm{Aut}(H) \arrow[d, "q"] \\
G \arrow[r, "\pi_0(E)=\psi_E"'] & \mathrm{Out}(H)
\end{tikzcd}$$

We will use this idea to give a much more general version of Question 2 and Question 3, but before doing this we need one more remark on categorical groups.

8.2 Remark.

1. Let $\mathbb{G}$ be a categorical group. The abelian group $\pi_1\mathbb{G}$ is a $\pi_0\mathbb{G}$-module under the action $\pi_0\mathbb{G} \times \pi_1\mathbb{G} \to \pi_1\mathbb{G}$ sending a pair $([X], f\colon I \to I)$ on the arrow

$$I \simeq X \otimes X^* \simeq X \otimes I \otimes X^* \xrightarrow{\mathrm{id} \otimes f \otimes \mathrm{id}} X \otimes I \otimes X^* \simeq X \otimes X^* \simeq I$$

2. Let $F\colon \mathbb{G} \to \mathbb{H}$ be an arrow in $\underline{CG}$. The pair of group homomorphisms

$$\pi_0(F)\colon \pi_0\mathbb{G} \to \mathbb{H}\ ,\ \ \pi_1(F)\colon \pi_1\mathbb{G} \to \pi_1\mathbb{H}$$

is equivariant

$$\begin{tikzcd}
\pi_0\mathbb{G} \times \pi_1\mathbb{G} \arrow[r] \arrow[d, "\pi_0(F)\times\pi_1(F)"'] & \pi_1\mathbb{G} \arrow[d, "\pi_1(F)"] \\
\pi_0\mathbb{H} \times \pi_1\mathbb{H} \arrow[r] & \pi_1\mathbb{H}
\end{tikzcd}$$

Now we can reformulate Question 2 and Question 3 more in general, using categorical groups instead of group extensions. Instead of the categorical groups $[G]_0$ and $\underline{\mathrm{Hol}}(H)$, consider two arbitrary categorical groups $\mathbb{G}$ and $\mathbb{H}$.

Question 2 bis: Given an equivariant pair of group homomorphisms

$$p\colon \pi_0\mathbb{G} \to \pi_0\mathbb{H}\ ,\ \ r\colon \pi_1\mathbb{G} \to \pi_1\mathbb{H}$$

does there exist an arrow $F\colon \mathbb{G} \to \mathbb{H}$ in $\underline{CG}$ such that $\pi_0(F) = p$ and $\pi_1(F) = r$?
Question 3 bis: How many different arrows $\mathbb{G} \to \mathbb{H}$ in $\underline{CG}$ can give the same equivariant pair (p, r)?

The answers to these questions are a generalization of the Schreier - Mac Lane theory of group extensions. They follows from two facts, a simple lemma and the fundamental Sinh's homotopy classification of categorical groups. In order to state these results, we need some constructions.

8.3 Definition.

1. The category *Mod* has, as objects, triples

$$(G \in Grp, A \in Ab, G \times A \to A \ \text{ an action})$$

Arrows are equivariant pairs (p, r) of group homomorphisms

$$\begin{tikzcd}
G \times A \arrow[r] \arrow[d, "p\times r"'] & A \arrow[d, "r"] \\
G' \times A(\arrow[r] & A'
\end{tikzcd}$$

2. The 2-category $\underline{\mathcal{Z}}^3$ has, as objects, pairs

$$((G, A) \in Mod, h \in \mathrm{H}^3(G, A))$$

Arrows are triples (p, r, g), with $(p, r)\colon (G, A) \to (G', A')$ an arrow in *Mod* and $g \in \mathrm{C}^2(G, A')$ such that $r_*(h) = p^*(h') + \partial(g)$

$$\begin{tikzcd}
\mathrm{C}^1(G,A) \arrow[r, "\partial"] \arrow[d, "r_*"'] & \mathrm{C}^2(G,A) \arrow[r, "\partial"] \arrow[d, "r_*"'] & \mathrm{Z}^3(G,A) \arrow[r, "q"] \arrow[d, "r_*"] & \mathrm{H}^3(G,A) \arrow[d, "r_*"] \\
\mathrm{C}^1(G,A') \arrow[r, "\partial"] & \mathrm{C}^2(G,A') \arrow[r, "\partial"] & \mathrm{Z}^3(G,A') \arrow[r, "q"] & \mathrm{H}^3(G,A') \\
\mathrm{C}^1(G',A') \arrow[r, "\partial"'] \arrow[u, "p^*"] & \mathrm{C}^2(G',A') \arrow[r, "\partial"'] \arrow[u, "p^*"] & \mathrm{Z}^3(G',A') \arrow[r, "q"'] \arrow[u, "p^*"'] & \mathrm{H}^3(G',A') \arrow[u, "p^*"']
\end{tikzcd}$$

Composition of arrows in $\underline{\mathcal{Z}}^3$ is as in the following diagram

$$\begin{tikzcd}
(G,A,h) \arrow[r, "{(p,r,g)}"] \arrow[rr, bend right, "{(p\cdot p', r\cdot r', p^*(g')+r'_*(g))}"'] & (G',A',h') \arrow[r, "{(p',r',g')}"] & (G'',A'',h'')
\end{tikzcd}$$

A 2-arrow $f\colon (p,r,g) \Rightarrow (p',r',g')\colon (G,A,h) \rightrightarrows (G',A',h')$ in $\underline{\mathcal{Z}}^3$ can exist only if $p = p'$ and $r = r'$, and is given by an element $f \in \mathrm{C}^1(G,A')$ such that $g' = g + \partial(f)$. All 2-arrows are invertible.

3. There is a forgetful 2-functor $\mathcal{U}\colon \underline{\mathcal{Z}}^3 \to \mathcal{M}od$

$$(p,r,g)\colon (G,A,h) \to (G',A',h') \quad \mapsto \quad (p,r)\colon (G,A) \to (G',A')$$

8.4 Definition. Let (G,A,h) and (G',A',h') be objects in $\underline{\mathcal{Z}}^3$ and consider an arrow $(p,r)\colon (G,A) \to (G',A')$ in $\mathcal{M}od$. The obstruction to the realizability of the arrow (p,r) is the element

$$\mathrm{obs}(p,r) = [r_*(h) - p^*(h')] \in \mathrm{H}^3(G,A')$$

8.5 Lemma. *Fix two objects (G,A,h) and (G',A',h') in $\underline{\mathcal{Z}}^3$ and an arrow $(p,r)\colon (G,A) \to (G',A')$ in $\mathcal{M}od$.*

1. *There exists an arrow in $\underline{\mathcal{Z}}^3$ of the form $(p,r,g)\colon (G,A,h) \to (G',A',h')$ if and only if $\mathrm{obs}(p,r) = 0$ in $\mathrm{H}^3(G,A')$.*

2. *Write*
$$\underline{\mathcal{Z}}^3_{(p,r)}((G,A,h),(G',A',h'))$$
for the set of arrows $(G,A,h) \to (G',A',h')$ in $\underline{\mathcal{Z}}^3$ sent on (p,r) by the 2-functor $\mathcal{U}\colon \underline{\mathcal{Z}}^3 \to \mathcal{M}od$. If $\mathrm{obs}(p,r) = 0$, then there is a transitive action

$$\mathrm{Z}^2(G,A') \times \underline{\mathcal{Z}}^3_{(p,r)}((G,A,h),(G',A',h')) \to \underline{\mathcal{Z}}^3_{(p,r)}((G,A,h),(G',A',h'))$$

$$t\ ,\ (p,r,g) \quad \mapsto \quad (p,r,g+t)$$

Moreover, such an action induces a simply transitive action

$$\mathrm{H}^2(G,A')\times\pi_0\underline{\mathcal{Z}}^3_{(p,r)}((G,A,h),(G',A',h'))\to\pi_0\underline{\mathcal{Z}}^3_{(p,r)}((G,A,h),(G',A',h'))$$

and, therefore, a bijection between the abelian group $\mathrm{H}^2(G,A')$ *and the set* $\pi_0(\underline{\mathcal{Z}}^3_{(p,r)}((G,A,h),(G',A',h')))$ *of 2-isomorphism classes of arrows.*

Proof. 1. Obvious, because $[r_*(h)-p^*(h')]=0$ if and only if there exists an element $g\in\mathrm{C}^2(G,A')$ such that $r_*(h)-p^*(h')=\partial(g)$, if and only if $(p,r,g)\colon(G,A,h)\to(G',A',h')$ is an arrow in $\underline{\mathcal{Z}}^3$.
2. The action is transitive: let $(p,r,g),(p,r,g')\colon(G,A,h)\to(G',A',h')$ be two arrows in $\underline{\mathcal{Z}}^3$. Since $\partial(g)=r_*(h)-p^*(h')=\partial(g')$ the element $t=g'-g$ is in $\mathrm{Z}^2(G,A')$.
The induced action is well-defined: if two elements $t,t'\in\mathrm{Z}^2(G,A')$ are such that $[t]=[t']$ in $\mathrm{H}^2(G,A')$, then there exists an element $f\in\mathrm{C}^1(G,A')$ such that $t'=t+\partial(f)$. If moreover $f'\colon(p,r,g)\Rightarrow(p,r,g')$ is a 2-arrow in $\underline{\mathcal{Z}}^3$, then $f'+f\colon(p,r,g+t)\Rightarrow(p,r,g'+t')$ is a 2-arrow in $\underline{\mathcal{Z}}^3$.
The induced action is simply transitive: consider elements t,t' in $\mathrm{Z}^2(G,A')$ and a 2-arrow $f\colon(p,r,g+t)\Rightarrow(p,r,g+t')$ in $\underline{\mathcal{Z}}^3$. We have that $g+t'=g+t+\partial(f)$, and then $t'=t+\partial(f)$. This means that $[t]=[t']$ in $\mathrm{H}^2(G,A')$. $\square$

8.6 Theorem. *There is a biequivalence of 2-categories over Mod*

$$\begin{tikzcd}
\underline{\mathcal{Z}}^3 \arrow[rr, "\mathcal{S}"] \arrow[dr, "\mathcal{U}"'] & & \underline{CG} \arrow[dl, "(\pi_0,\pi_1)"] \\
& Mod &
\end{tikzcd}$$

Proof. I'm going to give only the easy part of the proof, that is, the definition of the 2-functor $\mathcal{S}$ on objects. For the rest of the proof, see Chapter 5 in [5].
Let (G,A,h) be an object in $\underline{\mathcal{Z}}^3$. The categorical groups $\mathbb{G}=\mathcal{S}(G,A,h)$ has, as objects, the elements of the group G. The hom-set $\mathbb{G}(g_1,g_2)$ is empty if $g_1\neq g_2$, and is the abelian group A if $g_1=g_2$. Composition is addition in A, with the zero of A as identity arrows. The tensor product in $\mathbb{G}$ is defined by $g_1\otimes g_2=g_1g_2$ on objects, and by $a_1\otimes a_2=a_1+g\cdot a_2$ on arrows $a_1\colon g_1\to g_1$ and $a_2\colon g_2\to g_2$. The unit object is the unit of G and, for every object, the left and right constraints are identities. Finally, the associativity constraint is given by $a_{g_1,g_2,g_3}=h(g_1,g_2,g_3)\colon g_1g_2g_3\to g_1g_2g_3$. The triangle condition is the fact that $h\in\mathrm{Z}^3(G,A)$ is normalized, that is, $h(g_1,1,g_3)=0$, and the pentagon condition is exactly the cocycle condition $\partial(h)=0$. $\square$

The next corollary is the answer to Question 2 bis and Question 3 bis. If in the statement you put $\mathbb{G}=[G]_0$ and $\mathbb{H}=\underline{\mathrm{Hol}}(H)$, then you recover precisely Proposition 8.1, that is, the classical Schreier - Mac Lane theory of group extensions.

8.7 Corollary. *Let* $\mathbb{G}$ *and* $\mathbb{H}$ *be in* $\underline{CG}$ *and write*

$$\mathcal{S}^{-1}\colon\underline{CG}\to\underline{\mathcal{Z}}^3\ ,\quad \mathbb{G}\mapsto(\pi_0\mathbb{G},\pi_1\mathbb{G},h(\mathbb{G}))$$

Fix an arrow $(p,r)\colon (\pi_0\mathbb{G},\pi_1\mathbb{G}) \to (\pi_0\mathbb{H},\pi_1\mathbb{H})$ *in Mod and put*

$$\mathrm{Obs}(p,r) = [r_*(h(\mathbb{G})) - p^*(h(\mathbb{H}))] \in \mathrm{H}^3(\pi_0\mathbb{G},\pi_1\mathbb{H})$$

1. *There exists an arrow* $F\colon \mathbb{G}\to\mathbb{H}$ *in* $\underline{CG}$ *such that* $\pi_0(F) = p$ *and* $\pi_1(F) = r$ *if and only if* $\mathrm{Obs}(p,r) = 0$ *in* $\mathrm{H}^3(\pi_0\mathbb{G},\pi_1\mathbb{H})$.
2. *Write*

   $$\underline{CG}_{(p,r)}(\mathbb{G},\mathbb{H})$$

   for the set of arrows $\mathbb{G}\to\mathbb{H}$ *in* $\underline{CG}$ *sent on* (p,r) *by the 2-functor* $(\pi_0,\pi_1)\colon \underline{CG}\to Mod$. *If* $\mathrm{Obs}(p,r)=0$, *then there is a transitive action*

   $$\mathrm{Z}^2(\pi_0\mathbb{G},\pi_1\mathbb{H})\times\underline{CG}_{(p,r)}(\mathbb{G},\mathbb{H})\to\underline{CG}_{(p,r)}(\mathbb{G},\mathbb{H})$$

   Moreover, such an action induces a simply transitive action

   $$\mathrm{H}^2(\pi_0\mathbb{G},\pi_1\mathbb{H})\times\pi_0(\underline{CG}_{(p,r)}(\pi_0\mathbb{G},\pi_1\mathbb{H}))\to\pi_0(\underline{CG}_{(p,r)}(\pi_0\mathbb{G},\pi_1\mathbb{H}))$$

   and, therefore, a bijection between the abelian group $\mathrm{H}^2(G,A')$ *and the set* $\pi_0(\underline{CG}_{(p,r)}(\pi_0\mathbb{G},\pi_1\mathbb{H}))$ *of 2-isomorphism classes of arrows.*

Proof. This is the transcription of Lemma 8.5 using Theorem 8.6. qed

References for Chapter 8

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