A logical analysis of Banach's fixpoint theorem

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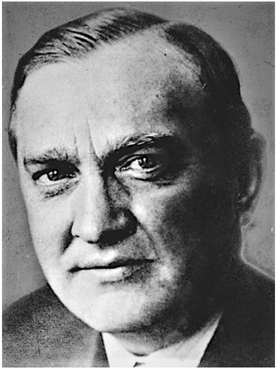
Banach: Fixpoint theorem (1922)



Stefan Banach (1892-1945)

Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*

publié dans Fund. Math. 3 (1922), p. 133–181.

§ 2. THÉORÈME 6. *Si*

1° *$U(X)$ est une opération continue dans E, le contre-domaine de $U(X)$ étant contenu dans E;*

2° *Il existe un nombre $0 < M < 1$ qui pour tout X' et X'' remplit l'inégalité*

$$\|U(X') - U(X'')\| \leqslant M \cdot \|X' - X''\|,$$

il existe un élément X tel que $X = U(X)$.

Démonstration. Y désignant un élément choisi d'une façon arbitraire, soit $\{X_n\}$ une suite qui satisfait aux conditions:

$$X_1 = Y \quad \text{et pour tout } n \ X_{n+1} = U(X_n).$$

Nous allons démontrer que la suite $\{X_n\}$ converge suivant la norme vers un certain élément X.

* Thèse présentée en juin 1920 à l'Université de Léopol pour obtenir le grade de docteur en philosophie.

Banach: Fixpoint theorem (modern version)

Let (X, d) be a complete metric space.

Let $f : X \to X$ be a contraction: $d(fx, fy) \leq k \cdot d(x, y)$ for some $0 < k < 1$.

(Note that f is a fortiori non-expansive.)

For any $x \in X$,

- infer from contractivity that the sequence $x, fx, f^2x, \ldots$ is Cauchy:

$$\lim d(f^n x, f^m x) = 0$$

- infer from completeness that the sequence converges, say to x^*:

$$\lim d(y, f^n x) = d(y, x^*)$$

- infer from non-expansiveness that $fx^* = x^*$:

$$0 = d(x^*, x^*) = \lim d(x^*, f^n x) \geq \lim d(fx^*, f^{n+1}x) = d(fx^*, x^*)$$

Infer from contractivity that the fixpoint is unique:

$$fx^* = x^*,\ fy^* = y^* \implies d(x^*, y^*) = d(fx^*, fy^*) \leq k \cdot d(x^*, y^*) \implies d(x^*, y^*) = 0.$$

Lawvere: Metric spaces are categories (1973)

Bill Lawvere (1937-2023)

METRIC SPACES, GENERALIZED LOGIC, AND CLOSED CATEGORIES

(*Conferenza tenuta il 30 marzo 1973*) *

By taking account of a certain natural generalization of category theory within itself, namely the consideration of strong categories whose hom-functors take their values in a given « closed category » $\mathcal{V}$ (not necessarily in the category $\mathcal{S}$ of abstract sets), we will show below that it is possible to regard a metric space as a (strong) category and that moreover by specializing the constructions and theorems of general category theory we can deduce a large part of *general* metric space theory.

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Quantale-enriched categories (1)

A **quantale** $Q = (Q, \bigvee, \circ, 1)$ is a closed (= residuated) monoidal complete lattice.

Closedness is equivalent to

$$a \circ (\bigvee_i b_i) = \bigvee_i (a \circ b_i) \quad \text{and} \quad (\bigvee_i a_i) \circ b = \bigvee_i (a_i \circ b)$$

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A Q-**distributor** $\Phi\colon \mathbb{A} \looparrowright \mathbb{B}$ is a matrix $\Phi\colon \mathbb{B}_0 \times \mathbb{A}_0 \to Q\colon (y, x) \mapsto \Phi(y, x)$ such that

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Q-categories and Q-functors form a category $\mathsf{Cat}(Q)$ in the obvious way.

Q-categories and Q-distributors form a (Sup-enriched) 2-category $\mathsf{Dist}(Q)$, in which composition is "matrix-like" and local order is "element-wise".

Bénabou: Distributors (1973)

Jean Bénabou (1932-2022)

LES DISTRIBUTEURS
d'après le cours de "Questions spéciales de mathématique"

par
J. BENABOU *rédigé par Jean-Roger ROISIN*

Rapport n° 33, janvier 1973
Séminaires de Mathématique Pure

Bâtiment Sc. I, Avenue du Cyclotron, 2 1348 Louvain-La-Neuve

Nous supposerons maintenant que $\mathcal{V}$ est un <u>cosmos</u> c'est-à-dire une <u>catégorie multiplicative symétrique fermée complète à gauche et à droite</u>.

Une flèche de $\mathcal{A}$ vers $\mathcal{B}$, appelée un distributeur, est un $\mathcal{V}$-bifoncteur vers $\mathcal{V}$, contravariant en $\mathcal{B}$ et covariant en $\mathcal{A}$.

4.3. <u>*Proposition*</u>.

Dist($\mathcal{V}$) est une bicatégorie fermée.

Quantale-enriched categories (2)

Examples of $\mathsf{Cat}(Q)$:

For $Q = (\{0,1\}, \vee, \wedge, 1)$: ordered sets and monotone maps.

$\mathbb{A}(x,y) = 1$ if $x \leq y$, 0 if $x \not\leq y$.

For $Q = ([0,\infty], \bigwedge, +, 0)$: (generalized) metric spaces and non-expansive maps.

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A *left-continuous t-norm* is a commutative, integral quantale $([0,1], \bigvee, *, 1)$, e.g. $x * y = \max\{x+y-1, 0\}$, used in many-valued logic.

For $Q = ([0,1], \bigvee, *, 1)$: "fuzzy" orders and "fuzzy" monotone maps.

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Compute $\Delta := ([0,\infty], \bigwedge, +, 0) \coprod ([0,1], \bigvee, *, 1)$ in $\mathsf{CMon}(\mathsf{Sup})$: its elements are *probability distributions* $u\colon [0,\infty] \to [0,1]$, with convolution product:

$$(u * v)(t) = \bigvee_{r+s=t} u(r) * v(s) \quad \text{and} \quad e(t) = \begin{cases} 0 \text{ if } t = 0 \\ 1 \text{ if } t \neq 0 \end{cases}$$

For $Q = (\Delta, \bigvee, *, e)$: probabilistic metric spaces and probability-increasing maps.

$\mathbb{A}(x,y)(t)$ is the probability that the distance from x to y is less than t.

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There are many more examples—in sheaf theory, non-commutative topology, monoidal topology, domain theory, quantum computing, automata theory...

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Any Q-functor $F\colon \mathbb{A} \to \mathbb{B}$ **represents** a left adjoint distributor ("the graph of F")

$$F_*\colon \mathbb{A} \multimap \mathbb{B} \text{ with elements } F_*(y,x) = \mathbb{B}(y, Fx).$$

Not every left adjoint distributor is thusly representable; however, whenever it is representable, then it is so by an essentially unique functor.

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$$\mathbb{1} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathbb{C} \quad \text{with elements} \quad \begin{cases} \phi(y) = \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \mathbb{C}(y, x_n) \\ \psi(y) = \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \mathbb{C}(x_n, y) \end{cases}$$

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The distributor $\phi\colon \mathbb{1} \,{-\!\!\circ\!\!\rightarrow}\, \mathbb{C}$ is representable, say by $F\colon \mathbb{1} \to \mathbb{C}\colon * \mapsto x^*$, if and only if

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$$1\!\!1 \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathbb{C} \quad \text{with elements} \quad \begin{cases} \phi(y) = \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \mathbb{C}(y, x_n) \\ \psi(y) = \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \mathbb{C}(x_n, y) \end{cases}$$

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The distributor $\phi\colon 1\!\!1 \xrightarrow{\circ} \mathbb{C}$ is representable, say by $F\colon 1\!\!1 \to \mathbb{C}\colon * \mapsto x^*$, if and only if

$$\bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \mathbb{C}(y, x_n) = \mathbb{C}(y, x^*) \text{ for all } y \in \mathbb{C}_0.$$

This is exactly the formula for convergence in a metric space:

$$\lim d(y, x_n) = d(y, x^*) \text{ for all } y \in X.$$

Quantale-enriched categories (3)

Any Q-functor $F\colon \mathbb{A} \to \mathbb{B}$ **represents** a left adjoint distributor ("the graph of F")

$$F_*\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\to} \mathbb{B} \text{ with elements } F_*(y,x) = \mathbb{B}(y, Fx).$$

Not every left adjoint distributor is thusly representable; however, whenever it is representable, then it is so by an essentially unique functor.

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("Categorical" Cauchy-completeness is stronger than "sequential" Cauchy-completenes, but under certain conditions on Q they coincide; see (Hofmann and Reis, 2013).)

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Proposition

Suppose that $F\colon \mathbb{C} \to \mathbb{C}$ *is a* Q*-functor on a Cauchy complete* $\mathbb{C}$*. If there is an* $x \in \mathbb{C}_0$ *such that* $(F^n x)_{n\in\mathbb{N}}$ *is Cauchy, then* F *has a fixpoint.*

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But can we provide ("contractivity") conditions on F to ensure the existence of a Cauchy sequence $(F^n x)_{n\in\mathbb{N}}$? And what about the uniqueness of a fixpoint?

Fixpoint theorem (2)

Definition

Say that $\varphi \colon Q \to Q$ is a **control function** and $F \colon \mathbb{C}_0 \to \mathbb{C}_0$ is a φ**-contraction**, if

- $\varphi(t) \geq t$ for all $t \in Q$,
- $\varphi(t) = t$ implies that either $t = 0$ or $1 \leq t$.
- $\mathbb{C}(Fx, Fy) \geq \varphi(\mathbb{C}(x, y))$ for any $x, y \in \mathbb{C}_0$.

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The "Banach case" for metric spaces: for

$$\varphi\colon [0, \infty] \to [0, \infty]\colon t \mapsto k \cdot t \quad \text{for some } 0 < k < 1$$

it is easily verified (recalling that $[0, \infty]$ comes with opposite order) that

$k \cdot t \leq t$,

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so a function $f\colon X \to X$ on a (generalized) metric space (X, d) is a φ-contraction if

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There are other non-trivial examples, e.g. for probabilistic metric spaces:

$$\text{define } \varphi\colon \Delta \to \Delta \text{ by } \varphi(u)(t) = \begin{cases} \frac{1}{2}(u(t) + 1) & \text{if } 0 < t \leq \infty \\ 0 & \text{if } t = 0 \end{cases}$$

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Proposition

If $\mathbb{C}$ is symmetric, then any two fixpoints of a φ-contraction are either isomorphic or in different summands of $\mathbb{C}$.

If $\mathbb{C}$ has no zero-homs, then any two fixpoints of a φ-contraction are always isomorphic.

Fixpoint theorem (3)

Proposition

Suppose that $F\colon \mathbb{C} \to \mathbb{C}$ is a φ-contraction on a Q-category. Suppose that Q is a ***continuous lattice*** *and that $\varphi\colon Q \to Q$ is a* ***lower-semicontinuous function****. Then, for any $x \in \mathbb{C}_0$ such that $\mathbb{C}(Fx, x) \neq 0 \neq \mathbb{C}(x, Fx)$, the sequence $(F^n x)_{n\in\mathbb{N}}$ is Cauchy.*

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It preserves (order and) directed suprema.

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These conditions are met by the previously mentioned examples, in particular the "Banach" case.

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These conditions are met by the previously mentioned examples, in particular the "Banach" case.

The result holds under weaker conditions, but it makes the statement more technically involved, so skipped here for convenience.

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Proof. Putting $C_{x,f} := \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \bigwedge_{m\geq N} \mathbb{C}(f^n x, f^m x) \in Q$, we recall from Subsection 1.2 that $\phi_{x,f} \dashv \psi_{x,f}$ if and only if $C_{x,f} \geq 1$. We shall show that $C_{x,f} \not\geq 1$ leads to a contradiction.

(i) Picking an $x \in \mathbb{C}_0$ such that $\mathbb{C}(x, fx) \neq 0 \neq \mathbb{C}(fx, x)$, we put $c_n := \mathbb{C}(f^n x, f^{n+1} x) \in Q$ for all $n \in \mathbb{N}$. By assumption, $0 < c_0 \leq 1$ and the conditions on φ imply that $c_0 \leq \varphi(c_0) \leq c_1$. Repeating the argument we find that $c_n \leq \varphi(c_n) \leq c_{n+1}$, so the sequence is increasing and strictly above 0. Therefore we can compute, using the conditions on φ, that:

$$\begin{aligned}\bigvee_{N\in\mathbb{N}} c_N &= \bigvee_{N\in\mathbb{N}} c_{N+1}\\ &= \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} c_{n+1}\\ &\geq \bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} \varphi(c_n)\\ &\geq \varphi(\bigvee_{N\in\mathbb{N}} \bigwedge_{n\geq N} c_n)\\ &= \varphi(\bigvee_{N\in\mathbb{N}} c_N)\\ &\geq \bigvee_{N\in\mathbb{N}} c_N\end{aligned}$$

We thus find a fixpoint of φ which is not 0, so it must satisfy $1 \leq \bigvee_{N\in\mathbb{N}} c_N$.

(ii) Similarly, the sequence $(a_n := \mathbb{C}(f^{n+1}x, f^n x))_{n\in\mathbb{N}}$ must also satisfy $1 \leq \bigvee_{\in\mathbb{N}} a_n$.

(iii) Next, suppose that $1 \not\leq C_{f,x}$; by continuity of the underlying complete lattice of Q, this means that there exists an $\epsilon \ll 1$ such that $\epsilon \not\leq C_{f,x}$ (and so in particular $\epsilon \neq 0$). Using the definition of $C_{f,x}$ as a sup-inf, we may infer:

$$\begin{aligned}\epsilon \not\leq \bigvee_{k\in\mathbb{N}} \left(\bigwedge_{n\geq k}\bigwedge_{m\geq k} \mathbb{C}(f^n x, f^m x)\right) &\Longrightarrow \forall k \in \mathbb{N} : \epsilon \not\leq \bigwedge_{n\geq k}\bigwedge_{m\geq k} \mathbb{C}(f^n x, f^m x)\\ &\Longrightarrow \forall k \in \mathbb{N}, \exists n_k, m_k \geq k : \epsilon \not\leq \mathbb{C}(f^{n_k} x, f^{m_k} x)\end{aligned}$$

In the last line above, it cannot be the case that $m_k = n_k$, because otherwise $\mathbb{C}(f^{n_k}x, f^{n_k}x) \geq 1$ (by the "identity" axiom for the Q-category $\mathbb{C}$), which would then also be above $\epsilon \ll 1$. So suppose that $n_k < m_k$, then we can replace m_k by

$$m'_k := \min\{m > n_k \mid \epsilon \not\leq \mathbb{C}(f^{n_k}x, f^m x)\}$$

and so we still have $\epsilon \not\leq \mathbb{C}(f^{n_k}x, f^{m'_k}x)$, but now we know also that $\epsilon \leq \mathbb{C}(f^{n_k}x, f^{m_k-1}x)$. Similarly, if $n_k > m_k$ then we may replace n_k by

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and we still have $\epsilon \not\leq \mathbb{C}(f^{n'_k}x, f^{m_k}x)$, but now we know also that $\epsilon \leq \mathbb{C}(f^{n'_k-1}x, f^{m_k}x)$. That is to say, we can always pick $n_k, m_k \geq k$ to ensure that

$$\epsilon \not\leq \mathbb{C}(f^{n_k}x, f^{m_k}x) \text{ and } \left\{\begin{array}{ll}\text{either} & \mathbb{C}(f^{n_k}x, f^{m_k-1}x) \geq \epsilon \quad (A)\\ \text{or} & \mathbb{C}(f^{n_k-1}x, f^{m_k}x) \geq \epsilon \quad (B)\end{array}\right.$$

Now denote, for each such pick of $n_k, m_k \geq k \in \mathbb{N}$,

$$d_k := \mathbb{C}(f^{n_k}x, f^{m_k}x);$$

and let us insist that $\epsilon \not\leq d_k$ for all $k \in \mathbb{N}$. In case condition (A) holds for d_k, then in particular $m_k > n_k$ so $m_k \geq 1$, and we can use the "composition" axiom in $\mathbb{C}$ to get

$$\begin{aligned}\epsilon \circ c_{m_k-1} &\leq \mathbb{C}(f^{n_k}x, f^{m_k-1}x) \circ \mathbb{C}(f^{m_k-1}x, f^{m_k}x)\\ &\leq \mathbb{C}(f^{n_k}x, f^{m_k}x)\\ &= d_k\end{aligned}$$

In case condition (B) holds for d_k we can similarly prove that

$$a_{n_k-1} \circ \epsilon \leq d_k.$$

Hence, using in $(*)$ that a continuous lattice is always meet-continuous, and that both sequences

$$\left(\bigwedge\{d_k \mid k \geq N \text{ and } (A) \text{ holds}\}\right)_{N\in\mathbb{N}} \text{ and } \left(\bigwedge\{d_k \mid k \geq N \text{ and } (B) \text{ holds}\}\right)_{N\in\mathbb{N}}$$

are increasing, we may compute that

$$\begin{aligned}\bigvee_{N\in\mathbb{N}}\bigwedge_{k\geq N} d_k &= \bigvee_{N\in\mathbb{N}_0}\bigwedge_{k\geq N} d_k\\ &= \bigvee_{N\in\mathbb{N}_0}\left(\bigwedge\{d_k \mid k \geq N \text{ and } (A) \text{ holds}\} \wedge \bigwedge\{d_k \mid k \geq N \text{ and } (B) \text{ holds}\}\right)\\ &\overset{(*)}{=} \left(\bigvee_{N\in\mathbb{N}_0}\bigwedge\{d_k \mid k \geq N \text{ and } (A) \text{ holds}\}\right)\\ &\qquad\qquad \wedge \left(\bigvee_{N\in\mathbb{N}_0}\bigwedge\{d_k \mid k \geq N \text{ and } (B) \text{ holds}\}\right)\\ &\geq \left(\bigvee_{N\in\mathbb{N}_0}\bigwedge\{\epsilon \circ c_{m_k-1} \mid k \geq N \text{ and } (A) \text{ holds}\}\right)\\ &\qquad\qquad \wedge \left(\bigvee_{N\in\mathbb{N}_0}\bigwedge\{a_{n_k-1} \circ \epsilon \mid k \geq N \text{ and } (B) \text{ holds}\}\right)\\ &\geq \left(\bigvee_{N\in\mathbb{N}}\bigwedge_{m\geq N} \epsilon \circ c_m\right) \wedge \left(\bigvee_{N\in\mathbb{N}}\bigwedge_{m\geq N} a_n \circ \epsilon\right)\\ &\geq \left(\epsilon \circ (\bigvee_{N\in\mathbb{N}}\bigwedge_{m\geq N} c_m)\right) \wedge \left((\bigvee_{N\in\mathbb{N}}\bigwedge_{m\geq N} a_n) \circ \epsilon\right)\\ &= \left(\epsilon \circ (\bigvee_{N\in\mathbb{N}} c_N)\right) \wedge \left((\bigvee_{N\in\mathbb{N}} a_N) \circ \epsilon\right)\\ &= (\epsilon \circ 1) \wedge (1 \circ \epsilon)\\ &= \epsilon\end{aligned}$$

So, even though $\epsilon \not\leq d_k$ (for all $k \in \mathbb{N}$), we do have that $0 \neq \epsilon \leq \bigvee_{N\in\mathbb{N}}\bigwedge_{k\geq N} d_k$.

(iv) Using the "composition" axiom in $\mathbb{C}$, we have for every $k \geq N \in \mathbb{N}$ (recall that $n_k, m_k \geq k$ too) that

$$d_k \geq c_{n_k} \circ \mathbb{C}(f^{n_k+1}x, f^{m_k+1}x) \circ a_{m_k} \geq c_{n_k} \circ \varphi(d_k) \circ a_{m_k} \geq c_N \circ \varphi(d_k) \circ a_N$$

and so we may compute that

$$\begin{aligned}\bigvee_{N\in\mathbb{N}}\bigwedge_{k\geq N} d_k &\geq \bigvee_{N\in\mathbb{N}}\bigwedge_{k\geq N} (c_N \circ \varphi(d_k) \circ a_N)\\ &\geq \bigvee_{N\in\mathbb{N}} \left(c_N \circ (\bigwedge_{k\geq N} \varphi(d_k)) \circ a_N\right)\\ &\overset{(*)}{=} \left(\bigvee_{N\in\mathbb{N}} c_N\right) \circ \left(\bigvee_{N\in\mathbb{N}}\bigwedge_{n\geq N} \varphi(d_k)\right) \circ \left(\bigvee_{N\in\mathbb{N}} a_N\right)\\ &= 1 \circ \left(\bigvee_{N\in\mathbb{N}}\bigwedge_{k\geq N} \varphi(d_k)\right) \circ 1\\ &\geq \varphi(\bigvee_{N\in\mathbb{N}}\bigwedge_{n\geq N} d_k)\\ &\geq \bigvee_{N\in\mathbb{N}}\bigwedge_{n\geq N} d_k\end{aligned}$$

where in $(*)$ we used once more the argument involving increasing sequences (explained in a previous footnote), but now for three sequences instead of two. This means that $\bigvee_{N\in\mathbb{N}}\bigwedge_{k\geq N} d_k$ is a fixpoint of φ which – as we showed earlier – is not 0, so we must have $1 \leq \bigvee_{N\in\mathbb{N}}\bigwedge_{k\geq N} d_k$.

(v) Since $\epsilon \ll 1 \leq \bigvee_{N\in\mathbb{N}}\bigwedge_{k\geq N} d_k$, and the latter supremum is directed, there must exist an $N_0 \in \mathbb{N}$ such that $\epsilon \leq \bigwedge_{k\geq N_0} d_k$. Yet, we established earlier that $\epsilon \not\leq d_k$ for all $k \in \mathbb{N}$. This is the announced contradiction. □

Fixpoint theorem (3)

Proposition

Suppose that $F\colon \mathbb{C} \to \mathbb{C}$ *is a* φ*-contraction on a* Q*-category. Suppose that* Q *is a* ***continuous lattice*** *and that* $\varphi\colon Q \to Q$ *is a* ***lower-semicontinuous function****. Then, for any* $x \in \mathbb{C}_0$ *such that* $\mathbb{C}(Fx, x) \neq 0 \neq \mathbb{C}(x, Fx)$*, the sequence* $(F^n x)_{n\in\mathbb{N}}$ *is Cauchy.*

Sketch of proof :

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Sketch of proof: If $(F^n x)_{n\in\mathbb{N}}$ is *not* Cauchy, then $1 \not\leq \bigvee_{N\in\mathbb{N}} \bigwedge_{m,n\geq N} \mathbb{C}(F^n x, F^m x)$.

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In the continuous lattice Q, there must then be an ϵ such that

$$\epsilon \ll 1 \quad \text{and} \quad \epsilon \not\leq \bigvee_{N\in\mathbb{N}} \bigwedge_{m,n\geq N} \mathbb{C}(F^n x, F^m x).$$

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Suppose that $F\colon \mathbb{C} \to \mathbb{C}$ is a φ-contraction on a Q-category. Suppose that Q is a ***continuous lattice*** *and that $\varphi\colon Q \to Q$ is a* ***lower-semicontinuous function****. Then, for any $x \in \mathbb{C}_0$ such that $\mathbb{C}(Fx, x) \neq 0 \neq \mathbb{C}(x, Fx)$, the sequence $(F^n x)_{n\in\mathbb{N}}$ is Cauchy.*

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Thus $0 \neq \epsilon \ll 1$ is such that

$$\text{for all } k \in \mathbb{N} \text{ there exist } m_k, n_k \geq k \text{ such that } \epsilon \not\leq \mathbb{C}(F^{n_k} x, F^{m_k} x).$$

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for all $k \in \mathbb{N}$ there exist $m_k, n_k \geq k$ such that $\epsilon \not\leq \mathbb{C}(F^{n_k} x, F^{m_k} x)$.

A "clever choice" of such indices $m_k, n_k \geq k$ can be made, say

$$d_k := \mathbb{C}(F^{n_k} x, F^{m_k} x),$$

so that, with lower-semicontinuity of φ, continuity of Q, and $\mathbb{C}(Fx, x) \neq 0 \neq \mathbb{C}(x, Fx)$,

$$\varphi\Big(\bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} d_k\Big) = \bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} d_k \neq 0.$$

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Thus, $1 \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} d_k$, which leads to $\epsilon \leq \bigwedge_{k\geq N_0} d_k$ for some N_0.

Fixpoint theorem (3)

Proposition

*Suppose that $F\colon \mathbb{C} \to \mathbb{C}$ is a φ-contraction on a Q-category. Suppose that Q is a **continuous lattice** and that $\varphi\colon Q \to Q$ is a **lower-semicontinuous function**. Then, for any $x \in \mathbb{C}_0$ such that $\mathbb{C}(Fx, x) \neq 0 \neq \mathbb{C}(x, Fx)$, the sequence $(F^n x)_{n\in\mathbb{N}}$ is Cauchy.*

Sketch of proof: If $(F^n x)_{n\in\mathbb{N}}$ is *not* Cauchy, then $1 \not\leq \bigvee_{N\in\mathbb{N}} \bigwedge_{m,n\geq N} \mathbb{C}(F^n x, F^m x)$.

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Thus $0 \neq \epsilon \ll 1$ is such that

$$\text{for all } k \in \mathbb{N} \text{ there exist } m_k, n_k \geq k \text{ such that } \epsilon \not\leq \mathbb{C}(F^{n_k} x, F^{m_k} x).$$

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Thus, $1 \leq \bigvee_{N\in\mathbb{N}} \bigwedge_{k\geq N} d_k$, which leads to $\epsilon \leq \bigwedge_{k\geq N_0} d_k$ for some N_0. Contradiction!

Fixpoint theorem (4)

Theorem

Suppose that $F\colon \mathbb{C} \to \mathbb{C}$ *is a* φ*-contraction on a Cauchy complete* Q*-category. Suppose that* Q *is a continuous lattice and that* $\varphi\colon Q \to Q$ *is a lower-semicontinuous morphism. If there exists an* $x \in \mathbb{C}_0$ *such that* $\mathbb{C}(Fx, x) \neq 0 \neq \mathbb{C}(x, Fx)$*, then* F *has a fixpoint.*

If $\mathbb{C}$ *is symmetric, then any two fixpoints of* F *are either isomorphic or in different summands of* $\mathbb{C}$*; if* $\mathbb{C}$ *has no zero-homs, then any two fixpoints of* F *are isomorphic.*

Fixpoint theorem (4)

Theorem

Suppose that $F\colon \mathbb{C} \to \mathbb{C}$ is a φ-contraction on a Cauchy complete Q-category. Suppose that Q is a continuous lattice and that $\varphi\colon Q \to Q$ is a lower-semicontinuous morphism. If there exists an $x \in \mathbb{C}_0$ such that $\mathbb{C}(Fx, x) \neq 0 \neq \mathbb{C}(x, Fx)$, then F has a fixpoint.

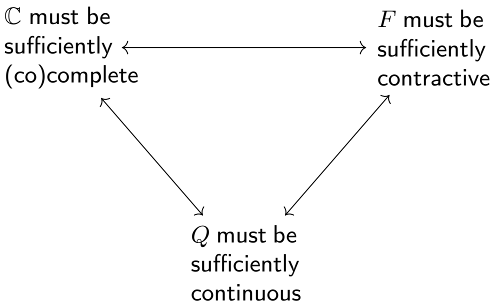
If $\mathbb{C}$ is symmetric, then any two fixpoints of F are either isomorphic or in different summands of $\mathbb{C}$; if $\mathbb{C}$ has no zero-homs, then any two fixpoints of F are isomorphic.

Examples:

- $Q = (\{0,1\}, \bigvee, \wedge, 1)$: the theorem trivializes for ordered sets.
- $Q = ([0,\infty], \bigwedge, +, 0)$: (generalized) Banach fixpoint theorem for (generalized) metric spaces, allowing for non-linear contractions (cf. (Boyd and Wong, 1969)).
- $Q = ([0,1], \bigvee, *, 1)$: a new fixpoint theorem for fuzzy orders, to be compared with e.g. (Coppola et al., 2008).
- $Q = (\Delta, \bigvee, *, e)$: a new fixpoint theorem for probabilistic metric spaces, encompassing certain known results (Hadžić and Pap, 2001).

Take-away message: an equilibrum of three

To formulate a fixpoint theorem for a φ-contraction $F\colon \mathbb{C} \to \mathbb{C}$ on a Q-category,



Our theorem captures known examples and produces new results. Yet, the literature abounds with fixpoint theorems. Further study is necessary!

Challenge: find a fixpoint theorem for *quantaloid*-enriched categories, to understand the situation for *partial* metric spaces (Hofmann and Stubbe, 2018).

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(More references in our paper.)