Cartesian Closed Double Categories

Susan Niefield

Union College
Schenectady, NY

Cartesian Closed Categories

Two approaches for a cartesian 1-category $\mathbb{C}$ to be closed

Objectwise: $(\ ) \times Y$ has a right adjoint, for all Y, denoted $(-)^Y$

As a bifunctor: $\mathbb{C}^{op} \times \mathbb{C} \xrightarrow{[\ ,\ ]} \mathbb{C}$ s.t. $\mathbb{C}(X \times Y, Z) \cong \mathbb{C}(X, [Y, Z])$

In [N 2020], we showed Y is lax exponentiable in $\mathbb{D}$ iff it is exponentiable in $\mathbb{D}_0$, for "glueing categories" $\mathbb{D}$, with examples $\mathbb{C}\mathrm{at}$, $\mathbb{P}\mathrm{os}$, $\mathbb{L}\mathrm{oc}$, and $\mathbb{T}\mathrm{op}$, so $\mathbb{C}\mathrm{at}$ and $\mathbb{P}\mathrm{os}$ are objectwise lax cc

We'll see that the two approaches differ for double categories

Outline

1. Double categories and examples
2. Lax double functors and adjoints
3. Lax cartesian closed double categories
4. Objectwise lax cartesian closed double categories
5. Lax locally cartesian closed double categories

Why lax?

Double Categories

A double category $\mathbb{D}$ is a pseudo internal category in CAT

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{D}_0 \qquad (\mathrm{id}^{\bullet} : \mathbb{D}_0 \to \mathbb{D}_1)$$

Objects X of $\mathbb{D}_0$, called objects of $\mathbb{D}$

Morphisms $X \xrightarrow{f} Y$ of $\mathbb{D}_0$, called horizontal morphisms of $\mathbb{D}$

Objects $X_s \overset{u}{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;\;} X_t$ of $\mathbb{D}_1$, called vertical morphism of $\mathbb{D}$

Morphisms $\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ u \big\downarrow\!\bullet & \varphi & \bullet\!\big\downarrow v \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$ of $\mathbb{D}_1$, called cells of $\mathbb{D}$

Examples

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ u \big\downarrow\!\!\bullet & \varphi & \bullet\!\!\big\downarrow v \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$$

$\mathbb{S}\mathrm{pan}$: sets, functions, spans, $\begin{array}{ccccc} & & X_s & \xrightarrow{f_s} & Y_s \\ & \nearrow & & \nearrow & \\ X & & \xrightarrow{f} & & Y \\ & \searrow & & \searrow & \\ & & X_t & \xrightarrow[f_t]{} & Y_t \end{array}$

Note: Also $\mathbb{S}\mathrm{pan}(\mathcal{D})$ and $\mathbb{C}\mathrm{osp}(\mathcal{D})$, for $\mathcal{D}$ with pbs and pos, resp.

$\mathbb{C}\mathrm{at}$: categories, functors, profunctors, $u \xrightarrow{\varphi} v(f_s, f_t)$

Q-$\mathbb{R}\mathrm{el}$: sets, functions, Q-valued relations, $u \leq v(f_s, f_t)$

for a quantale Q, where $X_s \times X_t \xrightarrow{u} Q$

Note: $\mathbb{R}\mathrm{el} \cong 2\text{-}\mathbb{R}\mathrm{el}$

Lax Functors

A lax functor $F : \mathbb{D} \longrightarrow \mathbb{E}$ consists of functors $F_0 : \mathbb{D}_0 \longrightarrow \mathbb{E}_0$ and $F_1 : \mathbb{D}_1 \longrightarrow \mathbb{E}_1$ compatible with s and t, and cells

$$\mathrm{id}^\bullet_{F_0 X} \longrightarrow F_1(\mathrm{id}^\bullet_X) \quad \text{and} \quad F_1 \bar{u} \odot F_1 u \longrightarrow F_1(\bar{u} \odot u)$$

satisfying naturality and coherence conditions

LxDbl denotes the 2-category double categories and lax functors

Oplax and pseudo functors are defined with the cells in the opposite direction and invertible, respectively

Definition (A 18)

$\mathbb{D}$ is called lax cartesian (AKA pre-cartesian) if the pseudo functors $\Delta : \mathbb{D} \longrightarrow \mathbb{D} \times \mathbb{D}$ and $! : \mathbb{D} \longrightarrow \mathbb{1}$ have right adoints $\times$ and 1 in LxDbl; and $\mathbb{D}$ is called cartesian if $\times$ and 1 are pseudo functors

Lax Cartesian Closed Double Categories

Definition

A lax cartesian double category $\mathbb{D}$ is lax cartesian closed if there is a lax functor $[-,-] : \mathbb{D}^{\text{op}} \times \mathbb{D} \longrightarrow \mathbb{D}$ with natural bijections $\mathbb{D}_0(X \times Y, Z) \cong \mathbb{D}_0(X, [Y, Z])$ and $\mathbb{D}_1(u \times v, w) \cong \mathbb{D}_1(u, [v, w])$

Note: $\times$ and $[-,-]$ are compatible with s and t, being lax functors

Theorem

Suppose $\mathbb{D}$ is cartesian a double category. Then $\mathbb{D}$ is lax cartesian closed if and only if $\mathbb{D}_0$ and $\mathbb{D}_1$ are cartesian closed categories satisfying $s[v, w] = {Z_s}^{Y_s}$ and $t[v, w] = {Z_t}^{Y_t}$

Proof.

($\Rightarrow$) Clear

($\Leftarrow$) To show $[-,-]$ is lax, use x oplax and the unit ε $\square$

Examples

$\mathbb{S}\text{pan}$ is lax cartesian closed via $[Y, Z]_0 = Z^Y$ and $[v, w]_1$ given by

$$
\begin{array}{ccc}
 & & Z_s^{Y_s} \\
 & \nearrow & \\
[v,w] & & \\
 & \searrow & \\
 & & Z_t^{Y_t}
\end{array}
\qquad \text{where} \qquad
[v,w] = \left\{ \begin{array}{ccccc}
 & & Y_s & \xrightarrow{\sigma_s} & Z_s \\
 & \overset{v_s}{\nearrow} & & \overset{w_s}{\nearrow} & \\
Y & \xrightarrow{\sigma} & Z & & \\
 & \underset{v_t}{\searrow} & & \underset{w_t}{\searrow} & \\
 & & Y_t & \xrightarrow[\sigma_t]{} & Z_t
\end{array} \right\}
$$

Since

$$
\begin{array}{ccccc}
 & & X_s \times Y_s & \xrightarrow{f_s} & Z_s \\
 & \overset{u_s \times v_s}{\nearrow} & & \overset{w_s}{\nearrow} & \\
X \times Y & \xrightarrow{f} & Z & & \\
 & \underset{u_t \times v_t}{\searrow} & & \underset{w_t}{\searrow} & \\
 & & X_t \times Y_t & \xrightarrow[f_t]{} & Z_t
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{ccccc}
 & & X_s & \xrightarrow{\hat{f}_s} & Z_s^{Y_s} \\
 & \overset{u_s}{\nearrow} & & \nearrow & \\
X & \xrightarrow{\hat{f}} & [v,w] & & \\
 & \underset{u_t}{\searrow} & & \searrow & \\
 & & X_t & \xrightarrow[\hat{f}_t]{} & Z_t^{Y_t}
\end{array}
$$

Note: Also, $\mathbb{S}\text{pan}(\mathcal{D})$, for cartesian closed $\mathcal{D}$ with equalizers

Examples, cont.

$\mathbb{C}\mathrm{osp}$ is not lax cartesian closed, since $s[v,w] \neq Z_s^{Y_s}$; as $\mathbb{C}\mathrm{osp}_1$ is a presheaf topos and one can show elements of $[v,w]_s$ correspond to

$$\begin{array}{ccccc} Y_s & \longrightarrow & Z_s & & \\ & \searrow^{v_s} & & \searrow^{w_s} & \\ & & Y & \longrightarrow & Z \\ & \nearrow & & \nearrow_{w_t} & \\ \emptyset & \longrightarrow & Z_t & & \end{array}$$

We'll see $\mathbb{C}\mathrm{osp}$ is "objectwise" lax cartesian closed

$\mathbb{C}\mathrm{at}$ is lax cartesian closed via $[Y,Z]_0 = Z^Y$ in $\mathbb{C}\mathrm{at}_0$ and

$$[v,w]_1(\sigma_s,\sigma_t) = \left\{ \begin{array}{ccc} Y_s & \xrightarrow{\sigma_s} & Z_s \\ {\scriptstyle v}\downarrow\!\bullet & \xrightarrow{\varphi} & \bullet\!\downarrow{\scriptstyle w} \\ Y_t & \xrightarrow[\sigma_t]{} & Z_t \end{array} \right\}$$

Examples, cont.

Q-$\mathbb{R}\mathrm{el}$ is lax cartesian closed, when Q is a locale, via

$$(u \times v)((x_s, y_s), (x_t, y_t)) = u(x_s, x_t) \wedge v(y_s, y_t)$$

with $[Y, Z]_0 = Z^Y$ in Sets and $[v, w]_1 \colon Z_s^{Y_s} \to\!\!\!\!\bullet\,\, Z_t^{Y_t}$ given by

$$[v, w]_1(\sigma_s, \sigma_t) = \bigwedge_{(y_s, y_t)} \big(v(y_s, y_t) \rightarrow w(\sigma_s y_s, \sigma_t y_t) \big)$$

defined using $\rightarrow$ in the locale Q

Then

$$\begin{array}{ccc} X_s \times Y_s & \xrightarrow{f_s} & Z_s \\ {\scriptstyle u\times v}\downarrow\bullet & \leq & \bullet\downarrow{\scriptstyle w} \\ X_t \times Y_t & \xrightarrow[f_t]{} & Z_t \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} X_s & \xrightarrow{\hat{f}_s} & Z_s^{Y_s} \\ {\scriptstyle u}\downarrow\bullet & \leq & \bullet\downarrow{\scriptstyle [v,w]_1} \\ X_t & \xrightarrow[\hat{f}_t]{} & Z_t^{Y_t} \end{array}$$

as $u \times v \leq w(f_s -, f_t -)$ iff $u \leq \big(v(y_s, y_t) \rightarrow w(f_s(-, y_s), f_t(-, y_t))\big)$, for all (y_s, y_t), iff $u \leq [v, w]_1(\hat{f}_s, \hat{f}_t)$

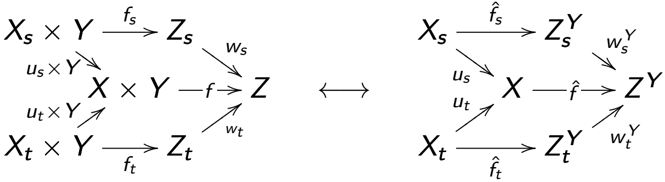
Objectwise Lax Cartesian Closed Double Categories

Definition

A double category $\mathbb{D}$ is called objectwise lax cartesian closed if $(-) \times Y : \mathbb{D} \longrightarrow \mathbb{D}$ exists and has a right adjoint in **LxDbl**, for all Y

Note: $\mathbb{C}\mathrm{at}$, $\mathbb{S}\mathrm{pan}(\mathcal{D})$, and Q-$\mathbb{R}\mathrm{el}$ are examples, for $\mathcal{D}$, Q as above, since lax cartesian closed implies objectwise lax cartesian closed

$\mathbb{C}\mathrm{osp}(\mathcal{D})$ is objectwise lax cartesian closed, for $\mathcal{D}$ a cartesian closed category with pushouts, since



Lax Locally Cartesian Closed Double Categories

Definition

$\mathbb{D}$ is called lax locally cartesian closed if every double slice $\mathbb{D}//B$, is lax cartesian closed, where

$$(\mathbb{D}//B)_0 = \mathbb{D}_0/B \text{ and } (\mathbb{D}//B)_1 = \mathbb{D}_1/\operatorname{id}^\bullet_B$$

- $\mathbb{C}\mathrm{at}$ is not, since $(\mathbb{C}\mathrm{at}//\mathbb{2})_1 \cong \mathrm{Cat}/(\mathbb{2} \times \mathbb{2})$
- $\mathbb{S}\mathrm{pan}(\mathcal{D})$ is lax locally cartesian closed, if $\mathcal{D}$ has pbs and is locally cartesian closed, since $\mathbb{S}\mathrm{pan}(\mathcal{D})//B \simeq \mathbb{S}\mathrm{pan}(\mathcal{D}/B)$
- Q-$\mathbb{R}\mathrm{el}$ is lax locally cartesian closed, if Q is a locale, since
  $$Q\text{-}\mathbb{R}\mathrm{el}//B \simeq (B^*Q)\text{-}\mathbb{R}\mathrm{el}(\mathrm{Sets}/B)$$
  where B^*Q is an internal locale in the topos Sets/B

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