Magnitude homology

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New results: paper in preparation with
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These slides: on my web page

Summary

Magnitude homology is a homology theory of enriched categories.

It specializes to a homology theory of metric spaces.

Main theorem (with Adrián Doña Mateo) *Two closed subsets of* $\mathbb{R}^N$ *have the same magnitude homology if and only if they are related by a certain concrete geometric condition.*

1. What is magnitude homology?

Richard Hepworth and Simon Willerton,
Categorifying the magnitude of a graph.

Tom Leinster and Michael Shulman, Magnitude
homology of enriched categories and metric spaces.

Warm-up: homology of an ordinary category

Any *unenriched* category $\boldsymbol{X}$ gives rise to a chain complex $C_*(\boldsymbol{X})$:

$$C_n(\boldsymbol{X}) = \coprod_{x_0,\ldots,x_n \in \boldsymbol{X}} \mathbb{Z} \cdot \big(\boldsymbol{X}(x_0, x_1) \times \cdots \times \boldsymbol{X}(x_{n-1}, x_n)\big)$$

where $\mathbb{Z} \cdot -$: **Set** → **Ab** is the free abelian group functor.

The differential ∂ is $\sum_{i=0}^{n}(-1)^i \partial_i$, where ∂_i composes at x_i (for $0 < i < n$) or forgets the first/last factor (for $i \in \{0, n\}$).

The homology $H_*(\boldsymbol{X})$ of $\boldsymbol{X}$ is the homology of $C_*(\boldsymbol{X})$.

Key ingredients here:

- (**Set**, $\times, 1$) is a monoidal category, whose unit object 1 is terminal.
- **Ab** is both abelian and monoidal.
- $\mathbb{Z} \cdot -$ is a strong monoidal functor.

The magnitude homology of an enriched category

Setup: Imitating the unenriched case, we start with:

- a monoidal category $\boldsymbol{V}$ whose unit object is terminal (generalizing **Set**)
- a monoidal abelian category $\boldsymbol{A}$ (generalizing **Ab**)
- a strong monoidal functor $\Sigma\colon \boldsymbol{V} \to \boldsymbol{A}$ (generalizing $\mathbb{Z} \cdot -$).

Let $\boldsymbol{X}$ be a $\boldsymbol{V}$-category.

Define a chain complex $C_*(\boldsymbol{X})$ in $\boldsymbol{A}$ by

$$C_n(\boldsymbol{X}) = \coprod_{x_0,\ldots,x_n \in \boldsymbol{X}} \Sigma\big(\boldsymbol{X}(x_0, x_1) \otimes \cdots \otimes \boldsymbol{X}(x_{n-1}, x_n)\big).$$

It has differential $\partial = \sum_{i=0}^{n} (-1)^i \partial_i$, where ∂_i either composes at x_i or forgets the first/last factor.

Definition The magnitude homology $MH_*(\boldsymbol{X})$ of $\boldsymbol{X}$ is the homology of $C_*(\boldsymbol{X})$.

The magnitude homology of a metric space

Metric spaces are categories enriched in $\boldsymbol{V} = (([0,\infty), \geq), +, 0)$.

To take the magnitude homology of metric spaces, we'll need:

- a monoidal abelian category $\boldsymbol{A}$
- a strong monoidal functor $\Sigma\colon [0,\infty) \to \boldsymbol{A}$.

We choose:

- $\boldsymbol{A} = \mathbf{Ab}^{[0,\infty)}$ with the convolution product: $(A \otimes B)_\ell = \coprod_{k+m=\ell} A_k \otimes B_m$
- $\Sigma\colon [0,\infty) \to \mathbf{Ab}^{[0,\infty)}$ to be the functor defined by

$$(\Sigma(\ell))(m) = \begin{cases} \mathbb{Z} & \text{if } \ell = m \\ 0 & \text{otherwise} \end{cases}$$

$(\ell, m \in [0,\infty))$.

The magnitude homology of a metric space, explicitly

Let X be a metric space.

The chain complex $C_*(X)$ in $\mathbf{Ab}^{[0,\infty)}$ is given by

$$C_n^\ell(X) = \mathbb{Z} \cdot \big\{ (x_0, \ldots, x_n) \ : \ d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell \big\}$$

$(n \in \mathbb{N}, \ell \in [0, \infty))$.

Equivalently, we can replace $C_*(X)$ by a normalized version, $\hat{C}_*(X)$:

$$\hat{C}_n^\ell(X) = \mathbb{Z} \cdot \big\{ (x_0, \ldots, x_n) \ : \ x_0 \neq \cdots \neq x_n, \ d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell \big\}.$$

The differential $\partial \colon \hat{C}_n(X) \to \hat{C}_{n-1}(X)$ is $\sum_{0<i<n} (-1)^i \partial_i$, where

$$\partial_i(x_0, \ldots, x_n) = \begin{cases} (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) & \text{if } x_i \text{ is between } x_{i-1} \text{ and } x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

Then $MH_*(X)$ is the homology of the chain complex $\hat{C}_*(X)$ in $\mathbf{Ab}^{[0,\infty)}$.

Magnitude homology is graded!

Magnitude homology of a metric space is a $[0, \infty)$-*graded* homology theory.

That is, when X is a metric space and n is a natural number, $MH_n(X)$ is not just an abelian group, but an object of $\mathbf{Ab}^{[0,\infty)}$ — a *family*

$$\left(MH_n^\ell(X)\right)_{\ell\in[0,\infty)}$$

of abelian groups.

(Compare Khovanov homology. . .)

Sample results

- 1st magnitude homology detects convexity:

Theorem *Let X be a closed subset of $\mathbb{R}^N$. Then*

$$X \text{ is convex} \iff MH_1^\ell(X) = 0 \text{ for all } \ell > 0.$$

- Work of Kyonori Gomi substantiates the slogan:

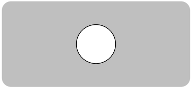
  The more geodesics are unique, the more magnitude homology is trivial.

- Ordinary homology detects the *existence* of holes.

Magnitude homology detects the *size* of holes.

Example (Ryuki Kaneta & Masahiko Yoshinaga) Let $r > 0$ and

$$X = \{x \in \mathbb{R}^N \ : \ \|x\| \geq r\}.$$



Then $r = \inf\{\ell/2n \ : \ MH_n^\ell(X) = 0\}$.

2. Preparation for the main theorem

What does it mean to “have the same homology”?

For any homology theory whatsoever, what does it mean for two objects X and Y to “have the same homology”? There are several interpretations. . .

Answer 1 *Crude:* $H_*(X) \cong H_*(Y)$.

Usually seen as unhelpful, too loose.

Unhelpful for us too. E.g. Emily Roff has exhibited metric spaces with the same magnitude homology (in this sense) but different topological homology.

Answer 2 *Quasi-isomorphism:* Declare X and Y to “have the same homology” if there is a map $X \to Y$ inducing an iso $H_*(X) \to H_*(Y)$.

Answer 3 *One step further:* Ask for existence of maps $X \rightleftarrows Y$ inducing mutually inverse maps $H_*(X) \rightleftarrows H_*(Y)$.

We follow Answer 3, where our objects are metric spaces and map means distance-decreasing (= 1-Lipschitz = weakly contractive = short) map.

Another preview of the main theorem

Theorem (with Adrián Doña Mateo) *Let X and Y be nonempty closed subsets of $\mathbb{R}^N$. The following are equivalent:*

- *there are distance-decreasing maps $X \rightleftarrows Y$ inducing mutually inverse maps $MH_n^*(X) \rightleftarrows MH_n^*(Y)$ for all $n \geq 1$*
- *X and Y are related by a certain concrete geometric condition.*

Next: that "concrete geometric condition".

The inner boundary of a space

Let X be a metric space.

Points $x, y \in X$ are adjacent if they are distinct and there is no point $z \in X$ strictly between them $(d(x,z) + d(z,y) = d(x,y) \Rightarrow z \in \{x, y\})$.

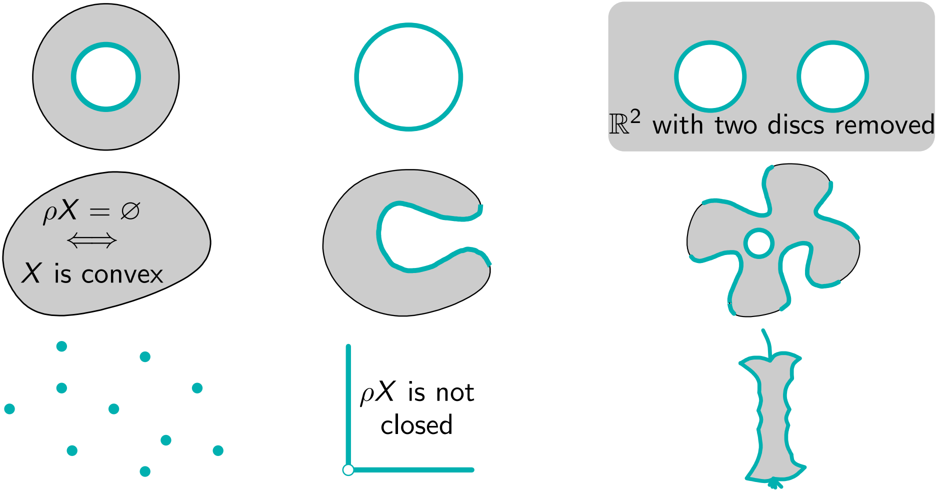
The inner boundary of X is

$$\rho X = \{x \in X \ : \ x \text{ is adjacent to some point of } X\}.$$

Note If $X \subseteq \mathbb{R}^N$ then $\rho X \subseteq \partial X$.

Examples of inner boundaries (all closed subsets of $\mathbb{R}^N$)

- ρX: inner boundary of X (the set of points adjacent to some other point)



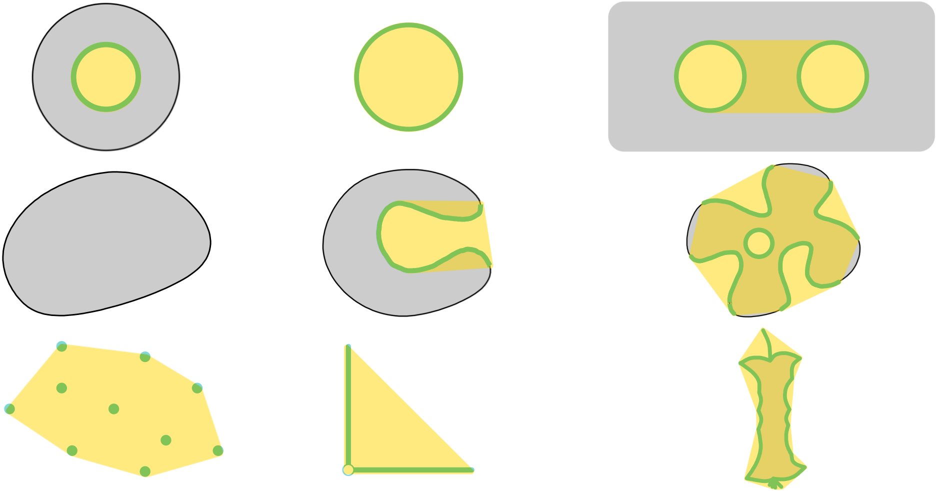
$\mathbb{R}^2$ with two discs removed

$\rho X = \varnothing \iff X$ is convex

ρX is not closed

The closed convex hull of the inner boundary

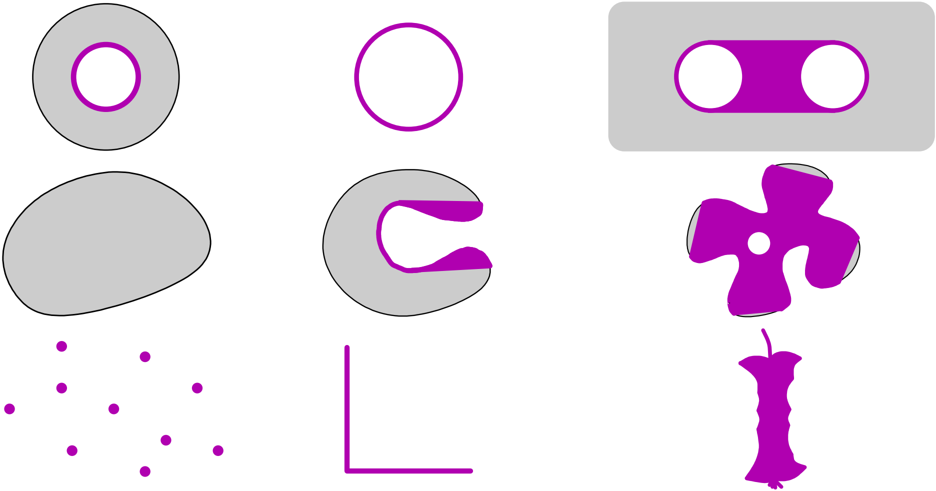
- ρX: inner boundary of X (the set of points adjacent to some other point)
- $\overline{\mathrm{conv}(\rho X)}$: closure of convex hull of ρX



The core of a subset of $\mathbb{R}^N$

- ρX: inner boundary of X (the set of points adjacent to some other point)
- $\overline{\mathrm{conv}(\rho X)}$: closure of convex hull of ρX
- $\mathrm{core}(X) = \overline{\mathrm{conv}(\rho X)} \cap X$

Fact: $\mathrm{core}(\mathrm{core}(X)) = \mathrm{core}(X)$



3. The main theorem

The main theorem

Theorem (with Adrián Doña Mateo) *Let X and Y be nonempty closed subsets of $\mathbb{R}^N$. The following are equivalent:*

(i) *there exist distance-decreasing maps $X \rightleftarrows Y$ inducing mutually inverse maps $MH_n^*(X) \rightleftarrows MH_n^*(Y)$ for* all $n \geq 1$

(ii) *there exist distance-decreasing maps $X \rightleftarrows Y$ inducing mutually inverse maps $MH_n^*(X) \rightleftarrows MH_n^*(Y)$ for* some $n \geq 1$

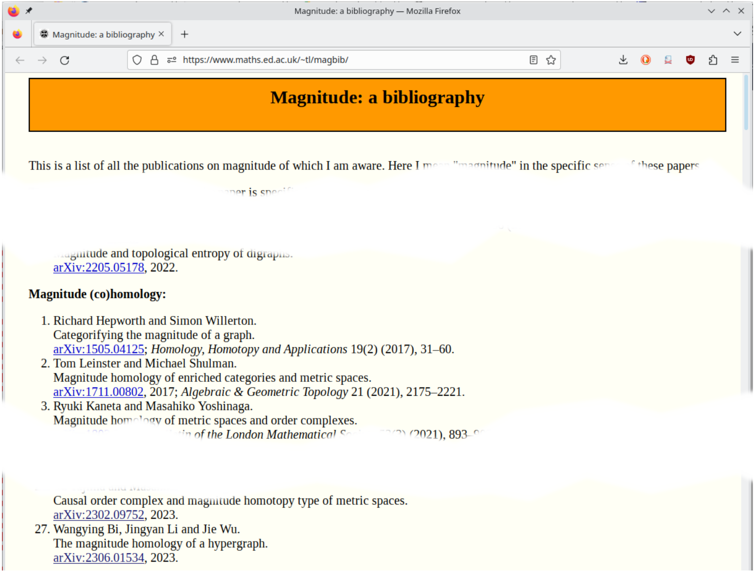
(iii) $\mathrm{core}(X)$ *and* $\mathrm{core}(Y)$ *are isometric.*

Proof A theorem of Kaneta and Yoshinaga + some convex geometry. ☐

In particular, X and $\mathrm{core}(X)$ have the same magnitude homology, for any X.

Magnitude homology equivalence
reduces to a concrete geometric condition,
for closed subsets of Euclidean space.

References



Magnitude: a bibliography

This is a list of all the publications on magnitude of which I am aware. Here I m "magnitude" in the specific se f these papers

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agnitude and topological entropy of digraphs.
arXiv:2205.05178, 2022.

Magnitude (co)homology:

1. Richard Hepworth and Simon Willerton.
   Categorifying the magnitude of a graph.
   arXiv:1505.04125; *Homology, Homotopy and Applications* 19(2) (2017), 31–60.
2. Tom Leinster and Michael Shulman.
   Magnitude homology of enriched categories and metric spaces.
   arXiv:1711.00802, 2017; *Algebraic & Geometric Topology* 21 (2021), 2175–2221.
3. Ryuki Kaneta and Masahiko Yoshinaga.
   Magnitude homology of metric spaces and order complexes.
   in of the London Mathematical S (2021), 893–

Causal order complex and magnitude homotopy type of metric spaces.
arXiv:2302.09752, 2023.

27. Wangying Bi, Jingyan Li and Jie Wu.
    The magnitude homology of a hypergraph.
    arXiv:2306.01534, 2023.

. . . plus paper with Adrián Doña Mateo in preparation.

Frequently asked questions

What about persistent homology?

Magnitude homology specializes to a homology theory of metric spaces.

Persistent homology is another homology theory of metric spaces.

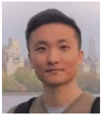
Both involve a real scale parameter.

How are they related?

Answer They capture quite different information about a space.

Two independent comparisons:

Nina Otter, Magnitude meets persistence: homology theories for filtered simplicial sets.



Simon Cho, Quantales, persistence, and magnitude homology.

Why the name “magnitude homology”?

Magnitude is a numerical invariant of enriched categories.

Magnitude homology is intended to be a categorification of magnitude, in the sense that the Euler characteristic of magnitude homology is magnitude.

This has been proved to be the case:

- for finite graphs by Hepworth and Willerton
- for *finite* metric spaces by Leinster and Shulman.

Does magnitude homology categorify magnitude?

For non-finite metric spaces, magnitude encodes a lot of geometric information.

Examples For compact $X \subseteq \mathbb{R}^N$, if you know the magnitude of tX for all $t > 0$, you can recover:

- the dimension of X (Mark Meckes)
- the volume of X (Juan Antonio Barceló and Tony Carbery)
- the surface area of X (Heiko Gimperlein and Magnus Goffeng).

In particular, different convex sets usually have different magnitude. . .
but they all have trivial magnitude homology!

Major challenge Refine the definition of magnitude homology so that for non-finite metric spaces too, it categorifies magnitude.