Enriched Accessible Categories

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joint with Giacomo Tendas

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1. Examples
2. Setting
3. Characterization theorems.

EXAMPLES

1 algebraic structures product theories

monoids, groups, rings, Lie alg

... ring with a module

① algebraic structures product theories

monoids, groups, rings, Lie alg

… ring with a module

② essentially alg. structures limit theories

categories, groupoids, cancellative monoids

… cat w a d.fibration

$$C_3 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; C_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; C_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; C_0$$

$C_1 \times_{C_0} C_1 \times_{C_0} C_1$ $C_1 \times_{C_0} C_1$

⇝ locally presentable categories (Gabriel–Ulmer)

① algebraic structures — product theories

monoids, groups, rings, Lie alg

... ring with a module

② essentially alg. structures — limit theories

categories, groupoids, cancellative monoids

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$$
\begin{tikzcd}
C_3 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & C_2 \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & C_1 \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=3] \arrow[r, shift right=3] & C_0 \arrow[l]
\end{tikzcd}
$$

$C_1 \times_{C_0} C_1 \times_{C_0} C_1$ $\quad$ $C_1 \times_{C_0} C_1$

⇝ locally presentable categories (Gabriel–Ulmer)

③ axiomatizable structures — (limit, colimit) sketches

fields, v.N regular rings, Hopf algebras

$$
\begin{tikzcd}
1 \arrow[r, "0"] & F & F^* \arrow[l, shift right=1, "inc"'] \arrow[l, shift left=1, "(\ )^{-1}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
E \arrow[r, "eq"] \arrow[rr, bend left, "\pi_1"] & \mathbb{R}^2 \arrow[r, shift left] \arrow[r, shift right] & \mathbb{R}
\end{tikzcd}
$$

$(x,y) \longmapsto xyx$

⇝ accessible categories (Lair)

① algebraic structures — product theories

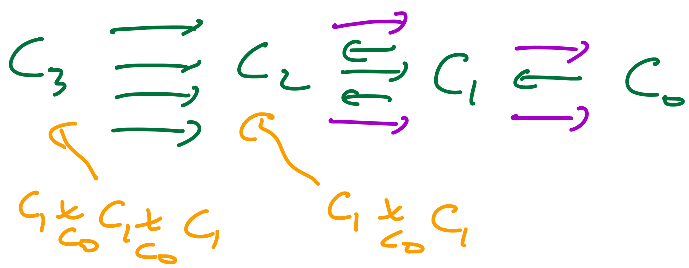
monoids, groups, rings, Lie alg

... ring with a module

② essentially alg. structures — limit theories

categories, groupoids, cancellative monoids

... cat w a d.fibration



⤳ locally presentable categories (Gabriel–Ulmer)

③ axiomatizable structures — (limit, colimit) sketches

fields, v.N regular rings, Hopf algebras

$$1 \xrightarrow{0} F \underset{(\ )^{-1}}{\overset{inc}{\leftleftarrows}} F^*$$

$$E \xrightarrow{eq} \mathbb{R}^2 \underset{(x,y) \mapsto xyx}{\overset{\pi_1}{\rightrightarrows}} \mathbb{R}$$

⤳ accessible categories (Lair)

④

loc. pres ⟺ acc & cocomplete

⟺ acc & complete.

2-dimensional version (categories with extra structure)

obj	mor	2-cells
• monoidal cats	strong monoidal functors	mon. n.t.
• symm. "	symm " "	"
• cats with f. lims	f.lim pres. functors	n.t.
• cats with f.coproducts	f.copr. " "	n.t.
• dist. cats	dist. functors	"
• regular cats	reg. "	"

pres. structure up to iso

Bourke: all examples of accessible 2-cats

not loc. pres!

Cat-enriched

2-dimensional version (categories with extra structure)

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• monoidal cats	strong monoidal functors	mon. n.t.
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• regular cats	reg. "	"

pres. structure up to iso

Bourke: all examples of accessible 2-cats

not loc. pres!

Cat-enriched

∞-version

quasicats with structure

∞-cats* with structure

examples of accessible { Set-categories
∞-cosmoi

SETTING

$\mathcal{V}$ symmetric monoidal closed (Kelly)
locally finitely presentable as a closed category
$\Rightarrow$ notion of α-small limit for reg. cardinal α
e.g. Set, Cat, Gpd, sSet, Ab, RMod, dgMod, any Gr. topos
write $\mathcal{P}\mathcal{C}$ for the free cocompletion of a $\mathcal{V}$-cat $\mathcal{C}$
i.e. the $\mathcal{V}$-category of small presheaves on $\mathcal{C}$

$$\mathcal{P}^{+}\mathcal{C} = \mathcal{P}(\mathcal{C}^{op})^{op} \text{ the free completion}$$

$\Rightarrow$ notion of α-flat
if $\mathcal{C}$ small, $\mathcal{C} \xrightarrow{G} \mathcal{V}$ is α-flat when $[\mathcal{C}^{op}, \mathcal{V}] \xrightarrow{\mathrm{Lan}_y G} \mathcal{V}$
pres α-small limits.

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$\mathcal{P}^{+}\mathcal{C} = \mathcal{P}(\mathcal{C}^{op})^{op}$ the free completion

$\Rightarrow$ notion of α-flat
if $\mathcal{C}$ small, $\mathcal{C} \xrightarrow{G} \mathcal{V}$ is α-flat when $[\mathcal{C}^{op}, \mathcal{V}] \xrightarrow{\mathrm{Lan}_y G} \mathcal{V}$
pres α-small limits.

Defn: $\mathcal{A}$ is accessible when it has the form
α-$\mathrm{Flat}(\mathcal{C}, \mathcal{V})$ for some α, some small $\mathcal{C}$.

CHARACTERIZATIONS

Locally presentable $\mathcal{V}$-categories. (Gabriel-Ulmer, Kelly)

<u>Thm</u>: For a $\mathcal{V}$-category $\mathcal{A}$, t.f.a.e.

(i) $\mathcal{A}$ is accessible with limits

(ii) $\mathcal{A}$ is accessible with colimits

(iii) $\mathcal{A}$ is <u>accessibly embedded</u> & reflective in some $[\mathcal{C}, \mathcal{V}]$

(iv) $\mathcal{A}$ is an <u>orthogonality class</u> in some $[\mathcal{C}, \mathcal{V}]$

(v) $\mathcal{A}$ is the $\mathcal{V}$-category of <u>models of a limit sketch</u>

Accessible $\mathcal{V}$-categories.

{ Lair
{ Borceux-Quinteiro-Rosický

Thm: For a $\mathcal{V}$-category $\mathcal{A}$, t.f.a.e.

(i) $\mathcal{A}$ is accessible ~~with limits~~

(v) $\mathcal{A}$ is the $\mathcal{V}$-category of models of a limit -colimit sketch

Accessible $\mathcal{V}$-categories.

Thm: For a $\mathcal{V}$-category $\mathcal{A}$, t.f.a.e.

(i) $\mathcal{A}$ is accessible ~~with limits~~

(ii) $\mathcal{A}$ is accessible with virtual colimits

(iii) $\mathcal{A}$ is accessibly embedded & virtually reflective in some $[\mathcal{C},\mathcal{V}]$

(iv) $\mathcal{A}$ is a~~n~~ virtual orthogonality class in some $[\mathcal{C},\mathcal{V}]$

(v) $\mathcal{A}$ is the $\mathcal{V}$-category of models of a limit-colimit sketch

cf. Guitart–Lair notion of "locally free diagram"

cf. Adamek–Rosicky notion of cone-injectivity

Accessible $\mathcal{V}$-categories.

Thm: For a $\mathcal{V}$-category $\mathcal{A}$, t.f.a.e.

(i) $\mathcal{A}$ is accessible ~~with limits~~

(ii) $\mathcal{A}$ is accessible with virtual colimits $\Leftrightarrow$ $P^{+}\mathcal{A}$ has colimits

(iii) $\mathcal{A}$ is accessibly embedded & virtually reflective in some $[\mathcal{C}, \mathcal{V}]$

(iv) $\mathcal{A}$ is a~~n~~ virtual orthogonality class in some $[\mathcal{C}, \mathcal{V}]$

(v) $\mathcal{A}$ is the $\mathcal{V}$-category of models of a limit-colimit sketch

cf. Guitart–Lair notion of
"locally free diagram"

cf. Adámek–Rosický notion of cone-injectivity

Locally multipresentable categories (Diers)

Thm: For a category $\mathcal{A}$, t.f.a.e.

(i) $\mathcal{A}$ is accessible with connected limits

(ii) $\mathcal{A}$ is accessible with multi colimits $\Leftrightarrow$ $\mathrm{Fam}^{+}(\mathcal{A})$ has colimits

(iii) $\mathcal{A}$ is accessibly embedded multi reflective in some $[\mathcal{C}, \mathcal{V}]$

(iv) $\mathcal{A}$ is ~~an~~ a multi orthogonality class in some $[\mathcal{C}, \mathcal{V}]$

(v) $\mathcal{A}$ is the category of models of a limit coproduct sketch

Accessible $\mathcal{V}$-categories with $\mathcal{I}$-limits

for suitable $\mathcal{A} \hookrightarrow \check{\mathcal{I}}(\mathcal{A})$

<u>Thm</u> (Schema) For a $\mathcal{V}$-category $\mathcal{A}$, t.f.a.e.

(i) $\mathcal{A}$ is accessible with $\mathcal{I}$-limits

(ii) $\mathcal{A}$ is accessible and $\check{\mathcal{I}}(\mathcal{A})$ has colimits

(iii) $\mathcal{A}$ is accessibly embedded in some $[\mathcal{C}, \mathcal{V}]$ and reflective relative to $\mathcal{A} \hookrightarrow \check{\mathcal{I}}(\mathcal{A})$.

$$\begin{tikzcd}
 & \check{\mathcal{I}}(\mathcal{A}) & \\
\mathcal{A} \arrow[ur, hook] \arrow[rr, hook] & & {[\mathcal{C}, \mathcal{V}]} \arrow[ul, dotted] \arrow[ull, phantom, "\Leftarrow"]
\end{tikzcd}$$

Accessible $\mathcal{V}$-categories with $\mathbb{I}$-limits

$\Phi, \mathbb{I}$ classes of weights

$\Phi(\mathbb{I}(\mathcal{C})) = \mathcal{P}(\mathcal{C}) \quad \forall \mathcal{C}.$

Thm: For a $\mathcal{V}$-category $\mathcal{A}$, t.f.a.e.

(i) $\mathcal{A}$ is accessible with $\mathbb{I}$-limits

(ii) $\mathcal{A}$ is accessible with Φ-virtual colimits $\Leftrightarrow$ $\Phi^{\dagger}(\mathcal{A})$ has colimits

(iii) $\mathcal{A}$ is accessibly embedded & Φ-virtually reflective in some $[\mathcal{C}, \mathcal{V}]$

(iv) $\mathcal{A}$ is ~~an~~ Φ-virtual orthogonality class in some $[\mathcal{C}, \mathcal{V}]$

(v) $\mathcal{A}$ is the $\mathcal{V}$-category of models of a limit Φ-colimit sketch

			$\mathcal{V} = \underline{Set}$
$\mathbb{I}$	all	none	connected
Φ	none	all	discrete
	loc. pres.	acc.	loc. multipres.

Locally polypresentable categories

Lamarche
Taylor
Hu–Tholen
Ageron

Thm: For a category $\mathcal{A}$, t.f.a.e.

(i) $\mathcal{A}$ is accessible with wide pullbacks

(ii) $\mathcal{A}$ is accessible with poly colimits $\Leftrightarrow$ $\mathcal{G}\mathcal{A}$ has colimits

(iii) $\mathcal{A}$ is accessibly embedded & poly reflective in some $[\mathcal{C}, \mathcal{V}]$

refl. relative $\mathcal{A} \longrightarrow \mathcal{G}\mathcal{A}$

(v) $\mathcal{A}$ is the category of models of a ~~fini~~ Galoisian sketch

Accessible 2-categories with PIE-limits

products, inserters, equifiers.

<u>Thm</u>: For a 2-category $\mathcal{A}$, t.f.a.e.

(i) $\mathcal{A}$ is accessible with PIE-limits

(ii) $\mathcal{A}$ is accessible and $S\mathcal{A}$ has colimits

(iii) $\mathcal{A}$ is accessibly embedded & reflective in some $[\mathcal{C}, \mathcal{V}]$ relative to $\mathcal{A} \to S\mathcal{A}$.

(v) $\mathcal{A}$ is the $\mathcal{V}$-category of models of a limit/PIE sketch

Weakly locally presentable categories

Adamek-Rosicky

Thm: For a category $\mathcal{A}$, t.f.a.e.

(i) $\mathcal{A}$ is accessible with products

(ii) $\mathcal{A}$ is accessible with weak colimits

(iii) $\mathcal{A}$ is accessibly embedded & weakly reflective in some $[\mathcal{C}, \mathbf{Cat}]$

(iv) —

(v) $\mathcal{A}$ is the category of models of a limit/epi sketch

Accessible 2-categories with flexible limits

<u>Thm</u>: For a 2-category $\mathcal{A}$, t.f.a.e.

(i) $\mathcal{A}$ is accessible with flexible limits

(ii) $\mathcal{A}$ is accessible with Σ-weak colimits

(iii) $\mathcal{A}$ is accessibly embedded & Σ-weakly reflective in some $[\mathcal{C}, \mathbf{Cat}]$

(iv) —

(v) $\mathcal{A}$ is the 2-category of models of a limit/RE sketch based on the papers

- Virtual concepts in the theory of accessible categories
- Accessible categories with a class of limits

both published in the JPAA.