Monadicity of strict ω-categories

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This note is about monadicity of ω**Cat** over presheaf subcategories of the category **Poly** of polygraphs.

Introduction

It is well known that the category of strict ω-categories ω**Cat** is monadic over globular and cellular sets. Both globular and cellular sets are presheaf subcategories of the category of all polygraphs **Poly**. This note is to announce that the category ω**Cat** is also monadic over the category of positive opetopic sets **pOpeSet** and the resulting monad is strongly Cartesian. The category **Poly** is not a presheaf category and it seems that ω**Cat** might fail to be monadic over **Poly** (both existing proofs seem to contain gaps [5],[4]).

The statement of the main results

The category of polygraphs **Poly** has as objects the levelwise free omega-categories (i.e., free ω-categories whose generators at level $n+1$ with their domains and codomains are added only after all cells at level n have been generated). Morphisms of polygraphs are ω-functors that send generators to generators.

Up to equivalence, the category of positive opetopic sets **pOpeSet** can be described as a full subcategory of **Poly** whose objects are polygraphs whose generators have generators as codomains and whose domains are non-identity cells. Positive opetopic sets form a presheaf category [7].

The category of positive opetopes **pOpe** is the exponent category of **pOpeSet**, and hence it has an embedding into the category of strict ω-categories ω**Cat**:

$$\mathbf{pOpe} \to \mathbf{pOpeSet} \to \omega\mathbf{Cat}$$

Even if this embedding is not full, it is full on isomorphisms. Thus we can consider various classes of ω-functors as morphisms between positive opetopes. For example, the ι-maps are omega-functors between opetopes that send generators to generators or identities on generators of lower dimension. It can be shown that the category $\mathbf{pOpe}_\iota$ of opetopes with ι-maps is a test category [9]. The category $\mathbf{pOpe}_{\iota,epi}$, i.e., the category of opetopes with ι-epis, is dually equivalent to the category of positive zoom complexes with embeddings [3], [8].

Theorem 1[7]
The embedding functor

$$i : \mathbf{pOpeSet} \to \omega\mathbf{Cat}$$

has a right adjoint $U_\omega : \omega\mathbf{Cat} \to \mathbf{pOpeSet}$ and the resulting monad $T_\omega = U_\omega \circ i$ is strongly Cartesian. In particular, T_ω is a famillialy representable, i.e., the functor part of T_ω is given as a coproduct of representable functors (so called positive opetopic cardinals).

Theorem 2[7]
The monad T_ω decomposes into two strongly Cartesian monads T_ι and T_c (generating identities and compositions), and a Cartesian distributive law combining them into the monad T_ω.

S. Henry in [2] gave a very general characterization of subcategories of Poly that are presheaf categories. Thus we can raise the following question.

Problem
What is the characterization of the presheaf subcategories $\mathcal{X}$ of the category of polygraphs **Poly** such that the embedding functor

$$\mathcal{X} \hookrightarrow \omega\mathbf{Cat}$$

has a right adoint which is monadic? When the resulting monad is strongly Cartesian?

Even if the definition of the category **pOpeSet** as a subcategory of the category of polygraphs **Poly** is fairly short and abstract, it is not very easy to work with it directly. In [7], definitions of **pOpe** and **pOpeSet** are given. They have a combinatorial flavour. These definition were used to prove the above facts. However, there are many other definitions of these categories. Some references to them can be found in the references below.

References

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