Flatness, weakly-lex colimits and free exact completions

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7 July 2023

Exact completion of lex categories

- Exact categories where introduced by Barr as an “ordinary” counterpart of abelian categories.
- Later free exact completions have been introduced:

Definition (Carboni–Magno)

Let $\mathcal{C}$ be lex. The free (Barr-)exact completion of $\mathcal{C}$ is an exact category $\mathcal{C}_{ex}$ t.w. $K\colon \mathcal{C} \hookrightarrow \mathcal{C}_{ex}$ for which Lan_K induces an equivalence:

$$\mathrm{Lex}(\mathcal{C}, \mathcal{E}) \simeq \mathrm{Ex}(\mathcal{C}_{ex}, \mathcal{E})$$

for any exact $\mathcal{E}$.

- $\mathcal{C}_{ex}$ is obtained by freely adding coequalizers of pseudo-equivalence relations to $\mathcal{C}$.
- $\mathcal{C} \hookrightarrow \mathcal{C}_{ex} \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ closure under finite limits and coequalizers of equivalence relations.

Φ-exact completion of lex categories

Garner–Lack introduce a general notion of Φ-exactness:

$\Phi =$ "class of colimits to which we impose exactness conditions".

Definition (Garner–Lack)

Let $\mathcal{C}$ be lex. The free Φ-exact completion of $\mathcal{C}$ is a Φ-exact category $\Phi_l\mathcal{C}$ t.w. $K\colon \mathcal{C} \hookrightarrow \Phi_l\mathcal{C}$ for which Lan_K induces an equivalence:

$$\mathrm{Lex}(\mathcal{C}, \mathcal{E}) \simeq \Phi\text{-}\mathrm{Ex}(\Phi_l\mathcal{C}, \mathcal{E})$$

for any Φ-exact $\mathcal{E}$.

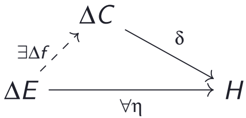
Note: $\mathcal{C} \hookrightarrow \Phi_l\mathcal{C} \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is the closure under finite limits and Φ-lex colimits.

Examples: Regular and Barr-exact categories, (infinitary) lextensive categories, pretopoi, etc.

Problem: does not capture all kinds of free exact completions.

Exact completion of weakly-lex categories

- A diagram $H\colon \mathcal{D} \to \mathcal{C}$ has a weak limit in $\mathcal{C}$ if there is C t.w. $\delta\colon \Delta C \to H$ such that



- If $\mathcal{C}$ has weak finite limits, then $\mathcal{C}_{ex}$, obtained by freely adding coequalizers of pseudo-equivalence relations, is exact. (Carboni–Vitale)

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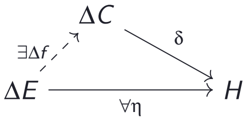

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Theorem (Carboni–Vitale)

Let $\mathcal{C}$ be weakly lex and $K\colon \mathcal{C} \hookrightarrow \mathcal{C}_{ex}$ be the inclusion. Then Lan_K induces an equivalence:

$$\mathrm{Lco}(\mathcal{C}, \mathcal{E}) \simeq \mathrm{Ex}(\mathcal{C}_{ex}, \mathcal{E})$$

for any exact $\mathcal{E}$.

But what is on the left-hand-side?

Left covering functors

Let $F\colon \mathcal{C} \to \mathcal{E}$ be a functor from a weakly lex category $\mathcal{C}$ to a regular category $\mathcal{E}$.

Definition (Carboni–Vitale/Hu)

We say that F is left covering if for any finite diagram $H\colon \mathcal{D} \to \mathcal{C}$ and any weak limit $C \in \mathcal{C}$ of H, the comparison map

$$FC \twoheadrightarrow \lim(FH)$$

is a regular epimorphism.

- if $\mathcal{C}$ is lex, then: left covering $=$ lex;
- if $\mathcal{E} = \mathbf{Set}$, then: left covering $=$ flat;

Questions:

- for general $\mathcal{C}$ and $\mathcal{E}$ do we have a "more formal" description?
- can we capture these in the context of Φ-lex colimits?

A notion of flatness

The following are equivalent for $F\colon \mathcal{C} \to \mathbf{Set}$:

1. F is flat (i.e. $\mathrm{El}(F)$ is filtered);
2. $\mathrm{Lan}_Y F\colon [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Set}$ is lex;
3. $\mathrm{Lan}_Y F\colon [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Set}$ preserves finite limits of representables.

- Replace **Set** with any lex $\mathcal{E}$;

Definition

A functor $F\colon \mathcal{C} \to \mathcal{E}$, into a lex category $\mathcal{E}$, is flat if and only if for any finite diagram $H\colon \mathcal{D} \to \mathcal{C}$, we have

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$$\mathrm{Lan}_Y F\,(\lim YH) \cong \lim\,(\mathrm{Lan}_Y F \circ YH)\,.$$

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Flatness and free Φ-exact completions

Some properties:

- if $\mathcal{E} = \mathbf{Set}$, then: flat = flat;
- if $\mathcal{C}$ is lex, then: flat = lex;
- if $\mathcal{E}$ is a Grothendieck topos, then: F is flat iff $\mathrm{Lan}_Y F \colon [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{E}$ is lex;

Back to Φ-lex colimits. Given a small $\mathcal{C}$, consider $\Phi_l\mathcal{C}$ to be the closure of $\mathcal{C}$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ under finite limits and Φ-lex colimits.

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Back to Φ-lex colimits. Given a small $\mathcal{C}$, consider $\Phi_l\mathcal{C}$ to be the closure of $\mathcal{C}$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ under finite limits and Φ-lex colimits.

Definition

The inclusion $K\colon \mathcal{C} \hookrightarrow \Phi_l\mathcal{C}$ exhibits $\Phi_l\mathcal{C}$ as the free Φ-exact completion of $\mathcal{C}$ if left Kan extending along K induces an equivalence

$$\mathrm{Flat}(\mathcal{C}, \mathcal{E}) \simeq \Phi\text{-}\mathrm{Ex}(\Phi_l\mathcal{C}, \mathcal{E})$$

for any Φ-exact category $\mathcal{E}$.

The main theorem

Given $\mathcal{C}$, define

$$\mathcal{C} \subseteq \Phi^{\diamond}[\mathcal{C}] \subseteq [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$$

by adding those M for which *M-weakly-lex colimits* exist in every Φ-exact $\mathcal{E}$.

⋆ In the exact case, objects of $\Phi^{\diamond}[\mathcal{C}]$ are coequalizers of pseudo equavelence relations between representables.

Theorem

The following are equivalent for a small category $\mathcal{C}$:

1. $K \colon \mathcal{C} \hookrightarrow \Phi_l\mathcal{C}$ *exhibits* $\Phi_l\mathcal{C}$ *as the free* Φ*-exact completion of* $\mathcal{C}$*;*
2. $\Phi^{\diamond}[\mathcal{C}] = \Phi_l\mathcal{C}$*;*
3. $\Phi^{\diamond}[\mathcal{C}]$ *has finite limits of diagrams landing in* $\mathcal{C}$*.*

⋆ In the exact case, finite limits in $\Phi^{\diamond}[\mathcal{C}]$ of diagrams landing in $\mathcal{C}$ are weak limits.

Examples

1. Φ_{reg} and Φ_{ex} for regular and exact categories;

The following are equivalent for a small Cauchy complete category $\mathcal{C}$:

- $\mathcal{C}$ has a free regular completion;
- $\mathcal{C}$ has a free exact completion;
- $\mathcal{C}$ is weakly lex.

For such a $\mathcal{C}$, a functor $F\colon \mathcal{C} \to \mathcal{E}$ into a regular category $\mathcal{E}$ is flat if and only if it is left covering: for any finite diagram $H\colon \mathcal{D} \to \mathcal{C}$ and any weak limit $C \in \mathcal{C}$ of H, the comparison map

$$FC \twoheadrightarrow \lim(FH)$$

is a regular epimorphism.

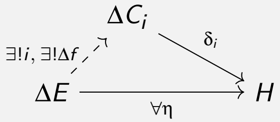
Examples

1. Φ_{reg} and Φ_{ex} for regular and exact categories;
2. Φ_{ilext} for infinitary lextensive categories;

The following are equivalent for a small Cauchy complete category $\mathcal{C}$:

- $\mathcal{C}$ has a free infinitary lextensive completion;
- $\mathcal{C}$ has finite multilimits.

$H\colon \mathcal{D} \to \mathcal{C}$ has a multilimit in $\mathcal{C}$ if there exists a family of objects $(C_i)_{i\in I}$ in $\mathcal{C}$ together with cones $\delta_i\colon \Delta C_i \to H$ for which:



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The following are equivalent for a small Cauchy complete category $\mathcal{C}$:

- $\mathcal{C}$ has a free infinitary lextensive completion;
- $\mathcal{C}$ has finite multilimits.

For such a $\mathcal{C}$, a functor $F\colon \mathcal{C} \to \mathcal{E}$ into an infinitary lextensive category $\mathcal{E}$ is flat if and only if it is finitely multicontinuous: for any finite diagram $H\colon \mathcal{D} \to \mathcal{C}$ with multilimit $(C_i)_{i\in I}$ the comparison

$$\textstyle\sum_{i\in I} FC_i \xrightarrow{\cong} \lim FH$$

is an isomorphism.

Examples

1. Φ_{reg} and Φ_{ex} for regular and exact categories;
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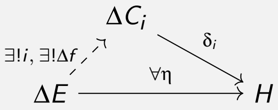
The following are equivalent for a small Cauchy complete category $\mathcal{C}$:

- $\mathcal{C}$ has a free lextensive completion;
- $\mathcal{C}$ has finite multi-finite limits.

$H\colon \mathcal{D} \to \mathcal{C}$ has a multi-finite limit in $\mathcal{C}$ if there is a finite family of objects $(C_i)_{i \leq n}$ in $\mathcal{C}$ together with cones $\delta_i\colon \Delta C_i \to H$ for which:



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The following are equivalent for a small Cauchy complete category $\mathcal{C}$:

- $\mathcal{C}$ has a free lextensive completion;
- $\mathcal{C}$ has finite multi-finite limits.

For such a $\mathcal{C}$, a functor $F\colon \mathcal{C} \to \mathcal{E}$ into a lextensive category $\mathcal{E}$ is flat if and only if for any finite diagram $H\colon \mathcal{D} \to \mathcal{C}$ with multi-finite limit $(C_i)_{i \leq n}$ the comparison

$$\textstyle\sum_{i\leq n} FC_i \xrightarrow{\cong} \lim FH.$$

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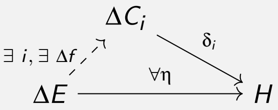
4. $\Phi_{\mathrm{pret}} = \Phi_{\mathrm{ex}} \cup \Phi_{\mathrm{lext}}$ for pretopoi;

The following are equivalent for a small Cauchy complete category $\mathcal{C}$:

- $\mathcal{C}$ has a free pretopos completion;
- $\mathcal{C}$ has finite fc-limits.

$H\colon \mathcal{D} \to \mathcal{C}$ has a fc-limit in $\mathcal{C}$ if there is a finite family of objects $(C_i)_{i \leq n}$ in $\mathcal{C}$ together with cones $\delta_i\colon \Delta C_i \to H$ for which:



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For such a $\mathcal{C}$, a functor $F\colon \mathcal{C} \to \mathcal{E}$ into a pretopos $\mathcal{E}$ is flat if and only if for any fc-limit $(C_i)_{i\leq n}$ of a finite diagram H in $\mathcal{C}$, the comparison

$$\textstyle\sum_{i\leq n} FC_i \twoheadrightarrow \lim(FH)$$

is a regular epimorphism.

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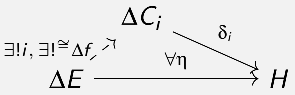
5. Φ_{gp} of free groupoid actions, for quasi-based categories;

The following are equivalent for a small Cauchy complete category $\mathcal{C}$:

- $\mathcal{C}$ has a free Φ-exact completion;
- $\mathcal{C}$ has finite polylimits.

polylimits = multilimits but the factorization is unique up to unique automorphism.



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- $\mathcal{C}$ has finite polylimits.

For such a $\mathcal{C}$, a functor $F\colon \mathcal{C} \to \mathcal{E}$ into a lextensive category $\mathcal{E}$ is flat if and only if it is finitely polycontinuous.

Examples

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4. $\Phi_{\text{pret}} = \Phi_{\text{ex}} \cup \Phi_{\text{lext}}$ for pretopoi;
5. Φ_{gp} of free groupoid actions, for quasi-based categories;
6. $\Phi_{\mathbb{D}}$ = $\mathbb{D}$-filtered diagrams, for a sound class $\mathbb{D}$.

The following are equivalent for a small Cauchy complete category $\mathcal{C}$:

- $\mathcal{C}$ has a free Φ-exact completion;
- $\text{Ind}_{\mathbb{D}}(\mathcal{C})$ has finite limits of diagrams in $\mathcal{C}$.

Thank You